CUBO A Mathematical Journal
Vol.16, $N^{\underline{o}}$02, (121–134). June 2014

Some Coupled Coincidence Point Theorems in Partially Ordered Uniform Spaces

Aris Aghanians, Kamal Fallahi, Kourosh Nourouzi
Faculty of Mathematics,
K. N. Toosi University of Technology,
P.O. Box 16315-1618, Tehran, Iran.
nourouzi@kntu.ac.ir

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland, Galway,
University Road, Galway, Ireland.

ABSTRACT

In this paper we investigate the existence of coupled coincidence points for some contractions in partially ordered separated uniform spaces under the mixed g-monotone property. We generalize a known result in partially ordered metric spaces to uniform spaces and give new types of contractions and results in partially ordered uniform spaces.

RESUMEN

En este artículo investigamos la existencia de puntos de coincidencia acoplados de algunas contracciones en espacios uniformes separados ordenados parcialmente bajo la propiedad g-monótona de mezcla. Generalizamos un resultado conocido en espacios métricos ordenados parcialmente a espacios uniformes y entregamos tipos nuevos de contracciones y resultados para espacios uniformes ordenados parcialmente.

Keywords and Phrases: Separated uniform space; Mixed g-monotone property; Coupled coincidence point.

2010 AMS Mathematics Subject Classification: 54H25, 54E15.

1 Introduction and Preliminaries

In [3], Gnana Bhaskar and Lakshmikantham investigated coupled fixed points for mappings having the mixed monotone property in metric spaces endowed with a partial order and they applied their coupled fixed point results to periodic boundary value problems. Lakshmikantham and Ćirić [4] generalized the results in [3] by considering coupled coincidence points and mappings having the mixed g-monotone property. Using compatible mappings in partially ordered metric spaces, Choudhury and Kundu [2] extended the coupled fixed point results in [3].

In this paper, we aim to give a new generalization of a fixed point result in [3] to partially ordered uniform spaces. Also, some new results on coupled coincidence points are presented.

We first start by recalling some notions in uniform spaces. An in-depth discussion of uniformity can be found in [6].

A sequence $\{x_n\}$ in a uniform space $(X, \mathcal{U})$ (briefly, X) is said to be convergent to a point $x \in X$, denoted by $x_n \to x$, if for each entourage $U \in \mathcal{U}$, there exists an $N > 0$ such that $(x_n, x) \in U$ for all $n \geq N$ and Cauchy if for each entourage $U \in \mathcal{U}$, there exists an $N > 0$ such that $(x_m, x_n) \in U$ for all $m, n \geq N$. The uniform space X is called sequentially complete if each Cauchy sequence in X is convergent to some point of X.

A uniformity $\mathcal{U}$ on a set X is separating if the intersection of all entourages in $\mathcal{U}$ is equal to the diagonal $\{(x, x) : x \in X\}$. In this case, X is is called a separated uniform space.

For any pseudometric ρ on X and any $r > 0$, we set

$$V(\rho, r) = \big\{(x, y) \in X \times X : \rho(x, y) < r\big\}.$$

Let $\mathcal{F}$ be a family of (uniformly continuous) pseudometrics on X that generates the uniformity $\mathcal{U}$ (see [1], Theorem 2.1). Denote by $\mathcal{V}$, the family of all sets of the form

$$\bigcap_{i=1}^{n} V(\rho_i, r_i),$$

where, $n \geq 1$ and $\rho_i \in \mathcal{F}$, $r_i > 0$ for each i. Then $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$, and the elements of $\mathcal{V}$ are called the basic entourages of X. If

$$V = \bigcap_{i=1}^{n} V(\rho_i, r_i) \in \mathcal{V},$$

then

$$\alpha V = \bigcap_{i=1}^{n} V(\rho_i, \alpha r_i) \in \mathcal{V},$$

for each positive number α.

Recall that for any two subsets U and V of $X \times X$, we denote by $U \circ V$ the set of all pairs $(x, z) \in X \times X$ for which $(x, y) \in V$ and $(y, z) \in U$ for some $y \in X$. We shall need the following lemma. For more details, the reader is referred to [1].

Lemma 1.1. [1] *Let* X *be a uniform space.*

i) *If* V *is a basic entourage of* X *and* $0 < \alpha \le \beta$*, then* $\alpha V \subseteq \beta V$.

ii) *If* ρ *is a pseudometric on* X *and* $\alpha, \beta > 0$*, then*

$$(x, y) \in \alpha V(\rho, r_1) \circ \beta V(\rho, r_2) \quad \textit{implies} \quad \rho(x, y) < \alpha r_1 + \beta r_2.$$

iii) *For each* $x, y \in X$ *and each basic entourage* V *of* X*, there exists a positive number* λ *such that* $(x, y) \in \lambda V$.

iv) *Each basic entourage* V *of* X *is of the form* $V(\rho, 1)$ *for some pseudometric* ρ *(the Minkowski's pseudometric of* V*) on* X.

Definition 1. [4] Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$ be two mappings.

i) The mapping F is said to have the mixed g-monotone property if F is g-nondecreasing and g-nonincreasing in its first and second arguments, respectively, that is,

$$g(x_1) \preceq g(x_2) \implies F(x_1, y) \preceq F(x_2, y) \qquad (x_1, x_2 \in X),$$

and

$$g(y_1) \preceq g(y_2) \implies F(x, y_2) \preceq F(x, y_1) \qquad (y_1, y_2 \in X),$$

for all $x, y \in X$.

ii) An element $(x, y) \in X \times X$ is called a coupled coincidence point for F and g if

$$F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y).$$

iii) The mappings F and g are called commutative if

$$F\big(g(x), g(y)\big) = g\big(F(x, y)\big) \qquad (x, y \in X).$$

Setting $g = I_X$ (the identity mapping of X) in Definition 1, we get the concepts of the mixed monotone property and coupled fixed point defined in [3].

2 Main Results

Throughout this section, we suppose that the nonempty set X is equipped with a separating uniformity $\mathcal{U}$ and a partial order $\preceq$ unless otherwise stated. Also, we consider a partial order $\sqsubseteq$ on $X \times X$ defined by

$$(x_1, y_1) \sqsubseteq (x_2, y_2) \iff x_1 \preceq x_2 \quad \text{and} \quad y_2 \preceq y_1.$$

By two comparable elements (x, y) and (u, v) of $X \times X$, we mean either $(x, y) \sqsubseteq (u, v)$ or $(u, v) \sqsubseteq (x, y)$. Furthermore, we assume that $\mathcal{F}$ is a family of (uniformly continuous) pseudometrics on X that generates the uniformity $\mathcal{U}$. We denote by $\mathcal{V}$, the family of all sets of the form $\bigcap_{i=1}^{n} V(\rho_i, r_i)$ in which for each i, $\rho_i \in \mathcal{F}$, $r_i > 0$ and $n \geq 1$.

We have the following lemma:

Lemma 2.1. *The Minkowski's pseudometric* ρ *of a basic entourage* V *is jointly continuous, i.e.,* $x_n \to x$ *and* $y_n \to y$ *imply* $\rho(x_n, y_n) \to \rho(x, y)$.

Proof. Let $\varepsilon > 0$ be given. Then there exists an $N > 0$ such that

$$(x_n, x) \in \frac{\varepsilon}{2}V \quad \text{and} \quad (y_n, y) \in \frac{\varepsilon}{2}V \qquad (n \geq N).$$

On the other hand, for each $n \geq 1$,

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y_n) + \rho(y_n, y). \tag{2.1}$$

Substituting x and y with x_n and y_n in (2.1), respectively, and combining the obtained inequalities yield

$$\left|\rho(x_n, y_n) - \rho(x, y)\right| \leq \rho(x_n, x) + \rho(y_n, y).$$

Hence, for $n \geq N$,

$$\left|\rho(x_n, y_n) - \rho(x, y)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\rho(x_n, y_n) \to \rho(x, y)$. □

To present our results, we need the following concept:

Definition 2. A mapping $g : X \to X$ is called sequentially continuous on X if for each $x \in X$ and each sequence $\{x_n\}$ in X converging to x, we have $g(x_n) \to g(x)$. Similarly, a mapping $F : X \times X \to X$ is called sequentially continuous on X if $x_n \to x$ and $y_n \to y$ imply $F(x_n, y_n) \to F(x, y)$.

Definition 3. A partially ordered uniform space X is called upper (lower) regular if for each nondecreasing (nonincreasing) sequence $\{x_n\}$ in X converging to x, one has $x_n \preceq x$ ($x \preceq x_n$) for all $n \geq 1$.

Hereafter, by a pair (F, g) we mean mappings $F : X \times X \to X$ and $g : X \to X$ such that F has the mixed g-monotone property, the range of g contains the range of F and $F(X \times X)$ or $g(X)$ is a sequentially complete uniform subspace of X unless otherwise stated.

We present some examples of such pairs.

Example 1. Consider $X = [0, +\infty)$ with the uniformity induced from the usual metric and define a partial order $\preceq$ by

$$x \preceq y \iff \Big(x = y \quad \text{or} \quad x, y \in [0, 1] \text{ with } x \leq y\Big).$$

Define $F : X \times X \to X$ and $g : X \to X$ by

$$F(x,y) = \begin{cases} \dfrac{x^2 - y^2}{3} & y \preceq x \\ \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(x) = x^2$$

for all $x, y \in X$. Then it is seen that the range of g contains the range of F since g is surjective on X, and because $g(x_1) \preceq g(x_2)$ implies $x_1 \preceq x_2$, it follows that F has the mixed g-monotone property.

Example 2. Let $X = \{1, 2, 3\}$ and $\mathcal{U}$ be the discrete uniformity on X, that is, $\mathcal{U} = \mathcal{P}(X \times X)$ and note that each uniform subspace of X is sequentially complete. Consider the partial order

$$\preceq = \big\{(1,1), (2,2), (3,3), (1,2)\big\}$$

on X and define F and g by

$$\begin{aligned} F \;&=\; \Big\{\big((1,1),1\big), \big((1,2),3\big), \big((1,3),1\big), \big((2,1),2\big), \big((2,2),3\big), \big((2,3),1\big), \\ &\qquad \big((3,1),2\big), \big((3,2),3\big), \big((3,3),2\big)\Big\}, \end{aligned}$$

and $g = \{(1,1), (2,3), (3,2)\}$. Observe that $F(X \times X) \subseteq g(X)$; furthermore, $g(x_1) \preceq g(x_2)$ implies either $x_1 = x_2$ or $x_1 = 1$ and $x_2 = 3$, and since

$$F(1,y) \preceq F(3,y) \quad \text{and} \quad F(x,3) \preceq F(x,1)$$

for all $x, y \in X$, it follows that F has the mixed g-monotone property. Here, $(1,1)$, $(1,3)$, $(3,1)$ and $(3,3)$ are the coupled coincidence points for F and g.

Example 3. Let X be a sequentially complete real topological vector space and C a pointed cone in X, that is, $C \cap (-C) = \{0\}$. It is well-known that the topology of a topological vector space can be derived by a unique uniformity, i.e., every topological vector space is "uniformizable" in a unique way (for the details, see [5]). Consider X with this uniformity and partial order $\preceq$ on X induced by C as

$$x \preceq y \iff y - x \in C.$$

Define mappings $F : X \times X \to X$ and $g : X \to X$ by

$$F(x,y) = x - y, \quad \text{and} \quad g(x) = \begin{cases} x & x \in C \\ 2x & x \notin C \end{cases}$$

for all $x, y \in X$. Then using the properties of a cone, it is easy to check that g is surjective on X. To see that F has the mixed g-monotone property, note that $g(x_1) \preceq g(x_2)$ implies $x_1 \preceq x_2$. Therefore, if $g(x_1) \preceq g(x_2)$, then

$$F(x_1, y) = x_1 - y \preceq x_2 - y = F(x_2, y) \qquad (y \in X).$$

Similarly, from $g(y_1) \preceq g(y_2)$ we get $F(x, y_2) \preceq F(x, y_1)$ for all $x \in X$. In this example, the coupled coincidence points for F and g are $(0,0)$ and all the pairs $(x, -x)$ with $x, -x \notin C$.

Theorem 2.1. *Suppose that the pair* (F, g) *satisfies the following conditions:*

i) *there exist* $\alpha, \beta > 0$ *with* $\alpha + \beta < 1$ *such that*

$$\big(F(x,y), F(u,v)\big) \in \alpha V_1 \circ \beta V_2 \tag{2.2}$$

if $V_1, V_2 \in \mathcal{V}$, $(g(x), g(u)) \in V_1$, $(g(y), g(v)) \in V_2$, *and the pairs* $(g(x), g(y))$ *and* $(g(u), g(v))$ *are comparable, where* $x, y, u, v \in X$*;*

ii) *there exist* $x_0, y_0 \in X$ *such that* $g(x_0) \preceq F(x_0, y_0)$ *and* $F(y_0, x_0) \preceq g(y_0)$.

Then F *and* g *have a coupled coincidence point if one of the following statements holds:*

$(*)$ F *and* g *are commutative and sequentially continuous on* X*;*

$(**)$ $g(X)$ *is upper and lower regular.*

Proof. Since $F(X \times X) \subseteq g(X)$, there exist $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. We can also choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, we get sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n) \qquad (n \geq 0).$$

By induction, we now see that $\{g(x_n)\}$ and $\{g(y_n)\}$ are nondecreasing and nonincreasing sequences in $g(X)$, respectively. In fact, $g(x_0) \preceq F(x_0, y_0) = g(x_1)$ and $g(y_1) \preceq g(y_0)$. If $g(x_{n-1}) \preceq g(x_n)$ and $g(y_n) \preceq g(y_{n-1})$ for $n \geq 1$, since F has the mixed g-monotone property, then

$$g(x_n) = F(x_{n-1}, y_{n-1}) \preceq F(x_n, y_{n-1}) \preceq F(x_n, y_n) = g(x_{n+1}).$$

Similarly, $g(y_{n+1}) \preceq g(y_n)$.

Now, let $V \in \mathcal{V}$ and suppose that ρ is the Minkowski's pseudometric of V. For given comparable elements $(g(x), g(y))$ and $(g(u), g(v))$ of $X \times X$, where $x, y, u, v \in X$, write $r_1 = \rho(g(x), g(u))$ and $r_2 = \rho(g(y), g(v))$ and take $\varepsilon > 0$. Then

$$\big(g(x), g(u)\big) \in (r_1 + \varepsilon)V \quad \text{and} \quad \big(g(y), g(v)\big) \in (r_2 + \varepsilon)V,$$

and, therefore, by (2.2), we have

$$\big(F(x,y), F(u,v)\big) \in \alpha(r_1 + \varepsilon)V \circ \beta(r_2 + \varepsilon)V.$$

From Lemma 1.1 we have

$$\rho\big(F(x,y), F(u,v)\big) < \alpha(r_1 + \varepsilon) + \beta(r_2 + \varepsilon) = \alpha r_1 + \beta r_2 + (\alpha + \beta)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\rho\big(F(x,y), F(u,v)\big) \leq \alpha\rho\big(g(x), g(u)\big) + \beta\rho\big(g(y), g(v)\big). \tag{2.3}$$

Next, by Lemma 1.1, let $\lambda > 0$ be such that

$$\big(g(x_1), g(x_0)\big), \big(g(y_1), g(y_0)\big) \in \lambda V.$$

Because $(g(x_n), g(y_n))$ and $(g(x_{n-1}), g(y_{n-1}))$ are comparable, by (2.3), we have

$$\begin{aligned} \rho\big(g(x_{n+1}), g(x_n)\big) &= \rho\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big) \\ &\le \alpha\rho\big(g(x_n), g(x_{n-1})\big) + \beta\rho\big(g(y_n), g(y_{n-1})\big), \end{aligned} \tag{2.4}$$

and similarly,

$$\begin{aligned} \rho\big(g(y_{n+1}), g(y_n)\big) &= \rho\big(F(y_n, x_n), F(y_{n-1}, x_{n-1})\big) \\ &\le \alpha\rho\big(g(y_n), g(y_{n-1})\big) + \beta\rho\big(g(x_n), g(x_{n-1})\big). \end{aligned} \tag{2.5}$$

Therefore, setting

$$\rho_n = \rho\big(g(x_{n+1}), g(x_n)\big) + \rho\big(g(y_{n+1}), g(y_n)\big) \quad n = 0, 1, \ldots,$$

from (2.4) and (2.5) we obtain

$$\begin{aligned} \rho_n &= \rho\big(g(x_{n+1}), g(x_n)\big) + \rho\big(g(y_{n+1}), g(y_n)\big) \\ &\le (\alpha + \beta)\Big(\rho\big(g(x_n), g(x_{n-1})\big) + \rho\big(g(y_n), g(y_{n-1})\big)\Big) \\ &= \delta\rho_{n-1}, \end{aligned}$$

where $\delta = \alpha + \beta < 1$. Thus, by induction, the inequality $\rho_n \le \delta^n \rho_0$ holds for all $n \ge 0$. Hence, for sufficiently large m and n with $m > n$, we have

$$\begin{aligned} \rho\big(g(x_m), g(x_n)\big) + \rho\big(g(y_m), g(y_n)\big) &\le \sum_{k=n+1}^{m} \Big[\rho\big(g(x_k), g(x_{k-1})\big) + \rho\big(g(y_k), g(y_{k-1})\big)\Big] \\ &= \rho_{m-1} + \cdots + \rho_n \\ &\le (\delta^{m-1} + \cdots + \delta^n)\rho_0 \\ &< \frac{\delta^n}{1-\delta} 2\lambda < 1, \end{aligned}$$

that is,

$$\big(g(x_m), g(x_n)\big), \big(g(y_m), g(y_n)\big) \in V.$$

Consequently, $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in $g(X)$, and so there exist $x, y \in X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$.

To see the existence of a coupled coincidence point for F and g, suppose first that $(*)$ holds. Since X is separated,

$$g^2(x_{n+1}) \to g^2(x),$$

with

$$g^2(x_{n+1}) = g\big(F(x_n, y_n)\big) = F\big(g(x_n), g(y_n)\big) \to F\big(g(x), g(y)\big),$$

implies that $g^2(x) = F(g(x), g(y))$. Similarly, $g^2(y) = F(g(y), g(x))$, that is, $(g(x), g(y))$ is a coupled coincidence point for F and g. On the other hand, if $(**)$ holds, then

$$g(x_n) \preceq g(x) \quad \text{and} \quad g(y) \preceq g(y_n),$$

for all $n \geq 0$. Thus, $(g(x), g(y))$ is comparable to each $(g(x_n), g(y_n))$. If $V \in \mathcal{V}$ and ρ is the Minkowski's pseudometric of V, then by (2.3) and Lemma 2.1, for sufficiently large n we have

$$\begin{aligned}
\rho\big(F(x,y), g(x)\big) &\leq \rho\big(F(x,y), g(x_{n+1})\big) + \rho\big(g(x_{n+1}), g(x)\big) \\
&= \rho\big(F(x,y), F(x_n, y_n)\big) + \rho\big(g(x_{n+1}), g(x)\big) \\
&\leq \alpha\rho\big(g(x), g(x_n)\big) + \beta\rho\big(g(y), g(y_n)\big) + \rho\big(g(x_{n+1}), g(x)\big) < 1,
\end{aligned}$$

that is, $(F(x,y), g(x)) \in V$. Since V is arbitrary and X is separated, we get $F(x,y) = g(x)$. Similarly, $F(y,x) = g(y)$ and so, in this case, (x,y) is a coupled coincidence point for F and g. □

Example 4. Let X be a nonzero real vector space and C be a pointed cone in X. Consider two arbitrary complete norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X and define

$$\rho_1\big((x_1,x_2),(y_1,y_2)\big) = \|x_1 - y_1\|_1,$$

and

$$\rho_2\big((x_1,x_2),(y_1,y_2)\big) = \|x_2 - y_2\|_2$$

for all $(x_1,x_2),(y_1,y_2) \in X^2 = X \times X$. It is easy to verify that the uniformity $\mathcal{U}$ on X^2 generated by the two pseudometrics ρ_1 and ρ_2 is separating and sequentially complete. Define a partial order $\preceq$ on X^2 by

$$(x_1,x_2) \preceq (y_1,y_2) \iff y_1 - x_1, x_2 - y_2 \in C \qquad \big((x_1,x_2),(y_1,y_2) \in X^2\big).$$

Since the family $\mathcal{F} = \{\rho_1, \rho_2\}$, which generates the uniformity $\mathcal{U}$ has finitely many elements, it follows that two mappings $F : X^2 \times X^2 \to X^2$ and $g : X^2 \to X^2$ defined by

$$F\big((x_1,x_2),(y_1,y_2)\big) = \Big(\frac{1}{3}(x_1 - y_1), \frac{1}{4}(x_2 - y_2)\Big),$$

and

$$g\big((x_1,x_2)\big) = (3x_1, 2x_2)$$

for all $(x_1,x_2),(y_1,y_2) \in X^2$ satisfy (2.2) since they satisfy the contractive condition

$$\begin{aligned}
\rho_i\Big(F\big((x_1,x_2),(y_1,y_2)\big), F\big((u_1,u_2),(v_1,v_2)\big)\Big) &\leq \frac{1}{4}\rho_i\Big(g\big((x_1,x_2)\big), g\big((u_1,u_2)\big)\Big) \\
&\quad + \frac{1}{4}\rho_i\Big(g\big((y_1,y_2)\big), g\big((v_1,v_2)\big)\Big)
\end{aligned}$$

for all $(x_1,x_2),(y_1,y_2),(u_1,u_2),(v_1,v_2) \in X^2$ such that the pairs $(g((x_1,x_2)), g((y_1,y_2)))$ and $(g((u_1,u_2)), g((v_1,v_2)))$ are comparable, and $i = 1,2$. Moreover, F and g commute and are sequentially continuous on X^2, the mapping F has the mixed g-monotone property and $F(X^2 \times X^2) \subseteq g(X^2) = X^2$. Therefore, setting $x_0 = (-2x^*, x^*)$ and $y_0 = (x^*, -2x^*)$ where $x^* \in C$, we see that the hypotheses of Theorem 2.1 are fulfilled and hence F and g have a coupled coincidence point, namely $(0, 0)$.

Setting $g = I_X$ in Theorem 2.1, the following generalization of the Gnana Bhaskar and Lakshmikantham's result [3] to partially ordered uniform spaces is obtained.

Corolary 1. *Suppose that X is sequentially complete and a mapping $F : X \times X \to X$ satisfies the following conditions:*

i) F *has the mixed monotone property;*

ii) *there exist $\alpha, \beta > 0$ with $\alpha + \beta < 1$ such that*

$$\big(F(x, y), F(u, v)\big) \in \alpha V_1 \circ \beta V_2$$

if $V_1, V_2 \in \mathcal{V}$, $(x, u) \in V_1$, $(y, v) \in V_2$, and the pairs (x, y) and (u, v) are comparable, where $x, y, u, v \in X$;

iii) *there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$.*

Then F has a coupled fixed point if one of the following statements holds:

a) F *is sequentially continuous on X;*

b) X *is upper and lower regular.*

Remark 1. In addition to the hypotheses of Theorem 2.1, suppose that $g(x_0) \preceq g(y_0)$. Suppose further that x and y are as in the proof of Theorem 2.1. Then $g(x) = g(y)$. To see this, we first show that $g(x_n) \preceq g(y_n)$ for all $n \geq 0$. If $g(x_n) \preceq g(y_n)$ for $n \geq 1$, since F has the mixed g-monotone property, it follows that

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(y_n, y_n) \preceq F(y_n, x_n) = g(y_{n+1}).$$

Thus, by induction, $g(x_n) \preceq g(y_n)$ for all $n \geq 0$.

Now, let $V \in \mathcal{V}$ and ρ be the Minkowski's pseudometric of V. Since $(g(x_n), g(y_n))$ and $(g(y_n), g(x_n))$ are comparable, by (2.3), we have

$$\begin{aligned}
\rho\big(g(x), g(y)\big) &\leq \rho\big(g(x), g(x_{n+1})\big) + \rho\big(g(x_{n+1}), g(y_{n+1})\big) + \rho\big(g(y_{n+1}), g(y)\big) \\
&= \rho\big(g(x), g(x_{n+1})\big) + \rho\big(F(x_n, y_n), F(y_n, x_n)\big) + \rho\big(g(y_{n+1}), g(y)\big) \\
&\leq \rho\big(g(x), g(x_{n+1})\big) + \alpha\rho\big(g(x_n), g(y_n)\big) \\
&\quad + \beta\rho\big(g(y_n), g(x_n)\big) + \rho\big(g(y_{n+1}), g(y)\big)
\end{aligned}$$

$$\begin{aligned}
&= \rho\big(g(x), g(x_{n+1})\big) + \delta\rho\big(g(x_n), g(y_n)\big) + \rho\big(g(y_{n+1}), g(y)\big) \\
&\leq \rho\big(g(x), g(x_{n+1})\big) + \delta\rho\big(g(x_n), g(x)\big) + \delta\rho\big(g(x), g(y)\big) \\
&\quad + \delta\rho\big(g(y), g(y_n)\big) + \rho\big(g(y_{n+1}), g(y)\big),
\end{aligned}$$

where $\delta = \alpha + \beta < 1$. Hence, the joint continuity of the Minkowski's pseudometrics yields

$$\begin{aligned}
\rho\big(g(x), g(y)\big) &\leq \frac{1}{1-\delta}\rho\big(g(x), g(x_{n+1})\big) + \frac{\delta}{1-\delta}\rho\big(g(x_n), g(x)\big) \\
&\quad + \frac{\delta}{1-\delta}\rho\big(g(y), g(y_n)\big) + \frac{1}{1-\delta}\rho\big(g(y_{n+1}), g(y)\big) < 1,
\end{aligned}$$

for sufficiently large n, that is, $(g(x), g(y)) \in V$. Since V is arbitrary and X is separated, we get $g(x) = g(y)$. In particular, if g is injective, then $F(x, x) = g(x)$.

We next present two coupled coincidence point theorems for two different types of contractions in partially ordered uniform spaces.

Theorem 2.2. *Suppose that a pair* (F, g) *satisfies the following conditions:*

i) *there exist positive numbers* α *and* β *with* $\alpha+\beta < 1$ *such that for all* $V_1, V_2 \in \mathcal{V}$*, if* $(F(x, y), g(x)) \in V_1$*,* $(F(u, v), g(u)) \in V_2$*, and* $(g(x), g(y))$ *and* $(g(u), g(v))$ *are comparable, then*

$$\big(F(x, y), F(u, v)\big) \in \alpha V_1 \circ \beta V_2, \tag{2.6}$$

where $x, y, u, v \in X$*;*

ii) *there exist* $x_0, y_0 \in X$ *such that* $g(x_0) \preceq F(x_0, y_0)$ *and* $F(y_0, x_0) \preceq g(y_0)$*.*

Then F *and* g *have a coupled coincidence point if* $(*)$ *or* $(**)$ *holds.*

Proof. Consider the sequences $\{x_n\}$ and $\{y_n\}$ with initial points x_0 and y_0 constructed in the proof of Theorem 2.1. Let $V \in \mathcal{V}$ and suppose that ρ is the Minkowski's pseudometric of V. For given comparable elements $(g(x), g(y))$ and $(g(u), g(v))$ of $X \times X$, where $x, y, u, v \in X$, write $r_1 = \rho(F(x, y), g(x))$ and $r_2 = \rho(F(u, v), g(u))$ and take $\varepsilon > 0$. Then

$$\big(F(x, y), g(x)\big) \in (r_1 + \varepsilon)V \quad \text{and} \quad \big(F(u, v), g(u)\big) \in (r_2 + \varepsilon)V.$$

Therefore, by (2.6),

$$\big(F(x, y), F(u, v)\big) \in \alpha(r_1 + \varepsilon)V \circ \beta(r_2 + \varepsilon)V.$$

By Lemma 1.1, we have

$$\rho\big(F(x, y), F(u, v)\big) < \alpha(r_1 + \varepsilon) + \beta(r_2 + \varepsilon) = \alpha r_1 + \beta r_2 + (\alpha + \beta)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\rho\big(F(x, y), F(u, v)\big) \leq \alpha\rho\big(F(x, y), g(x)\big) + \beta\rho\big(F(u, v), g(u)\big). \tag{2.7}$$

Next, by Lemma 1.1, choose a $\lambda > 0$ such that $(g(x_1), g(x_0)) \in \lambda V$. Because $(g(x_n), g(y_n))$ and $(g(x_{n-1}), g(y_{n-1}))$ are comparable, by (2.7), we have

$$\begin{aligned}
\rho\big(g(x_{n+1}), g(x_n)\big) &= \rho\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big)\\
&\leq \alpha\rho\big(F(x_n, y_n), g(x_n)\big) + \beta\rho\big(F(x_{n-1}, y_{n-1}), g(x_{n-1})\big)\\
&= \alpha\rho\big(g(x_{n+1}), g(x_n)\big) + \beta\rho\big(g(x_n), g(x_{n-1})\big).
\end{aligned}$$

Thus, for each $n \geq 1$, the inequality

$$\rho\big(g(x_{n+1}), g(x_n)\big) \leq \delta\rho\big(g(x_n), g(x_{n-1})\big)$$

holds, where $\delta = \frac{\beta}{1-\alpha}$. Clearly, $0 < \delta < 1$ and, by induction, we have

$$\rho\big(g(x_{n+1}), g(x_n)\big) \leq \delta^n\rho\big(g(x_1), g(x_0)\big) \qquad (n \geq 0).$$

Hence, for sufficiently large m and n with $m > n$ we have

$$\begin{aligned}
\rho\big(g(x_m), g(x_n)\big) &\leq \rho\big(g(x_m), g(x_{m-1})\big) + \cdots + \rho\big(g(x_{n+1}), g(x_n)\big)\\
&\leq \delta^{m-1}\rho\big(g(x_1), g(x_0)\big) + \cdots + \delta^n\rho\big(g(x_1), g(x_0)\big)\\
&< \frac{\delta^n}{1-\delta}\lambda < 1,
\end{aligned}$$

that is, $(g(x_m), g(x_n)) \in V$. Therefore, $\{g(x_n)\}$ is a Cauchy sequence in $g(X)$. Similarly, $\{g(y_n)\}$ is Cauchy, and so there exist $x, y \in X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$.

Now, if $(*)$ holds, then an argument similar to that in the proof of Theorem 2.1 establishes that $(g(x), g(y))$ is a coupled coincidence point if F and g. If $(**)$ holds, then

$$g(x_n) \preceq g(x) \quad \text{and} \quad g(y) \preceq g(y_n),$$

for all $n \geq 1$. Thus, $(g(x), g(y))$ is comparable to each $(g(x_n), g(y_n))$. Now, suppose $V \in \mathcal{V}$ and ρ is the Minkowski's pseudometric of V. Then, by (2.7), for each $n \geq 0$ we have

$$\begin{aligned}
\rho\big(F(x,y), g(x)\big) &\leq \rho\big(F(x,y), g(x_{n+1})\big) + \rho\big(g(x_{n+1}), g(x)\big)\\
&= \rho\big(F(x,y), F(x_n, y_n)\big) + \rho\big(g(x_{n+1}), g(x)\big)\\
&\leq \alpha\rho\big(F(x,y), g(x)\big) + \beta\rho\big(F(x_n, y_n), g(x_n)\big) + \rho\big(g(x_{n+1}), g(x)\big)\\
&= \alpha\rho\big(F(x,y), g(x)\big) + \beta\rho\big(g(x_{n+1}), g(x_n)\big) + \rho\big(g(x_{n+1}), g(x)\big).
\end{aligned}$$

Since the Minkowoski's pseudometrics are jointly continuous, hence for sufficiently large n we obtain

$$\rho\big(F(x,y), g(x)\big) \leq \frac{\beta}{1-\alpha}\rho\big(g(x_{n+1}), g(x_n)\big) + \frac{1}{1-\alpha}\rho\big(g(x_{n+1}), g(x)\big) < 1,$$

that is, $(F(x,y), g(x)) \in V$. Since V is arbitrary and X is separated, we get $F(x,y) = g(x)$. Similarly, $F(y,x) = g(y)$ and so, (x,y) is a coupled coincidence point for F and g. $\square$

We easily get the following consequence of Theorem 2.2 in partially ordered metric spaces:

Corolary 2. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Suppose that the pair (F, g) satisfies the following conditions:*

i) *there exist $\alpha, \beta > 0$ with $\alpha + \beta < 1$ such that*

$$\big(F(x, y), F(u, v)\big) \in \alpha V(d, r_1) \circ \beta V(d, r_2)$$

if $r_1, r_2 > 0$, $d(F(x, y), g(x)) < r_1$, $d(F(u, v), g(u)) < r_2$, and the pairs $(g(x), g(y))$ and $(g(u), g(v))$ are comparable, where $x, y, u, v \in X$;

ii) *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$.*

Then F and g have a coupled coincidence point if $(*)$ or $(**)$ holds.

Theorem 2.3. *Suppose that a pair (F, g) satisfies the following conditions:*

i) *there exist positive numbers α and β with $\alpha+\beta < 1$ such that for all $V_1, V_2 \in \mathcal{V}$, if $(F(x, y), g(u)) \in V_1$, $(F(u, v), g(x)) \in V_2$, and $(g(x), g(y))$ and $(g(u), g(v))$ are comparable, then*

$$\big(F(x, y), F(u, v)\big) \in \alpha V_1 \circ \beta V_2, \tag{2.8}$$

where $x, y, u, v \in X$;

ii) *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$.*

Then F and g have a coupled coincidence point if $(*)$ or $(**)$ holds.

Proof. Again, we construct the sequences $\{x_n\}$ and $\{y_n\}$ with initial points x_0 and y_0 as in the proof of Theorem 2.1. Since $\alpha + \beta < 1$, without loss of generality, we assume that $\alpha < \frac{1}{2}$. Let $V \in \mathcal{V}$ and suppose that ρ is the Minkowski's pseudometric of V. For given comparable elements $(g(x), g(y))$ and $(g(u), g(v))$ of $X \times X$, where $x, y, u, v \in X$, write $r_1 = \rho(F(x, y), g(u))$ and $r_2 = \rho(F(u, v), g(x))$ and take $\varepsilon > 0$. Then

$$\big(F(x, y), g(u)\big) \in (r_1 + \varepsilon)V \quad \text{and} \quad \big(F(u, v), g(x)\big) \in (r_2 + \varepsilon)V.$$

Therefore, by (2.8),

$$\big(F(x, y), F(u, v)\big) \in \alpha(r_1 + \varepsilon)V \circ \beta(r_2 + \varepsilon)V.$$

By Lemma 1.1, we have

$$\rho\big(F(x, y), F(u, v)\big) < \alpha(r_1 + \varepsilon) + \beta(r_2 + \varepsilon) = \alpha r_1 + \beta r_2 + (\alpha + \beta)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\rho\big(F(x, y), F(u, v)\big) \le \alpha\rho\big(F(x, y), g(u)\big) + \beta\rho\big(F(u, v), g(x)\big). \tag{2.9}$$

Now, by Lemma 1.1, let $\lambda > 0$ be such that $(g(x_1), g(x_0)) \in \lambda V$. Because $(g(x_n), g(y_n))$ and $(g(x_{n-1}), g(y_{n-1}))$ are comparable, by (2.9), we have

$$\begin{aligned}\rho\big(g(x_{n+1}), g(x_n)\big) &= \rho\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big)\\ &\leq \alpha\rho\big(F(x_n, y_n), g(x_{n-1})\big) + \beta\rho\big(F(x_{n-1}, y_{n-1}), g(x_n)\big)\\ &\leq \alpha\rho\big(F(x_n, y_n), g(x_n)\big) + \alpha\rho\big(g(x_n), g(x_{n-1})\big)\\ &= \alpha\rho\big(g(x_{n+1}), g(x_n)\big) + \alpha\rho\big(g(x_n), g(x_{n-1})\big).\end{aligned}$$

Thus, for each $n \geq 1$, the inequality

$$\rho\big(g(x_{n+1}), g(x_n)\big) \leq \delta\rho\big(g(x_n), g(x_{n-1})\big)$$

holds, where $\delta = \frac{\alpha}{1-\alpha}$. Since $\alpha < \frac{1}{2}$, hence $0 < \delta < 1$ and, by induction, we have

$$\rho\big(g(x_{n+1}), g(x_n)\big) \leq \delta^n\rho\big(g(x_1), g(x_0)\big) \qquad (n \geq 0).$$

Therefore, for sufficiently large m and n with $m > n$ we have

$$\begin{aligned}\rho\big(g(x_m), g(x_n)\big) &\leq \rho\big(g(x_m), g(x_{m-1})\big) + \cdots + \rho\big(g(x_{n+1}), g(x_n)\big)\\ &\leq \delta^{m-1}\rho\big(g(x_1), g(x_0)\big) + \cdots + \delta^n\rho\big(g(x_1), g(x_0)\big)\\ &< \frac{\delta^n}{1-\delta}\lambda < 1,\end{aligned}$$

that is, $(g(x_m), g(x_n)) \in V$. Consequently, $\{g(x_n)\}$ is a Cauchy sequence in $g(X)$. Similarly, $\{g(y_n)\}$ is Cauchy, and so there exist $x, y \in X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$.

If $(*)$ holds, then an argument similar to that in the proof of Theorem 2.1 establishes that $(g(x), g(y))$ is a coupled coincidence point if F and g. If $(**)$ holds, then

$$g(x_n) \preceq g(x) \quad \text{and} \quad g(y) \preceq g(y_n),$$

for all $n \geq 0$. Thus, $(g(x), g(y))$ is comparable to each $(g(x_n), g(y_n))$. If $V \in \mathcal{V}$ and ρ is the Minkowski's pseudometric of V, then by (2.9), for each $n \geq 1$ we have

$$\begin{aligned}\rho\big(F(x,y), g(x)\big) &\leq \rho\big(F(x,y), g(x_{n+1})\big) + \rho\big(g(x_{n+1}), g(x)\big)\\ &= \rho\big(F(x,y), F(x_n, y_n)\big) + \rho\big(g(x_{n+1}), g(x)\big)\\ &\leq \alpha\rho\big(F(x,y), g(x_n)\big) + \beta\rho\big(F(x_n, y_n), g(x)\big) + \rho\big(g(x_{n+1}), g(x)\big)\\ &\leq \alpha\rho\big(F(x,y), g(x)\big) + \alpha\rho\big(g(x), g(x_n)\big) + (\beta+1)\rho\big(g(x_{n+1}), g(x)\big).\end{aligned}$$

Since the Minkowoski's pseudometrics are jointly continuous, hence for sufficiently large n we get

$$\rho\big(F(x,y), g(x)\big) \leq \frac{\alpha}{1-\alpha}\rho\big(g(x), g(x_n)\big) + \frac{\beta+1}{1-\alpha}\rho\big(g(x_{n+1}), g(x)\big) < 1,$$

that is, $(F(x,y), g(x)) \in V$. Since V is arbitrary and X is separated, we have $F(x,y) = g(x)$. Similarly, $F(y,x) = g(y)$ and so, (x,y) is a coupled coincidence point for F and g. $\square$

Corolary 3. *Let* $(X, \preceq)$ *be a partially ordered set and* d *be a metric on* X. *Suppose that the pair* (F, g) *satisfies the following conditions:*

i) *there exist* $\alpha, \beta > 0$ *with* $\alpha + \beta < 1$ *such that*

$$\big(F(x,y), F(u,v)\big) \in \alpha V(d, r_1) \circ \beta V(d, r_2)$$

if $r_1, r_2 > 0$, $d(F(x,y), g(u)) < r_1$, $d(F(u,v), g(x)) < r_2$, *and the pairs* $(g(x), g(y))$ *and* $(g(u), g(v))$ *are comparable, where* $x, y, u, v \in X$;

ii) *there exist* $x_0, y_0 \in X$ *such that* $g(x_0) \preceq F(x_0, y_0)$ *and* $F(y_0, x_0) \preceq g(y_0)$.

Then F *and* g *have a coupled coincidence point if* $(*)$ *or* $(**)$ *holds.*

Received: December 2013. Revised: April 2014.

References

[1] S. P. Acharya, *Some results on fixed points in uniform spaces*, Yokohama Math. J. **22** (1974) 105-116.

[2] B. S. Choudhury, A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal. **73** (2010) 2524-2531.

[3] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006) 1379-1393.

[4] V. Lakshmikantham, L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009) 4341-4349.

[5] H. H. Schaefer, *Topological vector spaces*, Springer-Verlag, New York-Berlin, 1971.

[6] S. Willard, *General topology*, Addison-Wesley Publishing Co. Reading, Mass.-London-Don Mills, Ont., 1970.