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A Note on Inextensible Flows of Curves on Oriented Surface

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ABSTRACT

In this paper, we investigate a general formulation for inextensible flows of curves on an oriented surface in $\mathbb{R}^3$. We obtain necessary and sufficient conditions as partial differential equations involving the geodesic curvature and the geodesic torsion for inextensible curve flow lying on an oriented surface. Moreover, some special cases of inextensible curves on oriented surface are given.

RESUMEN

En este artículo investigamos una formulación general para flujos inextensibles de curvas sobre una superficie orientable en $\mathbb{R}^3$. Obtenemos condiciones necesarias y suficientes para las ecuaciones diferenciales parciales que involucran la curva geodésica y la torsión geodésica para curvas inextensibles fluyendo sobre superficies orientadas. Más aún, se entregan algunos casos especiales de curvas inextensibles sobre superficies orientadas.

Keywords and Phrases: Curvature flows, inextensible, oriented surface.

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1 Introduction

The flow of the curve is said to be inextensible if its arclength preserved. Curve design using splines is one of the most fundamental topic in CAGD. Inextensible flows of the curves have beautiful shapes preserving connection to their control polygon. On the other hand, physically inextensible curve and surface flows give rise to motion which no strain energy is induced. For example, the swinging motion of a cord of fixed length can be described by inextensible curve and surface flows. Many authors have studied geometric flow problems and applications of inextensible curve flows, [1]–[10]. An evolution equation for inelastic planar curves was derived by [9] and also, the general formulation of inextensible flows of curves and developable surfaces in $\mathbb{R}^3$ was exposed by [10].

In this paper, we derive a general formulation for inextensible flows of curves according to Darboux frame in $\mathbb{R}^3$. We give the necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equations involving the geodesic curvature and geodesic torsion.

2 Preliminaries

Let S be an oriented surface in three-dimensional Euclidean space E^3 and $\alpha(s)$ be a curve lying on the surface S. Suppose that the curve $\alpha(s)$ is spatial then there exists the Frenet frame $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$ at each points of the curve where $\overrightarrow{T}$ is unit tangent vector, $\overrightarrow{N}$ is principal normal vector and $\overrightarrow{B}$ is binormal vector, respectively. The Frenet equation of the curve $\alpha(s)$ is given by

$$\begin{aligned}
\overrightarrow{T}' &= \kappa\overrightarrow{N} \\
\overrightarrow{N}' &= -\kappa\overrightarrow{T} + \tau\overrightarrow{B} \\
\overrightarrow{B}' &= -\tau\overrightarrow{N}
\end{aligned}$$

where κ and τ are curvature and torsion of the curve $\alpha(s)$, respectively.

Since the curve $\alpha(s)$ lies on the surface S there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\left\{\overrightarrow{T}, \overrightarrow{g}, \overrightarrow{n}\right\}$. In this frame $\overrightarrow{T}$ is the unit tangent of the curve, $\overrightarrow{n}$ is the unit normal of the surface S and $\overrightarrow{g}$ is a unit vector given by $\overrightarrow{g} = \overrightarrow{n} \times \overrightarrow{T}$. Since the unit tangent $\overrightarrow{T}$ is common element of both Frenet frame and Darboux frame, the vectors $\overrightarrow{N}$, $\overrightarrow{B}$, $\overrightarrow{g}$ and $\overrightarrow{n}$ lie on the same plane. So that the relations between these frames can be given as follows

$$\begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{g} \\ \overrightarrow{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}$$

where φ is the angle between the vectors $\overrightarrow{g}$ and $\overrightarrow{N}$. The derivative formulae of the Darboux frame is

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{g}} \\ \dot{\vec{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix}$$

where k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsions, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

The relations between the geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows, [11]

$$k_g = \kappa \cos \varphi\,, \quad k_n = -\kappa \sin \varphi\,, \quad \tau_g = \tau + \tfrac{d\varphi}{ds}.$$

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of the curve $\alpha(s)$ can be calculated as follows, [11]

$$\begin{aligned} k_g &= \left\langle \tfrac{d\vec{\alpha}}{ds}, \tfrac{d^2\vec{\alpha}}{ds^2} \times \vec{n} \right\rangle \\ \tau_g &= \left\langle \tfrac{d\vec{\alpha}}{ds}, \vec{n} \times \tfrac{d\vec{n}}{ds} \right\rangle. \end{aligned}$$

In the differential geometry of surfaces, for any curve $\alpha(s)$ lying on a surface S the following relationships are well-known, [11]

i- $\alpha(s)$ is a geodesic curve if and only if $k_g = 0$,

ii- $\alpha(s)$ is an asymptotic line if and only if $k_n = 0$,

iii- $\alpha(s)$ is a principal line if and only if $\tau_g = 0$.

Through each point on a surface there passes, in general, a geodesic in every direction. A geodesic is uniquely determined by an initial point and tangent at that point. All straight lines on a surface are geodesics.

Along all curved geodesics the principal normal coincides with the surface normal. Along asymptotic lines osculating planes and tangent planes coincide, along geodesics they are normal. Through a point of a non-developable surface there are two asymptotic lines which can be real or imaginary.

3 Inextensible Flows of Curve Lying on Oriented Surface

Throughout this paper, we suppose that

$$\alpha\ : [0, l] \times [0, w) \to M \subset E^3$$

is a one parameter family of differentiable curves on orientable surface M in E^3, where l is the arclength of the initial curve. Let u be the curve parameterization variable, $0 \le u \le l$. If the speed of curve α is denoted by $v = \left\| \frac{\partial \vec{\alpha}}{\partial u} \right\|$ then the arclength of α is

$$S(u) = \int_0^u \left\| \frac{\partial \overrightarrow{\alpha}}{\partial u} \right\| du = \int_0^u v \, du. \tag{3.1}$$

The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}. \tag{3.2}$$

Thus, the arclength is $ds = v \, du$.

Definition 3.1. *Let M be an orientable surface and α be a differentiable curve on M in E^3. Any flow of the curve α can be expressed with respect to Darboux frame $\left\{ \overrightarrow{T}, \overrightarrow{g}, \overrightarrow{n} \right\}$ in the following form:*

$$\frac{\partial \overrightarrow{\alpha}}{\partial t} = f_1 \overrightarrow{T} + f_2 \overrightarrow{g} + f_3 \overrightarrow{n}. \tag{3.3}$$

Here, f_1, f_2 and f_3 are scalar speeds of the curve α. Let the arclength variation be

$$S(u,t) = \int_0^u v \, du. \tag{3.4}$$

In the Euclidean space the requirement that a curve not to be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} S(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0 \;, \quad u \in [0,1]. \tag{3.5}$$

Definition 3.2. *A curve evolution $\alpha(u,t)$ and its flow $\frac{\partial \overrightarrow{\alpha}}{\partial t}$ on the oriented surface M in E^3 are said to be inextensible if*

$$\frac{\partial}{\partial t} \left\| \frac{\partial \overrightarrow{\alpha}}{\partial u} \right\| = 0.$$

Now, we research the necessary and sufficient condition of a flow to be inextensible. For this reason, we need to the following Lemma.

Lemma 3.1. *In E^3, let M be an orientable surface and $\left\{ \overrightarrow{T}, \overrightarrow{g}, \overrightarrow{n} \right\}$ be a Darboux frame of α on M. There exists following relation between the scalar speed functions f_1, f_2, f_3 and the normal curvature k_n, geodesic curvature k_g of α the curve*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_g - f_3 v k_n. \tag{3.6}$$

Proof. Since $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute and $v^2 = \left\langle \frac{\partial \overrightarrow{\alpha}}{\partial u}, \frac{\partial \overrightarrow{\alpha}}{\partial u} \right\rangle$, we have

$$\begin{aligned} 2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \overrightarrow{\alpha}}{\partial u}, \frac{\partial \overrightarrow{\alpha}}{\partial u} \right\rangle \\ &= 2\left\langle \frac{\partial \overrightarrow{\alpha}}{\partial u}, \frac{\partial}{\partial u}\left(f_1 \overrightarrow{T} + f_2 \overrightarrow{g} + f_3 \overrightarrow{n} \right) \right\rangle \\ &= 2v\left(\frac{\partial f_1}{\partial u} - f_2 v k_g - f_3 v k_n \right). \end{aligned}$$

This completes the proof. □

If we consider the conditions of being geodesic and asymptotic of a curve and Lemma 3.1, we can give the following corollary.

Corollary 3.1. *If a curve is a geodesic curve or an asymptotic curve, then there are the following equations*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_3 v k_n$$

or

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_g,$$

respectively.

Theorem 3.1. *Let* $\left\{ \overrightarrow{T}, \overrightarrow{g}, \overrightarrow{n} \right\}$ *be the Darboux frame of a curve* α *on* M *and* $\frac{\partial \overrightarrow{\alpha}}{\partial t} = f_1 \overrightarrow{T} + f_2 \overrightarrow{g} + f_3 \overrightarrow{n}$ *be a differentiable flow of* α *in* $\mathbb{R}^3$. *Then the flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = f_2 k_g + f_3 k_n. \tag{3.7}$$

Proof. Suppose that the curve flow is inextensible. From the equations (3.4) and (3.6) for $u \in [0, l]$ we see that

$$\frac{\partial}{\partial t} S(u,t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left(\frac{\partial f_1}{\partial u} - f_2 v k_g - f_3 v k_n \right) du = 0. \tag{3.8}$$

Thus, it can be seen that

$$\frac{\partial f_1}{\partial u} = f_2 v k_g + f_3 v k_n. \tag{3.9}$$

Considering the last equation and (3.2), we reach

$$\frac{\partial f_1}{\partial s} = f_2 k_g + f_3 k_n.$$

Conversely, by following a similar way as above, the proof is completed. □

From Theorem 3.1, we have following corollary.

Corollary 3.2. *i- Let the curve α is a geodesic curve on M. Then the curve flow is inextensible if and only if $\frac{\partial f_1}{\partial s} = f_3 k_n$.*

ii- Let the curve α is an asymptotic line on M. Then the curve flow is inextensible if and only if $\frac{\partial f_1}{\partial s} = f_2 k_g$.

Now, we restrict ourselves to the arclength parameterized curves. That is, $\nu = 1$ and the local coordinate u corresponds to the curve arclength s. We require the following Lemma.

Lemma 3.2. *Let M be an orientable surface in E^3 and $\left\{\overrightarrow{T}, \overrightarrow{g}, \overrightarrow{n}\right\}$ be a Darboux frame of the curve α on M. Then, the differentiations of $\left\{\overrightarrow{T}, \overrightarrow{g}, \overrightarrow{n}\right\}$ with respect to t is*

$$\begin{aligned}
\frac{\partial \overrightarrow{T}}{\partial t} &= \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \overrightarrow{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \overrightarrow{n} \\
\frac{\partial \overrightarrow{g}}{\partial t} &= -\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \overrightarrow{T} + \psi \overrightarrow{n} \\
\frac{\partial \overrightarrow{n}}{\partial t} &= -\left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \overrightarrow{T} - \psi \overrightarrow{g}
\end{aligned}$$

where $\psi = \left\langle \frac{\partial \overrightarrow{g}}{\partial t}, \overrightarrow{n} \right\rangle$.

Proof. Since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are commutative, it seen that

$$\begin{aligned}
\frac{\partial \overrightarrow{T}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial \overrightarrow{\alpha}}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial \overrightarrow{\alpha}}{\partial t}\right) = \frac{\partial}{\partial s}\left(f_1 \overrightarrow{T} + f_2 \overrightarrow{g} + f_3 \overrightarrow{n}\right) \\
&= \frac{\partial f_1}{\partial s}\overrightarrow{T} + f_1 \frac{\partial \overrightarrow{T}}{\partial s} + \frac{\partial f_2}{\partial s}\overrightarrow{g} + f_2 \frac{\partial \overrightarrow{g}}{\partial s} + \frac{\partial f_3}{\partial s}\overrightarrow{n} + f_3 \frac{\partial \overrightarrow{n}}{\partial s}.
\end{aligned}$$

Substituting the equation (3.7) into the last equation and using Theorem 3.1, we have

$$\frac{\partial \overrightarrow{T}}{\partial t} = \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \overrightarrow{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \overrightarrow{n}.$$

Now, let us differentiate the Darboux frame with respect to t as follows;

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\left\langle \overrightarrow{T}, \overrightarrow{g} \right\rangle = \left\langle \frac{\partial \overrightarrow{T}}{\partial t}, \overrightarrow{g} \right\rangle + \left\langle \overrightarrow{T}, \frac{\partial \overrightarrow{g}}{\partial t} \right\rangle \\
&= \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) + \left\langle \overrightarrow{T}, \frac{\partial \overrightarrow{g}}{\partial t} \right\rangle \qquad (3.10) \\
0 &= \frac{\partial}{\partial t}\left\langle \overrightarrow{T}, \overrightarrow{n} \right\rangle = \left\langle \frac{\partial \overrightarrow{T}}{\partial t}, \overrightarrow{n} \right\rangle + \left\langle \overrightarrow{T}, \frac{\partial \overrightarrow{n}}{\partial t} \right\rangle \\
&= \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) + \left\langle \overrightarrow{T}, \frac{\partial \overrightarrow{n}}{\partial t} \right\rangle \qquad (3.11)
\end{aligned}$$

From (3.10) and (3.11), we have obtain

$$\frac{\partial \overrightarrow{g}}{\partial t} = -\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \overrightarrow{T} + \psi \overrightarrow{n}$$

and

$$\frac{\partial \vec{n}}{\partial t} = -\left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{T} - \psi \vec{g}$$

respectively, where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$. □

If we take into consideration last Lemma, we have following corollary.

Corollary 3.3. *Let M be an orientable surface in E^3.*

i- If the curve α is a geodesic curve, then

$$\begin{aligned}
\frac{\partial \vec{T}}{\partial t} &= \left(\frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{n}, \\
\frac{\partial \vec{g}}{\partial t} &= -\left(\frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{T} + \psi \vec{n}, \\
\frac{\partial \vec{n}}{\partial t} &= -\left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{T} - \psi \vec{g},
\end{aligned}$$

where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$.

ii- If the curve α is an asymptotic line, then

$$\begin{aligned}
\frac{\partial \vec{T}}{\partial t} &= \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{g} + \left(\frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{n}, \\
\frac{\partial \vec{g}}{\partial t} &= -\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{T} + \psi \vec{n}, \\
\frac{\partial \vec{n}}{\partial t} &= -\left(\frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{T} - \psi \vec{g},
\end{aligned}$$

where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$.

iii- If the curve is a curvature line, then

$$\begin{aligned}
\frac{\partial \vec{T}}{\partial t} &= \left(f_1 k_g + \frac{\partial f_2}{\partial s}\right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s}\right) \vec{n}, \\
\frac{\partial \vec{g}}{\partial t} &= -\left(f_1 k_g + \frac{\partial f_2}{\partial s}\right) \vec{T} + \psi \vec{n}, \\
\frac{\partial \vec{n}}{\partial t} &= -\left(f_1 k_n + \frac{\partial f_3}{\partial s}\right) \vec{T} - \psi \vec{g},
\end{aligned}$$

where $\psi = \left\langle \frac{\partial \vec{g}}{\partial t}, \vec{n} \right\rangle$.

Theorem 3.2. *Suppose that the curve flow $\frac{\partial \vec{\alpha}}{\partial t} = f_1 \vec{T} + f_2 \vec{g} + f_3 \vec{n}$ is inextensible on the orientable surface on M. In this case, the following partial differential equations are held:*

$$\begin{aligned}
\frac{\partial k_g}{\partial t} &= f_2 k_g^2 + f_3 k_g k_n + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2\frac{\partial f_3}{\partial s}\tau_g - f_3 \frac{\partial \tau_g}{\partial s} - f_1 k_n \tau_g - f_2 \tau_g^2 + \psi k_n, \\
\frac{\partial k_n}{\partial t} &= f_2 k_g k_n + f_3 k_n^2 + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + 2\frac{\partial f_2}{\partial s}\tau_g + f_2 \frac{\partial \tau_g}{\partial s} + f_1 k_g \tau_g - f_3 \tau_g^2 - \psi k_g, \\
\frac{\partial \tau_g}{\partial t} &= f_2 k_g \tau_g - \frac{\partial f_2}{\partial s} k_n + \frac{\partial f_3}{\partial s} k_g + f_3 k_n \tau_g + \frac{\partial \psi}{\partial s}\ .
\end{aligned}$$

Proof. Since $\frac{\partial}{\partial s}\frac{\partial \vec{T}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \vec{T}}{\partial s}$ we get

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \vec{T}}{\partial t} &= \frac{\partial}{\partial s}\left[\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \vec{g} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \vec{n}\right] \\
&= \left(\frac{\partial f_1}{\partial s} k_g + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s}\tau_g - f_3 \frac{\partial \tau_g}{\partial s}\right) \vec{g} + \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3 \tau_g\right) \frac{\partial \vec{g}}{\partial s} \\
&\quad + \left(\frac{\partial f_1}{\partial s} k_n + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s}\tau_g + f_2 \frac{\partial \tau_g}{\partial s}\right) \vec{n} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2 \tau_g\right) \frac{\partial \vec{n}}{\partial s}
\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \overrightarrow{T}}{\partial t} = &\left(\frac{\partial f_1}{\partial s}k_g + f_1\frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s}\tau_g - f_3\frac{\partial \tau_g}{\partial s}\right)\overrightarrow{g} + \left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3\tau_g\right)\left(-k_g\overrightarrow{T} + \tau_g\overrightarrow{n}\right)\\ &+\left(\frac{\partial f_1}{\partial s}k_n + f_1\frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial f_2}{\partial s}\tau_g + f_2\frac{\partial \tau_g}{\partial s}\right)\overrightarrow{n} + \left(f_1 k_n + \frac{\partial f_3}{\partial s} + f_2\tau_g\right)\left(-k_g\overrightarrow{T} - \tau_g\overrightarrow{g}\right)\end{aligned}$$

while

$$\frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s} = \frac{\partial}{\partial t}\left(k_g\overrightarrow{g} + k_n\overrightarrow{n}\right) = \frac{\partial k_g}{\partial t}\overrightarrow{g} + k_g\frac{\partial \overrightarrow{g}}{\partial t} + \frac{\partial k_n}{\partial t}\overrightarrow{n} + k_n\frac{\partial \overrightarrow{n}}{\partial t}.$$

Thus, from the both of above two equations, we reach

$$\frac{\partial k_g}{\partial t} = f_2 k_g^2 + f_3 k_g k_n + f_1\frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2\frac{\partial f_3}{\partial s}\tau_g - f_3\frac{\partial \tau_g}{\partial s} - f_1 k_n\tau_g - f_2\tau_g^2 + \psi k_n \tag{3.12}$$

and

$$\frac{\partial k_n}{\partial t} = f_2 k_g k_n + f_3 k_n^2 + f_1\frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + 2\frac{\partial f_2}{\partial s}\tau_g + f_2\frac{\partial \tau_g}{\partial s} + f_1 k_g\tau_g - f_3\tau_g^2 - \psi k_g. \tag{3.13}$$

Noting that $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{g}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \overrightarrow{g}}{\partial s}$, it is seen that

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \overrightarrow{g}}{\partial t} &= \frac{\partial}{\partial s}\left[-\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3\tau_g\right)\overrightarrow{T} + \psi\overrightarrow{n}\right]\\ &= -\left(\frac{\partial f_1}{\partial s}k_g + f_1\frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - \frac{\partial f_3}{\partial s}\tau_g - f_3\frac{\partial \tau_g}{\partial s}\right)\overrightarrow{T}\\ &\quad -\left(f_1 k_g + \frac{\partial f_2}{\partial s} - f_3\tau_g\right)\left(k_g\overrightarrow{g} + k_n\overrightarrow{n}\right)\\ &\quad +\frac{\partial \psi}{\partial s}n + \psi\left(-k_n\overrightarrow{T} - \tau_g\overrightarrow{g}\right)\end{aligned}$$

while

$$\frac{\partial}{\partial t}\frac{\partial \overrightarrow{g}}{\partial s} = \frac{\partial}{\partial t}\left(-k_g\overrightarrow{T} + \tau_g\overrightarrow{n}\right) = -\frac{\partial k_g}{\partial t}\overrightarrow{T} - k_g\frac{\partial \overrightarrow{T}}{\partial t} + \frac{\partial \tau_g}{\partial t}\overrightarrow{n} + \tau_g\frac{\partial \overrightarrow{n}}{\partial t}.$$

Thus, we obtain

$$\frac{\partial \tau_g}{\partial t} = f_2 k_g\tau_g - \frac{\partial f_2}{\partial s}k_n + \frac{\partial f_3}{\partial s}k_g + f_3 k_n\tau_g + \frac{\partial \psi}{\partial s}. \tag{3.14}$$

No other new formulas are obtained from the relation $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{n}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \overrightarrow{n}}{\partial s}$. □

Thus, we give the following corollary from last theorem.

Corollary 3.4. *Let M be an orientable surface in E^3.*

i- If the curve α is a geodesic curve on M, then we have

$$\frac{\partial k_n}{\partial t} = f_3 k_n^2 + f_1\frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} + 2\frac{\partial f_2}{\partial s}\tau_g + f_2\frac{\partial \tau_g}{\partial s} - f_3\tau_g^2$$

and

$$\frac{\partial \tau_g}{\partial t} = -\frac{\partial f_2}{\partial s}k_n + f_3 k_n\tau_g + \frac{\partial \psi}{\partial s}.$$

ii- If the curve α is an asymptotic line, we have

$$\frac{\partial k_g}{\partial t} = f_2 k_g^2 + f_1\frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2\frac{\partial f_3}{\partial s}\tau_g - f_3\frac{\partial \tau_g}{\partial s} - f_2\tau_g^2$$

and

$$\frac{\partial \tau_g}{\partial t} = f_2 k_g \tau_g + \frac{\partial f_3}{\partial s} k_g + \frac{\partial \psi}{\partial s}.$$

iii- If the curve α is a curvature line, then we have

$$\begin{aligned} \frac{\partial k_g}{\partial t} &= f_2 k_g^2 + f_3 k_g k_n + f_1 \frac{\partial k_g}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} + \psi k_n \\ \frac{\partial k_n}{\partial t} &= f_2 k_g k_n + f_3 k_n^2 + f_1 \frac{\partial k_n}{\partial s} + \frac{\partial^2 f_3}{\partial s^2} - \psi k_g. \end{aligned}$$

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