

## On the Poisson's equation $-\Delta u = \infty$ .

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### ABSTRACT

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain. We prove the existence of a bounded solution of the Poisson's equation  $-\Delta u = \infty$  on  $\Omega$ .

### RESUMEN

Sea  $\Omega \subset \mathbb{R}^N$  un dominio acotado. Probatmos la existencia de una solución acotada para la ecuación de Poisson  $-\Delta u = \infty$  en  $\Omega$ .

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<sup>1</sup>Dedicated to Professor Gaston M N'Guérékata on the occasion of his 60<sup>th</sup> birthday.

## 1 Introduction.

In [19] it is stated that

*Le mouvement d'un corps libre consiste dans le mouvement de translation de son centre de gravité et dans le changement de sa position autour de ce point. La recherche du mouvement du centre de gravité se réduit à déterminer le mouvement d'un point sollicité par des forces données; et, relativement aux corps célestes, ces forces sont le résultat des attractions de sphéroïdes dont la figure est supposée connue. Soient  $dm$  une molécule d'un sphéroïde;  $x', y', z'$  les trois coordonnées orthogonales de cette molécule;  $dm$  sera de la forme  $\xi dx' dy' dz'$ ,  $\xi$  étant fonction de  $x', y', z'$ . Soient encore  $x, y, z$  les coordonnées d'un point attiré, on aura*

$$V = \int \frac{\xi dx' dy' dz'}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}} \quad (1)$$

*cette intégrale étant prise relativement à toute l'étendue du sphéroïde. Ses limites étant indépendantes de  $x, y, z$  ainsi que les variables  $x', y', z'$ , il est clair qu'en différential l'expression de  $V$  par rapport à  $x, y, z$  il suffira, dans cette différentiation, d'avoir égard au radical que renferme cette expression, et alors il est facile de voir que l'on a*

$$0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \quad (2)$$

In modern interpretation of potential  $V$  of mass distributions, we have

$$V(x, y, z) = \int_G \frac{\xi(x', y', z') dx' dy' dz'}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}}. \quad (3)$$

where  $\xi(x', y', z')$  is the density of a mass distribution in the space  $x', y', z'$ . Then  $\nabla V$  furnishes the gravity field force and  $-\Delta V = 0$  on  $\mathbb{R}^3 - G$ .

In 1813 Poisson found that for a ball  $G$  the following equation is valid in the case of constant density  $\xi(x, y, z') = \rho$

$$-\Delta u = 4\pi\rho \text{ on } G \text{ Poisson's equation.}$$

Therefore a natural question is: there exists a *solution* for Poisson's equation with  $\rho = \infty$ ?. That kind of *solution* will be related to gravity potential of bodies with infinite density or black holes. The authors are not aware of a previous result deducing the existence of black holes using Newton gravity theory or the gravity potential inside of a black hole. The equation

$$-\Delta u = u^p, \quad (4)$$

for  $p$  a nonnegative real number and  $u > 0$  in a Ball of radius  $R$  in  $\mathbb{R}^3$ , with Dirichlet boundary conditions was introduced by Lane [18] for modelling both the temperature and the density of

mass on the surface of the sun. Today the problem (4) is named Lane-Emden-Fowler equation. It was used first in the mid-19th century in the study of internal structure of stars mainly by Chandrasekhar [4, 7, 9]. Singular Lane-Emden-Fowler equations ( $p < 0$ ) has been considered in a remarkable pioneering paper by Fulks and Maybe [10].

Eddigton [6] proposed the equation

$$-\Delta u = \frac{\exp(2u)}{1 + |x|^2} \text{ in } \mathbb{R}^3, \quad (5)$$

in order to represent the gravitational potential  $u$  of a globular cluster of stars.

Matukuma [20] introduced the equation

$$-\Delta u = \frac{u^r}{1 + |x|^2} \text{ in } \mathbb{R}^3, \quad (6)$$

where  $u$  is the gravity potential,  $\rho = (2\pi)^{-1}(1 + |x|^2)^{-1}u^r$  is the density and  $\int_{\mathbb{R}^3} \rho dx$  is the total mass to study the gravitational potential  $u$  of a globular cluster of stars. For the same problem Hénon [15] suggested

$$-\Delta u = |x|^l u^r \text{ in } \Omega \subset \mathbb{R}^3. \quad (7)$$

Black holes solutions means that the gravitational potential of the cluster behaves like  $\frac{1}{r}$  ( $r = |x|$ ) near the center.

Peebles [16, 17] gives for the first time a derivation of the steady state distribution of the star near a massive collapsed object. The question of the existence of black hole in a globular cluster is still open (1995). Core collapse does occur, for instance using Hubble Space Telescope, Bendinelli et.al. [2] reported the first detection of a collapsed core globular cluster in M31.

On May 25, 1994 astronomers at NASA headquarters announced the Hubble Space Telescope finding of a supermassive black hole in the heart of the giant galaxy M87, more than 50 million light-years.

The equation

$$-\Delta \frac{1}{|x - x_0|} = 4\pi\delta(x - x_0) \text{ in } \mathbb{R}^3,$$

has a deep insight because relate the formulation of the Laplace operator and the Dirac  $\delta$  function in a weak sense. The Laplace operator with point interaction in  $\mathbb{R}^3$  given by  $-\Delta + \alpha\delta$ ,  $\alpha \in \mathbb{R}$  has been widely study for your applications in quantum physics (see for expample [11]) and in seismic imaging [3].

Our purpose in this paper is to give a classical interpretation to the equation

$$-\Delta u = \infty \text{ in } \Omega \subset \mathbb{R}^N. \quad (8)$$

We define:

**Definition 1.1.** *The equation (8) has a classical solution if there exist two non decreasing sequences of functions  $\{u_j\}_{j=1}^\infty \in C(\bar{\Omega}) \cap C^2(\Omega)$  and  $\{f_j\}_{j=1}^\infty$  such that*

$$-\Delta u_j = f_j \quad \text{in } \Omega,$$

and  $\lim_{j \rightarrow \infty} f_j(x) = \infty$  for all  $x \in \Omega$  and  $\lim_{j \rightarrow \infty} u_j(x) = u(x) < \infty$  for all  $x \in \Omega$ .

Our main result in this article is as follows.

**Theorem 1.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 3$ . Then the problem*

$$-\Delta u = \infty \quad \text{in } \Omega, \tag{9}$$

has a non negative classical solution  $u$ .

Under the authors knowledge this is the first compactness result dealing with infinite on a non trivial domain (see for example [21] first chapter: direct methods in the calculus of variations). Similarly the theory of generalized functions not allow solutions to this kind of problem because every distribution is locally a Newtonian potential:

**Theorem 1.3.** *(page 277 [5]) Let  $\Omega$  be an open set of  $\mathbb{R}^N$ ,  $f \in \mathcal{D}'(\Omega)$  and  $u$  a solution (in the sense of distributions) of Poisson's equation  $\Delta u = f$  on  $\Omega$ . Then for every bounded open set  $\Omega_1$  with  $\bar{\Omega}_1 \subset \Omega$  there exists  $f_1 \in \mathcal{E}'$  the space of distributions on  $\mathbb{R}^N$  with compact support, such that  $f_1 = f$  on  $\Omega$  and  $u =$  the Newtonian potential of  $f_1$  on  $\Omega_1$ .*

Moreover if we study this problem using a weak formulation in Sobolev's spaces, the Georgi-Nash-Moser theory cannot be used to derive any comparable compactness result [14].

We will use a non linear singular elliptic approach as in [1, 8, 13] to obtain the result.

Our strategy is study the auxiliary problem

$$\begin{aligned} -\Delta u_{\epsilon, m} &= g_m(u_\epsilon) \quad \text{in } \Omega, \\ u_{\epsilon, m} &= \epsilon \quad \text{on } \partial\Omega, \end{aligned}$$

where  $g_m : (0, \infty) \rightarrow (0, \infty)$ ,  $m = 1, \dots, \infty$  is non increasing locally Hölder continuous function singular at the origin with the properties  $g_m(s) = g(s)$  for all  $s \geq 1$  and  $\lim_{m \rightarrow \infty} g_m(s) = \infty$  for all  $s \in (0, 1)$ ,  $m = 1, \dots, \infty$  and  $g : (0, \infty) \rightarrow (0, \infty)$  is strictly non increasing locally Hölder continuous function singular at the origin.

Our result 1.2 is obtained letting  $\lim_{m \rightarrow \infty, \epsilon \rightarrow 0^+} u_{\epsilon, m}$ . This limit by definition has not weak derivatives of first or second order.

## 2 Auxiliary results

**Theorem 2.1** ([1]). *Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^N$ ,  $N \geq 3$ ,  $g : (0, \infty) \rightarrow (0, \infty)$  is non increasing locally Hölder continuous function (that may be singular at the origin). Then the problem*

$$\begin{aligned} -\Delta u_\epsilon &= g(u_\epsilon) \quad \text{in } \Omega, \\ u_\epsilon &= \epsilon \quad \text{on } \partial\Omega, \end{aligned}$$

has a unique positive solution  $u \in C(\bar{\Omega}) \cap C^2(\Omega)$  for  $\epsilon \geq 0$ . Moreover  $u_{\epsilon_2} \geq u_{\epsilon_1}$  for  $\epsilon_2 \geq \epsilon_1$ .

We consider the the auxiliary problem

$$\begin{aligned} -\Delta u_m &= g_m(u_m) \quad \text{in } \Omega, \\ u_m &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{10}$$

**Lemma 2.2.** *Let  $u_m$  be a solution of the equation (10). Then  $u_{m+j} \geq u_m$ .*

*Proof.* Suppose that there exists  $x_0 \in \Omega$  such that  $u_m(x_0) > u_{m+j}(x_0)$ . Therefore for  $\tau > 0$  small enough we have the inequality  $u_m(x_0) > \tau + u_{m+j}(x_0)$ . Then by continuity in  $\Omega$  of the function  $F(x) = u_m(x) - \tau - u_{m+j}(x)$  there exist a non empty open set  $\Omega_\tau$  such that  $F(x) > 0$  for all  $x \in \Omega_\tau$  and  $F = 0$  on  $\partial\Omega_\tau$ . Using that  $u_m(x) > \tau + u_{m+j}(x)$  for all  $x \in \Omega_\tau$ , we deduce

$$g_m(u_m(x)) \leq g_{m+j}(u_m(x)) \leq g_{m+j}(\tau + u_{m+j}(x)) \leq g_{m+j}(u_{m+j}(x))$$

for all  $x \in \Omega_\tau$ . Then

$$\begin{aligned} -\Delta u_m &\leq -\Delta(u_{m+j} + \tau) \quad \text{in } \Omega_\tau, \\ u_m &= u_{m+j} + \tau \quad \text{on } \partial\Omega_\tau. \end{aligned}$$

and we obtain  $u_m \leq u_{m+j} + \tau$  in  $\Omega_\tau$  (Theorem 3.3 [14]) a contradiction.  $\square$

**Lemma 2.3.** *Let  $u_m$  be a solution of the equation (10). Then  $g_{m+j}(u_{m+j}(x)) \geq g_m(u_m(x))$ .*

*Proof.* Suppose that there exists  $x_0 \in \Omega$  such that  $g_m(u_m(x_0)) > g_{m+j}(u_{m+j}(x_0))$ . Then by continuity in  $\Omega$  of the function  $H(x) = g_m(u_m(x)) - g_{m+j}(u_{m+j}(x))$ , there exists  $\hat{\Omega} \subset \Omega$  such that  $H(x) > 0$  in  $\hat{\Omega}$  and  $H(x) = 0$  on  $\partial\hat{\Omega}$

$$\begin{aligned} -\Delta u_m &\geq -\Delta u_{m+j} \quad \text{in } \hat{\Omega}, \\ u_m &= u_{m+j} \quad \text{on } \partial\hat{\Omega}. \end{aligned}$$

We imply  $u_m \geq u_{m+j}$  in  $\hat{\Omega}$  (Theorem 3.3 [14]). Therefore  $g_m(u_m(x)) \leq g_m(u_{m+j}(x)) \leq g_{m+j}(u_{m+j}(x))$  for all  $x \in \hat{\Omega}$ . A contradiction.  $\square$

**Remark 2.4.** *In the proof of Lemmas 2.2 and 2.3 it is assumed only that  $g_m$  is a non increasing continuous function.*

### 3 Proof

*Proof of Theorem 1.2.* Let us consider the problem

$$\begin{aligned} -\Delta v &= g(v) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We introduce the equations

$$\begin{aligned} -\Delta e &= g(e) \quad \text{in } \Omega, \\ e &= 1 \quad \text{on } \partial\Omega. \end{aligned}$$

$$\begin{aligned} -\Delta w &= g(e) \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Using  $v \leq e$  (see Lemma 2.3 and 2.6 in [1]), we infer

$$\begin{aligned} -\Delta w &= g(e) \leq g(v) = -\Delta v \quad \text{in } \Omega, \\ w &= 0 = v \quad \text{on } \partial\Omega. \end{aligned}$$

Then  $w \leq v$  in  $\Omega$ . Setting  $g_0 = g$  and using the auxiliary results with the new sequence  $\{g_j\}_{j=0}^\infty$ , we conclude that  $w \leq u_m \leq e$  for  $m = 1, \dots, \infty$ . Using Lemma 2.2, we infer the existence of  $\lim_{m \rightarrow \infty} u_m(x) = u(x)$  for all  $x \in \Omega$ . We restrict ourselves to the situation  $\Omega = B_1(0)$  where  $B_1(0)$  is the ball of radius 1 with center at 0. Applying the main result of [12] we infer that  $u_m$  is a radial function with  $\frac{\partial u_m}{\partial r} < 0$ . Therefore  $u$  is also a radial non increasing function.

We proceed by contradiction, suppose that

$$\lim_{m \rightarrow \infty} g_m(u_m(x)) < \infty \quad \text{for all } 0 \leq \|x\| < 1.$$

Our first implication is that the function  $u$  is strictly non increasing, because if exists  $(r_1, r_2)$  with  $r_2 < 1$ , and  $u(r_1) = u(r_2)$ .

Then  $-\Delta u = 0$  on the annulus  $A(r_1, r_2)$ . Using Theorem 9.11 page 235 in [14], we deduce

$$\begin{aligned} \|u_m\|_{H^2, p(\Omega')} &\leq C(N, p, \Omega', A(r_1, r_2)) (\|u_m\|_{L^p(A(r_1, r_2))} + \|g(u_m)\|_{L^p(A(r_1, r_2))}) \\ &\leq C(N, p, \Omega', A(r_1, r_2)) (\|e\|_{L^p(A(r_1, r_2))} + \|\limsup_{m \rightarrow \infty} g_m(u_m(r_2))\|_{L^p(A(r_1, r_2))}), \end{aligned}$$

for all  $p > N$ , therefore by Sobolev's embedding theorem (Theo. 7.26 [14]) we deduce  $\|u_m\|_{C^{1, \alpha}(\Omega')} \leq C$ .

We use a non negative test function  $\varphi$  with support contained in  $\Omega'$ :

$$\begin{aligned} 0 &= \int_{\Omega'} \nabla u \cdot \nabla \varphi \, dx = \lim_{m \rightarrow \infty} \int_{\Omega'} \nabla u_m \cdot \nabla \varphi \, dx \\ &= \int_{\Omega'} g_m(u_m) \varphi \, dx \geq \int_{\Omega'} g_0(u_0) \varphi \, dx > 0. \end{aligned}$$

Contradiction, therefore we deduce that  $u$  is a strictly non increasing function. Moreover using again estimates in Theorems 9.11 and 9.12 in [14] we have  $u \in C_{loc}^{1,\alpha}(B_1(0))$ .

By assumption  $\limsup_{r \rightarrow 1} u(r) \geq 1$ , therefore  $u(r) > 1$  for  $0 \leq r < 1$ . By construction there exists  $0 < r_0 < 1$  such that  $g_0(u_0(r_0)) > g_0(1)$ .

Using Lemma 2.3, we derive  $g_0(u_0(r_0)) \leq g_m(u_m(r_0))$ . But  $\lim_{m \rightarrow \infty} u_m(r_0) = u(r_0) > 1$  and therefore for  $m$  big enough  $u_m(r_0) > 1$ . Moreover  $g_m(u_m(r_0)) = g_0(u_m(r_0)) < g_0(1)$  because  $g_0$  is strictly non increasing.

Contradiction. It follows that there exists  $0 \leq r_1 < 1$  such that

$$\lim_{m \rightarrow \infty} g_m(u_m(r_1)) = \infty.$$

Now, because  $u_m$  is a radial non increasing function, we infer that

$$g_m(u_m(r_1)) \leq g_m(u_m(r))$$

for all  $r_1 < r < 1$ . So

$$\lim_{m \rightarrow \infty} g_m(u_m(r)) = \infty \text{ for all } r_1 \leq r < 1.$$

Now for  $\Omega$  a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 3$  consider the transformation  $u_m(\frac{a+x}{R})$ . This ends the proof.  $\square$

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