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Existence of Entire Solutions for Quasilinear Elliptic Systems under Keller-Osserman Condition

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ABSTRACT

In this paper, we study the existence of entire solutions for the following elliptic system

$$\triangle_m u = p(x)f(v), \triangle_l v = q(x)g(u),\ x \in \mathbf{R}^N,$$

where $1 < m, l < \infty$, f, g are continuous and non-decreasing on $[0, \infty)$, satisfy $f(t) > 0, g(t) > 0$ for all $t > 0$ and the Keller-Osserman condition. We establish conditions on p and q that are necessary for the existence of positive solutions, bounded and unbounded, of the given equation.

RESUMEN

En este artículo estudiamos la existencia de soluciones enteras para el siguiente sistema elíptico

$$\triangle_m u = p(x)f(v), \triangle_l v = q(x)g(u),\ x \in \mathbf{R}^N,$$

donde $1 < m, l < \infty$, f, g son continuas y no-decrecientes en $[0, \infty)$, satisfaciendo $f(t) > 0, g(t) > 0$ para todo $t > 0$ y la condición de Keller-Osserman. Establecemos condiciones sobre p y qy que son necesarias para la existencia de soluciones positivas, acotadas y no acotadas de la ecuación dada.

Keywords and Phrases: quasi-linear elliptic system; sub/super-solution; large solution; existence.

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1 Introduction

In this paper, we investigate the following quasilinear elliptic system

$$\begin{cases} \triangle_m u = p(x)f(v), & x \in \mathbf{R}^N, \\ \triangle_l v = q(x)g(u), & x \in \mathbf{R}^N. \end{cases} \tag{1.1}$$

where $1 < m, l < \infty$, $N \geq \max\{m, l\} + 1, \triangle_m \cdot = \mathrm{div}(|\nabla \cdot|^{m-2}\nabla \cdot)$. Denote $d = \min\{m, l\}$, and see that $d > 1$. By an entire large solution (u, v), we mean a pair of functions $u, v \in C^1(\mathbf{R}^N)$ that satisfies (1.1) at every point of $\mathbf{R}^N$ and

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = \infty. \tag{1.2}$$

First, we introduce the assumptions below:

(H1) $p, q : \mathbf{R}^N \to [0, \infty)$ and $f, g : [0, \infty) \to [0, \infty)$ are continuous and nontrival functions;

(H2) f and g are nondecreasing on $[0, \infty)$ and $f(t) > 0, g(t) > 0$ for all $t > 0$;

(H3) $H(\infty) = \lim_{r\to\infty} H(r) = \infty$, where

$$H(r) = \int_c^r \frac{dt}{\sqrt[d]{F(t) + G(t)}}, \quad r \geq c > 0; \quad F(t) = \int_0^t f(s)ds, \; G(t) = \int_0^t g(s)ds.$$

and c is a positive constant. Notice that $H'(r) = \frac{1}{\sqrt[d]{F(r)+G(r)}} > 0, \forall\ r > c$, so H has the inverse function H^{-1} on $[0, \infty)$. Denote

$$\phi_1(r) := \max_{|x|=r} p(x), \quad \phi_2(r) := \min_{|x|=r} p(x),$$

$$\psi_1(r) := \max_{|x|=r} q(x), \quad \psi_2(r) := \min_{|x|=r} q(x).$$

Since 1980s, many important results have been obtained for quasilinear elliptic systems. We will introduce some results in the following. Existence and non-existence of solutions of the quasilinear elliptic system

$$\begin{cases} \mathrm{div}(|\nabla u|^{m-2}\nabla u) + f(u, v) = 0, & x \in \mathbf{R^N} \\ \mathrm{div}(|\nabla v|^{l-2}\nabla v) + g(u, v) = 0, & x \in \mathbf{R^N} \end{cases} \tag{1.3}$$

has gained much attention recently. See, for example, [3, 4, 10, 15, 19, 21, 22].

When $p = q = 2$, system (1.3) becomes

$$\begin{cases} \triangle u + f(u, v) = 0, & x \in \mathbf{R^N} \\ \triangle v + g(u, v) = 0, & x \in \mathbf{R^N} \end{cases}$$

for which the existence and the non-existence of positive solutions and positive boundary blow-up solutions have been investigated extensively. We list here, for example, [1,2,5,6,12-14,16] and refer to the references therein.

When $p = q = 2, f = -a(|x|)v^\alpha, g = -b(|x|)u^\beta$, system (1.3) becomes

$$\begin{cases} \triangle u = a(|x|)v^\alpha, & x \in \mathbf{R}^\mathbf{N} \\ \triangle v = b(|x|)u^\beta, & x \in \mathbf{R}^\mathbf{N} \end{cases} \tag{1.4}$$

for which existence results for positive boundary blow-up solutions can be found in a recent paper by Lair and Wood [12]. Lair and Wood established that all positive entire radial solutions of (1.4) are boundary blow-up provided that

$$\int_0^\infty ta(t)dt = \infty, \quad \int_0^\infty tb(t)dt = \infty.$$

If, on the other hand

$$\int_0^\infty ta(t)dt < \infty, \quad \int_0^\infty tb(t) < \infty,$$

then all positive entire radial solutions of (1.4) are bounded.

F. Cirstea and V.D. Radulescu [1], extended the above results to a larger class of systems

$$\begin{cases} \triangle u = a(|x|)g(v), & x \in \mathbf{R}^\mathbf{N} \\ \triangle v = b(|x|)f(u), & x \in \mathbf{R}^\mathbf{N} \end{cases}$$

In recent years, Zhijun Zhang et al.[23] studied the following semilinear elliptic systems

$$\begin{cases} \triangle u = p(x)f(v), & x \in \mathbf{R}^N, (N \geq 3), \\ \triangle v = q(x)g(u), & x \in \mathbf{R}^N. \end{cases} \tag{1.5}$$

They obtained the existence and nonexistence of solutions for (1.5) by considering a set of hypotheses on p, q, f and g.

Z.D. Yang [19], extended the above results to a class of systems

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) = a(|x|)g(v), & x \in \mathbf{R}^\mathbf{N}, \\ \operatorname{div}(|\nabla v|^{l-2}\nabla v) = b(|x|)f(u), & x \in \mathbf{R}^\mathbf{N}. \end{cases}$$

Motivated by the results of the papers [19-23]. In this paper, we consider the quasilinear elliptic system (1.1). We modify the method developed by Zhang et al.[23] and extend partial results of [23] to a quasilinear elliptic system (1.1).

2 Main Results

In order to establish our main result, we introduce the following hypotheses :

(H4) $r^{N-1}(\phi_1(r) + \psi_1(r))$ is nondecreasing for large r;

(H5) there exists a positive constant ε such that

$$\int_0^\infty t^{\frac{1+\varepsilon}{m-1}}(\phi_1(t)+\psi_1(t))^{\frac{1}{m-1}}dt<\infty,$$

and

$$\int_0^\infty t^{\frac{1+\varepsilon}{l-1}}(\phi_1(t)+\psi_1(t))^{\frac{1}{l-1}}dt<\infty.$$

Our main results are as the following:

Theorem 1. Under the hypotheses (H1)-(H5), equation (1.1) has a positive entire bounded solution (u,v).

From the above theorem, we get the following corollary

Corollary 1. Suppose that p and q are spherically symmetric(i.e. $p(x)=p(|x|,q(x)=q(|x|))$. Under hypotheses (H1)-(H3), (1.1) has one positive solution (u,v).
Suppose further that $P(\infty)=Q(\infty)=\infty$, where

$$P(\infty):=\lim_{r\to\infty}P(r),P(r):=\int_0^r(t^{1-N}\int_0^t s^{N-1}p(s)ds)^{\frac{1}{m-1}}dt,r\geq 0;$$

$$Q(\infty):=\lim_{r\to\infty}Q(r),Q(r):=\int_0^r(t^{1-N}\int_0^t s^{N-1}q(s)ds)^{\frac{1}{l-1}}dt,r\geq 0$$

Then every positive radial entire solution (u,v) of (1.1) is large and satisfies

$$u(r)\geq u(0)+f(v(0))P(r),v(r)\geq v(0)+g(u(0))Q(r).\quad \forall r\geq 0.$$

Corollary 2. Under the assumption (H1)-(H4), if (1.1) has a non-negative radial entire large solution, then at least one of the following two equations hold:

$$\int_0^\infty r^{\frac{1+\varepsilon}{m-1}}(p(r)+q(r))^{\frac{1}{m-1}}dr=\infty,\quad \forall\varepsilon>0.$$

$$\int_0^\infty r^{\frac{1+\varepsilon}{l-1}}(p(r)+q(r))^{\frac{1}{l-1}}dr=\infty,\quad \forall\varepsilon>0.$$

Remark 1. By (H1) and (H3), we have

$$\int_a^\infty \frac{ds}{\sqrt[d]{F(s)}}=\int_a^\infty \frac{ds}{\sqrt[d]{G(s)}}=\infty.$$

Remark 2. When $2\leq d<\infty$, $\int_0^\infty r^{\frac{1}{d-1}}(p(x)+q(x))^{\frac{1}{d-1}}dr=\infty$ implies

$$\int_0^\infty(t^{1-N}\int_0^t s^{N-1}(p(x)+q(x))(s)ds)^{\frac{1}{d-1}}dt=\infty$$

.

Remark 3. If $\int_a^\infty \frac{ds}{\sqrt[d]{F(s)}} < \infty, a > 0$, then $\int_a^\infty \frac{dt}{(f(t))^{\frac{1}{d-1}}} < \infty$. In other words, if $\int_a^\infty \frac{dt}{(f(t))^{\frac{1}{d-1}}} = \infty$, then $\int_a^\infty \frac{ds}{\sqrt[d]{F(s)}} = \infty$.

Proof. We only need to prove

$$\frac{(f(t))^{\frac{1}{d-1}}}{s} > \delta^d \tag{1.6}$$

for $\forall\ \delta > 0$. Then $F(s) \equiv \int_0^s f(t)dt \le sf(s) \le \frac{f^{\frac{d}{d-1}}(s)}{\delta^d}$, and $(F(s))^{-\frac{1}{d}} \ge \frac{\delta}{(f(s))^{\frac{1}{d-1}}}$. We suppose that (1.6) is not true, then $\exists$ an increasing sequence $\{s_j\}$, $\lim_{j\to\infty} s_j = \infty$ such that $\frac{(f(s_j))^{\frac{1}{d-1}}}{s_j} < \frac{1}{j}$, which equals to $f(s_j) \le (\frac{s_j}{j})^{d-1}$, then $(f(s_j))^{-\frac{1}{d}} \ge (\frac{s_j}{j})^{-\frac{d-1}{d}}$. Since f is nondecreasing, we get $f(s) \le f(s_j)$ for all $s \in [0, s_j]$, so $F(s) \le sf(s) \le sf(s_j)$ for all $s \in [0, s_j]$, and

$$\begin{aligned}
&\int_{s_1}^{s_j} (F(s))^{-\frac{1}{d}} ds \ge \int_{s_1}^{s_j} (sf(s_j))^{-\frac{1}{d}} ds \\
&\ge (\frac{s_j}{j})^{-\frac{d-1}{d}} \int_{s_1}^{s_j} s^{-\frac{1}{d}} ds = j^{1-\frac{1}{d}}(1 - (\frac{s_1}{s_j})^{1-\frac{1}{d}}) \to \infty
\end{aligned}$$

This is a contradiction.

In order to prove the Theorem 1, we give the following lemma.

Lemma 1. For any nonnegative a and b, we have

$$(a+b)^\alpha \le a^\alpha + b^\alpha, \quad \alpha \in (0,1]$$

$$(a+b)^\beta \le 2^{\beta-1}(a^\beta + b^\beta), \quad \beta \in [1,\infty)$$

Proof of Theorem 1. First, we have to find a pair of super-solution, $(\bar{u}, \bar{v})$ and sub-solution,$(\underline{u}, \underline{v})$, which satisfy $\underline{u} \le \bar{u}$ and $\underline{v} \le \bar{v}$. Consider the following system of integral equation:

$$\begin{aligned}
\underline{u}(r) &= \beta + \int_0^r (t^{1-N} \int_0^t s^{N-1} \phi_1(s) f(\underline{v}(s)) ds)^{\frac{1}{m-1}} dt, r \ge 0 \\
\underline{v}(r) &= \beta + \int_0^r (t^{1-N} \int_0^t s^{N-1} \psi_1(s) g(\underline{u}(s)) ds)^{\frac{1}{l-1}} dt, r \ge 0
\end{aligned} \tag{1}$$

where $\beta \ge c > 0$, c is in (H3). Let $\{\underline{v}_k\}_{k\ge0}$ and $\{\underline{u}_k\}_{k\ge1}$ be the sequence of positive continuous functions defined on $[0,\infty)$ by $\underline{v}_0 = \beta$,

$$\begin{aligned}
\underline{u}_k(r) &= \beta + \int_0^r (t^{1-N} \int_0^t s^{N-1} \phi_1(s) f(\underline{v}_{k-1}(s)) ds)^{\frac{1}{m-1}} dt, r \ge 0 \\
\underline{v}_k(r) &= \beta + \int_0^r (t^{1-N} \int_0^t s^{N-1} \psi_1(s) g(\underline{u}_{k-1}(s)) ds)^{\frac{1}{l-1}} dt, r \ge 0
\end{aligned} \tag{2}$$

Then, $\underline{v_0} \le \underline{v_1}$, $\underline{u}_k(r) \ge \beta$, and $\underline{v}_k(r) \ge \beta$ for all $r \ge 0$, $k \in N$. Using the non-decreasing property of f and g, we get $\underline{u}_1(r) \le \underline{u}_2(r)$ for all $r \ge 0$, then $\underline{v}_1(r) \le \underline{v}_2(r)$ for all $r \ge 0$. Continuing this line, we obtain that the sequence $\{\underline{u}_k\}$ and $\{\underline{v}_k\}$ are increasing with respect to k for $r \in [0, \infty)$. Besides,

$$\underline{u}_k'(r) = (r^{1-N} \int_0^r s^{N-1} \phi_1(s) f(\underline{v}_{k-1}(s)) ds)^{\frac{1}{m-1}} \ge 0,$$

$$\underline{v}_k'(r) = (r^{1-N} \int_0^r s^{N-1} \psi_1(s) g(\underline{u}_{k-1}(s)) ds)^{\frac{1}{l-1}} \ge 0,$$

for each $r > 0$, and

$$(r^{N-1} |\underline{u}_k'|^{m-2} \underline{u}_k')' = r^{N-1} \phi_1(r) f(\underline{v}_{k-1}(r)) \le r^{N-1} \phi_1(r) f(\underline{v}_k(r)) \tag{3}$$

let

$$\Theta(r) = \max_{0 \le t \le r} (\phi_1(t) + \psi_1(t)),$$

using this and the fact that $\underline{u}_k' \ge 0$, we note that (3) yields

$$((\underline{u}_k'(r))^{m-1})' \le \Theta(r) f(\underline{v}_k(r)),$$

Multiply this by $\underline{u}_k'$ and integrate to get

$$(\underline{u}_k'(r))^m \le \frac{m}{m-1} \Theta(r) \int_{2\beta}^{v_k(r)+u_k(r)} f(s) ds$$

In the same way,

$$(\underline{v}_k'(r))^l \le \frac{l}{l-1} \Theta(r) \int_{2\beta}^{v_k(r)+u_k(r)} g(s) ds$$

Then from the inequality $(\underline{u}_k' + \underline{v}_k')^d \le 2^{d-1}((\underline{u}_k')^d + (\underline{v}_k')^d)$, where $d = \min\{m, l\}$, and the above two inequalities, we get

$$\begin{aligned} (\underline{u}_k' + \underline{v}_k')^d &\le 2^{d-1}((\underline{u}_k')^d + (\underline{v}_k')^d) \\ &\le 2^{d-1}((\underline{u}_k')^m + (\underline{v}_k')^l + 1) \\ &\le 2^{d-1}(\frac{d}{d-1} \Theta(r) \int_{2\beta}^{v_k(r)+u_k(r)} (f(s) + g(s)) ds + 1) \\ &\le 2^{d-1}(\frac{d}{d-1} \Theta(r)(F(\underline{u} + \underline{v}) + G(\underline{u} + \underline{v})) + 1) \end{aligned} \tag{4}$$

which yields

$$\begin{aligned} \underline{u}_k' + \underline{v}_k' &\le 2^{\frac{d-1}{d}} (\frac{d}{d-1} \Theta(r)(F(\underline{u}_k + \underline{v}_k) + G(\underline{u}_k + \underline{v}_k)) + 1)^{\frac{1}{d}} \\ &\le \sqrt[d]{\frac{2^{d-1} d}{d-1} \Theta(r)} (F(\underline{u}_k + \underline{v}_k) + G(\underline{u}_k + \underline{v}_k))^{\frac{1}{d}} + 2^{\frac{d-1}{d}} \end{aligned} \tag{5}$$

Integrating the above inequality, we get

$$\begin{aligned}
&\int_0^r \frac{\underline{u}_k'(t)+\underline{v}_k'(t)}{(F(\underline{u}_k(t)+\underline{v}_k(t))+G(\underline{u}_k(t)+\underline{v}_k(t)))^{\frac{1}{d}}}dt\\
&=\int_{2\beta}^{v_k(r)+u_k(r)} \frac{d\tau}{\sqrt[d]{F(\tau)+G(\tau)}}\\
&\le \int_0^r (\sqrt[d]{\frac{2^{d-1}d\Theta(t)}{d-1}}+C)dt
\end{aligned}$$

where $C=\frac{2^{\frac{d-1}{d}}}{F(2\beta)+G(2\beta)}$. We can easily get

$$H(\underline{u}_k(r)+\underline{v}_k(r))\le H(2\beta)+\int_0^r(\sqrt[d]{\frac{2^{d-1}d\Theta(t)}{d-1}}+C)dt$$

As we know that H^{-1} is increasing on $[0,\infty)$, so

$$u_k(r)+v_k(r)\le H^{-1}(H(2\beta)+\int_0^r\sqrt[d]{\frac{2^{d-1}d\Theta(t)}{d-1}}+C)dt,\quad \forall r\ge 0$$

Following by the definition of $\underline{u}_k(r)$ and $\underline{v}_k(r)$ and (H3), we get that the sequence $\{\underline{u}_k\}$ and $\{\underline{v}_k\}$ are bounded and equi-continuous on $[0, C_0]$ for arbitrary $C_0>0$. By Arzela-Ascoli theorem, $\{\underline{u}_k\}$ and $\{\underline{v}_k\}$ have subsequence converging uniformly to $\underline{u}$ and $\underline{v}$ on $[0, C_0]$. By the arbitrariness of $C_0>0$,we see that $(\underline{u},\underline{v})$ is a positive entire solution of

$$\Delta_m\underline{u}=\phi_1(r)f(\underline{v})\ge p(x)f(\underline{v}),\quad x\in\mathbf{R}^N \tag{6}$$

$$\Delta_l\underline{v}=\psi_1(r)g(\underline{u})\ge q(x)g(\underline{v}),\quad x\in\mathbf{R}^N \tag{7}$$

Then, we take conclusion that $(\underline{u},\underline{v})$ is a positive entire sub-solution of (1.1).

In order to prove $(\underline{u},\underline{v})$ is bounded, choosing $R>0$, so that $r^{d(N-1)}(\phi_1(r)+\psi_1(r))$ is non-decreasing on $[R,\infty)$ and $\underline{u}(r)>0, \underline{v}(r)>0$. This is possible because of (H4). Since $(\underline{u},\underline{v})$ satisfies

$$(r^{N-1}(\underline{u}')^{m-1})'=r^{N-1}\phi_1(r)f(\underline{v}(r)), \tag{8}$$

$$(r^{N-1}(\underline{v}')^{l-1})'=r^{N-1}\psi_1(r)g(\underline{u}(r)). \tag{9}$$

$\underline{u}'(r)\ge 0$ and $\underline{v}'(r)\ge 0$ for $r\ge 0$, and (H2) hold, multiplying (8) and (9) by $\underline{u}'$ and $\underline{v}'$, respectively, and integrating from R to r. Take (8) as an example,

$$\int_R^r(s^{N-1}(\underline{u}')^{m-1})'\underline{u}'(s)ds=\int_R^r s^{N-1}\phi_1(s)f(\underline{v}(s))\underline{u}'(s)ds,$$

which implies that

$$\frac{m-1}{m}r^{N-1}(\underline{u}'(r))^m-\frac{m-1}{m}R^{N-1}(\underline{u}'(R))^m+\frac{N-1}{m}\int_R^r s^{N-2}(\underline{u}'(s))^m ds=\int_R^r s^{N-1}\phi_1(s)f(\underline{v}(s))\underline{u}'(s)ds$$

It follows that

$$r^{N-1}(\underline{u}'(r))^m \leq R^{N-1}(\underline{u}'(R))^m + \frac{m}{m-1}\int_R^r s^{N-1}\phi_1(s)f(\underline{v}(s))\underline{u}'(s)ds$$

Using the monotonicity of $t^{N-1}(\phi_1(t)+\psi_1(t))$ for $t \geq 0$, we get

$$r^{N-1}(\underline{u}'(r))^m \leq \bar{C} + \frac{m}{m-1}r^{N-1}(\phi_1(r)+\psi_1(r))(F(\underline{u}(r)+\underline{v}(r)) + G(\underline{u}(r)+\underline{v}(r)))$$

for $r > R$, where $\bar{C} = R^{N-1}(\underline{u}'(R))^m + R^{N-1}(\underline{v}'(R))^l$.

which yields

$$\underline{u}'(r) \leq \sqrt[m]{\bar{C}}r^{\frac{1-N}{m}} + \sqrt[m]{\frac{m}{m-1}(\phi_1(r)+\psi_1(r))}(F(\underline{u}+\underline{v}) + G(\underline{u}+\underline{v}))^{\frac{1}{m}}$$

So

$$\begin{aligned}&\underline{u}'(r)+\underline{v}'(r) \leq C_1(r^{\frac{1-N}{m}} + r^{\frac{1-N}{l}})\\ &+(\sqrt[m]{\frac{m}{m-1}(\phi_1(r)+\psi_1(r))} + \sqrt[l]{\frac{l}{l-1}(\phi_1(r)+\psi_1(r))})(2(F(\underline{u}+\underline{v}) + G(\underline{u}+\underline{v}))^{\frac{1}{d}}+1)\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dr}\int_{\underline{u}(R)+\underline{v}(R)}^{\underline{u}(r)+\underline{v}(r)} \frac{d\tau}{\sqrt[d]{F(\tau)+G(\tau)}}\\ &\leq C_1(r^{\frac{1-N}{m}} + r^{\frac{1-N}{l}})(F(\underline{u}+\underline{v}) + G(\underline{u}+\underline{v}))^{-\frac{1}{d}} + h(r)(2+(F(\underline{u}+\underline{v}) + G(\underline{u}+\underline{v}))^{-\frac{1}{d}})\end{aligned} \tag{10}$$

where $h(r) = \sqrt[m]{\frac{m}{m-1}(\phi_1(r)+\psi_1(r))} + \sqrt[l]{\frac{l}{l-1}(\phi_1(r)+\psi_1(r))}$.
We notice the fact that

$$F(\underline{u}(r)+\underline{v}(r)) + G(\underline{u}(r)+\underline{v}(r)) \geq F(\underline{u}(R)+\underline{v}(R)) + G(\underline{u}(R)+\underline{v}(R)) = C_2$$

for all $r \geq R$, and

$$\sqrt[m]{\frac{m}{m-1}(\phi_1(r)+\psi_1(r))} \leq \frac{m}{m-1}\sqrt[m]{r^{1+\varepsilon}(\phi_1(r)+\psi_1(r))r^{-1-\varepsilon}}$$

Using Young's inequality, we get

$$\sqrt[m]{\frac{m}{m-1}(\phi_1(r)+\psi_1(r))} \leq \frac{1}{m-1}r^{-1-\varepsilon} + r^{\frac{1+\varepsilon}{m-1}}(\phi_1(r)+\psi_1(r))^{\frac{1}{m-1}} \quad \text{for } \varepsilon > 0.$$

In the same way,

$$\sqrt[l]{\frac{l}{l-1}(\phi_1(r)+\psi_1(r))} \leq \frac{1}{l-1}r^{-1-\varepsilon} + r^{\frac{1+\varepsilon}{l-1}}(\phi_1(r)+\psi_1(r)))^{\frac{1}{l-1}} \quad \text{for } \varepsilon > 0.$$

Then integrate (10) from R to r, $r \geq R$,

$$H(\underline{u}(r)+\underline{v}(r)) - H(\underline{u}(R)+\underline{v}(R))$$
$$\leq C_3 + C_4\left(\left(\frac{1}{m-1}+\frac{1}{l-1}\right)\frac{R^{-\varepsilon}}{\varepsilon} + \int_R^r t^{\frac{1+\varepsilon}{m-1}}(\phi_1+\psi_1)^{\frac{1}{m-1}}dt + \int_R^r t^{\frac{1+\varepsilon}{l-1}}(\phi_1+\psi_1)^{\frac{1}{l-1}}dt\right)$$

where $C_3 = \sqrt[d]{C_2}C_1\left(\frac{mR^{\frac{1+m-N}{m}}}{N-m-1} + \frac{lR^{\frac{1+l-N}{l}}}{N-l-1}\right)$, $C_4 = 2 + (F(2R)+G(2R))^{-\frac{1}{d}}$.

From (H5), we know

$$\int_R^r t^{\frac{1+\varepsilon}{m-1}}(\phi_1+\psi_1)^{\frac{1}{m-1}}dt < \infty,$$

and

$$\int_R^r t^{\frac{1+\varepsilon}{l-1}}(\phi_1+\psi_1)^{\frac{1}{l-1}}dt < \infty,$$

So

$$H(\underline{u}(r)+\underline{v})(r) < \infty$$

Letting $r \to \infty$, since H satisfies (H3), we find that $(\underline{u},\underline{v})$ is bounded .

By now, we have find a pair of bounded sub-solution to (1.1). We still have to find $(\bar{u},\bar{v})$, which is a bounded super-solution of (1.1), and $\underline{u}(r) \leq \bar{u}(r)$, $\underline{v}(r) \leq \bar{v}(r)$ for all $r \geq 0$. Actually, since $(\underline{u},\underline{v})$ is nondecreasing and bounded, we have

$$\lim_{r\to\infty}\underline{u}(r) = M_1 > 0, \quad \lim_{r\to\infty}\underline{v}(r) = M_2 > 0.$$

Let $\bar{u}(0) = \bar{v}(0) = \max\{M_1, M_2\}$, $\bar{u}'(0) = \bar{v}'(0) = 0$, then, the following system

$$\Delta_m\bar{u}(x) = \phi_2(r)f(\bar{v}(r)) \quad r > 0$$
$$\Delta_l\bar{v}(x) = \psi_2(r)g(\bar{u}(r)) \quad r > 0$$

has a bounded solution $(\bar{u},\bar{v})$ by the same argument, and it is a supersolution for (1.1). From the above process, we get conclusion that

$$\underline{u}(r) \leq M_1 \leq \bar{u}(r), \quad \underline{v}(r) \leq M_2 \leq \bar{v}(r). \quad \forall r \geq 0.$$

The standard super-sub solution principle [18,20] implies that (1.1) has a bounded solution (u,v) satisfying $\underline{u}(x) \leq u(x) \leq \bar{u}$ and $\underline{v}(x) \leq v(x) \leq \bar{v}$ on $\mathbf{R}^N$, which is the desired solution. This completes the proof.

3 Conclusion

The boundary value quasilinear differential equation systems (1.1) are mathematical models occurring in the studies of the m-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium. When $m \neq 2$, the problem becomes more complicated since certain nice properties in herent to the case $m = 2$ seem to be lost or at least difficult to verify. The main differences between $m = 2$ and $m \neq 2$ can be founded in [8,9]. When $m = 2$, it is well known that all the positive solutions in $C^2(B_R)$ of the problem

$$\begin{cases} \triangle u + f(u) = 0 \text{ in } B_R \\ u(x) = 0 \text{ on } \partial B_R \end{cases}$$

are radially symmetric solutions for very general f(see [7]). Unfortunately, this result does not apply to the case $m \neq 2$. Kichenassary and Smoller showed that there exist many positive nonradial solutions of the above problem for some f(see [11]). The major stumbling block in the case of $m \neq 2$ is that certain nice features inherent to the case $m = 2$ seem to be lost or at least difficult to verify. In this paper, we first give some necessary preliminary knowledge. Secondly, we further study the existence of positive solutions to problem (1.1) which the right hand side functions are more general based on the method of sub-supersolution.

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