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Trisectors like Bisectors with equilaterals instead of Points

Spiridon A. Kuruklis
Eurobank,
Group IT Security and Risk Management Division
Athens, GREECE
`skuruklis@gmail.com`

ABSTRACT

It is established that among all Morley triangles of $\triangle ABC$ the only equilaterals are the ones determined by the intersections of the proximal to each side of $\triangle ABC$ trisectors of either interior, or exterior, or one interior and two exterior angles. It is showed that these are in fact equilaterals, with uniform proofs. It is then observed that the intersections of the interior trisectors with the sides of the interior Morley equilateral form three equilaterals. These along with Pasch's axiom are utilized in showing that Morley's theorem does not hold if the trisectors of one exterior and two interior angles are used in its statement.

RESUMEN

Se establece que entre todos los triángulos de Morley de $\triangle ABC$, los únicos equiláteros son theones determinados por las intersecciones del proximal a cada lado de los trisectores $\triangle ABC$ de ángulos interior, o exterior, o uno interior y dos exteriores. Se muestra que estos están en triángulos equiláteros de facto con demostraciones uniformes. Luego, se observa que las intersecciones de trisectores interiores con los lados de un equilátero Morley interior forman tres triángulos equiláteros. Junto con el axioma de Pasch, se utilizan para probar que el Teorema de Morley no se satisface si se usan los trisectores de un ángulo exterior y dos interiores.

Keywords and Phrases: Angle trisection, Morley's theorem, Morley trisector theorem, Morley triangle, Morley interior equilateral, Morley central equilateral, Morley exterior equilateral, Pasch's axiom, Morley's magic, Morley's miracle, Morley's mystery.

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1 Introduction

The systematic study of the angle trisectors in a triangle starts after 1899, when *Frank Morley*, a Cambridge mathematician, who had just been recently appointed professor at Haverford College, U.S.A. while investigating certain geometrical properties using abstract algebraic methods, made the following astonishing observation, known since then as *Morley's theorem.*

In any triangle the trisectors of its angles, proximal to the three sides respectively, meet at the vertices of an equilateral.

A *Morley triangle* of $\triangle ABC$ is formed by the three points of intersection of pairs of angle trisectors connected by each triangle side. Obviously for a particular side there are four possibilities for pairing trisectors since there are four of them that the side connects. Thus Morley's theorem claims that *a Morley triangle of* $\triangle ABC$ *is equilateral, if it is formed by the intersections of trisectors proximal to the three sides of* $\triangle ABC$ *respectively.*

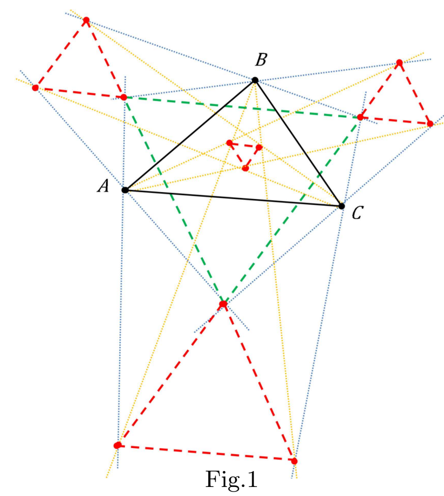

Fig.1

It should be noted that Morley's theorem, as it is stated, is subject to interpretation as the term angle could mean either interior or exterior angle, or even a combination of both for the different instances of the term in the statement.

According to the angle meaning, Morley's theorem gives the following Morley equilaterals of $\triangle ABC$. The intersections of the proximal trisectors of the interior angles form the *interior Morley equilateral of* $\triangle ABC$. Also the intersections of the proximal trisectors of the exterior angles form the *central Morley equilateral of* $\triangle ABC$. In addition the intersections of the proximal trisectors of one interior and two exterior angles form an *exterior Morley equilateral of* $\triangle ABC$, and thus there are three exterior Morley equilaterals of $\triangle ABC$. Fig.1 depicts the above Morley equilaterals. Proofs that the above Morley triangles are in fact equilaterals are given in Part 3 of this work.

But so an obvious question, that several authors have raised, begs for an answer. *In a* $\triangle ABC$ *are there other Morley equilaterals besides the interior, the central and the three exterior Morley equilaterals?*

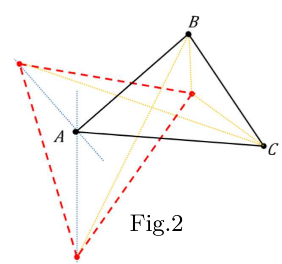

Fig.2

Apparently the requirement of Morley's theorem is satisfied by three more Morley triangles formed by combinations of proximal trisectors of an exterior and two interior angles. One of them is portrayed in Fig.2. Some experimentation using computer generated graphs for these triangles has tempted the belief that Morley's theorem holds for them as well [14]. But in Part 5, it will be proved that these are *not* equilaterals.

After the examination of all Morley triangles it will be shown that the equilateral ones are exactly the interior, the central and the three exterior Morley equilaterals.

This enables the establishment of an analogy between the structures of the angle bisectors and the angle trisectors in a triangle. Namely, the structure of trisectors resembles the structure of bisectors with the inner and the exterior Morley equilaterals of $\triangle ABC$ corresponding to the incenter and the excenters of $\triangle ABC$ respectively, while the central Morley equilateral corresponds to the triangle with vertices the excenters of $\triangle ABC$.

Morley's theorem is considered among the most surprising discoveries in mathematics as it went curiously unnoticed across the ages. Ancient Greeks studied the triangle geometry in depth and they could find it. But curiously they did not and it was overlooked during the following two thousand years.

Angle trisectors exist regardless of how they can be constructed. If the structure of angle trisectors maintains the regularity which characterizes the triangle geometry then theorems must exist for expressing it.

The first observation about this regularity may have forgotten. Morley didn't publish it until 25 years later by providing a sketchy proof, when the theorem had become already famous. But Morley, excited by his discovery, travelled back to England to mention it to his expert friends. In turn mathematical gossip spread it over the world and several journals proposed it for a proof.

Obviously, the simplicity of the theorem statement creates the expectation of an equally simple proof. This simplicity challenges the mathematical talent.

The vast majority of publications on Morley's theorem has treated only the trisectors of the interior angles and gave proofs for the <u>interior</u> Morley equilateral. In the preface of the first publication on the subject, by Taylor and Marr [12], it is recognized that the Morley's work on vector analysis, from which the above theorem follows, holds for both interior and exterior trisectors. The paper's treatment of the theorem with only the interior trisectors is explained as "Morley's work never published and it was only the particular case of internal trisectors that reached the authors". The very respectable given effort has produced proofs of many kinds, exploiting a variety of features. Trigonometric, analytic and algebraic proofs supplement the proofs of a purely geometric kind. Site *Cut the Knot* [13] presents 27 different proofs of Morley's theorem from many more available. Notably, *Roger Penrose* [9] used a tiling technique, *Edsger Dijkstra* applied the rule of sines three times and then the monotonicity of the function $y = \sin(x)$ in the first quadrant [3], *Alain Connes* offered a proof in Algebraic Geometry [1], *John Conway* showed it in plane geometry like a jigsaw puzzle solution [2], while *Richard Guy* proved that it is a consequence of his Lighthouse theorem [5]. However, a geometric, concise and logically transparent proof is still desirable.

Richard Guy notes: "There are a few hints that there is more than one Morley triangle, but Hosberger [p. 98] asks the reader to show that Morley's theorem holds also in the case of the trisection of the exterior angles of a triangle " [5]. Rose [10] and Spickerman [11] have proposed proofs, using different methods, for the central Morley equilateral. In Parts 3 and 4 proofs for the exterior Morley equilaterals will be offered.

The most popular technique for proving Morley's theorem is encountered as *indirect, backwards* or *reverse construction method* and fits in the following scheme.

Given a triangle assume that its angles are trisected and equal to 3α, 3β *and* 3γ, *respectively, where* $\alpha+\beta+\gamma=60^o$. *In order to show that one of its Morley triangles is equilateral, start with an equilateral* $\triangle A'B'C'$ *and construct a* $\triangle ABC$ *with angles* 3α, 3β *and* 3γ, *so that* $\triangle A'B'C'$ *is the appropriate (interior, central or exterior) Morley triangle of* $\triangle ABC$. *Thus* $\triangle ABC$ *would be similar to the given triangle and so would be their corresponding Morley triangles.*

Proofs of the above method most often construct $\triangle ABC$ by erecting $\triangle B'AC'$, $\triangle C'BA'$ and $\triangle A'CB'$ with proper choice of the angles formed on the sides of $\triangle A'B'C'$. However repeated requests have been recorded in geometry discussion forums for an explanation of the particular, seemingly arbitrary, choice of angles made at the beginning of these proofs. Of course the reasoning of the choice is not necessary for their validity. But the readers unfulfilling understanding may have encouraged the mathematical folklore the use of words "mystery", "magic" or "miracle" for referring to Morley's theorem. This is not justifiable as there is nothing mathematically extraordinary related to the theorem.

The presented proofs for showing that the interior, the central and the exterior Morley triangles are equilaterals use the classical *Analysis and Synthesis method*. They exploit the inherent symmetries of the problem and characterized by their uniform structure, logical transparency, remarkable shortness and the distinct aesthetics of the Euclidean geometry. The Synthesis part follows the previous method scheme. But it is empowered by two simple observations, supplying necessary and sufficient conditions for a point to be the incenter or one of the excenters of a given triangle. Even though they are almost trivial have a subtlety that enables to confront the messy complexity of the triangle trisectors by enforcing clean simplicity and create proofs by harnessing the power of the triangle angle bisector theorem. In addition these proofs reveal fundamental properties of the Morley equilaterals stated as Corollaries. Besides their extensive use for showing Morley triangles as not equilaterals, their fertility is demonstrated by proving the following: *(1) The two sides extensions of the inner Morley equilateral meet the corresponding inner trisectors at two points which with the two sides common vertex form an equilateral. (2) The sides of Morley equilaterals are collinear or parallel. (3) In any triangle the exterior trisectors of its angles, proximal to the three sides respectively, meet at the vertices of an equilateral, if and only if, the interior trisectors of an angle and the exterior trisectors of the other two angles, proximal the three sides respectively, meet at the vertices of an equilateral.*

In short, this work advocates that for Morley's observation a natural theoretical setting is Euclidean geometry.

2 Notation and Counting of all Morley triangles

In a Morley triangle of $\triangle ABC$ each vertex is the intersection of two trisectors, each of which is either proximal or distal to a side of $\triangle ABC$. Hence a vertex is called proximal, distal or mix with respect to the triangle side it belongs in the case the trisectors are both proximal, both distal, or one proximal and one distal to the side, respectively.

So we may denote a proximal, distal or mix vertex with respect to a side by using as superscripts p, d or $*$ to the letter of the corresponding angle of $\triangle ABC$ opposite to the side.

Thus A^p, A^d and A^* denote the proximal, distal and mix vertex of a Morley triangle with respect to BC respectively. In Fig.3 the notations for all intersections of the interior trisectors of $\triangle ABC$ are showed. Notice that a Morley triangle may have either proximal vertices, or distal vertices, or exactly two mix vertices.

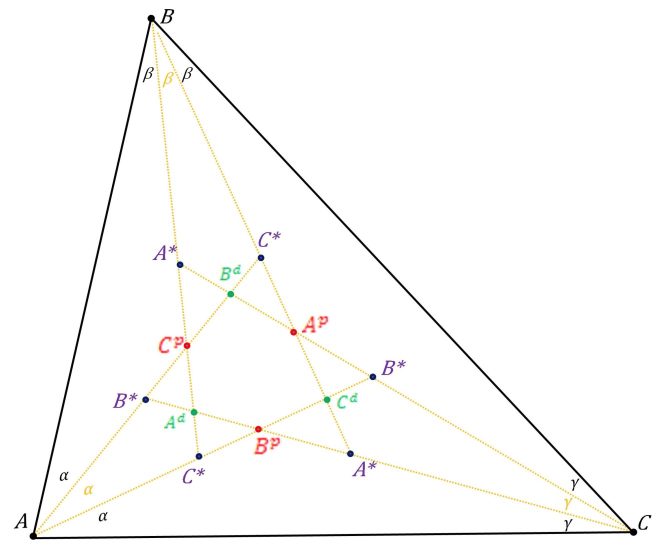

Fig.3

Specifically, $\triangle A^pB^pC^p$ denotes the inner Morley triangle of proximal vertices, which is the inner Morley triangle determined by the intersections of proximal to each side trisectors. Also there is just one Morley triangle with distal vertices which is denoted by $\triangle A^dB^dC^d$. In addition there are three Morley triangles with one vertex proximal and two vertices mix. They are denoted by $\triangle A^pB^*C^*$, $\triangle B^pC^*A^*$ and $\triangle C^pA^*B^*$

Moreover there are three more Morley triangles with one vertex distal and two vertices mix. They are denoted by $\triangle A^dB^*C^*$, $\triangle B^dC^*A^*$ and $\triangle C^dA^*B^*$. Notice that a proximal or a distal vertex is uniquely determined but a mix vertex is not as there are two such denoted by the same letter. However in a Morley triangle with a pair of mix vertices, given its proximal or distal vertex, the mix vertices are uniquely specified due to the choice restrictions in pairing trisectors for the second and then for the third vertex. Hence there are 8 *interior Morley triangles* formed by the trisectors of the interior angles of $\triangle ABC$.

Similarly the trisectors of the exterior angles of $\triangle ABC$ form Morley triangles. These are denoted by $\triangle A_A^p B_B^p C_C^p$, for the Morley triangle of proximal vertices, $\triangle A_A^d B_B^d C_C^d$, for the Morley triangle of distal vertices, $\triangle A_A^p B_B^* C_C^*$, $\triangle B_B^p C_C^* A_A^*$ and $\triangle C_C^p A_A^* B_B^*$, for the Morley triangles with one proximal and two mix vertices, $\triangle A_A^d B_B^* C_C^*$, $\triangle B_B^d C_C^* A_A^*$ and $\triangle C_C^d A_A^* B_B^*$ for the Morley triangles of one distal and two mix vertices. In this notation we use subscripts in order to distinguish a vertex determined by the interior trisectors from the vertex of the same type determined by the exterior trisectors. Hence, in general, there are 8 Morley triangles formed by the trisectors of the exterior angles of $\triangle ABC$. Their vertices are in the exterior of $\triangle ABC$ and due to their rather central location with respect to $\triangle ABC$ are called *central Morley triangles.*

There are two more possibilities for the formation of a Morley triangle. One is by combining the trisectors of an interior angle with the trisectors of the other two exterior angles of $\triangle ABC$. Another is by combining the trisectors of two interior angles with the trisectors of the third exterior angle of $\triangle ABC$.

The Morley triangles formed by combining the trisectors of the interior $\angle A$ with the trisectors of the exterior $\angle B$ and $\angle C$ are denoted by $\triangle A_A^p b_A^p c_A^p$, for the Morley triangle of proximal vertices, $\triangle A_A^d b_A^d c_A^d$, for the Morley triangle of distal vertices, $\triangle C_C^p a_C^* b_C^*$, $\triangle a_C^p b_C^* C_C^*$ and $\triangle b_C^p C_C^* a_C^*$, for the Morley triangles with one proximal and two mix vertices, and $\triangle A_A^d b_A^* c_A^*$, $\triangle b_A^d c_A^* A_A^*$, $\triangle c_A^d A_A^* b_A^*$ for the Morley triangles with one distal and two mix vertices. The use of a small letter is for denoting the intersection of an interior and an exterior trisector of $\triangle ABC$. The vertices of these 8 triangles formed by the trisectors of the interior $\angle A$ with the trisectors of the exterior $\angle B$ and $\angle C$ are in the exterior of $\triangle ABC$ and thus they are called *exterior Morley triangles relative to* $\angle A$.

Similarly are denoted the Morley triangles relative to $\angle B$, which are formed by combining the trisectors of the interior angle $\angle B$ with the trisectors of the exterior $\angle C$ and $\angle A$, and also the ones relative to $\angle C$ formed by combining the trisectors of the interior $\angle C$ with the trisectors of the exterior $\angle A$ and $\angle B$. Hence, in general, there are 24 exterior Morley triangles determined by the intersections of trisectors of an interior and two exterior angles of $\triangle ABC$.

The Morley triangles formed by combining the trisectors of the interior $\angle B$ and $\angle C$ with the trisectors of the exterior $\angle A$ are denoted by $\triangle A^p b_A^p c_A^p$, for the Morley triangle of proximal vertices, $\triangle A^d b_A^d c_A^d$, for the Morley triangle of distal vertices, $\triangle A^p b_A^* c_A^*$, $\triangle b_A^p c_A^* A^*$, $\triangle c_A^p A^* b_A^*$, for the Morley triangles of one proximal vertex and two mix, $\triangle A^d b_A^* c_A^*$, $\triangle b_A^d c_A^* A^*$, $\triangle c_A^d A^* b_A^*$, for the Morley triangles of one distal vertex and two mix. It should be remarked that in this notation the same symbol for the intersection of an interior with an exterior trisector may refer to two different points, an ambiguity which is clarified in a Morley triangle since one of its vertices specifies its type and so the vertex that the symbolism refers. Hence, there are 8 Morley triangles relative to the exterior $\angle A$, which obviously have one vertex inside and two outside $\triangle ABC$. In general, there are 24 Morley triangles of $\triangle ABC$ determined by the intersections of trisectors of one exterior and two interior angles of $\triangle ABC$.

Conclude that in total there are, in general, 64 Morley triangles of $\triangle ABC$.

3 Uniform Proofs for all Morley Equilaterals

In this part we will prove that five Morley triangles are equilaterals. The proofs are uniform and utilize two basic observations for *determining the incenter or an excenter of* $\triangle ABC$ *using only one of its bisectors.*

Observe that the incenter I is lying on a unique arc passing through two vertices and I. In Fig.4 the unique arc passing through A, B and I is depicted. Obviously

$$\angle AIB{=}180^0 - \tfrac{1}{2}\angle ABC = \tfrac{1}{2}\angle BAC = 90^o + \tfrac{1}{2}\angle ACB.$$

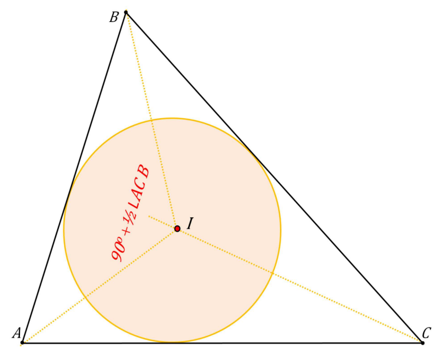

Fig.4

Thus I may be characterized as the intersection in the interior of $\triangle ABC$ of a bisector with the arc of size $90^o + \frac{1}{2}\angle ACB$ passing through A and B. Clearly an analogous result holds for the other two pairs of vertices of $\triangle ABC$. We refer to this as *the Incenter Lemma*.

If I_C is the excenter relative to $\angle C$ then

$$\angle AI_CB = 90^o - \tfrac{1}{2}\angle ACB,\ \angle BI_CC = \tfrac{1}{2}\angle BAC \text{ and } \angle CI_CA = \tfrac{1}{2}\angle CBA.$$

Thus I_C is determined by the intersection in the exterior of $\triangle ABC$ of a bisector, either of the interior $\angle C$ or the exterior $\angle A$ or $\angle B$ with the arc of size $90^o - \frac{1}{2}\angle ACB$ passing through A and B, or with the arc passing through B and C of size $\frac{1}{2}\angle BAC$, or with the arc of size $\frac{1}{2}\angle CBA$ passing through C and A. Evidently analogous results hold for the other two excenters I_A and I_B. We refer to this as *the Excenter Lemma*.

Theorem 1. *In any triangle the interior trisectors of its angles, proximal to the sides, meet at the vertices of an equilateral.*

Proof.
Analysis: Let $\triangle ABC$ be a triangle with $\angle A = 3\alpha, \angle B = 3\beta$ and $\angle C = 3\gamma$, where $\alpha+\beta+\gamma = 60^o$. Suppose that $\triangle A^pB^pC^p$ is equilateral, where A^p, B^p and C^p are the intersections of the trisectors proximal to the sides BC, CA and AB respectively. The aim of this step is to calculate the angles formed by the sides of $\triangle A^pB^pC^p$ and the trisectors of $\triangle ABC$. See Fig.5.
Let C^d be the intersection of AB^p and BA^p. Since AC^p and BC^p are angle bisectors in $\triangle AC^dB$, C^p is the incenter.

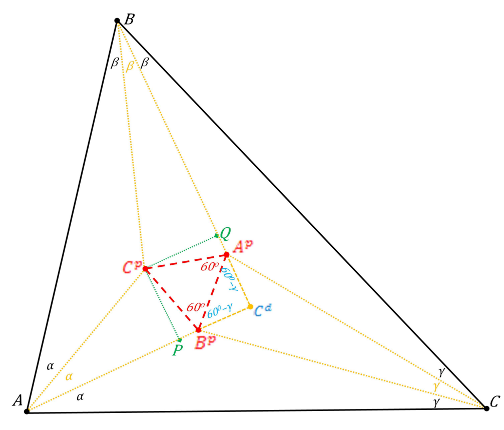

Fig.5

Let P and Q be the orthogonal projections of C^p on AC^d and BC^d respectively. Thus $C^pP = C^pQ$ and $C^dP = C^dQ$. But so $\triangle C^pPB^p = \triangle C^pQB^p$ as right triangles having two pairs of sides equal. Hence $B^pP = A^pQ$. Then $C^dA^p = C^dB^p$ and so $\triangle A^pC^dB^p$is isosceles. Now from $\triangle AC^dB$ we have $\angle AC^dB = 180^o - (2\alpha + 2\beta) = 60^o + 2\gamma$.Therefore $\angle C^dB^pA^p = \angle C^dA^pB^p = \frac{1}{2}[180^o - (60^o + 2\gamma)] = 60^o - \gamma$. Consequently

$$\angle C^pB^pA = \angle C^pA^pB = 180^o - 60^o - (60^o - \gamma) = 60^o + \gamma = \gamma^+.$$

Let A^d be the intersection of BC^p and CB^p. Also let B^d be the intersection of CA^p and AC^p. Then from $\triangle BA^dC$ and $\triangle CB^dA$ find similarly

$$\angle A^pC^pB = \angle A^pB^pC = \alpha^+ \text{ and } \angle B^pA^pC = \angle B^pC^pA = \beta^+.$$

Synthesis: Suppose that a triangle is given and assume that its angles are trisected and equal to 3α, 3β and 3γ, respectively, where $\alpha + \beta + \gamma = 60^o$. Then around an equilateral $\triangle A^pB^pC^p$ will construct $\triangle ABC$ with angles 3α, 3β and 3γ so that A^p, B^p and C^p will be the intersections of the proximal to the sides interior trisectors.

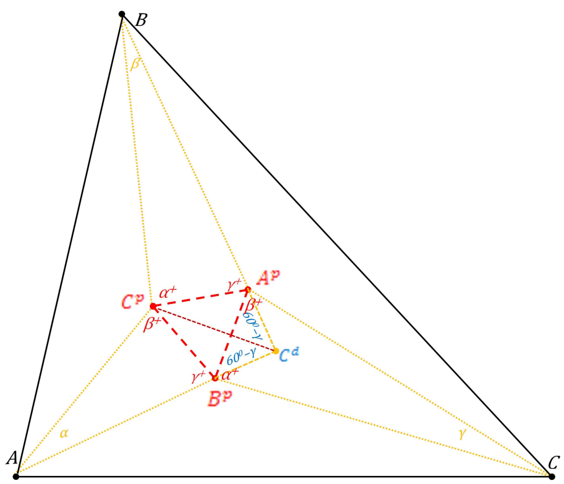

Fig.6

On the side B^pC^p erect $\triangle B^pAC^p$ with adjacent angles $\gamma^+ = \gamma + 60^o$ and $\beta^+ = \beta + 60^o$.
Similarly, erect $\triangle C^pBA^p$ and $\triangle A^pCB^p$ on the sides C^pA^p and A^pB^p respectively with corresponding angles as shown in Fig.6, which were found in the Analysis step.

Let C^d be the intersection AB^p and BA^p. Notice that $\triangle A^pC^dB^p$ is isosceles as two of its angles are $180^o - 60^o - \gamma^+ = 60^o - \gamma$.
Thus

$$C^dA^p = C^dB^p \ (1) \text{ and } \angle A^pC^dB^p = 180^o - 2(60^o - \gamma) = 60^o + 2\gamma \ (2)$$

Since $\triangle A^pB^pC^p$ has been taken equilateral, $C^pA^p = C^pB^p$. Combine this with (1) and infer that C^p is on the A^pB^p bisector and so on the $\angle AC^dB$ bisector. Moreover from (2) $\angle AC^pB = 360^0 - (\alpha^+ + 60^o + \beta^+) = 180^o - (\alpha + \beta) = 90^o + \frac{1}{2}(60^o + 2\gamma) = 90^o + \frac{1}{2}\angle A^pC^dB^p = 90^o + \frac{1}{2}\angle AC^dB$.

Hence, by the Incenter Lemma, C^p is the incenter of $\triangle AC^dB$.

Similarly it is shown that A^p and B^p are the incenters of $\triangle BA^dC$ and $\triangle CB^dA$, respectively, where A^d is the intersection of BC^p and CB^p, while B^d is the intersection of CA^p and AC^p. Thus $\angle C^pAB = \angle C^pAB^p = \angle CAB^p$ and so AB^p, AC^p are trisectors of $\angle A$.

Also the choice of angles in the construction of $\triangle B^p A C^p$ implies $\angle C^p A B^p = \alpha$. Hence $\angle A = 3\alpha$. Likewise infer that BC^p, BA^p are trisectors of $\angle B$ with $\angle B = 3\beta$ and CA^p, CB^p are trisectors of $\angle C$ with $\angle C = 3\gamma$.

Corollary 1. *a) The angles between the trisectors of* $\triangle ABC$ *and the sides of its inner Morley equilateral* $\triangle A^p B^p C^p$ *are:* $\angle A^p B^p C = \angle A^p C^p B = \alpha^+$, $\angle B^p C^p A = \angle B^p A^p C = \beta^+$, $\angle C^p A^p B = \angle C^p B^p A = \gamma^+$.

b) The heights of the equilateral $\triangle A^p B^p C^p$ *are:* $A^p A^d$, $B^p B^d$ *and* $C^p C^d$.

Theorem 2. *In any triangle the exterior trisectors of its angles, proximal to the sides, meet at the vertices of an equilateral.*

Proof.

<u>*Analysis:*</u> Let $\triangle ABC$ be a triangle with $\angle A = 3\alpha, \angle B = 3\beta$ and $\angle C = 3\gamma$, where $\alpha + \beta + \gamma = 60^o$. Let A_A^p, B_B^p and C_C^p be the intersections of the exterior trisectors proximal to the sides BC, CA and AB respectively. Suppose $\triangle A_A^p B_B^p C_C^p$ is equilateral. The aim of this step is to calculate the angles formed by the sides of $\triangle A_A^p B_B^p C_C^p$ and the exterior trisectors of $\triangle ABC$.

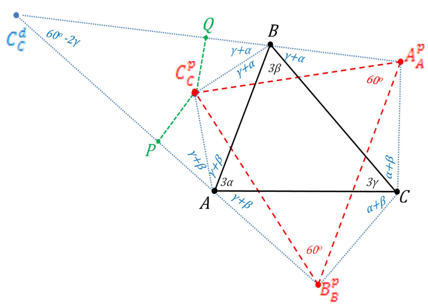

Fig.7a ($\gamma < 30^o$)

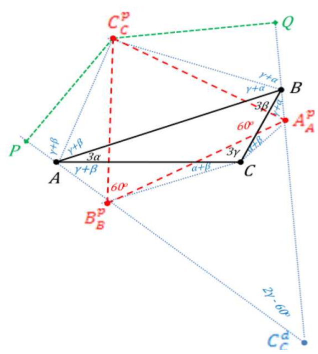

Fig.7b ($\gamma > 30^o$)

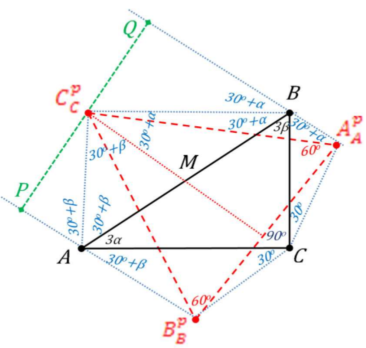

Fig.7c ($\gamma = 30^o$)

Let P and Q be the orthogonal projections of C_C^p on AB_B^p and BA_A^p respectively.

Notice that AB_B^p and BA_A^p may intersect each other or be parallel since

$$\angle PAB + \angle QBA = 2(\beta + \gamma) + 2(\gamma + \alpha) = 120^o + 2\gamma.$$

If AB_B^p and BA_A^p intersect each other let C_C^d be their intersection. Next consider all possible cases.

▷ If $\gamma < 30^o$ then

C_C^d and C_C^p are at the same side of AB. In $\triangle AC_C^d B$, AC_C^p and BC_C^p are interior angle bisectors and so C_C^p is the incenter, while it is calculated $\angle AC_C^d B = 60^o - 2\gamma$. Fig.7a.

▷ If $\gamma > 30^o$ then
C_C^d and C_C^p are on different sides of AB. In $\triangle AC_C^dB$, AC_C^p and BC_C^p are exterior angle bisectors and so C_C^p is the excenter relative to $\angle C_C^d$, while it is calculated $\angle AC_C^dB = 2\gamma - 60^o$. Fig.7b.

Hence in both the above cases ($\gamma \neq 30^o$) it holds $C_C^pP = C_C^pQ$. Thus $\triangle C_C^pPB_B^p = \triangle C_C^pQA_A^p$, as right triangles having two pairs of sides equal. Consequently $\angle C_C^pB_B^pP = \angle C_C^pA_A^pQ$ and so $\triangle A_A^pC_C^dB_B^p$ is isosceles. Thus:
◇ If $\gamma > 30^o$ then
$\angle C_C^pB_B^pP = \angle C_C^pA_A^pQ = \frac{1}{2}[180^o - \angle A_A^pC_C^dB_B^p] - 60^o = \frac{1}{2}[180^o - (60^o - 2\gamma)] - 60^o = \gamma.$
◇ If $\gamma < 30^o$ then
$\angle C_C^pB_B^pP = \angle C_C^pA_A^pQ = 180^o - 60^o - \frac{1}{2}[180^o - \angle A_A^pC_C^dB_B^p] = 180^o - 60^o \frac{1}{2}[180^o - (2\gamma - 60^o)] = \gamma.$
Deduce that for $\gamma \neq 30^o$ it holds $\angle C_C^pB_B^pA = \angle C_C^pA_A^pB = \gamma$.

▷ If $\gamma\ = 30^o$ then $\alpha + \beta = 30^o$ and $AB_B^p//BA_A^p$. Fig.7c.
Notice $\angle AC_C^pB = 180^o - (30^o + \beta) - (30^o + \alpha)$ and so $\angle AC_C^pB = 90^o$.
Let M be the midpoint of AB. Since $\triangle AC_C^pB$ is right triangle, $C_C^pM = MA = MB$. Then $C_C^pM = MA$ gives $\angle C_C^pAM = \angle MC_C^pA$. But AC_C^p is the $\angle PAB$ bisector and thus $\angle C_C^pAM = \angle C_C^pAP$. Hence $\angle MC_C^pA = \angle C_C^pAP$ and so $B_B^pA//C_C^pM//A_A^pB$.
Since $MA = MB$, C_C^pM bisects $A_A^pB_B^p$ and so $C_C^pM \perp A_A^pB_B^p$, as $\triangle A_A^pB_B^pC_C^p$ is equilateral.
Then AB_B^p, $BA_A^p \perp A_A^pB_B^p$. Therefore $\angle AB_B^pA_A^p = 90^o$ and given that $\angle C_C^pB_B^pA_A^p = 60^o$ infer $\angle C_C^pB_B^pA = 30^o$. Similarly infer $\angle C_C^pA_A^pB = 30^o$.
Deduce that for $\gamma = 30^o$ it holds $\angle C_C^pB_B^pA = \angle C_C^pA_A^pB = \gamma$.

○ Conclude that for any value of γ it holds $\angle C_C^pB_B^pA = \angle C_C^pA_A^pB = \gamma$.

Then from $\triangle C_C^pAB_B^p$ and $\triangle C_C^pBA_A^p$ deduce $\angle B_B^pC_C^pA = \beta$ and $\angle A_A^pC_C^pB = \alpha$ respectively. Also from $\triangle BA_A^pC$ and $\triangle CB_B^pA$ infer $\angle B_B^pA_A^pC = \beta$ and $\angle A_A^pB_B^pC = \alpha$.

Synthesis: Let a triangle be given in which its angles are equal to 3α, 3β and 3γ respectively, where $\alpha + \beta + \gamma = 60^o$. Then around an equilateral, which is denoted by $\triangle A_A^pB_B^pC_C^p$, will construct a $\triangle ABC$ with angles 3α, 3β and 3γ so that A_A^p, B_B^p and C_C^p will be the meeting points of the exterior angle trisectors proximal to the sides of $\triangle ABC$. On the side $B_B^pC_C^p$ erect $\triangle B_B^pAC_C^p$ with adjacent angles γ and β which were calculated in the Analysis step. Similarly, erect $\triangle C_C^pBA_A^p$ and $\triangle A_A^pCB_B^p$ on the sides $C_C^pA_A^p$ and $A_A^pB_B^p$ with corresponding angles as they are depicted in Fig.8. Hence $\triangle ABC$ has been determined. So it remains to be proved that the resulting $\triangle ABC$ has angles 3α, 3β and 3γ respectively and the erected sides are the trisectors of its exterior angles.

Let P and Q be the orthogonal projections of C_C^p on the extensions of AB_B^p and BA_A^p respectively. Next consider all cases regarding AB_B^p and BA_A^p.

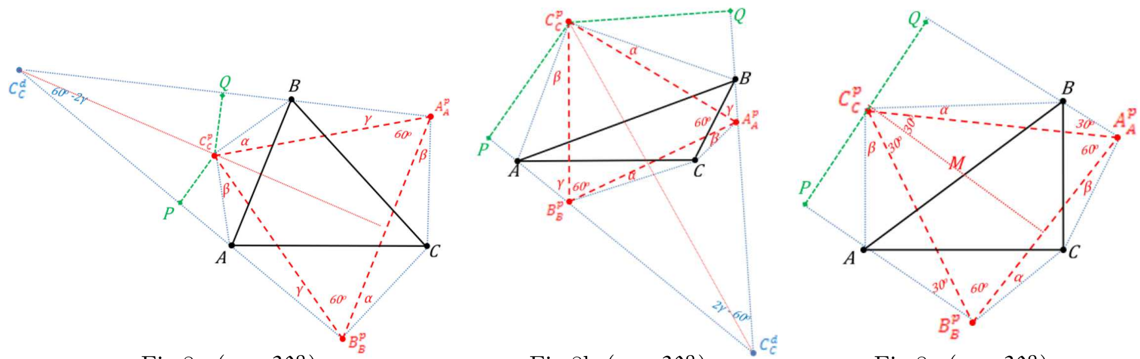

Fig.8a ($\gamma < 30^o$) Fig.8b ($\gamma > 30^o$) Fig.8c ($\gamma = 30^o$)

$\triangleright$ Assume $\gamma \neq 30^o$. Set $s = (30^o - \gamma)/|30^o - \gamma|$ and let C_C^d be the meeting point of AB_B^p and $A_A^p B$. The choice of angles in the erection of $\triangle B_B^p A C_C^p$ and $\triangle C_C^p B A_A^p$ implies:

C_C^p is lying on the $\angle A_A^p C_C^d B_B^p$ bisector (1) and $\angle AC_C^p B = 90^o + \frac{1}{2} s \angle AC_C^d B$ (2)

To verify (1) notice that $\triangle A_A^p C_C^d B_B^p$ is isosceles, as two of its angles are by construction either $60^o + \gamma$ ($\gamma < 30^o$) or $120^o - \gamma$ ($\gamma > 30^o$). But $\triangle A_A^p B_B^p C_C^p$ is assumed equilateral and so $C_C^p A_A^p = C_C^p B_B^p$. Thus $C_C^d C_C^p$ bisects side $A_A^p B_B^p$ of the isosceles $\triangle A_A^p C_C^d B_B^p$ and so C_C^p is lying on the $\angle A_A^p C_C^d B_B^p$ bisector.

To verify (2) notice that in the isosceles $\triangle A_A^p C_C^d B_B^p$ either $\angle AC_C^d B = 180^o - 2(60^o + \gamma) - 60^o - 2\gamma$ ($\gamma < 30^o$) or $\angle AC_C^d B = 180^o - 2(60^o + \gamma) = 2\gamma - 60^o$ ($\gamma > 30^o$). Thus $\angle AC_C^d B = s(60^o - 2\gamma)$. Hence $\angle AC_C^p B = \beta + 60^o + \beta = 60^o + (60^o - \gamma) = 90^o + \frac{1}{2} s(60^o - 2\gamma) = 90^o + s\angle AC_C^d B$.

Therefore from (1) and (2), by the Incenter Lemma ($\gamma < 30^o, s = 1$) or the Excenter Lemma ($\gamma > 30^o, s = -1$), C_C^p is the incenter or the excenter of $\triangle AC_C^d B$ respectively.
Thus AC_C^p and BC_C^p are bisectors (interior or exterior) in $\triangle AC_C^d B$. So, using $\triangle C_C^p A B_B^p$ and $\triangle C_C^p B A_A^p$, deduce $\angle C_C^p AB = \angle C_C^p AP = \gamma + \beta$ and $\angle C_C^p BA = \angle C_C^p BQ = \gamma + \beta$. Consequently $\angle C_C^p AB = \gamma + \beta$ and $\angle C_C^p BA = \gamma + \beta$ while AC_C^p and BC_C^p bisect the angles formed by AB and the extensions of AB_B^p and BA_A^p respectively.

$\triangleright$ Assume $\gamma = 30^o$. Then $\alpha + \beta = 30^o$. Notice AB_B^p , $BA_A^p \perp A_A^p B_B^p$ and so $A_A^p B // B_B^p A$. Also $\angle AC_C^p B = \beta + 60^o + \alpha = 90^o$ and so $\triangle AC_C^p B$ is right triangle. Let M be the midpoint of AB. Hence $C_C^p M$=MA=MA. But $C_C^p M = MA$ implies $\angle C_C^p AM = \angle MC_C^p A$. Since $\angle C_C^p AM = \angle C_C^p AP$, $\angle C_C^p AM = \angle AC_C^p M$. Consequently $C_C^p M // AP$. Thus $A_A^p B // C_C^p M // B_B^p A$ and since $MA = MA$, $C_C^p M$ passes through the midpoint of $A_A^p B_B^p$. As a result $C_C^p M$ is a height of the equilateral $\triangle A_A^p B_B^p C_C^p$ and so $\angle A_A^p C_C^p M = \angle B_B^p C_C^p M = 30^o$. Therefore

$$\angle C_C^p AB = \angle MC_C^p A = \angle MC_C^p B_B^p = \angle B_B^p C_C^p A = 30^o + \beta = \gamma + \beta.$$

Similarly it is shown $\angle C_C^p BA = 30^o + \alpha = \gamma + \alpha$.

Also $A_A^p B // C_C^p M // B_B^p A$ implies $\angle C_C^p AP = \angle AC_C^p M$ and $\angle QBC_C^p = \angle BC_C^p M$. So

$$\angle C_C^p AP = \angle C_C^p AB \text{ and } \angle C_C^p BQ = \angle C_C^p BA.$$

∘ Conclude for any γ it holds $\angle C^p_C AB = \gamma+\beta$ and $\angle C^p_C BA = \gamma+\alpha$, while AC^p_C and BC^p_C bisect the angles formed by AB and the extensions of AB^p_B and BA^p_A respectively.

The rest cases are treated similarly. Considering BC^p_C and CB^p_B it shown that $\angle A^p_A BC = \alpha+\gamma$ and $\angle A^p_A CB = \alpha+\beta$ while BA^p_A and CA^p_A bisect the angles formed by BC and the extensions of BC^p_C and CB^p_B respectively, and eventually considering CA^p_A and AC^p_C it is proved that $\angle B^p_B CA = \beta+\alpha$ and $\angle B^p_B AC = \beta+\gamma$ while CB^p_B and AB^p_B bisect the angles formed by AC and the extensions of AC^p_C and CA^p_A respectively. Conclude that AB^p_B bisects the angle between AC and the extension of AC^p_C, while AC^p_C bisects the angle between AB and the extension of AB^p_B. Thus AB^p_B and AC^p_C are trisectors of the exterior $\angle A$.

Also $\angle C^p_C AB = \angle B^p_B AC = \gamma+\beta$. Hence $\angle A = \frac{1}{2}[360^o - 6(\beta+\gamma)] = 180^o - 3(\beta+\gamma) = 3\alpha$. Similarly it is shown that BC^p_C, BA^p_A are trisectors of the exterior $\angle B$ with $\angle B = 3\beta$ and CA^p_A, CB^p_B are trisectors of the exterior $\angle C$ with $\angle C = 3\gamma$.

Corollary 2. *a) The angles between the exterior trisectors of* $\triangle ABC$ *and the sides of its central Morley equilateral* $\triangle A^p_A B^p_B C^p_C$ *are:* $\angle A^p_A B^p_B C = \angle A^p_A C^p_C B = \alpha$, $\angle B^p_B C^p_C A = \angle B^p_B A^p_A C = \beta$, $\angle C^p_C A^p_A B = \angle C^p_C B^p_B A = \gamma$.
b) The heights of the equilateral $\triangle A^p_A B^p_B C^p_C$ *are:* $A^p_A A^d_A$, $B^p_B B^d_B$ *and* $C^p_C C^d_C$.

Theorem 3. *In any triangle the interior trisectors of an angle and the exterior trisectors of the other two angles, proximal the three sides respectively, meet at the vertices of an equilateral.*

Proof.
Analysis: Let $\triangle ABC$ be a triangle with $\angle A = 3\alpha$, $\angle B = 3\beta$ and $\angle C = 3\gamma$, where $\alpha+\beta+\gamma = 60^o$. Let C^p_C be the intersection of the exterior trisectors of $\angle B$ and $\angle C$, proximal to AB, while a^p_C and b^p_C are the intersections of the interior with the exterior trisectors proximal to BC and CA respectively. Suppose that $\triangle a^p_C C^p_C b^p_C$ is equilateral. The aim of this step is to calculate the angles between the sides of $\triangle a^p_C C^p_C b^p_C$ and the interior trisectors of $\angle C$ and also the exterior trisectors of $\angle A$ and $\angle B$.

Let P and Q be the orthogonal projections of C^p_C on Ab^p_C and Ba^p_C, respectively. It was observed in the course of the Analysis Step of Theorem 2 that the trisectors Ab^p_C and Ba^p_C intersect each other iff $\gamma \neq 30^o$. Recall that if $\gamma \neq 30^o$ C^p_C is the incenter ($\gamma < 30^o$) or the excenter ($\gamma > 30^o$) of $\triangle BC^d_C C$ while for $\gamma = 30^o$ $Ab^p_C // Ba^p_C$. But it was shown that in either case it holds $C^p_C P = C^p_C Q$ and hence $\triangle C^p_C P b^p_C = \triangle C^p_C Q a^p_C$, as right triangles having two pairs of sides equal. This implies $\angle Ab^p_C C^p_C = \angle Ba^p_C C^p_C$ (1). Consequently:

▷ If $\gamma \neq 30^o$ then C^d_C is determined. Hence (1) implies $\angle b^p_C a^p_C C^d_C = \angle a^p_C b^p_C C^d_C$. Thus $\triangle a^p_C C^d_C b^p_C$ is isosceles. However from $\triangle AC^d_C B$ it is calculated that

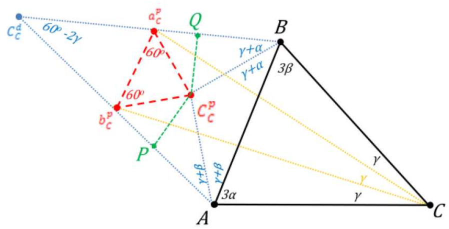

Fig.9a ($\gamma < 30^o$)

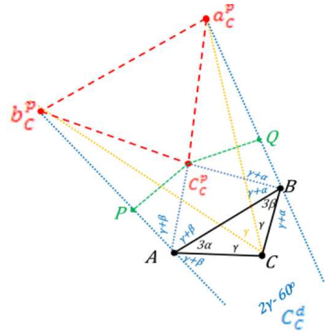

Fig.9b ($\gamma > 30^o$)

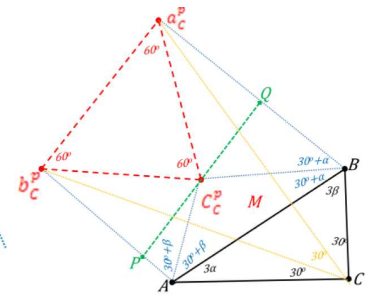

Fig.9c ($\gamma = 30^o$)

$$\angle AC_C^dB = 60^o - 2\gamma \ (\gamma < 30^o) \text{ or } \angle AC_C^dB = 2\gamma - 60^o \ (\gamma > 30^o).$$

Then we have respectively.

◇ For $\gamma < 30^o$, C_C^d is on the other side of $a_C^pb_C^p$ from A, B and

$$\angle Ab_C^pa_C^p = \angle Ba_C^pb_C^p = 180^o - \tfrac{1}{2}[180^o - (60^o - 2\gamma)] = 120^o - \gamma.$$

◇ For $\gamma > 30^o$, C_C^d is on the same side of $a_C^pb_C^p$ with A,B and

$$\angle Ab_C^pa_C^p = \angle Ba_C^pb_C^p = \tfrac{1}{2}[180^o - (2\gamma - 60^o)] = 120^o - \gamma.$$

In either case $\angle C_C^pa_C^pB = \angle C_C^pb_C^pA = 60^o - \gamma = \alpha + \beta$.

▷ If $\gamma = 30^o$ then $Ab_C^p // Ba_C^p$. Also C_C^pP, C_C^pQ are collinear and $\alpha + \beta = 30^o$. Thus $\angle a_C^pC_C^pb_C^p = 180^o - (30^o + \beta) - (30^o + \alpha) = 60^o$ and so by (1) $\angle a_C^pC_C^pQ = \angle b_C^pC_C^pP = \frac{1}{2}(180^o - \angle a_C^pC_C^pb_C^p) = \frac{1}{2}(180^o - 60^o) = 60^o$. Hence $\angle C_C^pa_C^pB = \angle C_C^pb_C^pA = 30^o = \alpha + \beta$.

○ In conclusion for any γ it holds $\angle Ab_C^pC_C^p = \angle Ba_C^pC_C^p = 60^o - \gamma = \alpha + \beta$.

Finally from $\triangle Ba_C^pC$ and $\triangle Cb_C^pA$ we find $\angle Ba_C^pC = \alpha$ and $\angle Ab_C^pC = \beta$ respectively, and so $\angle C_C^pa_C^pC = \beta$ and $\angle C_C^pb_C^pC = \alpha$. Yet from $\triangle b_C^pAC_C^p$ and $\triangle C_C^pBa_C^p$ calculate that $\angle b_C^pC_C^pA = (\gamma + \alpha)^+$ and $\angle a_C^pC_C^pB = (\gamma + \beta)^+$.

Synthesis: Suppose that a triangle is given with angles equal to 3α, 3β and 3γ, respectively, where $\alpha + \beta + \gamma = 60^o$. Then from an equilateral, which we denote $\triangle a_C^pC_C^pb_C^p$, will construct a $\triangle ABC$ with angles 3α, 3β and 3γ so that the sides of the erected triangles are the proper angle trisectors of the resulting $\triangle ABC$. On the side $a_C^pb_C^p$ erect $\triangle a_C^pCb_C^p$ with adjacent angles $\beta^+ = 60^o + \beta$ and $\alpha^+ = 60^o + \alpha$ so that C_C^p is inside $\triangle a_C^pCb_C^p$. Next on the side $b_C^pC_C^p$ erect $\triangle b_C^pAC_C^p$ with adjacent angles $\alpha + \beta$ and $(\gamma + \alpha)^+$. Finally on the side $C_C^pa_C^p$ erect $\triangle a_C^pBC_C^p$ with adjacent angles $\alpha + \beta$ and $(\gamma + \beta)^+$. Thus $\triangle ABC$ has been determined. See Fig.10 for the corresponding value of γ. So it remains to be proved that the resulting $\triangle ABC$ has angles 3α, 3β and 3γ, respectively and the erected sides Ca_C^p, Cb_C^p are trisectors of $\angle C$, while Ab_C^p, AC_C^p, and

BC^p_C, Ba^p_C are trisectors of the exterior angles $\angle A$ and $\angle B$ respectively.

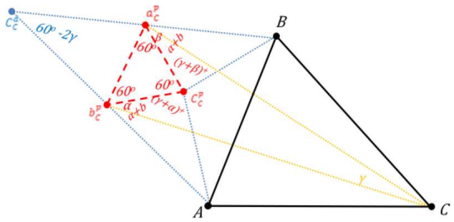

Fig.10a ($\gamma < 30^o$)

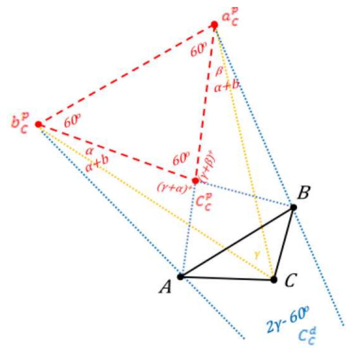

Fig.10b ($\gamma > 30^o$)

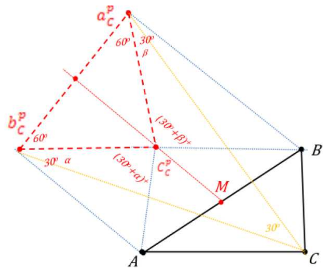

Fig.10c ($\gamma = 30^o$)

Notice that if either $\gamma < 30^o$ or $\gamma > 30^o$ then Ab^p_C and Ba^p_C intersect each other, while for $\gamma = 30^o$ $Ab^p_C // Ba^p_C$.

<u>First</u> we deal with the erected sides Ca^p_C and Cb^p_C and prove that they are trisectors of $\angle C$. We also show that $\angle C = 3\gamma$. See Fig.11.

Let b^d_C be the intersection of AC^p_C and Ca^p_C. Notice the choice of angles in the construction of $\triangle a^p_C Cb^p_C$ and $\triangle a^p_C BC^p_C$ yields $\angle Ca^p_C C^p_C = \beta$ and $\angle a^p_C C^p_C b^d_C = 180^o - (\gamma + \alpha)^+ - 60^o = \beta$. Hence $\triangle a^p_C b^d_C C^p_C$ is isosceles. Thus $\angle Ab^d_C C = 2\beta$ and $\angle Ab^p_C C = \frac{1}{2}\angle Ab^d_C C$.

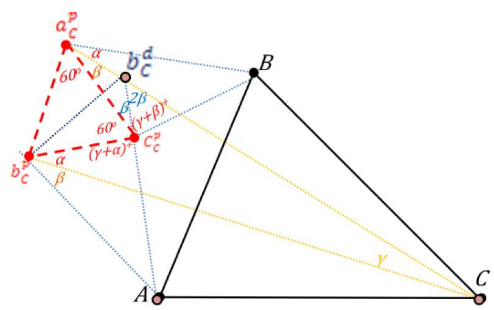

Fig.11

Since $\triangle a^p_C b^d_C C^p_C$ is isosceles and from the assumption $\triangle a^p_C C^p_C b^p_C$ is equilateral, infer that $b^d_C b^p_C$ bisects $a^p_C C^p_C$ and so $b^d_C b^p_C$ is the $\angle a^p_C b^d_C C^p_C$ bisector. Hence b^d_C is lying on the exterior bisector of $\triangle Ab^d_C C$. Thus, by the Excenter Lemma, b^p_C is the excenter of $\triangle Ab^d_C C$ relative to $\angle C$. But so Cb^p_C is the $\angle ACa^p_C$ bisector.

Similarly show that Ca^p_C is the $\angle BCb^p_C$ bisector.

Therefore Cb^p_C and Ca^p_C are trisectors of $\angle C$. Also

$$\angle a^p_C Cb^p_C = 180^o - \angle Cb^p_C a^p_C - \angle Ca^p_C b^p_C = 180^o - \alpha^+ - \beta^+ = \gamma.$$

Conclude $\angle C = 3\gamma$.

<u>Next</u> we deal with the erected sides AC^p_C, Ab^p_C and BC^p_C, Ba^p_C and prove that are trisectors of $\angle A$ and $\angle B$. We also show that $\angle A = 3\alpha$ and $\angle B = 3\beta$.

▷ Assume $\gamma \neq 30^o$. Set $s = (30^o - \gamma)/|30^o - \gamma|$ and let C_C^d be the meeting point of Ab_C^p and Ba_C^p. The choice of angles in the erection of $\triangle C_C^p Ab_C^p$and $\triangle C_C^p Ba_C^p$ implies:

C_C^p is lying on the $\angle AC_C^d B$ bisector (1) and $\angle AC_C^p B = 90^o + \frac{1}{2}s\angle AC_C^d B$ (2)

To verify (1) notice that $\triangle a_A^p C_C^d b_C^p$ is isosceles, as two of its angles are either $120^o - (\alpha + \beta)$ ($\gamma < 30^o$) or $60^o + (\alpha + \beta)$ ($\gamma > 30^o$). Using the fact that $\triangle a_C^p C_C^p b_C^p$ is equilateral, infer that $C_C^d C_C^p$ bisects $a_C^p b_C^p$ and so C_C^p is lying on the $\angle a_C^p C_C^d b_C^p$ bisector.

To verify (2) notice that in the isosceles $\triangle a_C^p C_C^d b_C^p$
⋄ if $\gamma < 30^o$ then $\angle AC_C^d B = 180^o - 2[120^o - (\alpha + \beta)] = 60^o - 2\gamma = s(60^o - 2\gamma)$,
⋄ if $\gamma > 30^o$ then $\angle AC_C^d B = 180^o - 2[60^o + (\alpha + \beta)] = 2\gamma - 60^o - s(60^o - 2\gamma)$.
So for the cases $\gamma < 30^o$ and $\gamma > 30^o$ have respectively:
⋄ $\angle AC_C^d B = s(60^o - 2\gamma)$ and
⋄ $\angle AC_C^p B = 360^o - (\alpha + \gamma)^+ - (\beta + \gamma)^+ = 120^o - \gamma = 90^o + \frac{1}{2}(60^o - 2\gamma) = 90^o + \frac{1}{2}s\angle AC_C^d B$.

Therefore from (1) and (2), by the Incenter Lemma ($\gamma < 30^o$, $s = 1$) or the Excenter Lemma ($\gamma > 30^o$, $s = -1$), C_C^p is the incenter or the excenter of $\triangle AC_C^d B$ respectively. Thus

$$\angle C_C^p Ab_C^p = \angle C_C^p AB \text{ and } \angle C_C^p Ba_C^p = \angle C_C^p BA.$$

Moreover from $\triangle C_C^p Ab_C^p$ and $\triangle C_C^p Ba_C^p$ infer $\angle C_C^p Ab_C^p = \gamma + \beta$ and $\angle C_C^p Ba_C^p = \gamma + \alpha$.
Deduce for $\gamma \neq 30^o$ it holds $\angle C_C^p Ab_C^p = \angle C_C^p AB = \gamma + \beta$ and $\angle C_C^p Ba_C^p = \angle C_C^p BA = \gamma + \alpha$.

▷ Assume $\gamma = 30^o$ and so $\alpha + \beta = 30^o$. Then the choice of angles in construction of $\triangle C_C^p Ba_C^p$ and $\triangle C_C^p Ab_C^p$ implies $\angle C_C^p b_C^p A = \angle C_C^p a_C^p B = \alpha + \beta = 30^o$. Hence $Ab_C^p, Ba_C^p \perp a_C^p b_C^p$ and thus $Ab_C^p // Ba_C^p$. Draw from C_C^p the height of the equilateral $\triangle a_C^p b_C^p C_C^p$ meeting AB at M. Hence $C_C^p M // Ab_C^p // Ba_C^p$, and also MC_C^p bisects $a_C^p b_C^p$. But so M is the midpoint of AB. Also notice that $\triangle AC_C^p B$ is right triangle as $\angle AC_C^p B = 360^o - 60^o - (\beta + 30^o)^+ - (\alpha + 30^o)^+ = 90^o$. Then $C_C^p M = MA = MB$. Now $C_C^p M = MA$ implies $\angle C_C^p AM = \angle MC_C^p A$. Yet $C_C^p M // Ab_C^p$ implies $\angle C_C^p Ab_C^p = \angle AC_C^p M$. Thus $\angle C_C^p Ab_C^p = \angle C_C^p AM$ and so AC_C^p is the $\angle b_C^p AB$ bisector.

Similarly it is shown that BC_C^p is the $\angle a_C^p BA$ bisector.

Also the choice of angles in the construction of $\triangle C_C^p Ab_C^p$ gives

$$\angle C_C^p Ab_C^p = 180^o - 30^o - (\alpha + 30^o)^+ = 30^o + \beta.$$

Deduce for $\gamma = 30^o$ it holds $\angle C_C^p Ab_C^p = \angle C_C^p AB = 30^o + \beta = \gamma + \beta$.
Similarly it is shown that $\angle C_C^p Ba_C^p = \angle C_C^p BA = \gamma + \alpha$.

∘ Conclude for all γ it holds

$$\angle C_C^p Ab_C^p = \angle C_C^p AB = 30^o + \beta = \gamma + \beta \text{ and } \angle C_C^p Ba_C^p = \angle C_C^p BA = \gamma + \alpha.$$

From $\triangle Ab_C^p C$ it follows $\angle b_C^p AC = 180^o - \beta - \gamma$, since from the construction choice of angles $\angle Ab_C^p C = \beta$ and $\angle ACb_C^p = \gamma$, as found in the first step. But clearly $\angle b_C^p AC = \angle b_C^p AC_C^p + \angle C_C^p AB + \angle CAB = 2(\gamma + \beta) + \angle A$. Then $\angle A = 3\alpha$ and similarly $\angle B = 3\beta$. Therefore the angles of $\triangle ABC$ are 3α, 3β and 3γ. Since $\angle C_C^p Ab_C^p = \angle C_C^p AB = \gamma + \beta$ it follows that Ab_C^p and AC_C^p

are trisectors of $\angle A$ in $\triangle ABC$.
Similarly it shown that Ba_C^p and BC_C^p are trisectors of $\angle B$ in $\triangle ABC$.

Corollary 3. *a) In any $\triangle ABC$, the angles formed by the side's of the exterior Morley equilateral $\triangle a_C^p C_C^p b_C^p$ relative to the $\angle C$ and the exterior trisectors of $\angle A$ and $\angle B$ are: $\angle a_C^p C_C^p B = (\gamma+\beta)^+$, $\angle b_A^p C_C^p A = (\gamma+\alpha)^+$, $\angle C_C^p a_C^p B = \angle C_C^p b_A^p A = \alpha+\beta$, while with the interior trisectors of $\angle C$ are: $\angle C_C^p a_C^p C = \beta$ and $\angle C_C^p b_C^p C = \alpha$.*

b) The heights of the equilateral $\triangle a_C^p C_C^p b_C^p$ are: $a_C^p a_C^d$, $b_C^p b_C^d$ and $C_C^p C_C^d$.

4 Implications

4.1 Companion Equilaterals of the inner Morley equilateral

Theorem 4. *The two sides' extensions of the inner Morley equilateral meet the corresponding inner trisectors at two points which with the two sides' common vertex form an equilateral.*

Proof. As usually $\triangle A^p B^p C^p$ denotes the interior Morley equilateral of $\triangle ABC$. Let S_A be the intersection of the extension of side $A^p C^p$ with the trisector CB^p and let K_A be the intersection of the extension of side $A^p B^p$ with the trisector BC^p. Moreover let A^d be the intersection of the trisectors BC^p and CB^p. By Corollary 1a, $\angle A^p B^p C = \angle C^p A^p B = \alpha^+$ and so $\triangle B^p A^d C^p$ is isosceles. Thus $A^p A^d$ is bisector of both $\angle B^p A^p C^p$ and $\angle B^p A^d C^p$.
Hence $\angle B^p A^p A^d = \angle C^p A^p A^d$ and $\angle B^p A^d A^p = \angle C^p A^d A^p$.
Also $\angle S_A A^d A^p = \angle K_A A^d A^p$ as obviously $\angle S_A A^d C^p = \angle K_A A^d B^p$ and $\angle S_A A^d A^p = \angle S_A A^d C^p + \angle C^p A^d A^p$ while $\angle K_A A^d A^p = \angle K_A A^d A^p + \angle B^p A^d A^p$.
Therefore $\triangle A^p S_A A^d$ and $\triangle A^p K_A A^d$ are equal because in addition they have side $A^p A^d$ in common. Consequently $\triangle S_A A^p K_A$ is equilateral.

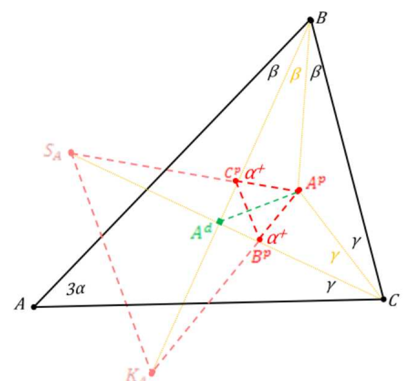

Fig.12

The previous equilateral is named *companion equilateral relative to vertex* A^p and it will be denoted by $\triangle S_A A^p K_A$. Obviously there are two more companion equilaterals relative to vertices B^p and C^p, denoted by $\triangle S_B B^p K_B$ and $\triangle S_C C^p K_C$, respectively.

Corollary 4. *For the companion equilateral relative to vertex A^p, $\triangle S_A A^p K_A$, it holds*
$$\angle BAS_A = \angle CAK_A = |\beta+\gamma-\alpha|.$$

In fact, the points S_A *and* K_A *, for* $\alpha < 30^o$ *are outside* $\triangle ABC$*, for* $\alpha = 30^o$ *are on* AB *and* AC*, respectively, and for* $\alpha > 30^o$ *are inside* $\triangle ABC$*.*

Proof. Corollary 1a asserts $\angle C^p A^p B = \gamma^+$ and $\angle B^p A^p C = \beta^+$. Thus from $\triangle K_A A^p B$ and $\triangle K_A A^p B$ it is calculated $\angle C^p S_A B^p = \alpha$ and $\angle C^p K_A B^p = \alpha$, respectively. Then the points B^p, C^p, S_A and K_A are cyclic as $B^p C^p$ is seen from S_A and K_A with the same angle. Thus $\angle AS_A K_A = \angle AB^p K_A$ and $\angle AK_A S_A = \angle AC^p S_A$. From $\triangle BC^p A$ and $\triangle CB^p A$ infer that $\angle AC^p K_A = \beta + \gamma$ and $\angle AB^p S_A = \alpha + \gamma$, respectively. Hence
$\angle AS_A K_A = \beta + \gamma$ and $\angle AK_A S_A = \alpha + \gamma$.

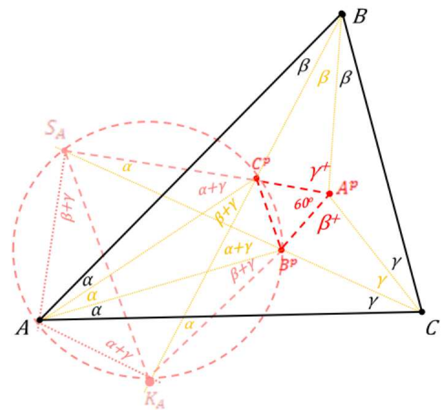

Fig.13

Next notice that $\angle S_A AC^p = 180^o - \angle AS_A C^p - \angle S_A C^p A = 180^o - (60^o + \beta + \gamma) - (\alpha + \gamma) = \beta + \gamma$
and similarly $\angle K_A AC = \beta + \gamma$.
However

$$\angle S_A AC^p = \angle S_A AB \pm \angle BAC^p \text{ and}$$

$$\angle K_A AC = \angle K_A AC \pm \angle CAB^p,$$

where $\pm$ may be either $+$ or $-$ depending on the location of S_A and K_A with respect to AB and AC respectively. Therefore $\angle S_A AB = \angle K_A AC = |\beta + \gamma - \alpha|$.

Because $\alpha + \beta + \gamma = 60^o$ for $\alpha < 30^o$ the points S_A and K_A are outside $\triangle ABC$, for $\alpha = 30^o$ the points S_A and K_A are on AB and AC, respectively and for $\alpha > 30^o$ the points S_A and K_A are inside $\triangle ABC$.

4.2 Relation of the Morley equilaterals

Theorem 5. *In any triangle the sides of Morley equilaterals are either collinear or parallel.*

Proof. Corollary 2a claims $\angle BC^p_C A^p_A = \alpha$. Also Corollary 3a confirms $\angle BC^p_C a^p_C = (\beta + \gamma)^+$. Hence $\angle B^p_B C^p_C a^p_C = \angle B^p_B C^p_C A^p_A + \angle A^p_A C^p_C B + \angle BC^p_C a^p_C =$
$= 60^o + \alpha + (\beta + \gamma)^+ = 180^o$.

Thus $a^p_C C^p_C$ is extension of $B^p_B C^p_C$.
Similarly it is shown that $b^p_C C^p_C$ is extension of $A^p_A C^p_C$.

As $\angle b^p_C a^p_C C^p_C = \angle A^p_A B^p_B C^p_C = 60^o$, it follows

$$a^p_C b^p_C // A^p_A B^p_B.$$

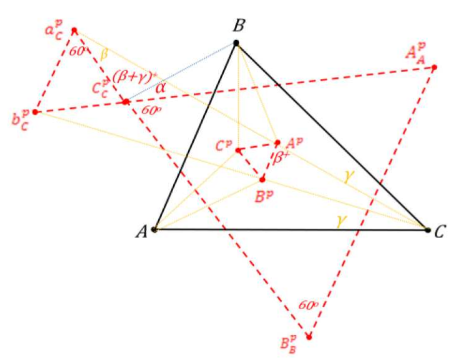

Fig.14

Since $\angle b^p_C a^p_C C = \angle b^p_C a^p_C C^p_C + \angle C^p_C a^p_C C = 60^o + \beta$

and $\angle B^p A^p C = \beta^+$ then

$$a_C^p b_C^p // A^p B^p.$$

The previous result is mentioned as a fact in [8] and it might be in print elsewhere. At any event, it inspires the next theorem.

4.3 Interrelationship between central and exterior Morley equilaterals

Theorem 6. *In any triangle, the exterior trisectors of its angles, proximal to the three sides respectively, meet at the vertices of an equilateral, if and only if, the interior trisectors of an angle and the exterior trisectors of the other two angles, proximal the three sides respectively, meet at the vertices of an equilateral.*

Proof. Let $\triangle ABC$ be given with $\angle A = 3\alpha$, $\angle B = 3\beta$ and $\angle C = 3\gamma$, where $\alpha + \beta + \gamma = 60^0$.

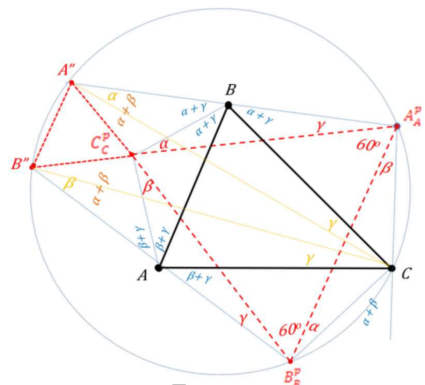

Fig.15a

($\Longrightarrow$) Assume that $\triangle A_A^p B_B^p C_C^p$ is the central Morley equilateral formed by the intersections of the exterior trisectors, proximal to the sides of $\triangle ABC$. See Fig.15a.

Extend $A_A^p C_C^p$ and $B_B^p C_C^p$ to meet the extensions of $A_A^p B$ and $B_B^p A$ at A'' and B'' respectively. Then AB'' and BA'' are trisectors of the exterior $\angle A$ and $\angle B$ respectively, as extensions of the corresponding trisectors. Thus it suffices to show that $\triangle A'' C_C^p B''$ is equilateral and also that CA'' and CB'' are trisectors of the interior $\angle C$.

First show that $\triangle A'' C_C^p B''$ is equilateral.

Notice that $\triangle A'' A_A^p B_B^p = \triangle B'' A_A^p B_B^p$ because they have $A_A^p B_B^p$ common, $\angle A'' B_B^p A_A^p = \angle B'' A_A^p B_B^p = 60^o$ and $\angle A'' A_A^p B_B^p = \angle B'' B_B^p A_A^p = 60^o + \gamma$ by Corollary 2a. Thus $A'' B_B^p = B'' A_A^p$. But so $C_C^p A'' = C_C^p B''$, as $C_C^p A_A^p = C_C^p B_B^p$ since $\triangle A_A^p B_B^p C_C^p$ is equilateral. Also $\angle A'' C_C^p B'' = \angle A_A^p C_C^p B_B^p = 60^o$ and hence $\triangle A'' B_B^p C_C^p$ is equilateral.

Next show that CA'' and CB'' are trisectors of the interior $\angle C$.
From $\triangle A_A^p A'' B_B^p$ and $\triangle A_A^p B'' B_B^p$ it is easily calculated that $\angle A_A^p A'' B_B^p = \alpha + \beta$ and $\angle AB'' B_B^p = \alpha + \beta$. Hence A_A^p, B_B^p, A'', B'' are cyclic. But $\angle B_B^p A_A^p C = \beta$ and $\angle A_A^p B_B^p C = \alpha$, by Corollary 2a. So from $\triangle A_A^p C B_B^p$, $\angle A_A^p C B_B^p = 180^o - (\alpha + \beta)$. But so C is also on this circle. Thus $\angle CA'' A_A^p = \angle A_A^p B_B^p C = \alpha$ and $\angle CB'' B_B^p = \angle B_B^p A_A^p C = \beta$. Finally note that in $\triangle BA''C$ and $\triangle CB''A$ it holds respectively $\angle A''CB = (\alpha + \gamma) - \alpha = \gamma$ and $\angle B''CA = (\beta + \gamma) - \beta = \gamma$.

Conclude CA'' and CB'' are trisectors of the interior $\angle C$, as $\angle C = 3\gamma$.

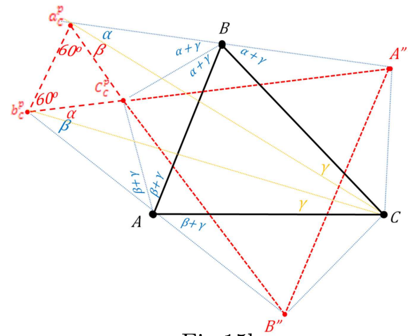

Fig.15b

($\Longleftarrow$) Assume that the exterior triangle $\triangle a_C^p C_C^p b_C^p$ is equilateral formed by the intersections of the trisectors of the interior $\angle C$ and the exterior $\angle A$ and $\angle B$. See Fig.15b.

Extend $a_C^p C_C^p$ and $b_C^p C_C^p$ to meet the extensions of $a_C^p B$ and b_C^p A at A'' and B'' respectively. Then AB'' and BA'' are trisectors of the exterior $\angle A$ and $\angle B$ respectively, as extensions of the corresponding trisectors.

Thus it suffices to show that $\triangle A''C_C^p B''$ is equilateral and that CA'' and CB'' are trisectors of the interior $\angle C$.

Note $\triangle A''a_C^p b_C^p = \triangle B''a_C^p b_C^p$, because they have $a_C^p b_C^p$ common, $\angle A''B_B^p A_A^p = \angle B''A_A^p B_B^p = 60^o$ and $\angle A''A_A^p B_B^p = \angle B''B_B^p A_A^p = 60^0 + \alpha + \beta$ by Corollary 3a. Thus $A''b_C^p = B''a_C^p$. Since $\triangle a_C^p C_C^p b_C^p$ is equilateral then $C_C^p A'' = C_C^p B''$ and so $\triangle A''C_C^p B''$ is equilateral.

Also from $\triangle a_C^p BC$ and $\triangle b_C^p CA$ it is calculated that $\angle a_C^p CB = \angle b_C^p CA = \gamma$ and so Ca_C^p and Cb_C^p are trisectors of $\angle C$.

5 The non Equilateral Morley Triangles

For a given $\triangle ABC$ there are in general 64 Morley triangles, as the trisectors of its three angles meet at many points. Among them are the inner, the central and the exterior Morley equilaterals.

A number of authors (see for example [4] or [5]) have wondered: *Are there more Morley equilaterals for $\triangle ABC$?*

This part examines all the remaining Morley triangles of $\triangle ABC$ systematically and shows that none of them is equilateral.

In the sequel the following easily proved lemma is used.

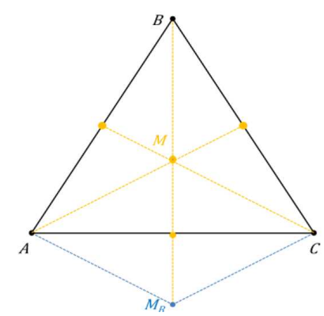

Fig.16

The Equilateral Center Lemma. *The incenter of an equilateral is the unique interior point from which its sides are seen with* 120^o. *Similarly the excenter relative to an angle is the unique exterior point from which the side opposite to the angle is seen with* 120^o *while the other two sides are seen with* 60^o.

5.1 Morley triangles by trisectors of interior angles

This section treats the non equilateral Morley triangles formed by the trisectors of the interior angles of $\triangle ABC$. The proximal to the sides trisectors meet at A^p, B^p and C^p and $\triangle A^pB^pC^p$

denotes the inner Morley equilateral.

5.1.1 *The Interior Morley triangle of distal vertices*

The interior Morley triangle of distal vertices is denoted by $\triangle A^dB^dC^d$ where A^d, B^d and C^d are the meeting points of the distal trisectors with respect to the sides BC, CA and AB, respectively, as shown in Fig.17. If $\triangle ABC$ is equilateral then $\triangle A^dB^dC^d$ is equilateral as well. Thus in the following we assume that $\triangle ABC$ is not equilateral.

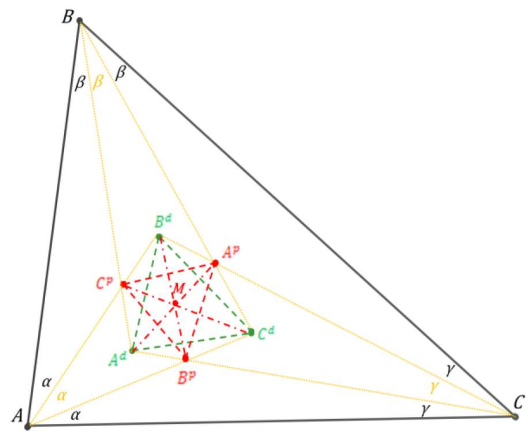

Fig.17

From Corollary 1b, we have that A^pA^d, B^pB^d and C^pC^d are the heights of the inner Morley equilateral $\triangle A^pB^pC^p$.

Let M be the center of $\triangle A^pB^pC^p$. Thus $\angle A^pMB^d = \angle B^dMC^p = \angle C^pMA^d = \angle A^dMB^p$

$$= \angle B^pMC^d = \angle C^dMA^p = 60^o$$

So $\angle A^dMB^d = \angle B^dMC^d = \angle C^dMA^d = 120^o$. Hence the sides of $\triangle A^dB^dC^d$ are seen from M with 120^o.

Assume towards a contradiction that $\triangle A^dB^dC^d$ is equilateral. Then, by the Equilateral Center Lemma, A^pA^d is a height of $\triangle A^dB^dC^d$. Thus A^pA^d bisects B^dC^d. Hence A^pA^d bisects $\angle B^dA^pC^d$ and so $\angle BA^pC$. Since A^p is the incenter of $\triangle BA^dC$, A^pA^d bisects also $\angle BA^dC$. But so the exterior angles of $\triangle A^dA^pB$ and $\triangle A^dA^pC$ at vertex A^p are $\frac{1}{2}\angle BA^pC = \frac{1}{2}\angle BA^dC + \beta$ and $\frac{1}{2}\angle BA^pC = \frac{1}{2}\angle BA^dC + \gamma$. Hence $\beta = \gamma$. Similarly it is shown that $\alpha = \beta$. Thus $\triangle ABC$ is equilateral contrary to the assumption.

Conclude that $\triangle A^dB^dC^d$ cannot be equilateral (if $\triangle ABC$ is not equilateral).

5.1.2 *Interior Morley triangles with one proximal and two mix vertices*

There are three interior Morley triangles with one proximal and two mix vertices denoted by $\triangle A^pB^*C^*$, $\triangle B^pC^*A^*$ and $\triangle C^pA^*B^*$. We will study only $\triangle A^pB^*C^*$ as the other two are similar.

Since A^p is the intersection of the proximal trisectors, B^* must be the intersection of the remaining trisector CB^p (proximal to CA) with AC^p as distal. Then C^* is the intersection of the left trisectors BC^p (distal to AB) and AB^p (proximal). So B^* is on AC^p and C^* is on AB^p. See Fig.18. Corollary 1a asserts $\angle AC^pB^p = \beta^+$ and $\angle AB^pC^p = \gamma^+$. Hence $\angle AC^pA^p = 60^o + \beta^+ < 180^o$ and $\angle AB^pA^p = 60^o + \gamma^+ < 180^o$. Therefore the quadrangle $AB^pA^pC^p$ is convex and so $\angle B^*A^pC^*$ is inside $\angle B^pA^pC^p$.

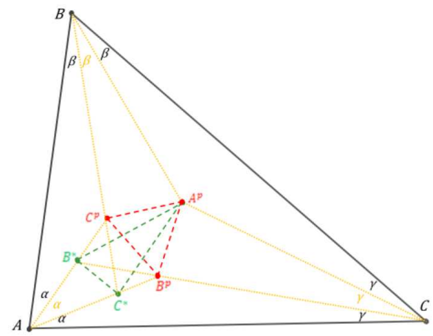

Fig.18

Conclude that $\angle B^*A^pC^* < 60^o$ and thus $\triangle A^pB^*C^*$ cannot be equilateral.

5.1.3 *Interior Morley triangles with one distal and two mix vertices*

There are three interior Morley triangles with one distal and two mix vertices which are denoted by $\triangle A^dB^*C^*$, $\triangle B^dC^*A^*$ and $\triangle C^dA^*B^*$. We will study only $\triangle A^dB^*C^*$ as the other two are similar. Since A^d is the intersection of the distal trisectors, B^* must be the intersection of the remaining trisectors CA^p (distal to CA) and AB^p, as proximal. So C^* is the intersection of the left trisectors AC^p and BA^p (mix to AB).

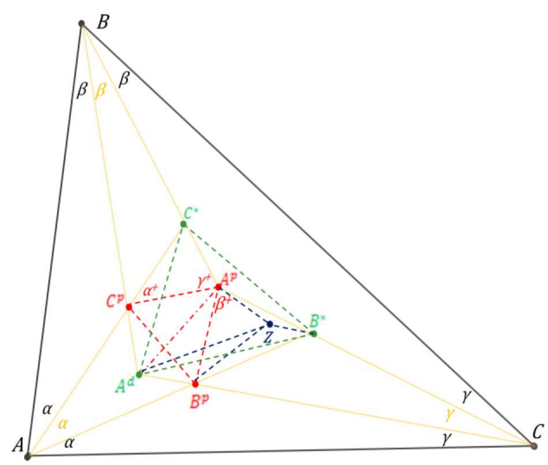

Fig.19

First notice that if $\beta = \gamma$ then $\triangle A^dB^*C^*$ is isosceles, because Corollary 1a with $\beta = \gamma$ yields $\triangle B^pA^pB^* = \triangle C^pA^pC^*$ and so $A^pB^* = A^pC^*$ which implies $\triangle A^dA^pB^* = \triangle A^dA^pC^*$.

Thus if $\alpha = \beta = \gamma$ then $\triangle A^dB^*C^*$ is equilateral.

Assume $\triangle ABC$ is not equilateral. Then it has two sides not equal and thus in the following we may assume $\gamma < \beta$. Fig.19.

Suppose, towards a contradiction, that $A^dB^* = A^dC^*$. Let Z be the symmetric point of C^* with respect to A^pA^d. We will fist show that Z is inside $\triangle A^dA^pB^*$.
From Corollary 1b, A^pA^d is height of the equilateral $\triangle A^pB^pC^p$ and so B^p and C^P are symmetric with respect to A^pA^d. Consequently $\angle B^pA^pZ = \angle C^pA^pC^*$ and since $\angle C^pA^pC^* = \gamma^+$ infer $\angle B^pA^pZ = \gamma^+$. Since $\angle B^pA^pB^* = \beta^+$ and by assumption $\gamma < \beta$, deduce $\angle B^pA^pZ < \angle B^pA^pB^*$. Moreover $\angle A^pB^pZ = \angle A^pC^pC^* = \alpha + (\alpha + \beta) = \alpha + \gamma$ while $\angle A^pB^pB^* = \alpha + (\alpha + \gamma) = \alpha + \beta$. So $\angle A^pB^pZ < \angle A^pB^pB^*$.

Since Z is inside $\triangle A^dA^pB^*$ then $\angle A^dZB^* > \angle A^dA^pB^* = \angle A^dA^pB^p + \angle B^pA^pB^* = 30^o + \beta^+ = 90^o + \beta$. However, the assumption $A^dB^* = A^dC^*$ implies $A^dZ = A^dB^*$ and so $\angle A^dZB^* = \angle A^dB^*Z$. Thus $\angle A^dZB^* + \angle A^dB^*Z > 2(90^o + \beta) > 180^o$. Hence two angles of $\triangle A^dZB^*$ have sum greater than 180^o, which is a contradiction.

Conclude that $\triangle A^dB^*C^*$ cannot be equilateral (if $\triangle ABC$ is not equilateral).

5.2 Morley triangles by trisectors of exterior angles

This section treats the non equilateral Morley triangles formed by the trisectors of the exterior angles. So throughout this section trisectors mean trisectors of the exterior angles of $\triangle ABC$.

The proximal trisectors meet at the points A_A^p, B_B^p and C_C^p and so $\triangle A_A^pB_B^pC_C^p$ denotes the central Morley equilateral. Notice that the trisectors BC_C^p and CB_B^p are parallel iff

$$\angle B_B^pCB + \angle C_C^pBC = 180^o \Leftrightarrow 2(\alpha+\beta) + 2(\alpha+\gamma) = 180^o.$$

So for $\alpha = 30^o$ $BC_C^p//CB_B^p$. In this case the distal trisectors with respect to BC do not intersect and hence the distal to BC vertex A_A^d is not determined. Also if $30^o > \alpha$ then A_A^d and A_A^p are on the same side of BC while $\angle BA_A^dC = 60^o - 2\alpha$. If $30^o < \alpha$ then A_A^d and A_A^p are on different sides of BC while $\angle BA_A^dC = 2\alpha - 60^o$. See Fig.20.

5.2.1 *The Central Morley triangle of distal vertices*

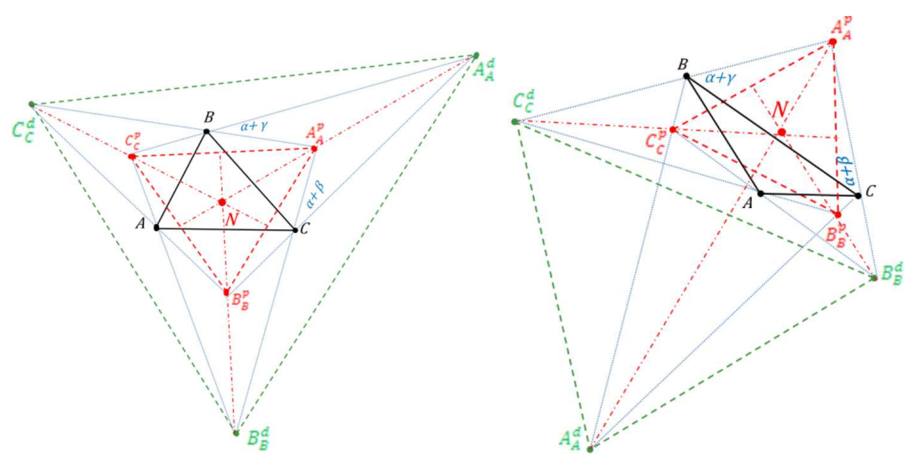

Fig.20a ($\gamma < 30^o$) Fig.20b ($\gamma > 30^o$)

The Morley triangle of distal vertices is denoted by $\triangle A_A^dB_B^dC_C^d$, where A_A^d, B_B^d and C_C^d are the meeting points of the distal trisectors of the exterior angles with respect to the sides BC, CA and AB, respectively. In Fig.20a and Fig.20b the different locations of $\triangle A_A^dB_B^dC_C^d$ with respect to $\triangle A_A^pB_B^pC_C^p$ are illustrated. Note that $A_A^pA_A^d$, $B_B^pB_B^d$ and $C_C^pC_C^d$, by Corollary 1b, are the heights of the central Morley equilateral $\triangle A_A^pB_B^pC_C^p$ and let N be their intersection.

Notice that if $\triangle ABC$ is equilateral then $\triangle A_A^dB_B^dC_C^d$ is equilateral as well. Thus in the following we assume that $\triangle ABC$ is not equilateral. Also suppose towards a contradiction that $\triangle A_A^d B_B^d C_C^d$ is equilateral.

▷ If $\triangle ABC$ is an acute triangle the equilateral $\triangle A_A^p B_B^p C_C^p$ is inside $\triangle A_A^d B_B^d C_C^d$. Hence
$$\angle A_A^d N B_B^d = \angle B^d N C_C^d = \angle C_C^d N A_A^d = 120^o.$$

▷ If $\triangle ABC$ is an obtuse triangle (assume $\alpha > 30^o$) A_A^d and A_A^p are on different sides of BC. Hence
$$\angle A_A^d N B_B^d = \angle A_A^d N C_C^d = 60^o \text{ and } \angle B_B^d N C_C^d = 120^o.$$

Thus, by the Equilateral Center Lemma, N is the incenter (acute) or the excenter (obtuse) of the assumed equilateral $\triangle A_A^d B_B^d C_C^d$. Hence $A_A^p A_A^d$ is a height of $\triangle A_A^d B_B^d C_C^d$ and so $A_A^p A_A^d$ bisects $B_B^d C_C^d$ and $\angle B_B^d A_A^p C_C^d$.

▷ In the case of the acute triangle, notice in $\triangle B A_A^d C$ that A_A^p is the incenter while the bisector $A_A^p A_A^d$ bisects $\angle B A_A^p C$. As a result $\angle A_A^d BC = \angle A_A^d CB \Leftrightarrow 2(\alpha+\gamma) = 2(\alpha+\beta) \Leftrightarrow \gamma = \beta$.

▷ In the case of the obtuse triangle, note that since $A_A^p A_A^d$ bisects $B_B^d C_C^d$ it bisects $\angle B A_A^p C$. Also it bisects $\angle B A_A^d C$ as a height of $\triangle A_A^p B_B^p C_C^p$. Then $\triangle A_A^p B A_A^d = \triangle A_A^p C A_A^d$ and so $A_A^p B = A_A^p C$.
Therefore $\alpha+\gamma = \alpha+\beta$ and so $\gamma = \beta$. Similarly we show that $\alpha = \beta$.
Deduce that $\triangle ABC$ is equilateral contrary to the assumption that it is not.

Conclude that $\triangle A_A^d B_B^d C_C^d$ cannot be equilateral (if $\triangle ABC$ is not equilateral).

5.2.2 *The Central Morley triangles with one proximal and two mix vertices*

There are three Morley triangles of $\triangle ABC$ formed by exterior trisectors with one proximal and two mix vertices denoted by $\triangle A_A^p B_B^* C_C^*$, $\triangle B_B^p C_C^* A_A^*$ and $\triangle C_C^p A_A^* B_B^*$. We will study only $\triangle A_A^p B_B^* C_C^*$ as the other two are similar.

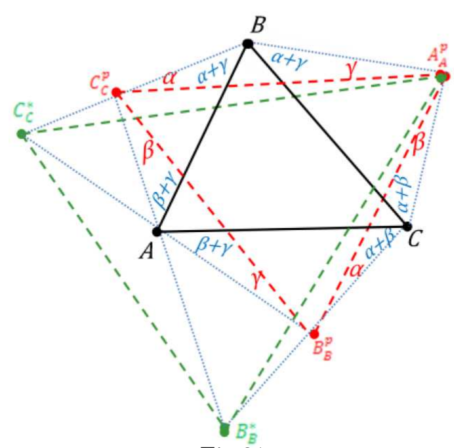

Fig.21

As vertex A_A^p is the intersection of the proximal to BC trisectors, vertex B_B^* is the intersection of the remaining trisector CB_B^p (proximal to CA) with AC_C^p, as distal. Then vertex C_C^* is the intersection of the left trisectors AB_B^p (distal to AB) and BC_C^p (proximal). Thus B_B^* is on CB_B^p while C_C^* is on BC_C^p. Using the angle values between the sides of $\triangle A_A^p B_B^p C_C^p$ and the trisectors of $\triangle ABC$ given by Corollary 2a it is easily deduced that all the angles of the quadrangles $B_B^* B_B^p A_A^p C_C^p$ and

$C_C^* C_C^p A_A^p B_B^p$ are less than 180^o and so they are convex.

For instance

$$\angle C_C^* C_C^p A_A^p = 180^o - \angle B C_C^p A_A^p = 180^o - \alpha,$$

while

$$\angle A B_B^p A_A^p = 60^o + \alpha.$$

Since $B_B^* B_B^p A_A^p C_C^p$ is convex infer $A_A^p C_C^*$ is inside $\angle B_B^p A_A^p C_C^p$. Also since $C_C^* C_C^p A_A^p B_B^p$ is convex infer $A_A^p B_B^*$ is inside $\angle B_B^p A_A^p C_C^p$.

Conclude that $\angle B_C^* A_A^p C_C^* < L B_B^p A_A^p C_C^p = 60^o$ and so $\triangle A_A^p B_B^* C_C^*$ cannot be equilateral.

5.2.3 *The Central Morley triangles with one distal and two mix vertices*

There are three Morley triangles formed by exterior trisectors with one distal and two mix vertices which are denoted by $\triangle A_A^d B_B^* C_C^*$, $\triangle B_B^d C_C^* A_A^*$ and $\triangle C_C^d A_A^* B_B^*$. We will study only $\triangle A_A^d B_B^* C_C^*$ as the other two are similar.

Since A_A^d is the intersection of the distal to BC trisectors BC_C^p and CB_B^p, B_B^* is the intersection of the remaining trisector CA_A^p (proximal to CA) with AB_B^p as distal. Then C_C^* is the intersection of the left trisectors AC_C^p (proximal to AB) and BA_A^p (distal).

If $\alpha = 30^o$ then A_A^d is not determined. If $30^o > \alpha$ then A_A^d is on the same side of BC with A_A^p and $\angle B A_A^d C = 60^o - 2\alpha$. If $30^o < \alpha$ then A_A^d and A_A^p are on different sides of BC while $\angle B A_A^d C = 2\alpha - 60^o$.

Consider the case $\beta = \gamma$. Then $\triangle B_B^p A_A^p B_B^* = \triangle C_C^p A_A^p C_C^*$ by Corollary 2a. So $A_A^p B_B^* = A_A^p C_C^*$ and in turn $\triangle A_A^d A_A^p B_B^* = \triangle A_A^d A_A^p C_C^*$. Thus $\triangle A_A^d B_B^* C_C^*$ is isosceles. So if $\triangle ABC$ is equilateral then $\triangle A_A^d B_B^* C_C^*$ is equilateral.

Next consider that $\triangle ABC$ is not equilateral and let $\beta > \gamma$.

Since the location of A_A^d depends on the value of α, assume that $30^o > \alpha$. Suppose, towards a contradiction that $\triangle A_A^d B_B^* C_C^*$ is equilateral. Let W be the symmetric point of C_C^* with respect to $A_A^p A_A^d$ and let V be the intersection of $A_A^d W$ and $B_B^* B_B^p$. In the

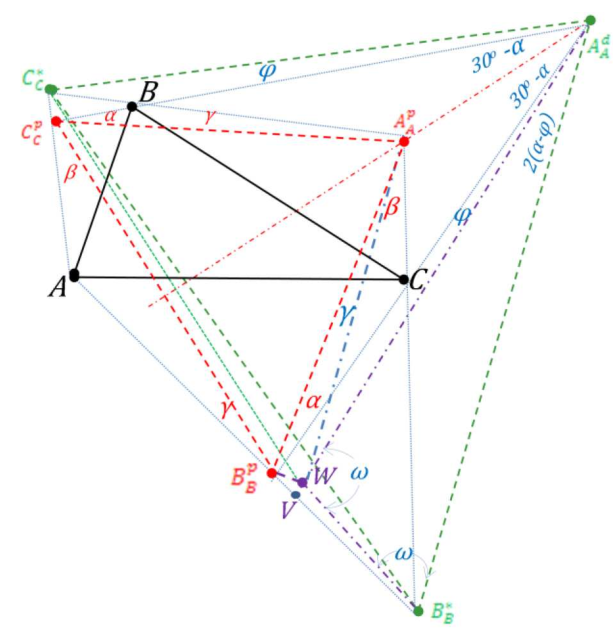

Fig.22

sequel we find the location of W.

From Corollary 2b, $A_A^p A_A^d$ is height of the equilateral $\triangle A_A^p B_B^p C_C^p$ and so A_A^p and B_B^p are symmetric with respect to $A_A^p A_A^d$.

Also by Corollary 2a the angles between the trisectors of $\triangle ABC$ and the central Morley equilateral are as depicted in Fig.22. Consequently,
$\angle A_A^d C_C^p W = \angle A_A^d C_C^p C_C^* = 180^o - 60^o - \alpha - \beta = 60^o + \gamma$ and $\angle A_A^d B_B^p B_B^* = 180^o - 60^o - \alpha - \beta = 60^o + \beta$. Therefore $\angle B_B^* B_B^p W < \angle A_A^d B_B^p B_B^*$ and the line $B_B^p W$ is inside $\angle C B_B^p B_B^*$.

Furthermore $\angle B_B^p A_A^p W = \angle C_C^p A_A^p C_C^* = \gamma$ and since $\beta > \gamma$, $A_A^p W$ is inside $\angle B_B^p A_A^p B_B^*$. Thus W is inside $\triangle C B_B^p B_B^*$.
Clearly C_C^* is outside $\triangle ABC_C^p$. Thus we may set $\angle C_C^p A_A^d C_C^* = \varphi$. Then by symmetry $\angle B_B^p A_A^d W = \varphi$. Also set $\angle A_A^d W B_B^* = \omega$. Since $\triangle A_A^d B_B^* C_C^*$ is assumed equilateral then $\angle C_C^* A_A^d B_B^* = 60^o$ and so $\angle W A_A^d B_B^* = 2(\alpha - \varphi)$.

However $A_A^d B_B^* = A_A^d W$ and so $A_A^d C_C^* = A_A^d B_B^*$ implies $A_A^d W = A_A^d B_B^*$. Thus $\angle W B_B^* A_A^d = \omega$ and from $\triangle A_A^d W B_B^*$, $2\omega + 2(\alpha - \varphi) = 180^o \Leftrightarrow \omega = 90^o + \varphi - \alpha$. Given that W is inside $\triangle C B_B^p B_B^*$ infer $\triangle C B_B^p B_B^*$, $\omega > \angle A_A^d V B_B^*$.

But from $\triangle A_A^d V B_B^*$ we deduce $\angle A_A^d V B_B^* = \angle A_A^d B_B^p B_B^* + \angle B_B^p A_A^d V = 60^o + \beta + \varphi$.

Thus $\omega > 60^o + \beta + \varphi \Longrightarrow \omega > 60^o + \beta + (\omega + \alpha - 90^o) \Longrightarrow \alpha + \beta < 30^o$ which contradicts the assumption $30^o > \alpha$.

The case $\alpha > 30^o$ is similar and it is omitted.

Conclude that $\triangle A_A^d B_B^* C_C^*$ cannot be equilateral (if $\triangle ABC$ is not equilateral).

5.3 Morley triangles by trisectors of one interior and two exterior angles

This section deals with the non equilateral Morley triangles formed by the trisectors of one interior and two exterior angles. Even crude figures of these triangles indicate clearly that they are too asymmetric to be equilaterals. Nevertheless it must be shown rigorously that they are not. We will consider only those formed by the interior trisectors of $\angle C$ and the exterior trisectors of $\angle A$ and $\angle B$, as the other two cases are similar. $\triangle a_c^p C_C^p b_c^p$ denotes the exterior Morley equilateral relative to $\angle C$.

Notice that the trisectors Ab_C^p and Ba_C^p are parallel iff

$$\angle a_C^p BA + \angle b_C^p AB = 180^o \Leftrightarrow 2(\alpha + \gamma) + 2(\beta + \gamma) = 180^o \Leftrightarrow \gamma = 30^o.$$

In this case the distal trisectors with respect to AB do not intersect and hence the distal vertex C_C^d is not determined. Also if $30^o > \gamma$ then C_C^d and C_C^p are on the same side of AB with $\angle AC_C^d B = 60^o - 2\gamma$. If $30^o < \gamma$ then C_C^d and C_C^p are on different sides of AB with $\angle AC_C^d B = 2\gamma - 60^o$. See Fig.23.

Futhermore note that from Corollary 3a the trisectror Ca_C^p is inside $\angle C_C^p a_C^p B$ and so Ca_C^p

intersects the trisector BC_C^p between B and C_C^p. Moreover Corollary 3a implies that the extension of AC_C^p is inside $\angle a_C^p C_C^p B$ and so Ca_C^p intersects AC_C^p inside $\triangle C_C^p B a_C^p$. In addition AC_C^p intesects Ba_C^p between B and a_C^p. Similarly the trisectror Cb_C^p intersects the trisector AC_C^p between A and C_C^p and the trisector BC_C^p inside $\triangle C_C^p A b_C^p$. Also BC_C^p intersects Ab_C^p between A and b_C^p.

5.3.1 *The Morley triangle of distal vertices*

This is denoted by $\triangle a_C^d C_C^d b_C^d$. Vertex C_C^d is the intersection of the distal to AB trisectors Ab_C^p and Ba_C^p and is determined iff $\gamma \neq 30^o$. Vertex a_C^d is the intersection of the distal to BC trisectors BC_C^p and Cb_C^p and hence it is inside $\triangle C_C^p A b_C^p$. Vertex b_C^d is the intersection of the distal to AC trisectors Ca_C^p and AC_C^p and hence it is inside $\triangle C_C^p B a_C^p$. See Fig.23.

Thus, for $\angle C \neq 90^o$ C_C^d is determined while a_C^d and b_C^d are inside $\angle AC_C^d B$. But so $\angle a_C^d C_C^d b_C^d < \angle AC_C^d B$. However $\angle AC_C^d B = |60^o - 2\gamma|$. Hence $\angle AC_C^d B < 60^o$. Therefore $\angle a_C^d C_C^d b_C^d < 60^o$.

Conclude $\triangle a_C^d C_C^d b_C^d$ is not equilateral.

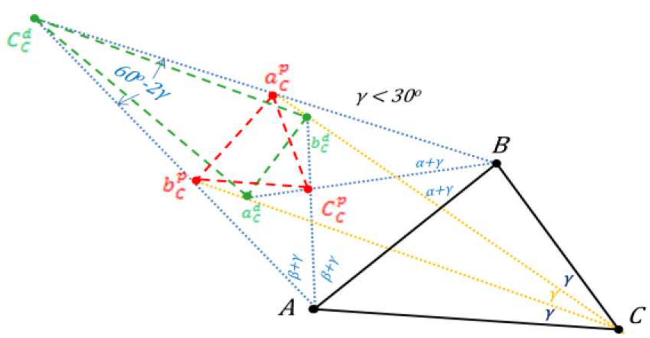

Fig.23a ($\gamma < 30^o$)

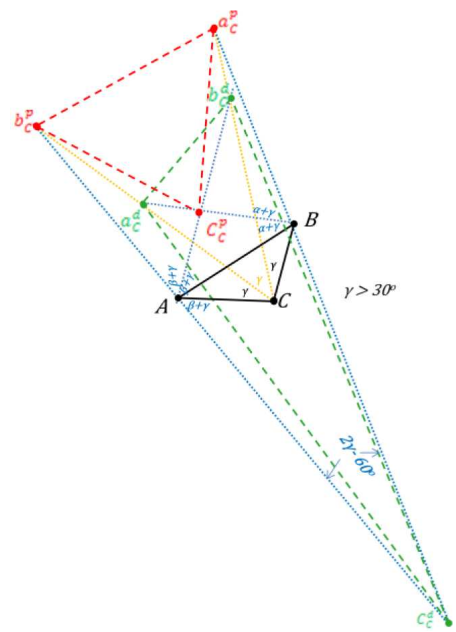

Fig.23b ($\gamma > 30^o$)

5.3.2 *The Morley triangles with one proximal and two mix vertices*

There are three such triangles denoted by $\triangle a_C^p b_C^* C_C^*$, $\triangle b_C^p a_C^* C_C^*$ and $\triangle C_C^p a_C^* b_C^*$.

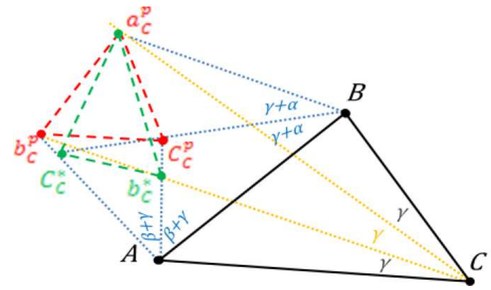

Fig.24

a. $\triangle a_C^p b_C^* C_C^*$ **:** Vertex a_C^p is the intersection of the proximal to BC trisectors Ba_C^p and Ca_C^p. Vertex b_C^* must be the intersection of the remaining interior trisector Cb_C^p (proximal to) with the exterior trisector Ab_C^p, as distal. Hence C_C^* is the intersection of the left trisectors, BC_C^p (proximal to AB) and Ab_C^p (distal). See Fig.24.

So b_C^* is on AC_C^p and it is between A and C_C^p. Also C_C^* is on Ab_C^p and it is between A and b_C^p. Notice $\angle a_C^p b_C^p C_C^p = \angle a_C^p C_C^p b_C^p = 60^o$ while, by Corollary 3a, $\angle C_C^p b_C^p A = \alpha+\beta$ and $\angle b_C^p C_C^p A = (\gamma+\alpha)^+$. Thus $\angle a_C^p C_C^p A < 180^o$ and $\angle a_C^p b_C^p A < 180^o$. Hence the quadrangle $Ab_C^p a_C^p C_C^p$ is convex. Therefore $\angle b_C^* a_C^p C_C^*$ is inside $\angle b_C^p a_C^p C_C^p$ and so $\angle b_C^* a_C^p C_C^* < 60^o$.

Conclude that $\triangle a_C^p b_C^* C_C^*$ is not equilateral.

b. $\triangle b_C^p a_C^* C_C^*$ **:** It is shown as above that it is not equilateral.

c. $\triangle C_C^p a_C^* b_C^*$ **:** Vertex C_C^p is the intersection of the proximal to AB exterior trisectors. Thus a_C^* is the intersection of the remaining exterior trisector Ba_C^p (proximal to BC) with the interior trisector Cb_C^p, as distal. Then b_C^* is the intersection of the left trisectors Ca_C^p (distal to AC) and Ab_C^p (proximal).
So a_C^* and C_C^p are on the same side of BC iff

$$\angle BCb_C^p + \angle CBa_C^p < 180^o \Leftrightarrow 2\gamma + 2(\alpha+\gamma) + 3\beta < 180^o \Leftrightarrow \gamma < \alpha.$$

If $\gamma = \alpha$ then a_C^* is not determined as $Ba_C^p // Cb_C^p$.
Also b_C^* and C_C^p are on different sides of AC iff

$$\angle ACa_C^p + \angle CAb_C^p < 180^o \Leftrightarrow 2\gamma + 2(\beta+\gamma) + 3\alpha < 180^o \Leftrightarrow \gamma < \beta.$$

If $\gamma = \beta$ then b_C^* is not determined as $Ab_C^p // Ca_C^p$.
Since a_C^*, b_C^* and C_C^p are outside $\triangle ABC$ while a_C^* and b_C^* are on Ab_C^p and Ba_C^p respectively we deduce

$$a_C^* \text{ and } C_C^p \text{ are on the same side of } AB \text{ iff } \gamma < \alpha$$

while

$$b_C^* \text{ and } C_C^p \text{ are on the same side of } AB \text{ iff } \gamma < \beta.$$

The above conditions correlate the ranges of $\alpha, \beta C_C^p$ and γ with the different locations of a_C^* and b_C^* and vise versa.
Recall that Ab_C^p and Ba_C^p intersect at C_C^d iff $\gamma \neq 30^o$, with $\angle AC_C^d B = |60^o - 2\gamma|$, while C_C^p and C_C^d are on the same side of AB iff $\gamma < 30^o$.

Next all the different locations of a_C^* and b_C^* are considered.

<u>Case 1</u>: a_C^* and b_C^* are on the other side of AB from C_C^p.
This happens iff $\gamma \leq 30^o$ (and so $\alpha < \gamma$ and $\beta < \gamma$) or $\gamma < 30^o$ with $\alpha < \gamma$ and $\beta < \gamma$. Fig.25a,b.

If $\gamma = 30^o$ then $Ab_C^p // Ba_C^p$, while for $\gamma \neq 30^o$ Ab_C^p and Ba_C^p meet at C_C^d.

But so, a_C^* and b_C^* are on the extensions (to the other side of AB from C_C^p) of Cb_C^p, Ba_C^p and $Ca_C^p, b_C^p A$ respectively.

Since C_C^p is inside $\triangle a_C^p C_C^p b_C^p$, then $\angle a_C^* C_C^p b_C^*$ encompasses $\angle a_C^* C b_C^*$. Therefore $\angle a_C^* C_C^p b_C^* < \angle a_C^* C b_C^*$. However $\angle a_C^* C b_C^* = \gamma$. Deduce $\angle a_C^* C_C^p b_C^* < 60^o$.

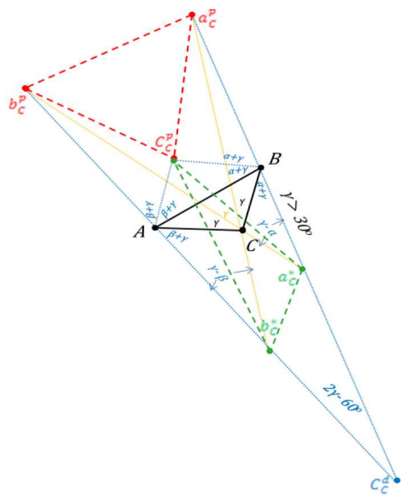

Fig.25a ($\gamma > 30^o$)

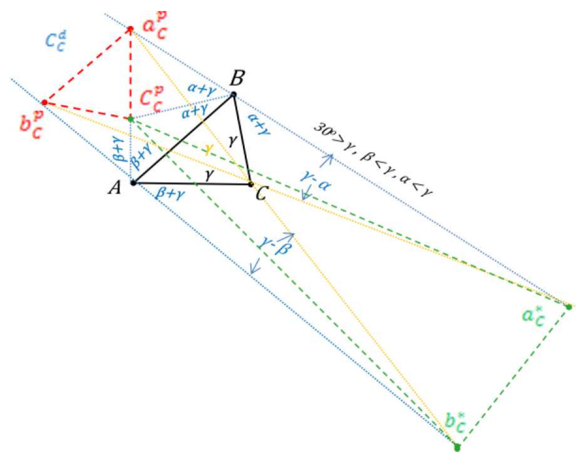

Fig.25b ($\gamma < 30^o, \beta < \gamma, \alpha < \gamma$)

<u>Case 2</u> : a_C^* and b_C^* are on the same side of AB with C_C^p.
This happens iff $\gamma < 30^o$, $\beta > \gamma$ and $\alpha > \gamma$. Fig.25c.
Then C_C^d and C_C^p are on the same side of AB. Note that Ca_C^p intersects sides AB and BC_C^d of $\triangle AC_C^d B$ internally and so, by Pasch's axiom, it intersects the third side AC_C^d externally. Thus b_C^* is on the extension of AC_C^d. Similarly a_C^* is on the extension of BC_C^d. Since C_C^p is inside $\triangle AC_C^d B$ then $\angle a_C^* C_C^p b_C^*$ encompasses $\angle a_C^* C_C^d b_C^*$. Therefore $\angle a_C^* C_C^p b_C^* < \angle a_C^* C_C^d b_C^*$. But $\angle a_C^* C_C^d b_C^* = \angle AC_C^d B = 60^o - 2\gamma$. Deduce $\angle a_C^* C_C^p b_C^* < 60^o$.

<u>Case 3</u>: a_C^* and b_C^* are on different sides of AB. Fig.25d.

This happens iff $\gamma < 30^o$ with $\beta > \gamma$ and $\alpha < \gamma$ or with $\beta < \gamma$ and $\alpha > \gamma$.
Next consider the case $\gamma < 30^o$ with $\beta > \gamma$ and $\alpha < \gamma$.
Then C_C^p is on the same side with C_C^d. Hence C_C^p is inside $\triangle AC_C^d B$ and also C_C^p is inside $\triangle a_C^p C b_C^p$. By Pasch's axiom on $\triangle AC_C^d B$, since Ca_C^p intersects sides AB and BC_C^d at interior points, infer Ca_C^p intersects the third side AC_C^d at an exterior point. Thus b_C^* is on extensions of AC_C^d and Ba_C^p on the same side of AB with C_C^p. So C_C^p is inside $\triangle b_C^p C b_C^*$ on the other side of $a_C^p b_C^p$ from C_C^d and

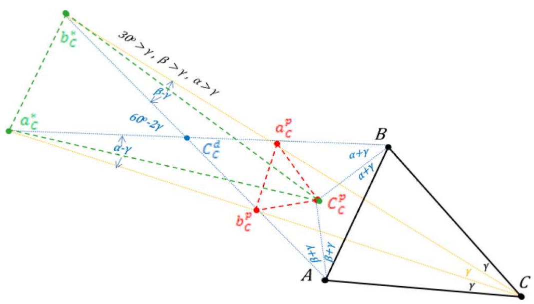

Fig.25c $(\gamma < 30^o, \beta > \gamma, \alpha > \gamma)$

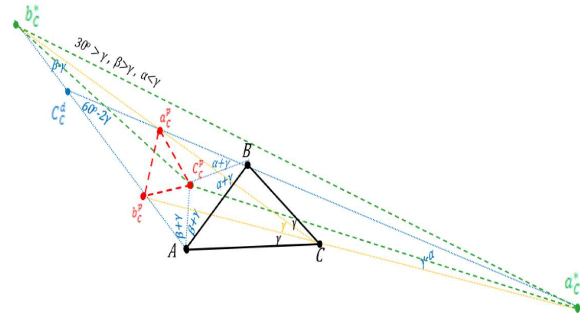

Fig.25d $(\gamma < 30^o,\ \beta < \gamma,\ \alpha < \gamma)$

b_C^*. Consequently $a_C^p C_C^d$ intersects $b_C^* C_C^p$ between b_C^* and C_C^p. Hence $a_C^* a_C^p$ is inside $\triangle a_C^* C_C^p b_C^*$. Consequently a_C^p and B are inside $\angle a_C^* C_C^p b_C^*$. Therefore $\angle a_C^* C_C^p b_C^*$ encompasses $\angle a_C^p C_C^p B$ and so $\angle a_C^* C_C^p b_C^* > \angle a_C^p C_C^p B$.

However Corollary 3a asserts $\angle a_C^p C_C^p B = (\beta + \gamma)^+$. Deduce $\angle b_C^* C_C^p a_C^* > 60^o$.

The case $\gamma < 30^o$ with $\beta > \gamma$ and $\alpha < \gamma$ is similar and it is omitted.

Conclude that $\triangle C_C^p a_C^* b_C^*$ is not equilateral.

5.3.3 *The Morley triangles with one distal and two mix vertices*

These Morley triangles are denoted by $\triangle C_C^d a_C^* b_C^*$, $\triangle b_C^d C_C^* a_C^*$ and $\triangle a_C^d c_C^* B_C^*$.

a. $\triangle C_C^d a_C^* b_C^*$**:** Vertex C_C^d is the intersection of the distal to AB trisectors Ab_C^p and Ba_C^p. Thus vertex a_C^* is determined by the intersection of the remaining trisector C_C^p , distal to BC, with Ca_C^p, as proximal. Vertex b_C^*, is determined by the left trisectors AC_C^p and Cb_C^p which are distal and proximal to CA, respectively.

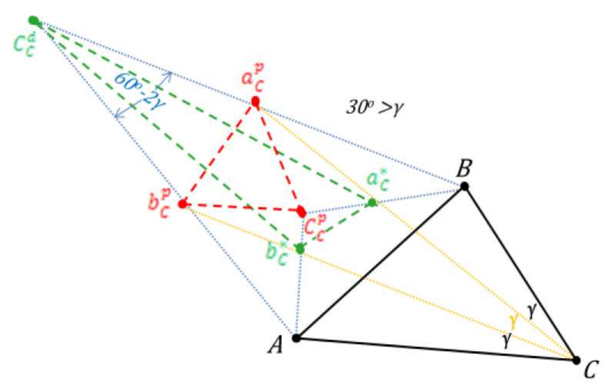

Fig.26

Vertex C_C^d is determined iff $\gamma \neq 30^o$ with C_C^p and C_C^d to be on the same side of AB iff $\gamma > 30^o$.

Moreover a_C^* is always located between B and C_C^p while b_C^* is always located between A and C_C^p. Fig.39 depicts the case for C_C^p and C_C^d to be on the same side of AB. Regardless the location of C_C^d, vertices a_C^* and b_C^* are inside $\angle AC_C^d B$. Thus $\angle a_C^* C_C^d b_C^* < \angle AC_C^d B$. Since $\angle AC_C^d B = |60^o - 2\gamma| < 60^o$ then $\angle a_C^* C_C^d b_C^* < 60^o$.

Conclude that $\triangle C_C^d a_C^* b_C^*$ is not equilateral.

b. $\triangle b_C^d C_C^* a_C^*$**:** Vertex b_C^d is the intersection of the distal to CA trisectors Ca_C^p and AC_C^p. Hence vertex C_C^* is determined by the intersection of the remaining trisector b_C^p, distal to AB, with BC_C^p, as proximal.

Vertex a_C^* is determined by the intersection of the left trisectors Cb_C^p and Ba_C^p which are distal and proximal to BC, respectively.
Trisectors Ca_C^p and AC_C^p always intersect each other and so b_C^d is located on the same side of AB with C_C^p. Also Ab_C^p and BC_C^p always intersect each other and so C_C^* is located on the same side of AB with C_C^p. However a_C^* is not always determined as $Cb_C^p // Ba_C^p$ iff $\angle b_C^p CB + \angle b_C^p C a_C^p = 180^o$ $\Leftrightarrow \gamma = \alpha$. In fact a_C^* is on the same side with b_C^d and C_C^* iff $\alpha = \gamma$. It should also be noted that $30^o < \gamma$ implies $\alpha < \gamma$. See Fig.27a.

For establishing that $\triangle b_C^d a_C^* C_C^*$ is not equilateral we will show that $\angle b_C^d a_C^* C_C^* < 60^o$. Recall that b_C^d is inside $\triangle a_C^p B C_C^p$.

Thus $\angle B a_C^* C_C^*$ encompasses $\angle b_C^d a_C^* C_C^*$. Hence for proving $\angle b_C^d a_C^* C_C^* < 60^o$ it suffices to show $\angle B a_C^* C_C^* < 60^o$.

Notice that $\angle B a_C^* C_C^* = \angle B a_C^* a_C^d + \angle a_C^d a_C^* C_C^*$ and $\angle B a_C^* a_C^d = \angle B a_C^* C$. From $\triangle B a_C^* C$ it is calculated $\angle B a_C^* C = |\gamma - \alpha|$ regardless the location of a_C^*. Hence $\angle B a_C^* C_C^* = |\gamma - \alpha| + \angle a_C^d a_C^* C_C^*$.

Moreover:

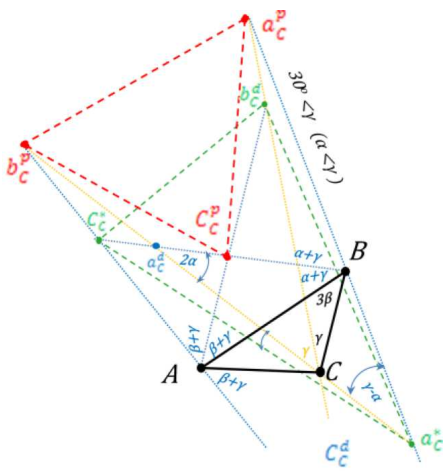

Fig.27a ($30^o < \gamma$, $\alpha < \gamma$)

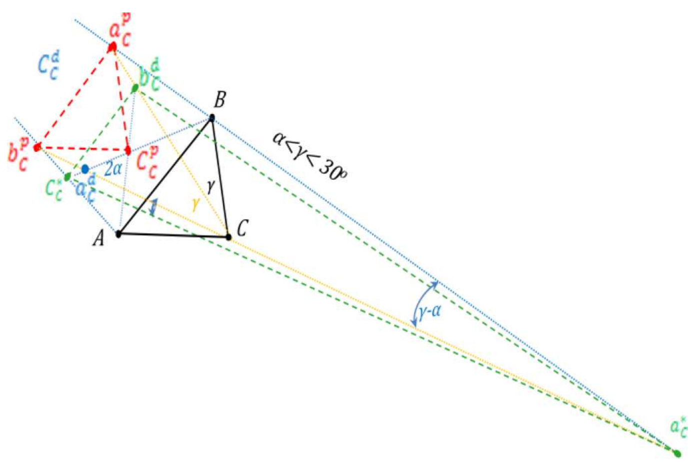

Fig.27b ($\alpha < \gamma < 30^o$)

▷ If $\alpha < \gamma$ then a_C^* is on the other side of AB from a_C^d and C_C^*. See Fig.27a,b. Thus $\angle B a_C^d C$ is exterior angle in $\triangle a_C^d a_C^* C_C^*$. Hence $\angle a_C^d a_C^* C_C^* < \angle B a_C^d C$. In $\triangle B a_C^d C$ it is calculated $\angle C_C^p a_C^d C = 2\alpha$ and so $\angle a_C^d a_C^* C_C^* < 2\alpha$. Infer $\angle B a_C^* C_C^* < (\gamma - \alpha) + 2\alpha = \gamma + \alpha < 60^o$.

▷ If $\gamma < \alpha$ then a_C^* is on the same side of AB, with a_C^d and C_C^*. See Fig.27c.

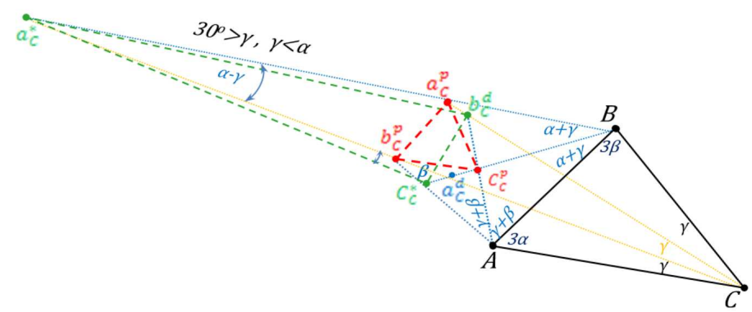

Fig.27c ($\gamma < 30^o$, $\gamma < \alpha$)

Thus $\angle \mathfrak{a}_C^d \mathfrak{b}_C^p C_C^*$ is exterior angle in $\triangle \mathfrak{b}_C^p \mathfrak{a}_C^* C_C^*$. Hence $\angle \mathfrak{b}_C^p \mathfrak{a}_C^* C_C^* < \angle \mathfrak{a}_C^d \mathfrak{b}_C^p C_C^*$. But $\angle \mathfrak{a}_C^d \mathfrak{b}_C^p C_C^* = \angle C\mathfrak{b}_C^p A$. In $\triangle C\mathfrak{b}_C^p A$ it is calculated $\angle C\mathfrak{b}_C^p A = \beta$ and so $\angle \mathfrak{b}_C^p \mathfrak{a}_C^* C_C^* < \beta$. Infer $\angle B\mathfrak{a}_C^* C_C^* < (\alpha - \gamma) + \beta < 60^o$.

Conclude that $\triangle \mathfrak{b}_C^d \mathfrak{a}_C^* C_C^*$ is not equilateral.

c. $\triangle \mathfrak{a}_C^d B_C^* \mathfrak{c}_C^*$**:** It is shown as above that it is not equilateral.

5.4 Morley triangles by trisectors of one exterior and two interior angles

Eventually the non equilateral Morley triangles formed by trisectors of one exterior and two interior angles are treated. Obviously these Morley triangles have one vertex in the interior and two in the exterior of $\triangle ABC$. As previously we will consider only those formed by the trisectors of the exterior $\angle A$ combined with the interior trisectors of $\angle B$ and $\angle C$ as the other two cases are similar.

5.4.1 *The Morley triangle of proximal vertices*

This is denoted by $\triangle A^p \mathfrak{b}_A^p \mathfrak{c}_A^p$. Vertex A^p is the intersection of the proximal to BC interior trisectors, vertex $\mathfrak{b}_A^p$ is the intersection of CB^p with the exterior trisector of $\angle A$ proximal to AC, while vertex $\mathfrak{c}_A^p$ is the intersection of BC^p with the exterior trisector of $\angle A$ proximal to AB.

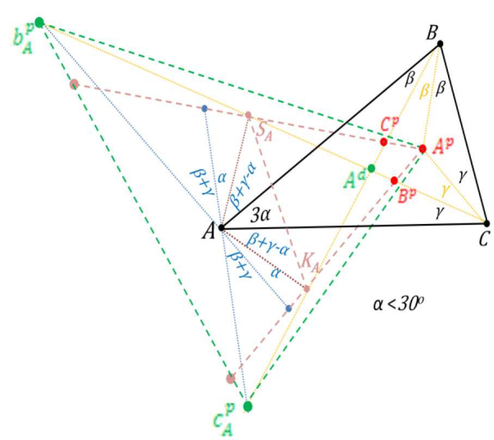

Fig.28

Consider the companion equilateral $\triangle S_A A^p K_A$ relative to vertex A. In Fig.28 the case $\alpha < 30^o$ is depicted for which Corollary 4 asserts that $\angle BAS_A = \angle CAK_A = \beta + \gamma - \alpha$ while S_A and K_A are outside of $\triangle ABC$. Consider the intersections of line A^pC^p with the sides of $\triangle b^p_A A^d c^p_A$. Then A^pC^p intersects side $A^d c^p_A$ at C^p and so externally, while it intersects side $A^d b^p_A$ at S_A internally since $\angle BAb^p_A = 2(\beta + \gamma)$ and $\angle BAS_A = \beta + \gamma - \alpha$. Thus, by Pasch's axiom, A^pC^p intersects the third side $b^p_A c^p_A$ internally. Similarly line A^pB^p intersects $b^p_A c^p_A$ internally. Thus $\angle b^p_A A^p c^p_A$ encompasses $\angle S_A A^p K_A$ and so $\angle S_A A^p K_A < Lb^p_A A^p c^p_A$.

But $\angle S_A A^p K_A = 60^o$ and so $\angle b^p_A A^p c^p_A > 60^o$. Therefore for $\alpha < 30^o$ the $\triangle A^p b^p_A c^p_A$ is not equilateral. The cases $\alpha > 30^o$ and $\alpha = 30^o$ are similar.

Conclude that $\triangle A^p b^p_A c^p_A$ cannot be equilateral.

▷ *Note that the non equilateral $\triangle A^p b^p_A c^p_A$ fails the original statement of Morley's theorem.*

5.4.2 *The Morley triangle of distal vertices*

This is denoted by $\triangle A^d b^d_A c^d_A$. Vertex b^d_A is the intersection of CA^p, the distal to CA trisector of the interior $\angle C$, and the distal to CA trisector of the exterior $\angle A$. Vertex c^d_A is the intersection of BA^p, the remaining trisector of $\angle B$ (distal to AB) with the distal to AB trisector of the exterior $\angle A$. Also it is easily seen that b^d_A and c^d_A are determined iff $\beta \neq \gamma$. Hence, for $\beta \neq \gamma$, A^d is inside $\angle Ac^d_A B$ and $\angle Cb^d_A A$. From $\triangle Cb^d_A A$ it is calculated that
$\angle Cb^d_A A = 180^o - 3\alpha - (\beta+\gamma) - 2\gamma = 2\beta$ and similarly from $\triangle Ac^d_A B$, $\angle Ac^d_A B = 2\gamma$. Since $\alpha + \beta + \gamma = 60^o$, at least one of β and γ is less than 30^o. Thus either $\angle b^d_A c^d_A A^d$ or $\angle c^d_A b^d_A A^d$ is less than 60^o.
Conclude that $\triangle A^d b^d_A c^d_A$ is not equilateral.

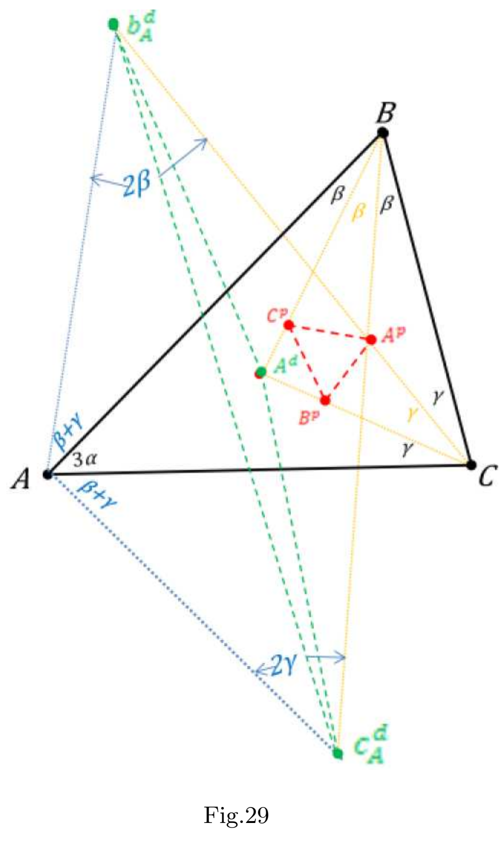

Fig.29

5.4.3 *The Morley triangles with a proximal and two mix vertices*

These triangles are denoted by $\triangle A^p b_A^* c_A^*$, $\triangle b_A^p A^* c_A^*$ and $\triangle c_A^p A^* b_A^*$.

a. $\triangle A^p b_A^* c_A^*$ **:** Vertex A^p is the intersection of the proximal to BC interior trisectors. Hence b_A^* and c_A^* are the intersections of the two remaining interior trisectors, CB^p and BC^p, with the trisectors of the exterior $\angle A$. Since each of these interior trisectors is proximal to the side it belongs, it must be paired with the distal to the corresponding side exterior trisector.

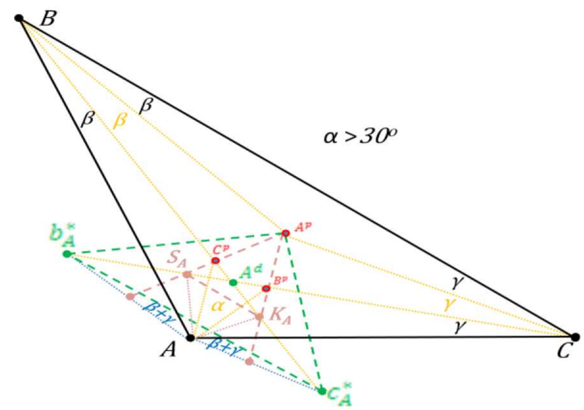

Fig.30

Consider the companion equilateral $\triangle S_A A^p K_A$ relative to vertex A^p. In Fig.30 the case $\alpha > 30^o$ is depicted for which Corollary 4 asserts that vertices S_A and K_A are inside $\triangle ABC$.

Consider the intersections of line A^pC^p with the sides of $\triangle b_A^p A^d c_A^p$. A^pC^p intersects side $A^d c_A^p$ at C^p and so externally, while A^pC^p intersects side $A^d b_A^p$ at S_A and so internally. Thus, by Pasch's axiom, A^pC^p intersects the third side Ab_A^* internally. Similarly A^pB^p intersects Ac_A^* internally. Hence $\angle b_A^p A^p c_A^p$ encompasses $\angle S_A A^p K_A$ and so $\angle S_A A^p K_A < \angle b_A^p A^p c_A^p$. But $\angle S_A A^p K_A = 60^o$ and so $\angle b_A^* A^p c_A^* > 60^o$. Therefore for $\alpha > 30^o$ $\triangle A^p b_A^* c_A^*$ is not equilateral.

The cases $\alpha < 30^o$ and $\alpha = 30^o$ are similar and they are omitted.

Conclude that $\triangle A^p b_A^* c_A^*$ is not equilateral.

b. $\triangle b_A^p A^* c_A^*$: Vertex b_A^p is the intersection of the proximal to AC trisectors, which are CB^p and the corresponding trisector of the exterior $\angle A$. Thus A^* is the intersection of the remaining interior trisector of $\angle C$, CA^p, which is proximal to BC, with BC^p as distal to BC. Then c_A^* is the intersection of the left trisectors BA^p, which is distal to AB, with the proximal to AB trisector of the exterior $\angle A$.

Notice that the last two trisectors are parallel iff $2\beta = \beta + \gamma \Leftrightarrow \beta = \gamma$. Thus c_A^* exists iff $\beta \neq \gamma$.

Also if $\beta > \gamma$ then b_A^p and c_A^* are on the same side of AC, while for $\beta < \gamma$, b_A^p and c_A^* are on different sides of AC.

Case $\beta > \gamma$: We will show $\angle b_A^p c_A^* A^* < 60^o$.

Notice that A^* is inside $\triangle ABA^p$ and so $c_A^* A^p$ is a right bound for the right side $c_A^* A^*$ of $\angle b_A^p c_A^* A^*$.
In following we will find a left bound for the left side $c_A^* b_A^*$ of $\angle b_A^p c_A^* A^*$.

Let b_A^* be the intersection of CA^p with Ab_A^p and note that the points A, A^p, c_A^*, and b_A^* are cyclic, because from $\triangle BA^pC$ it follows that $b_A^* A^p c_A^* = \beta + \gamma$ and so $b_A^* c_A^*$ is seen from A and A^p with angle $\beta + \gamma$.

The extension of A^pC^p meets the exterior trisector Ab_A^* between A and b_A^*. Then it crosses the circle, say at T. We will show that a left bound for side $c_A^* b_A^*$ of $\angle b_A^p c_A^* A^*$ is the bisector of $\angle b_A^* c_A^* A$.

Note that in a triangle the bisector of an angle crosses its opposite side at a point which is between the side's middle point and the side's common vertex with the shortest of the other two sides.

Let G be the intersection of the $\angle b_A^* c_A^*$ A bisector with Ab_A^*. Also let M be the middle of Ab_A^*.
$\triangleright$ Cb_A^p is angle bisector in $\triangle ACb_A^*$. It is easily calculated from $\triangle ACb_A^*$ that $\angle Ab_A^* C = \beta + \gamma$ and so $\angle Ab_A^* C < \angle b_A^* AC$. Thus $CA < Cb_A^*$. Hence b_A^p is between A and M.

$\triangleright$ $c_A^* G$ is angle bisector in $\triangle Ab_A^* c_A^*$. Obviously $\angle b_A^* A c_A^* = \beta + \gamma$. Also $\angle c_A^* b_A^* A = \angle c_A^* b_A^* C + \angle Cb_A^* A$ while $\angle c_A^* b_A^* A = \angle c_A^* b_A^* C + \angle Cb_A^* A$. But $\angle c_A^* b_A^* C = \angle c_A^* AA^p = \angle c_A^* AB + \angle BAA^p = (\beta + \gamma) + \angle BAA^p$. So $\angle b_A^* A c_A^* < \angle c_A^* b_A^* A$ and thus $b_A^* c_A^* < b_A^* A$. Hence G is between b_A^* and M. Therefore b_A^p is on Ab_A^* and it is between A and G. So $\angle Gc_A^* A^p$ encompasses $\angle b_A^p c_A^* A^*$ and thus

$$\angle b_A^p c_A^* A^* < \angle G c_A^* A^p \qquad .$$

In the sequel we calculate $\angle Gc_A^* A^p$. Notice that $\angle Gc_A^* A^p = \angle Gc_A^* A + \angle Ac_A^* A^p$ whereas $\angle Ac_A^* A^p = \beta - \gamma$ and $\angle Gc_A^* A = \frac{1}{2}\angle b_A^* c_A^* A = \frac{1}{2}\angle b_A^* A^p A = \frac{1}{2}[\angle b_A^* A^p T + \angle TA^p A]$.

But

$$\angle b_A^* A^p T = \angle BA^p C^p - \angle BA^p A^* = \gamma^+ - (\beta + \gamma) = \alpha + \gamma \qquad .$$

Also

$$\angle TAb_A^* = \angle TA^p b_A^* = \alpha + \gamma \text{ and } \angle A^p TA = \angle Ac_A^* A^p = \beta - \gamma.$$

Then from $\triangle TAA^p$ it is calculated

$$\angle TA^p A = \alpha + \gamma - \angle C^p AA^p \qquad .$$

Thus

$$\angle Gc_A^* A = \alpha + \gamma - \tfrac{1}{2}\angle TA^p A$$

and so $\angle Gc_A^* A^p = \alpha + \beta - \frac{1}{2}\angle C^p AA^p$ where $0 < \angle C^p AA^p < \alpha$. Hence $\angle Gc_A^* A^p < 60^o$. Therefore $\angle b_A^p c_A^* A^* < 60^o$.

Conclude that for $\beta > \gamma$ $\triangle b_A^p A^* c_A^*$ is not equilateral.

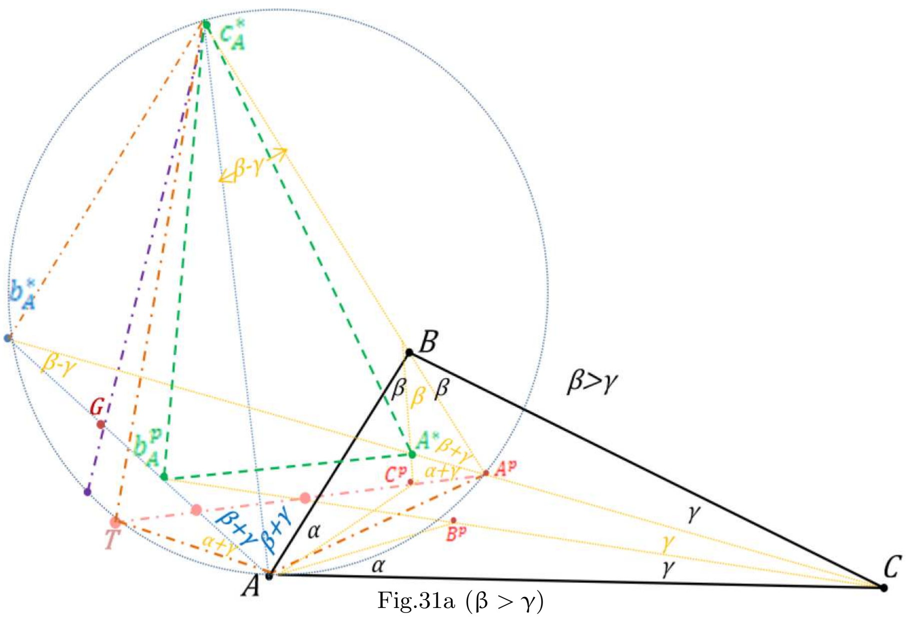

Fig.31a ($\beta > \gamma$)

Case $\beta < \gamma$: We will show that $\angle b_A^p A^* c_A^* > 60^o$.

Consider the intersections of Ac_A^* with BC^p and CB^p denoted by c_A^p and b_A^* respectively. Notice that $\angle b_A^p A^* c_A^*$ encompasses $\angle b_A^* A^* c_A^p$. So it suffices to show that $\angle b_A^* A^* c_A^p > 60^o$.

Observe that b_A^*, A^*, C and c_A^p are cyclic, because side $b_A^* A^*$ is seen from c_A^p and C with angle γ as from $\triangle Ac_A^p B$ it is calculated $\angle Ac_A^p B = \gamma$. Thus

$$\angle b_A^* A^* c_A^p = \angle b_A^* C c_A^p.$$

Moreover $\angle b_A^* C c_A^p = \angle A^d C c_A^p$, since Cb_A^* passes through A^d. But
$\angle A^d C c_A^p = \angle A^d CA + \angle ACc_A^p = \gamma + \angle ACc_A^p$ and so

$$\angle b_A^* C c_A^p = \gamma + \angle ACc_A^p.$$

In adition A, A^d, C and c_A^p are also cyclic since AA^d is seen from C and c_A^p with angle γ.

Consequently $\angle ACc_A^p = \angle AA^d c_A^p$ and so

$$\angle b_A^* C c_A^p = \gamma + \angle AA^d c_A^p.$$

Yet from $\triangle AA^d B$ infer

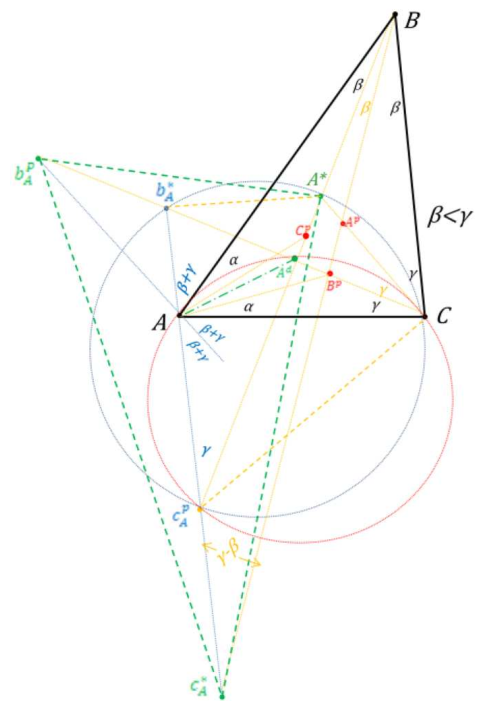

Fig.31b

$$\angle AA^d c_A^p = \angle ABA^d + \angle A^d AB.$$

However $\angle A^d AB > \angle C^p AB$ and thus
$\angle AA^d c_A^p > \angle ABA^d + \angle C^p AB = \beta + \alpha$.
Therefore $\angle b_A^* A^* c_A^p > \gamma + \beta + \alpha = 60^o$.

Conclude that for $\beta < \gamma$, $\triangle b_A^p A^* c_A^*$ is not equilateral.

c. $\triangle c_A^p A^* b_A^*$ **:** It is showed that it is not equilateral similarly as $\triangle b_A^p A^* c_A^*$.

5.4.4 *The Morley triangles with one distal and two mix vertices*

These triangles are denoted by $\triangle A^d b_A^* c_A^*$, $\triangle c_A^d A^* b_A^*$ and $\triangle b_A^d A^* c_A^*$.

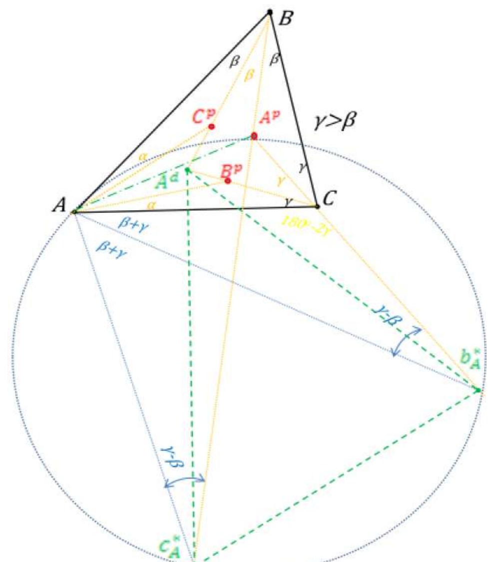

Fig.32

a. $\triangle A^d b_A^* c_A^*$**:** Obviously A^d is the intersection of CB^p and BC^p. Then b_A^* is the intersection of the remaining interior trisectors CA^p (distal to AC) with the proximal to AC exterior trisector of $\angle A$. Notice these two lines are parallel iff $\beta + \gamma = 2\gamma \Leftrightarrow \beta = \gamma$. Moreover c_A^* is the intersection of the left trisectors, the interior BC^p (proximal) with the distal to AB trisector of the exterior $\angle A$. Notice these lines are parallel iff $\beta = \gamma$. Therefore $\triangle A^d b_A^* c_A^*$ is determined iff $\beta \neq \gamma$. From $\triangle Ac_A^* B$ and $\triangle Ab_A^* C$ it follows $\angle A^p b_A^* A = \angle A^p b_A^* A = |\beta - \gamma|$ and so b_A^* and c_A^* are on the same side of AC.

We will consider only the case $\gamma > \beta$ as the other one is similar.

Notice that A^p is inside $\angle BA^d C$. Thus $\angle A^d c_A^* b_A^*$ encompasses $\angle A^p c_A^* b_A^*$ and so

$$\angle A^d c_A^* b_A^* > \angle A^p c_A^* b_A^*.$$

Also notice that A, A^p, b_A^* and c_A^* are cyclic as AA^p is seen from b_A^* and c_A^* with angle $\gamma - \beta$. Thus $\angle A^p c_A^* b_A^* = \angle A^p A b_A^*$. But $\angle A^p A b_A^* = \angle A^p AC + \angle CA b_A^* = \angle A^p AC + (\beta + \gamma)$. Moreover $\angle A^p AC > \angle B^p AC = \alpha$ and so $\angle A^p A b_A^* > \alpha + (\beta + \gamma) = 60^o$. Therefore $\angle A^d c_A^* b_A^* > 60^o$.

Conclude that $\triangle A^d b_A^* c_A^*$ is not equilateral.

b. $\triangle b_A^d A^* c_A^*$ **:** Obviously b_A^d is the intersection of CA^p with the distal to AC exterior trisector of $\angle A$. Then A^* is the intersection of the remaining trisector CB^p (proximal to AB)

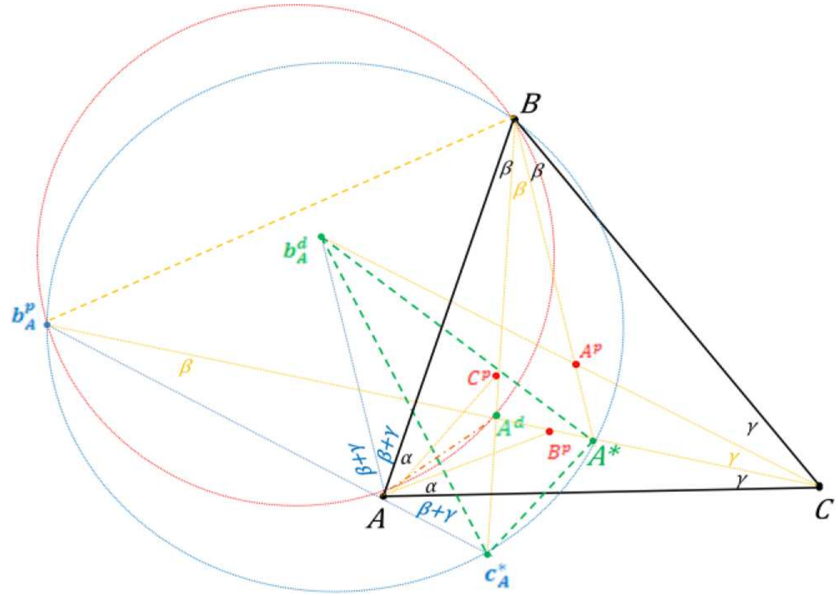

Fig.33

with the BA^p (distal). Thus c_A^* is the intersection of the left trisectors, BC^p and the distal to AB exterior trisector of $\angle A$.

Notice that b_A^d and c_A^* exist iff $\beta \neq \gamma$. We will show that $\angle b_A^d A^* c_A^* > 60^o$.

Note that B^p is inside $\triangle b_A^d A^* c_A^*$ and so $\angle b_A^d A^* c_A^*$ encompasses $\angle b_A^p A^* c_A^*$. Thus it suffices to prove $\angle b_A^p A^* c_A^* > 60^o$.

For this we use a symmetric argument to the proof of $\angle b_A^* A^* c_A^p > 60^o$ (5.4.4.a case $\beta < \gamma$).

Let b_A^p be the intersection of Ac_A^* with CA^*. Notice that c_A^*, A^*, B, b_A^p are cyclic as $c_A^* A^*$ is seen from B and b_A^p with angle β. Thus $\angle c_A^* A^* b_A^p = \angle c_A^* B b_A^p$. Moreover $\angle c_A^* A^* b_A^p = \angle A^d B b_A^p$, since Bc_A^* passes through A^d. But $\angle A^d B b_A^p = \angle A^d BA + \angle ABb_A^p = \beta + \angle ABb_A^p$ and so $\angle c_A^* A^* b_A^p = \beta + \angle ABb_A^p$.

However A, A^d, B, b_A^p are cyclic as AA^d is seen from B and b_A^p with angle β. Consequently
$$\angle ABb_A^p = \angle AA^d b_A^p \text{ and so } \angle c_A^* A^* b_A^p = \beta + \angle AA^d b_A^p.$$

Yet from $\triangle AA^dC$ infer $\angle AA^d b_A^p = \angle ACA^d + \angle A^d AC$. However $\angle A^d AC > \angle B^p AB$ and thus $\angle AA^d c_A^p > \angle ACA^d + \angle B^p AB = \gamma + \alpha$. Therefore $\angle c_A^* A^* b_A^p > \beta + \gamma + \alpha = 60^o$.

Conclude that $\triangle b_A^d A^* c_A^*$ is not equilateral.

c. $\triangle c_A^d A^* b_A^*$**:** This case is similar to the above and it is omitted.

6 Analogy between Bisectors and Trisectors in a triangle

The essence of the previous work is portrayed in the following two figures illustrating the analogy between the (well understood) structure of angle bisectors and the (under study) structure of angle trisectors in a triangle.

The structure of angle bisectors	**The structrure of angle trisectors**

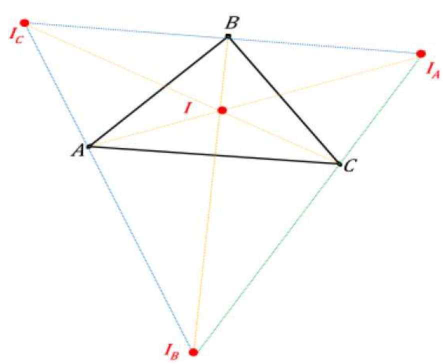

Fig.34a

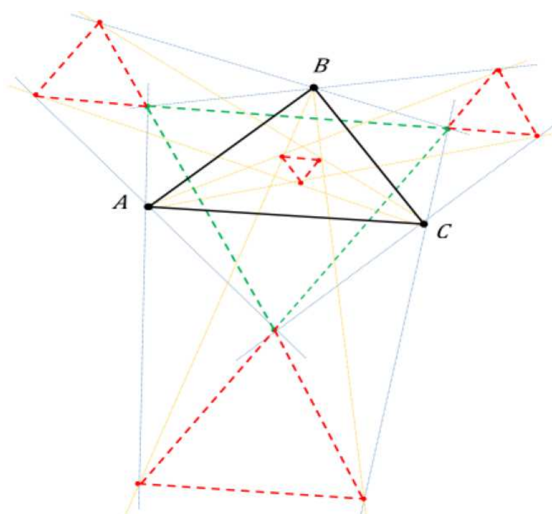

Fig.34b

The interior angle bisectors pass through a unique point (incenter).	*The interior angle trisectors proximal to the triangle sides pass through the vertices of a unique equilateral (inner Morley equilateral).*
The bisector of an interior angle and the bisectors of the other two exterior angles pass through a unique point (excenter).	*The trisectors of an interior angle and the trisectors of the other two exterior angles proximal to the triangle sides pass through the vertices of a unique equilateral (exterior Morley equilateral).*
The exterior bisectors pass through the vertices of a unique triangle with orthocenter the interior angle bisectors common point (incenter). $^{+}$	*The exterior trisectors proximal to the triangle sides pass through the vertices of a unique equilateral (central Morley equilateral).* $^{+}$

$^{+}$ This fact follows from the previous one

This structural similarity suggests that the triangle trisectors with the proper pairing meet at equilaterals which correspond to the triangle bisectors common points. The perception that trisectors behave like bisectors with equilaterals instead of points invites further exploration. New results could be inspired from the vast variety of the angle bisectors' point-line-circle theorems revealing more exciting analogies between the two structures.

This work owes gratitude to John Conway, Gerry Ladas, George Metakides, Stanley Tennenbaum, Thanasis Fokas, Fotis Fragos and foremost to Frank Morley. Its title is the instantaneous Ladas response to the showing of the last figure, when the project was trying to take off. If it revealed any of the Morley triangles' mystery hopefully it has left their charm untouched.

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