CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$03, (83–91). October 2008

Many-Ended Complete Minimal Surfaces Between Two Parallel Planes in $\mathbb{R}^3$

Francisco Brito*
Universidade Federal de Pernambuco – UFPE,
CCEN-Departamento de Matemática,
50.740.540, Recife-PE, Brazil
email: `brito@dmat.ufpe.br`

ABSTRACT

We use some special convergent Hadamard gap series to provide examples of complete minimal surfaces of many different conformal types between two parallel planes in three dimensional Euclidean space.

RESUMEN

Nosotros usamos algunas series convergentes especiales de Hadamard para dar ejemplo de superficies mínimas completas de varios diferentes tipos conforme entre dos planos paralelos en espacios euclideanos de dimensión tres.

Key words and phrases: *Complete minimal surfaces, Hadamard gap series.*

Math. Subj. Class.: *53A10.*

---

*I am grateful to Profs. Marcus V. M. Wanderley and Maria Elisa Oliveira for helpful hints and discussions.

1 Introduction

In [7] F. Xavier and L. P. M.Jorge established the existence of complete non-planar minimal surfaces between two parallel planes in $\mathbb{R}^3$. Their technique consisted of an artful use of Runge's Theorem to prove the existence of holomorphic functions on the unit disc $\mathbb{D}$ with the right properties they needed. Later their method was adapted by others to produce new surfaces as above with new features, like having cylindrical type as in [6] or being non-orientable as in [3].

Another way for rendering complete minimal surfaces between two parallel planes in $\mathbb{R}^3$ was developed in [1]. This method consisted mainly in proving the existence of bounded holomorphic functions h in $\mathbb{D}$, given by lacunary power series, and such that $\displaystyle\int_\gamma |h'(z)|^2|dz| = \infty$ for all divergent paths γ in the unit disc.

In this paper we intend to show the flexibility of the second method by producing examples of complete minimal surfaces between two parallel planes in $\mathbb{R}^3$ of the following conformal types:

1. A disc with finitely many points removed.
2. Any annulus, $0 < r < |z| < R$.
3. Any annulus as above with finitely many points removed.

This work is organized as follows: In §2 we give some definitions and prove the lemmas that will be needed in the other sections. In the three remaining sections we describe the examples of the types above.

Remark 1.1 *This work was written about fourteen years ago and circulated as a preprint for some time. Meanwhile N.Nadirashvili proved in [4] the existence of bounded complete minimal surfaces in* $\mathbb{R}^3$.

2 Some definitions and lemmas

Lacunary power series were defined in [1] with the restriction that they would have radius of convergence 1. This is just a mild technical point. Here we use any positive real number R as radius of convergence and make the necessary changes for having an analogue of Theorem 2 of [1].

Definition 2.1 *A convergent power series* $\displaystyle\sum_{k=0}^{\infty} a_k z^{n_k}$ *is lacunary if there exists a real number* $q > 1$ *such that* $\dfrac{n_{k+1}}{n_k} \geq q$ *for all* $k = 0, 1, \ldots$.

Lemma 2.2 *Let* $h(z) = \displaystyle\sum_{k=0}^{\infty} a_k z^{n_k}$ *be a lacunary power series of radius of convergence* $R > 0$*, and suppose that the following three conditions hold:*

a) $\displaystyle\sum_{k=0}^{\infty}|a_k|R^{n_k}$ *converges.*

b) $\displaystyle\lim_{k\to\infty} R^{n_k}|a_k|min\left\{\frac{n_{k+1}}{n_k}\ ,\ \frac{n_k}{n_{k-1}}\right\}=\infty.$

c) $\displaystyle\sum_{k=0}^{\infty}R^{2n_k}|a_k|^2 n_k$ *diverges.*

Then h is bounded in $\mathbb{D}_R=\{z\in\mathbb{C};|z|<R\}$, and for all divergent paths γ in $\mathbb{D}_R$, one has that $\displaystyle\int_\gamma |h'(z)|^2|dz|=\infty$.

Proof. The change of variable $z=Rw$ together with (a) show that h is bounded. The same change of variable and Theorem 2 of [1] finish the proof. ∎

Lemma 2.3 *If h satisfies the conditions of the above lemma in $\mathbb{D}_R$, $H(z)=h(z^{-1})$ is a bounded holomorphic function in $A_R=\{z\in\mathbb{C};|z|>R^{-1}\}$, and for all divergent paths γ in A_R tending to a point of $|z|=R^{-1}$ one has that* $\displaystyle\int_\gamma |H^{'}(z)|^2|dz|=\infty$.

Proof. The change of variable $z=w^{-1}$ and Lemma 2.2 prove this assertion. ∎

Now, given $r,R\in\mathbb{R}$, $0<r<R$, let $\Omega_{R,r}$ be the annulus $r<|z|<R$.

Lemma 2.4 *Suppose that* $h_1(z)=\displaystyle\sum_{k=0}^{\infty}a_k z^{n_k}$ *and* $h_2(z)=\displaystyle\sum_{k=0}^{\infty}b_k z^{m_k}$ *are lacunary power series that satisfy the conditions of Lemma 2.2, and have radii of convergence R and r^{-1} respectively, with $0<r<R$. Suppose further that $h_1^{'}(z)$ and $h_2^{'}(z^{-1})$ do not vanish in $|z|=r$ and $|z|=R$ respectively. Then, for all divergent paths γ in $\Omega_{R,r}$ one has that* $\displaystyle\int_\gamma |h_1^{'}(z)|^2|(h_2(z^{-1}))'|^2|dz|=\infty.$

Proof. A divergent path in $\Omega_{R,r}$ either approach $|z|=r$ or $|z|=R$. Suppose that γ is a divergent path that approaches $|z|=r$. Since $h_1^{'}$ is holomorphic in a neighborhood of that circle and does not vanish at any point of it, it follows that there is a perhaps smaller neighborhood U of $|z|=r$ having compact closure $\overline{U}$, and such that $\displaystyle\inf_{z\in\gamma\cap\overline{U}}\{|h_1^{'}(z)|^2\}=A>0$. Consequently, if $\tilde\gamma$ denotes the portion of γ inside $\overline{U}$ one has that

$$\int_\gamma |h_1^{'}(z)|^2|(h_2(z^{-1}))^{'}|^2|dz|\geq A^2\int_{\tilde\gamma}|(h_2(z^{-1}))^{'}|^2|dz|=\infty$$

by Lemma 2.3. The rest of the proof follows in a similar way. ∎

In the next sections we will use mainly the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ (see [5]) and the three lemmas above.

3 Minimal immersions of a disc with finitely many points removed

Let $\Omega = \mathbb{D}_R - \{a_1, a_2, \ldots, a_n\}$, where $\mathbb{D}_R$ is the open disc of radius R centered at the origin and a_1,a_2,...,a_n are distinct points of $\mathbb{D}_R$.

Theorem 3.1 *There exist complete minimal immersions $\mathcal{M}$ of Ω between two parallel planes of $\mathbb{R}^3$. Furthermore the ends of $\mathcal{M}$ corresponding to the points a_1,a_2,...,a_n are all planar and have index one.*

Proof. To avoid notational inconveniences we will prove separately the cases $n = 1$ and $n > 1$. In the first case we take $\Omega = \mathbb{D}_R - \{a\}$, for some $a \in \mathbb{D}_R$. Using the Weierstrass representation, set

$$f(z) = (z - a)^{-2} \text{ and } g(z) = (z - a)^2 h'(z)$$

where h is any function as in Lemma 2.2. Clearly the data above defines a minimal immersion $\mathcal{M}$ of Ω in $\mathbb{R}^3$. Moreover, $\mathcal{M}$ is also complete for the metric is given by

$$\lambda(z)|dz| = \tfrac{1}{2}\{|z - a|^{-2} + |z - a|^2 |h'(z)|^2\}|dz|,$$

and because $x_3(z) = \mathrm{Re}(h(z))$, it follows from the properties of h that the third coordinate of $\mathcal{M}$ is bounded.

The end corresponding to a is of course planar because fg is holomorphic at that point and have index one because f has a pole of order two at a and fg^2 vanishes at that same point. For information on the behavior of ends of complete minimal surfaces see [2]. ■

Now we consider the case $n > 1$. In the Weierstrass representation for $\mathcal{M}$ set

$$f(z) = F(z)^{-1} \exp\left\{\sum_{j=1}^{n} A_j F_j(z)\right\} \quad \text{and} \quad g(z) = h'(z)F(z),$$

where h is a function as in Lemma 1.1, $F(z) = \displaystyle\prod_{j=1}^{n}(z - a_j)^2$ the functions F_j satisfy $(z - a_j)^2 F_j^{'}(z) = F(z)$, and the A_j are constants to be chosen so that

$$\int_\sigma f(z)dz = 0 \text{ for all closed curves } \sigma \text{ in } \Omega.$$

Since fg and fg^2 have holomorphic extensions to all of $|z| < R$, it follows that this will be enough to exclude the possibility of real periods appearing in the Weierstrass representation of $\mathcal{M}$. An easy computation shows that the choice $A_j = F_j^{''}(a_j)(F_j^{'}(a_j))^{-2}$, $j = 1, \ldots, n$ solves the problem.

Observe that the metric $\lambda(z)|dz|$ on $\mathcal{M}$ is given by

$$2\lambda(z)|dz| = \{|F(z)|^{-1} + |F(z)||h^{'}(z)|^2\} \left| \exp\left\{ \sum_{k=1}^{n} A_k F_k(z) \right\} \right| |dz|,$$

and since there is a positive real C such that $\left| \exp\left\{ \sum_{k=1}^{n} A_k F_k(z) \right\} \right| \geq C$, $z \in \Omega$, it follows from the properties of h and that f has poles of order 2 at the a_j that $\mathcal{M}$ is complete. Also, $x_3(z) = \mathrm{Re} \int h^{'}(z) \exp\{\sum_{j=1}^{n} A_j F_j(z)\} dz$ is bounded in Ω. This can be seen in the following way: $\exp\left\{ \sum_{k=1}^{n} A_k F_k(z) \right\}$ and its derivatives as well as h are all bounded holomorphic functions in Ω. As a matter of fact they are bounded in $\mathbb{D}_R$, so, by integration by parts, it follows that $\int h^{'}(z) \exp\{\sum_{j=1}^{n} A_j F_j(z)\} dz$ is bounded in $\mathbb{D}_R$, so x_3 is also bounded.

By an argument similar to the one done in the case $n = 1$ we conclude that the ends corresponding to the points a_j are all planar and have index one. ■

4 Complete minimal annuli between two parallel planes in $\mathbb{R}^3$

All the examples of complete minimal annuli in [6] have the conformal type of an annulus of the form $R^{-1} < |z| < R$. Here we give examples of all possible annuli $0 < r < |z| < R < \infty$.

Theorem 4.1 *Given any annulus $\Omega_{R,r}$ there is a complete minimal immersion of it in $\mathbb{R}^3$ with one coordinate bounded.*

Proof. Take any two functions h_1 and h_2 as in Lemma 1.3, say $h_1(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ and $h_2(z) = \sum_{l=0}^{\infty} a_l z^{m_l}$, with radii of convergence R and r^{-1} respectively, and such that all the n_k and m_l are simultaneously either even or odd. Then in the Weierstrass representation we set in $\mathbb{D}_{R,r}$,

$$f(z) = 1 \quad \text{and} \quad g(z) = h_1^{'}(z) H_2^{'}(z),$$

where $H_2(z) = h_2(z^{-1})$. Because f is constant, and g is an even function where defined, it follows that for all closed curves γ in $\Omega_{R,r}$ one has that

$$\int_\gamma f dz = \int_\gamma f g dz = \int_\gamma f g^2 dz = 0.$$

Thus, the minimal surface so obtained is in fact well defined. Furthermore, since the metric $\lambda(z)|dz|$ is given by

$$2\lambda(z)|dz| = \left(1 + |h_1'(z)|^2 |H_2'(z)|^2\right) |dz|$$

it follows from Lemma 1.3 that it is complete. It remains to prove only that one of the coordinates of that immersion is bounded.

Since $x_3(z) = Re\int g dz = Re\int h_1'(z)H_2'(z)dz$, it is enough to prove that $\int g dz$ is a bounded holomorphic function in $\Omega_{R,r}$. Consider $\rho \in \mathbb{R}$ such that $r < \rho < R$ and define the sets

$$A_1 = \Omega_{R,r} \cap \{z \in \mathbb{C} \text{ such that } |z| \leq \rho\} \text{ and } A_2 = \Omega_{R,r} \cap \{z \in \mathbb{C} \text{ such that } |z| \geq \rho\}.$$

We then observe that $h_1'(z)$, its derivatives and H_2 are bounded in A_1 and the same happens to H_2', its derivatives and h_1 in A_2. So, by integration by parts we can conclude that $\int g dz$ is bounded in both A_1 and A_2, hence in $\Omega_{R,r}$. ■

5 The case of a annulus with finitely many points removed

Let $A = \{a_1, \ldots, a_n\}$ be a set of distinct points of $\Omega_{R,r}$ such that $A \bigcap (-A) = \emptyset$, and set $\Omega = \Omega_{R,r} - A$.

Theorem 5.1 *There is a complete minimal immersion of Ω between two parallel planes of $\mathbb{R}^3$. Furthermore, the ends corresponding to $a_1, \ldots, a_n$ are planar.*

Proof. First suppose $n = 1$ and take $\Omega = \Omega_{R,r} - \{a\}$ with $a \in \Omega_{R,r}$. Consider holomorphic functions h_1 and h_2 as in Theorem 3.1, and define the surface $\mathcal{M}$ using the Weierstrass representation by taking

$$f(z) = (z - a)^{-2} \quad \text{and} \quad g(z) = (z - a)^2 h_1'(z) H_2'(z),$$

keeping the notation of Theorem 3.1. Then f is holomorphic in a neighborhood of $|z| \leq r$, and has residue zero at a, so $\int_\sigma f dz = 0$, for all closed curves σ inside Ω.

Because $fg = h_1' H_2'$ is an even holomorphic function in $\Omega_{R,r}$ it follows that $\int_\sigma fg dz = 0$ for all closed curves σ in Ω too.

Now we must make one more choice in order to have $\int_\sigma f g^2 dz = 0$ for all closed curves σ in Ω. The idea is to start with h_1 and h_2 having no low powers in their power series expansions. Since $f(z)g^2(z) = (z-a)^2 (h_1'(z))^2 (H_2'(z))^2$ is holomorphic in Ω, by expanding $(z-a)^2$, it is clear that the only term that may cause problems is $-2az(h_1'(z))^2(H_2'(z))^2$ because the other two terms are even functions. Hence if the functions h_1 and h_2 are chosen to satisfy Theorem 3.1 and have the form

$$h_1(z) = \sum_{k=1}^{\infty} a_k z^{1+2^{2^k}} \ \text{ and } \ h_2(z) = \sum_{k=1}^{\infty} b_k z^{1+2^{2^k}}$$

there is no term in z^{-1} in the Laurent expansion of $z(h_1^{'}(z))^2(H_2^{'}(z))^2$. Hence, for all closed curves σ in Ω, $\displaystyle\int_\sigma fg^2 dz = 0$ as wanted. Besides, as in Theorem 2.1, the end corresponding to the point a is planar and have index one.

In order to study the case $n > 1$ define the following holomorphic functions in Ω: $F(z) = \displaystyle\prod_{j=1}^{n}(z-a_j)^2$, $F_k(z) = (z-a_k)^{-2}F(z)$ and $G_k^{'}(z) = zF_k(z)F_k(-z)$ for $k = 1, \ldots, n$, and finally $H(z) = \displaystyle\sum_{j=1}^{n} A_j G_j(z)$, where the constants A_j are to be determined so that, if the immersion $\mathcal{M}$ of Ω is defined in terms of the Weierstrass representation by setting

$$f(z) = (F(z))^{-1} \exp H(z) \ \text{ and } \ g(z) = F(z) \exp\left\{-\frac{1}{2}H(z)\right\} h_1^{'}(z)H_2^{'}(z),$$

then f has residue zero at all the points a_j. As before, the functions h_1 and h_2 are chosen satisfying the conditions of Lemma 1.3 and the exponents are chosen so that $\displaystyle\int_{|z|=\rho} f(z)g(z)^2 dz = 0$, for $r < \rho < R$.

It must be pointed out that once these constants A_j are determined the rest is done quite easily as follows: First we observe that H is an even holomorphic function in $\Omega_{R,r}$, and the same happens to $h_1^{'}$ and $H_2^{'}$, thus fg is an even holomorphic function in $\Omega_{R,r}$ and so $\displaystyle\int_\sigma fg dz = 0$ for all closed curves σ in $\Omega_{R,r}$ as wanted.

Furthermore,

$$2\lambda(z)|dz| = |F(z)|^{-1}|\exp H(z)| + |F(z)||h_1^{'}(z)|^2|H_2^{'}(z)|^2,$$

hence, repeating the reasoning in the proof of Theorem 2.1 we conclude that $\lambda(z)|dz|$ is complete and x_3 is bounded.

Now we determine the constants A_j. First, we observe that all the poles of f have order two and that for $j = 1, \ldots, n$, $(z-a_j)^2 f(z) = \dfrac{\exp H(z)}{F_j(z)}$, thus

$$\begin{aligned} \frac{d}{dz}\{(z-a_j)^2 f(z)\} &= \frac{\exp H(z)}{F_j^2(z)}\left[H^{'}(z)F_j(z) - F_j^{'}(z)\right] \\ &= \frac{\exp H(z)}{F_j^2(z)}\left[F_j(z)\left\{\sum_{k=1}^{n} A_k z F_k(z)F_k(-z)\right\} - F_j^{'}(z)\right]. \end{aligned}$$

So, the residue of f at a_j is zero if and only if

$$F_j(a_j)\left\{\sum_{k=1}^{n} A_k a_j F_k(a_j)F_k(-a_j)\right\} - F_j^{'}(a_j) = 0.$$

Since $F_k(a_j) = 0$ for $k \neq j$, $F_j(a_j)$, $F_j(-a_j) \neq 0$ and $a_j \neq -a_k$, for $1 \leq j,k \leq n$, it follows that $a_j F_j^2(a_j)F_j(-a_j)A_j - F_j^{'}(a_j) = 0$, for each j, $1 \leq j \leq n$, hence

$$A_j = \frac{F_j^{'}(a_j)}{a_j F_j^2(a_j)F_j(-a_j)}, \text{ for } j = 1, \ldots, n.$$

To finish the proof it is enough to show that we can choose h_1 and h_2 in such a way that $\displaystyle\int_{|z|=\rho} f(z)g(z)^2 dz = 0$, for $r < \rho < R$.

Since $f(z)g^2(z) = F(z)[h_1^{'}(z)H_2^{'}(z)]^2$, and F has degree $2n$, if we define

$$h_1(z) = \sum_{k=1}^{\infty} a_k z^{1+2n+2^{2^k}} \text{ and } h_2(z) = \sum_{k=1}^{\infty} b_k z^{1+2n+2^{2^k}}$$

we are done. Also the observations about the ends in the other cases are valid here without change. ∎

The assumption that $(-A) \cap A = \emptyset$ is not really needed. It is just a technical difficulty that can be easily overcome as follows:

Corollary 5.2 *If A is any finite subset of $\Omega_{R,r}$ there is a complete minimal immersion of $\Omega = \Omega_{R,r} - A$ between two parallel planes of $\mathbb{R}^3$. Furthermore, the ends corresponding to the points of A are planar.*

Proof. Induction on the number of elements of A shows that there exists a transformation

$$\pi : \Omega \longrightarrow \Omega^{'},$$

where $\pi(z) = z^{2^p}$ for some positive integer p and $\Omega^{'}$ satisfies the condition of Theorem 5.1.

It is clear that $(\pi\,, \Omega)$ is an unramified covering of $\Omega^{'}$, and by Theorem 5.1, there is a complete minimal immersion X of $\Omega^{'}$ between two parallel planes of $\mathbb{R}^3$. So $X\circ\pi$ is also a complete minimal immersion of Ω in $\mathbb{R}^3$ with the same properties as before. ∎

Received: February 2008. Revised: June 2008.

References

[1] F.F. de Brito, *Power series with hadamard gaps and hyperbolic complete minimal surfaces*, Duke Math. Journal **68** (1992), 297–300.

[2] L.P.M. Jorge and W. Meeks, *The topology of complete minimal surfaces opf finite Gaussian curvature*, Topology **22** (1983), 203–221.

[3] F. Lopez, *A non orientable complete minimal surface in $\mathbb{R}^3$ between two parallel planes*, Proceedings of the Amer. Math. Soc **103** (1988), 913–917.

[4] N. Nadirashvili, *Hadamard and Calabi-Yau's conjectures on negatively curved and minimal surfaces*, Invent. Math. **126** (1996), 457–465.

[5] R. Osserman, *A survey of minimal surfaces*, Van Nostrand, New York, 1969.

[6] H. Rosenberg and E. Toubiana, *A cylindrical type minimal surface in a slab in $\mathbb{R}^3$*, Bull. Sc. Math. **111** (1987), 241–245.

[7] F. Xavier and L.P.M. Jorge, *A complete minimal surface in $\mathbb{R}^3$ between two parallel planes*, Ann. of Math. **112** (1980), 203–206.