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Voronovskaya type asymptotic expansions for multivariate quasi-interpolation neural network operators

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ABSTRACT

Here we study further the multivariate quasi-interpolation of sigmoidal and hyperbolic tangent types neural network operators of one hidden layer. We derive multivariate Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

RESUMEN

Aquí estudiamos extensiones de la cuasi-interpolación multivariada de operadores de redes neuronales de tipo sigmoidal y tangente hiperbólica de una capa oculta. Obtenemos expansiones asintóticas del tipo Voronovskaya para el error de aproximación de estos operadores para el operador unidad.

Keywords and Phrases: Multivariate Neural Network Approximation, multivariate Voronovskaya type asymptotic expansion.

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1 Background

Here we follow [5], [6].

We consider here the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1+e^{-x_i}}, \quad x_i \in \mathbb{R},\ i = 1, ..., N;\ x := (x_1, ..., x_N) \in \mathbb{R}^N,$$

each has the properties $\lim_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, ..., N$.

These functions play the role of activation functions in the hidden layer of neural networks.

As in [7], we consider

$$\Phi_i(x_i) := \frac{1}{2}\left(s_i(x_i+1) - s_i(x_i-1)\right), \quad x_i \in \mathbb{R},\ i = 1, ..., N.$$

We notice the following properties:

i) $\Phi_i(x_i) > 0$, $\forall\, x_i \in \mathbb{R}$,

ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1$, $\forall\, x_i \in \mathbb{R}$,

iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1$, $\forall\, x_i \in \mathbb{R}$; $n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi_i(x_i)\, dx_i = 1$,

v) Φ_i is a density function,

vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i)$, $x_i \geq 0$, for $i = 1, ..., N$.

We see that ([7])

$$\Phi_i(x_i) = \left(\frac{e^2-1}{2e^2}\right) \frac{1}{(1+e^{x_i-1})(1+e^{-x_i-1})}, \quad i = 1, ..., N.$$

vii) Φ_i is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$, $i = 1, ..., N$.

Let $0 < \beta < 1$, $n \in \mathbb{N}$. Then as in [6] we get

viii)

$$\sum_{\begin{cases} k_i = -\infty \\ : |nx_i - k_i| > n^{1-\beta} \end{cases}}^{\infty} \Phi_i(nx_i - k_i) \leq 3.1992 e^{-n^{(1-\beta)}}, \quad i = 1, ..., N.$$

Denote by $\lceil\cdot\rceil$ the ceiling of a number, and by $\lfloor\cdot\rfloor$ the integral part of a number. Consider here $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right) \subset \mathbb{R}^N$, $N \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$; $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

As in [6] we obtain

ix)
$$0 < \frac{1}{\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} \Phi_i(nx_i - k_i)} < \frac{1}{\Phi_i(1)} = 5.250312578,$$

$\forall\, x_i \in [a_i, b_i]$, $i = 1, ..., N$.

x) As in [6], we see that
$$\lim_{n\to\infty} \sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} \Phi_i(nx_i - k_i) \neq 1,$$
for at least some $x_i \in [a_i, b_i]$, $i = 1, ..., N$.

We will use here
$$\Phi(x_1, ..., x_N) := \Phi(x) := \prod_{i=1}^{N} \Phi_i(x_i), \quad x \in \mathbb{R}^N. \tag{1}$$

It has the properties:

(i)' $\Phi(x) > 0$, $\forall\, x \in \mathbb{R}^N$,

We see that
$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} \Phi(x_1 - k_1, x_2 - k_2, ..., x_N - k_N) =$$
$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\prod_{i=1}^{N}\Phi_i(x_i - k_i) = \prod_{i=1}^{N}\left(\sum_{k_i=-\infty}^{\infty}\Phi_i(x_i - k_i)\right) = 1.$$

That is

(ii)'
$$\sum_{k=-\infty}^{\infty}\Phi(x - k) := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\Phi(x_1 - k_1, ..., x_N - k_N) = 1,$$
$k := (k_1, ..., k_N)$, $\forall\, x \in \mathbb{R}^N$.

(iii)'
$$\sum_{k=-\infty}^{\infty}\Phi(nx - k) :=$$
$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\Phi(nx_1 - k_1, ..., nx_N - k_N) = 1,$$
$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)'

$$\int_{\mathbb{R}^N} \Phi(x)\, dx = 1,$$

that is Φ is a multivariate density function.

Here $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil, ..., \lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor). \end{aligned}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, fixed $x \in \mathbb{R}^N$, have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx - k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx - k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx - k).$$

In the last two sums the counting is over disjoint vector of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, $r \in \{1, ..., N\}$.

It holds

(v)'

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx - k) \leq 3.1992 e^{-n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

Furthermore it holds

(vi)'

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx - k)} < (5.250312578)^N,$$

$\forall\, x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is clear also that

(vii)'

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \Phi(nx - k) \leq 3.1992 e^{-n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

By (x) we obviously see that

(viii)'

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx - k) \neq 1$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i] \right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We define the multivariate positive linear neural network operator

$\left(x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N} [a_i, b_i] \right) \right)$

$$G_n(f, x_1, ..., x_N) := G_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx - k)} \tag{2}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Phi_i(nx_i - k_i) \right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i) \right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

Notice here that for large enough $n \in \mathbb{N}$ we get that

$$e^{-n^{(1-\beta)}} < n^{-\beta j}, \ \ j = 1, ..., m \in \mathbb{N}, \ 0 < \beta < 1.$$

Thus be given fixed $A, B > 0$, for the linear combination $\left(An^{-\beta j} + Be^{-n^{(1-\beta)}} \right)$ the (dominant) rate of convergence to zero is $n^{-\beta j}$. The closer β is to 1 we get faster and better rate of convergence to zero.

By $AC^m\left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $m, N \in \mathbb{N}$, we denote the space of functions such that all partial derivatives of order $(m-1)$ are coordinatewise alsolutely continuous functions, also $f \in C^{m-1}\left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Let $f \in AC^m\left(\prod_{i=1}^N [a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, and $|\alpha| := \sum_{i=1}^N \alpha_i = l$, where $l = 0, 1, ..., m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is order l.

We denote

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m}\{\|f_\alpha\|_\infty\}, \tag{3}$$

where $\|\cdot\|_\infty$ is the supremum norm.

We assume here that $\|f_\alpha\|_{\infty,m}^{\max} < \infty$.

Next we follow [3], [4].

We consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}(\tanh(x+1) - \tanh(x-1)) > 0, \quad \forall\, x \in \mathbb{R}.$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 1. *([3]) $\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x\to-\infty} \Psi(x) = 0 = \lim_{x\to\infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0) = 0.3809297$.

Theorem 2. *([3]) We have that $\sum_{i=-\infty}^{\infty} \Psi(x - i) = 1$, $\forall\, x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx - i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Also it holds

$$\sum_{i=-\infty}^{\infty} \Psi(x + i) = 1, \quad \forall x \in \mathbb{R}.$$

Theorem 3. *([3]) It holds $\int_{-\infty}^{\infty} \Psi(x)\, dx = 1$.*

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 4. *([3]) Let* $0 < \alpha < 1$ *and* $n \in \mathbb{N}$. *It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\alpha)}}.$$

Theorem 5. *([3]) Let* $x \in [a, b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < \frac{1}{\Psi(1)} = 4.1488766.$$

Also by [3] we get that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1,$$

for at least some $x \in [a, b]$.

In this article we will use

$$\Theta(x_1, ..., x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{4}$$

It has the properties:

(i)* $\Theta(x) > 0$, $\forall \ x \in \mathbb{R}^N$,

(ii)*

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(x_1-k_1, ..., x_N-k_N) = 1,$$

where $k := (k_1, ..., k_N)$, $\forall \ x \in \mathbb{R}^N$.

(iii)*

$$\sum_{k=-\infty}^{\infty} \Theta(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(nx_1-k_1, ..., nx_N-k_N) = 1,$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)*

$$\int_{\mathbb{R}^N} \Theta(x) \, dx = 1,$$

that is Θ is a multivariate density function.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \prod_{i=1}^{N} \Psi(nx_i - k_i) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \prod_{i=1}^{N} \Psi(nx_i - k_i) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Psi(nx_i - k_i) \right).$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Theta(nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Theta(nx-k).$$

In the last two sums the counting is over disjoint vector of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, $r \in \{1, ..., N\}$.

Il holds

(v)*

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Theta(nx-k) \leq e^4 \cdot e^{-2n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Also it holds

(vi)*

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k)} < \frac{1}{(\Psi(1))^N} = (4.1488766)^N,$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is clear that

(vii)*

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \Theta(nx-k) \leq e^4 \cdot e^{-2n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

Also we get

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta(nx-k)\neq 1,$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We define the multivariate positive linear neural network operator $(x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right))$

$$F_n(f,x_1,...,x_N):=F_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Theta(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta(nx-k)}\tag{5}$$

$$:=\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Psi(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Psi(nx_i-k_i)\right)}.$$

Our considered neural networks here are of one hidden layer.

In this article we find Voronovskaya type asymptotic expansions for the above described neural networks quasi-interpolation normalized operators $G_n(f,x)$, $F_n(f,x)$, where $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ is fixed but arbitrary. For other neural networks related work, see [2], [3], [4], [5], [6] and [7]. For neural networks in general, see [8], [9] and [10].

Next we follow [1], pp. 284-286.

About Taylor formula -Multivariate Case and Estimates

Let Q be a compact convex subset of $\mathbb{R}^N$; $N\geq 2$; $z:=(z_1,...,z_N)$, $x_0:=(x_{01},...,x_{0N})\in Q$.

Let $f:Q\to\mathbb{R}$ be such that all partial derivatives of order $(m-1)$ are coordinatewise absolutely continuous functions, $m\in\mathbb{N}$. Also $f\in C^{m-1}(Q)$. That is $f\in AC^m(Q)$. Each m^{th} order partial derivative is denoted by $f_\alpha:=\frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$ and $|\alpha|:=\sum_{i=1}^{N}\alpha_i=m$. Consider $g_z(t):=f(x_0+t(z-x_0))$, $t\geq 0$. Then

$$g_z^{(j)}(t)=\left[\left(\sum_{i=1}^{N}(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_{01}+t(z_1-x_{01}),...,x_{0N}+t(z_N-x_{0N})),\tag{6}$$

for all $j=0,1,2,...,m$.

Example 6. *Let* $m=N=2$. *Then*

$$g_z(t)=f(x_{01}+t(z_1-x_{01}),x_{02}+t(z_2-x_{02})),\quad t\in\mathbb{R},$$

and

$$g_z'(t)=(z_1-x_{01})\frac{\partial f}{\partial x_1}(x_0+t(z-x_0))+(z_2-x_{02})\frac{\partial f}{\partial x_2}(x_0+t(z-x_0)).\tag{7}$$

Setting

$$(*) = (x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})) = (x_0 + t(z - x_0)),$$

we get

$$g_z''(t) = (z_1 - x_{01})^2 \frac{\partial f^2}{\partial x_1^2}(*) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1}(*) +$$

$$(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2}(*). \tag{8}$$

Similarly, we have the general case of $m, N \in \mathbb{N}$ *for* $g_z^{(m)}(t)$.

We mention the following multivariate Taylor theorem.

Theorem 7. *Under the above assumptions we have*

$$f(z_1, ..., z_N) = g_z(1) = \sum_{j=0}^{m-1} \frac{g_z^{(j)}(0)}{j!} + R_m(z, 0), \tag{9}$$

where

$$R_m(z, 0) := \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{m-1}} g_z^{(m)}(t_m)\, dt_m \right) ... \right) dt_1, \tag{10}$$

or

$$R_m(z, 0) = \frac{1}{(m-1)!} \int_0^1 (1 - \theta)^{m-1} g_z^{(m)}(\theta)\, d\theta. \tag{11}$$

Notice that $g_z(0) = f(x_0)$.

We make

Remark 8. *Assume here that*

$$\|f_\alpha\|_{\infty, Q, m}^{\max} := \max_{|\alpha| = m} \|f_\alpha\|_{\infty, Q} < \infty.$$

Then

$$\left\| g_z^{(m)} \right\|_{\infty, [0,1]} = \left\| \left[\left(\sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^m f \right] (x_0 + t(z - x_0)) \right\|_{\infty, [0,1]} \leq \tag{12}$$

$$\left(\sum_{i=1}^N |z_i - x_{0i}| \right)^m \|f_\alpha\|_{\infty, Q, m}^{\max},$$

that is

$$\left\| g_z^{(m)} \right\|_{\infty, [0,1]} \leq \left(\|z - x_0\|_{l_1} \right)^m \|f_\alpha\|_{\infty, Q, m}^{\max} < \infty. \tag{13}$$

Hence we get by (11) that

$$|R_m(z, 0)| \leq \frac{\left\| g_z^{(m)} \right\|_{\infty, [0,1]}}{m!} < \infty. \tag{14}$$

And it holds

$$|R_m(z,0)| \leq \frac{\left(\|z-x_0\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,Q,m}^{\max}, \tag{15}$$

$\forall\ z, x_0 \in Q$.

Inequality (15) will be an important tool in proving our main results.

2 Main Results

We present our first main result

Theorem 9. *Let* $0 < \beta < 1$, $x \in \prod_{i=1}^N [a_i, b_i]$, $n \in \mathbb{N}$ *large enough,* $f \in AC^m\left(\prod_{i=1}^N [a_i, b_i]\right)$, $m, N \in \mathbb{N}$. *Assume further that* $\|f_\alpha\|_{\infty,m}^{\max} < \infty$. *Then*

$$G_n(f,x) - f(x) =$$

$$\sum_{j=1}^{m-1} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) G_n \left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) + o\left(\frac{1}{n^{\beta(m-\varepsilon)}} \right), \tag{16}$$

where $0 < \varepsilon \leq m$.

If $m = 1$, *the sum in (16) collapses.*

The last (16) implies that

$$n^{\beta(m-\varepsilon)} \left[G_n(f,x) - f(x) - \sum_{j=1}^{m-1} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) G_n \left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right] \tag{17}$$

$$\to 0, \quad \text{as } n \to \infty,\ 0 < \varepsilon \leq m.$$

When $m = 1$, *or* $f_\alpha(x) = 0$, *for* $|\alpha| = j$, $j = 1, ..., m-1$, *then we derive that*

$$n^{\beta(m-\varepsilon)} [G_n(f,x) - f(x)] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq m$.

Proof. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$; $x_0, z \in \prod_{i=1}^N [a_i, b_i]$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0N} + t(z_N - x_{0N})), \tag{18}$$

for all $j = 0, 1, ..., m$.

By (9) we have the multivariate Taylor's formula

$$f(z_1, ..., z_N) = g_z(1) = \sum_{j=0}^{m-1} \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} g_z^{(m)}(\theta)\, d\theta. \tag{19}$$

Notice $g_z(0) = f(x_0)$. Also for $j = 0, 1, ..., m-1$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i=j}} \left(\frac{j!}{\prod_{i=1}^N \alpha_i!}\right)\left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i}\right) f_\alpha(x_0). \tag{20}$$

Furthermore

$$g_z^{(m)}(\theta) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i=m}} \left(\frac{m!}{\prod_{i=1}^N \alpha_i!}\right)\left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i}\right) f_\alpha(x_0 + \theta(z - x_0)), \tag{21}$$

$0 \le \theta \le 1$.

So we treat $f \in AC^m\left(\prod_{i=1}^N [a_i, b_i]\right)$ with $\|f_\alpha\|_{\infty,m}^{\max} < \infty$.

Thus, by (19) we have for $\frac{k}{n}, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ that

$$f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) - f(x) = \sum_{j=1}^{m-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i=j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)\left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i\right)^{\alpha_i}\right) f_\alpha(x) + R, \tag{22}$$

where

$$R := m\int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)\cdot$$

$$\left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i\right)^{\alpha_i}\right) f_\alpha\left(x + \theta\left(\frac{k}{n} - x\right)\right) d\theta. \tag{23}$$

By (15) we obtain

$$|R| \le \frac{\left(\left\|x - \frac{k}{n}\right\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}. \tag{24}$$

Notice here that

$$\left\|\frac{k}{n} - x\right\|_\infty \le \frac{1}{n^\beta} \Leftrightarrow \left|\frac{k_i}{n} - x_i\right| \le \frac{1}{n^\beta},\ i = 1, ..., N. \tag{25}$$

So, if $\left\|\frac{k}{n} - x\right\|_\infty \le \frac{1}{n^\beta}$ we get that $\left\|x - \frac{k}{n}\right\|_{l_1} \le \frac{N}{n^\beta}$, and

$$|R| \le \frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max}. \tag{26}$$

Also we see that

$$\left\|x - \frac{k}{n}\right\|_{l_1} = \sum_{i=1}^N \left|x_i - \frac{k_i}{n}\right| \le \sum_{i=1}^N (b_i - a_i) = \|b - a\|_{l_1},$$

therefore in general it holds

$$|R| \leq \frac{\left(\|b-a\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}. \tag{27}$$

Call

$$V(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k).$$

Hence we have

$$U_n(x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) R}{V(x)} = \tag{28}$$

$$\frac{\sum_{\begin{cases} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k) R}{V(x)} + \frac{\sum_{\begin{cases} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k) R}{V(x)}.$$

Consequently we obtain

$$|U_n(x)| \leq \left(\frac{\sum_{\begin{cases} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k)}{V(x)} \right) \left(\frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max} \right) +$$

$$\frac{1}{V(x)} \left(\sum_{\begin{cases} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k) \right) \frac{\left(\|b-a\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max} \overset{\text{(by (v)', (vi)')}}{\leq}$$

$$\frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max} + (5.250312578)^N (3.1992) e^{-n^{(1-\beta)}} \frac{\left(\|b-a\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}. \tag{29}$$

Therefore we have found

$$|U_n(x)| \leq \frac{\|f_\alpha\|_{\infty,m}^{\max}}{m!} \left\{ \frac{N^m}{n^{m\beta}} + (5.250312578)^N (3.1992) e^{-n^{(1-\beta)}} \left(\|b-a\|_{l_1}\right)^m \right\}. \tag{30}$$

For large enough $n \in \mathbb{N}$ we get

$$|U_n(x)| \leq \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \left(\frac{1}{n^{m\beta}} \right). \tag{31}$$

That is

$$|U_n(x)| = O\left(\frac{1}{n^{m\beta}} \right), \tag{32}$$

and

$$|U_n(x)| = o(1). \tag{33}$$

And, letting $0 < \varepsilon \leq m$, we derive

$$\frac{|U_n(x)|}{\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right)} \leq \left(\frac{2\|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}\right)\frac{1}{n^{\beta\varepsilon}} \to 0, \tag{34}$$

as $n \to \infty$.

I.e.

$$|U_n(x)| = o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right). \tag{35}$$

By (22) we observe that

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)}{V(x)} - f(x) =$$

$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)\right)\frac{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right)}{V(x)} + \tag{36}$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)R}{V(x)}.$$

The last says

$$G_n(f,x) - f(x) - \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)G_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right) = U_n(x). \tag{37}$$

The proof of the theorem is complete. ☐

We present our second main result

Theorem 10. *Let* $0 < \beta < 1$, $x \in \prod_{i=1}^{N}[a_i,b_i]$, $n \in \mathbb{N}$ *large enough,* $f \in AC^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m, N \in \mathbb{N}$. *Assume further that* $\|f_\alpha\|_{\infty,m}^{\max} < \infty$. *Then*

$$F_n(f,x) - f(x) =$$

$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)F_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right) + o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right), \tag{38}$$

where $0 < \varepsilon \leq m$.

If $m = 1$, *the sum in (38) collapses.*

The last (38) implies that

$$n^{\beta(m-\varepsilon)}\left[F_n(f,x) - f(x) - \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)F_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right)\right] \tag{39}$$

$$\to 0, \quad as\ n \to \infty,\ 0 < \varepsilon \leq m.$$

When $m = 1$*, or* $f_\alpha(x) = 0$*, for* $|\alpha| = j$*,* $j = 1, ..., m-1$*, then we derive that*

$$n^{\beta(m-\varepsilon)} [F_n(f, x) - f(x)] \to 0,$$

as $n \to \infty$*,* $0 < \varepsilon \leq m$*.*

Proof. Similar to Theorem 9, using the properties of $\Theta(x)$, see (4), (i)*-(vii)* and (5). □

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