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# Wave Front Sets Singularities of Homogeneous Sub-Riemannian Three Dimensional Manifolds

Víctor Ayala[1]
*Universidad Católica del Norte – Departamento de Matemáticas*
*Avenida Angamos 0610, Casilla 1280*
*Antofagasta – Chile*
*email:* vayala@ucn.cl

Marcos M. Diniz, José C.P. Lima, José M.M. Veloso
*Universidade Federal do Pará – CCEN – Departamento de Matemática,*
*Rua Augusto Corrêa, 01, CEP: 66075-110,*
*Belém – Pará, Brasil*
*email:* mdiniz@ufpa.br, pojo@ufpa.br, veloso@ufpa.br
*and*
Ivan Tribuzy
*Universidade Federal do Amazonas – Departamento de Matemática*
*Japiim, Manaus – Amazonas, 69077-000, Brasil*
*email:* argo@ufam.br

## ABSTRACT

A graphic study of wave front sets of exponential sub-Riemannian maps is performed for homogeneous three dimensional sub-Riemannian manifolds. We verify that depending on dimension of the sub-Riemannian isometry group of the manifold, the first singularities of wave front sets are of two types. If the group is four dimensional, the singularity is a conjugate point. If the group is three dimensional, there are two conjugate points and the wave front set intersects along a segment which connects both points.

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**RESUMEN**

Un estudio gráfico del conjunto fuente de ondas de la aplicación exponencial sub-Riemanniana es presentada para variedades sub-Riemannianas tri-dimensionales homogéneas. Verificamos que, dependiendo del grupo de isometria subRiemanniano de la variedad, las primeras singularidades de los conjuntos frente de onda son de dos tipos. Si el grupo es de dimensión cuatro la singularidad es un punto conjugado. Si el grupo es tri-dimensional, hay dos puntos conjugados y el conjunto frente de ondas intercepta a lo largo un segmento que une ambos puntos.

**Key words and phrases:** *sub-Riemannian geometry, exponential map, wave front sets, singularities, three dimensional manifolds.*

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# 1 Introduction

A Sub-Riemannian (SR) manifold is a smooth manifold $M$ with a distribution $D \subset TM$ and a smooth fiber inner product $\langle\, , \rangle$ on $D$. We will restrict the study to the three dimensional case $M$ and when $D$ is a two dimensional contact distribution. We define the notion of a homogeneous SR manifold, and describe the classification of simply connected homogeneous manifolds of dimension three obtained by Diniz in [3]. Depending on $G = \mathrm{Isom}(M)$, the group of SR isometries of $M$, there are two classes: $\dim G = 4$ or $\dim G = 3$. In the first case the manifolds are the Heisenberg group $H^3$, the sphere $S^3_r$ and the quadric $Q^3_r$. The second case contains several Lie subgroups of the Lie group $GL(2, \mathbf{C})$ of the complex matrices of order 2. In fact, this class includes the special unitary group $SU(2)$, the universal covering $\widetilde{E}(2)$ of the affine group $E(2)$, the universal covering $\widetilde{SU}(1,1)$, of $SU(1,1)$, and two special groups $G_1$ and $G_2$ which we describe later. The classification of contact homogeneous three dimensional subRiemanian manifolds depend on a sub-Riemannian connection introduced in [5], through 4 parameters: $\lambda \geq 0$, the curvature $K$ and $W_1 \geq 0$ and $W_2$ the torsion components of the connection.

The SR geodesics are minimizing curves tangent to $D$. This fact allows to define the SR exponential map notion. Unlike the Riemannian case, the SR exponential map is singular at the origin. We determine the SR geodesic equations and apply the equations to SR homogeneous manifolds to obtain graphics of wave front sets (wfs) of SR exponential maps by using software MATHEMATICA. It turns out that wfs are images of two dimensional cylinders, and they have infinite auto intersections. Furthermore, conjugate points are points where the differential of a wfs of SR exponential maps is singular. Generic results about wave front sets in three-dimensional contact SR-manifolds can be found in [1] , [2] , and [4].

If $\dim G = 4$ there is only one point at the first auto intersection of wfs and this point is conjugate, as in Figure 1. If $\dim G = 3$ the set of the first auto intersection is a segment, and the extremals of this segment are conjugate points, as in Figure 2. It is a conjecture that all wfs of SR exponential maps are as Figure 1 or Figure 2.

# 2 Exponential Map

**Definition 2.1** *A sub-Riemannian three dimensional contact manifold consists of a manifold $M$, together with a two dimensional contact distribution $D$ (a vector sub-bundle $D \subset TM$ of the tangent bundle of $M$) endowed with a fiber inner product $\langle\, ,\, \rangle$.*

As $D$ is contact, we have $D + [D, D] = TM$.

**Definition 2.2** *An* adapted basis $(e_{11}, e_{12}, e_{21})$ *is a basis of $TM$ such that $(e_{11}, e_{12})$ is an orthonormal basis of $D$. A coreferential $\theta^{11}, \theta^{12}, \theta^{21}$ is* adapted *if its dual basis is adapted.*

**Proposition 2.1** *There exists a unique form $\theta^{21}$, unless of sign, defined on $M$ such that* $\ker \theta^{21} = D$ *and $d\theta^{21}$ is the volume form on $D$.*

The hypothesis that $D$ is a contact distribution implies by the well known Chow's theorem, that any two points belonging to the same connected component of $M$ can be joined by an *horizontal curve*. That is to say there exists an absolutely continuous curve $c(t)$ such that $\dot{c}(t) \in D_{c(t)}$, for all $t$ where $\dot{c}(t)$ exists.

The length of a smooth horizontal curve $c : [a, b] \to M$ is defined by

$$l(c) = \int_a^b \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle}\mathrm{d}t.$$

The *distance* between two points is given by the infimum of the horizontal curves lengths joining these two points.

A sub-Riemannian *geodesic* is an horizontal curve that locally realize the distance between its points. They are solutions of the Hamilton-Jacobi equations.

We start by given a description of the geodesics in terms of a local adapted frame. Let us write

$$\begin{array}{rcl} \mathrm{d}\theta^{11} & = & a^{11}_{(11)(12)}\, \theta^{11} \wedge \theta^{12} + a^{11}_{(11)(21)}\, \theta^{11} \wedge \theta^{21} + a^{11}_{(12)(21)}\, \theta^{12} \wedge \theta^{21} \\ \mathrm{d}\theta^{12} & = & a^{12}_{(11)(12)}\, \theta^{11} \wedge \theta^{12} + a^{12}_{(11)(21)}\, \theta^{11} \wedge \theta^{21} + a^{12}_{(12)(21)}\, \theta^{12} \wedge \theta^{21} \\ \mathrm{d}\theta^{21} & = & a^{21}_{(11)(12)}\, \theta^{11} \wedge \theta^{12} + a^{21}_{(11)(21)}\, \theta^{11} \wedge \theta^{21} + a^{21}_{(12)(21)}\, \theta^{12} \wedge \theta^{21}. \end{array}$$

It follows from Proposition 2.1 that

$$a^{21}_{(11)(21)} = 1.$$

Let $\mu \in T^*_p M$ determined by

$$\mu = \mu_{11}\, \theta^{11}_p + \mu_{12}\, \theta^{12}_p + \mu_{21}\, \theta^{21}_p \quad .$$

Given $\lambda \in T_p^*M$ we can associate canonically a vector $g(\lambda) \in D_p$ by $\langle g(\lambda), v\rangle = \lambda(v)$, for every $v \in D_p$. If we write $\mu_1 = g(\mu)$, then we have

$$\mu_1 = \mu_{11}\, e_{11} + \mu_{12}\, e_{12}.$$

In the sequel we denote by $(M, D, g)$ a sub-Riemannian manifold $(M, D)$ with its associated application $g$.

Next we get the equations for the geodesics. We follow [6] and [7]. Let $\pi : T^*M \to M$ the canonical projection and $d\pi : T(T^*M) \to TM$ its differential. Define a one form $\omega$ on $T^*M$ by $\iota(v)\omega_\lambda = \iota(d\pi(v))\lambda$ where $\lambda \in T^*M$ and $v \in T_\lambda(T^*M)$. The *canonical sympletic form* on $T^*M$ is the two-form $\Omega = d\omega$.

If $H$ is a Hamiltonian function on $T^*M$, we define the *Hamiltonian vector field* $\overrightarrow{H}$ on $T^*M$ by $\Omega(v, \overrightarrow{H}) = \iota(v)dH$. The *bicharacteristics* of $\overrightarrow{H}$ are the absolutely continuous curves $C(t)$ on $T^*M$ such that $\dot{C}(t) = \overrightarrow{H}(C(t))$ for almost every $t$. A curve $c(t)$ is say to be *characteristic* if $c(t) = \pi(C(t))$ on $M$, for some bicharacteristic curve $C(t)$ on $T^*M$.

Take a coordinate system $(x^j)$ on a neighborhood of $p \in M$. Given a vector $\lambda \in T^*M$ we can write $\lambda = \Sigma\lambda_{\alpha i}\theta^{\alpha i}$, where $\alpha i = 11, 12, 21$. Then $(x^j, \lambda_{\alpha i})$ is a coordinate system for $T^*M$, and the vector fields $(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial \lambda_{\alpha i}})$ form a basis of $T(T^*M)$. In particular, it is possible to suspend the vector fields $e_{\alpha i}$ to $T^*M$, and we denote them by the same symbols. In the same way, we denote by $\theta^{\alpha i}$ the one-forms $\pi^*(\theta^{\alpha i})$. With this notation, we obtain

$$\omega = \sum_{\alpha i} \lambda_{\alpha i}\theta^{\alpha i} \ \textit{and} \ \ \Omega = \sum_{\alpha i} \left(\mathrm{d}\lambda_{\alpha i} \wedge \theta^{\alpha i} + \lambda_{\alpha i} d\theta^{\alpha i}\right).$$

The Hamiltonian function associated to the sub-Riemannian manifold $M$ is given by

$$H(x, \lambda) = H(x^j, \lambda_{\alpha i}) = \frac{1}{2}(\lambda_{11}^2 + \lambda_{12}^2),$$

and it is a straightforward calculus to show that

$$\overrightarrow{H}(x, \lambda) = \sum_{i=1}^{2} \lambda_{1i}(e_{1i} - \sum_{\alpha j}\sum_{\beta l} a_{(1i)(\alpha j)}^{\beta l}\lambda_{\beta l}\frac{\partial}{\partial \lambda_{\alpha j}}).$$

Taking account that the geodesics are the characteristics curves, we get the following results:

**Proposition 2.2** *Given a point $p \in M$ and $\mu \in T_p^*M \setminus D_p^\perp$, the* normal geodesic *with initial conditions $(p, \mu)$ is the curve $c$, solution of the ordinary differential system:*

$$\begin{cases} \dot{c} = \Sigma_{i=1}^2 \lambda_{1i} e_{1i} \\ \dot{\lambda}_{\alpha j} + \Sigma_{i=1}^2 \lambda_{1i}\, \Sigma_{\beta l} \lambda_{\beta l}\, a_{(1i)(\alpha j)}^{\beta l} = 0 \\ c(0) = p \\ \lambda_{\alpha j}(0) = \mu_{\alpha j} \quad , \end{cases} \tag{1}$$

*for all $(\alpha j)$, where $\mu = \Sigma_{\alpha i}\mu_{\alpha i}\theta_p^{\alpha i}$.*

The stablished formulation of the normal geodesic equations are already in a desirable form for our purposes.

The adapted referential $(e_{11}, e_{12}, e_{21})$ writing in coordinates $(x_1, x_2, x_3)$ read as

$$\begin{cases} e_{11} = E^1_{11}\dfrac{\partial}{\partial x_1} + E^2_{11}\dfrac{\partial}{\partial x_2} + E^3_{11}\dfrac{\partial}{\partial x_3} \\ e_{12} = E^1_{12}\dfrac{\partial}{\partial x_1} + E^2_{12}\dfrac{\partial}{\partial x_2} + E^3_{12}\dfrac{\partial}{\partial x_3} \\ e_{21} = E^1_{21}\dfrac{\partial}{\partial x_1} + E^2_{21}\dfrac{\partial}{\partial x_2} + E^3_{21}\dfrac{\partial}{\partial x_3} \quad . \end{cases}$$

In our case the geodesic equations in 1 are given by the following system:

$$\begin{cases} \dot{x}_1 = \lambda_{11}E^1_{11} + \lambda_{12}E^1_{12} \\ \dot{x}_2 = \lambda_{11}E^2_{11} + \lambda_{12}E^2_{12} \\ \dot{x}_3 = \lambda_{11}E^3_{11} + \lambda_{12}E^3_{12} \\ \dot{\lambda}_{11} + \lambda_{12}\,(\lambda_{1,1}\,a^{11}_{(12)(11)} + \lambda_{12}\,a^{12}_{(12)(11)} + \lambda_{21}\,a^{21}_{(12)(11)}) = 0 \\ \dot{\lambda}_{12} + \lambda_{11}\,(\lambda_{1,1}\,a^{11}_{(11)(12)} + \lambda_{12}\,a^{12}_{(11)(12)} + \lambda_{21}\,a^{21}_{(11)(12)}) = 0 \\ \dot{\lambda}_{21} + \lambda_{11}\,(\lambda_{1,1}\,a^{11}_{(11)(21)} + \lambda_{12}\,a^{12}_{(11)(21)} + \lambda_{21}\,a^{21}_{(11)(21)}) + \\ \qquad +\lambda_{12}\,(\lambda_{1,1}\,a^{11}_{(12)(21)} + \lambda_{12}\,a^{12}_{(12)(21)} + \lambda_{21}\,a^{21}_{(12)(21)}) = 0 \\ x_j(0) = x_j(p) \\ \lambda_{\alpha j}(0) = \mu_{\alpha j} \quad . \end{cases} \tag{2}$$

It is an easy verification that for any positive constant $a$, $c(t, p, a\mu) = c(at, p, \mu)$. On the other hand, at each point $p \in M$ there exists a neighborhood $V$ of $0 \in T^*_pM$ such that

$$c(1, p, \cdot\,) : U = V \setminus D^{\perp}_p \to M$$

given by $c(1, p, \mu)$, $\mu \in U$, is well defined.

**Definition 2.3** *Let $p \in M$ and $U$ as above.*

1. The *exponential map* at the point $p$ is the map $\exp_p : U \to M$ given by

$$\exp_p(\mu) = c(1, p, \mu).$$

2. Let $\epsilon > 0$, the *wave front set* of radius $\epsilon$ at $p$ is defined by the cylinder image under the exponential map: $exp_p\{\mu \in U : \mu^2_{11} + \mu^2_{12} = \epsilon^2\}$

3. A vector $\nu \in U$ is a *conjugate point* if $\mathrm{d}_\nu(\exp_p) : T_\nu U \to T_{\exp_p(\nu)}M$ is degenerated.

Our goal is to give information about the wave front set and the first conjugate point of the homogeneous three dimensional SR-manifolds, through computational graphic images of the exponential map.

# 3 Homogeneous Sub-Riemannian Three Dimensional Contact Manifolds

We start the section with the notion of sub-Riemannian isometry.

**Definition 3.1** *If $f : M \to N$ is a (local) diffeomorphism between two sub-Riemannian manifolds $(M, D, g)$ and $(N, D', g')$ such that for every $X, Y \in T_pM$, and $p \in M$*

*i)* $df_p(D_p) = D'_{f(p)}$;

*ii)* $g'(df_p(X), df_p(Y)) = g(X, Y)$

*then we say that $f$ is a* (local) sub-Riemannian isometry *between $M$ and $N$. Furthermore, is $f$ is a bijection, we say that $f$ is a* sub-Riemannian isometry.

**Definition 3.2** *A sub-Riemannian manifold $M$ is* homogeneous *if the group*

$$\mathrm{Isom}(M) = \{f : M \to M | f \textit{ is a sub-Riemannian isometry}\}$$

*acts transitively on $M$.*

Diniz in [3] obtained a classification of contact homogeneous three dimensional sub-Riemannian manifolds, which we describe next. In terms of the sub-Riemannian connection introduced in [5], it is possible to choose vector fields $e_{11}, e_{12}, e_{21}$ generating a Lie algebra as below:

$$\begin{array}{rcl} [e_{11}, e_{12}] & = & -\Gamma_1 e_{11} - \Gamma_2 e_{12} - 2e_{21} \\ [e_{12}, e_{21}] & = & \Gamma e_{11} + \lambda e_{12} \\ [e_{21}, e_{11}] & = & \lambda e_{11} + \Gamma e_{12} \end{array} \tag{3}$$

where $\lambda, \Gamma_1, \Gamma_2, \Gamma$ are constants and $\lambda \geq 0$, such that:

$$\begin{cases} K = -(2\Gamma + \Gamma_1^2 + \Gamma_2^2) \\ W_1 = 2\Gamma_2\Gamma = 2\lambda\Gamma_1 \\ W_2 = -2\Gamma_1\Gamma = -2\lambda\Gamma_2 \\ W_1 = \pm W_2 \ . \end{cases}$$

Here, $K$ is the curvature and $W_1$ and $W_2$ are the components of the torsion, with $W_1 \geq 0$. There are two possibilities for the dimension of $\mathrm{Isom}(M)$: 3 or 4. In the case of $\dim \mathrm{Isom}(M) = 4$ we have $\lambda = 0$ and $W_1 = W_2 = 0$. On the other hand, if $\dim \mathrm{Isom}(M) = 3$, we get $\lambda > 0$ and

$$\begin{array}{rcl} [e_{11}, e_{12}] & = & -\frac{W_1}{2\lambda} e_{1,1} + \frac{W_2}{2\lambda} - 2e_{21} \\ [e_{12}, e_{21}] & = & \Gamma e_{11} + \lambda e_{12} \\ [e_{21}, e_{11}] & = & \lambda e_{11} + \Gamma e_{12}. \end{array} \tag{4}$$

### 3.1 The case $\lambda = 0$ and $W_1 = W_2 = 0$.

Under the condition $\lambda = 0$ and $W_1 = W_2 = 0$, we have three different situations.

1. The first one is $K = 0$. In this case we get

$$\begin{array}{rcl} [e_{11}, e_{12}] & = & -2e_{21} \\ [e_{12}, e_{21}] & = & 0 \\ [e_{21}, e_{11}] & = & 0 \quad . \end{array} \tag{5}$$

These relations corresponding to the Lie algebra of the three dimensional Heinsenberg Lie group $H^3$. Furthermore, $H^3$ is diffeomorphics to $\mathbf{R}^3$. We consider the vector fields

$$\begin{cases} e_{11} = \dfrac{\partial}{\partial x^1} - x^2 \dfrac{\partial}{\partial x^3} \\ e_{12} = \dfrac{\partial}{\partial x^2} + x^1 \dfrac{\partial}{\partial x^3} \\ e_{21} = -\dfrac{\partial}{\partial x^3} \quad . \end{cases} \tag{6}$$

2. The second possibility is $K > 0$. It turns out that for any $r > 0$, the associated Lie algebra

$$\begin{array}{rcl} [e_{11}, e_{12}] & = & -2e_{21} \\ [e_{12}, e_{21}] & = & -2e_{11} \\ [e_{21}, e_{11}] & = & -2e_{12} \quad . \end{array} \tag{7}$$

can be represented on the $r$-sphere of dimension three

$$S_r^3 = \{(y_1, y_2, y_3, y_4) \in \mathbf{R}^4 : y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2\}$$

by the vector fields

$$\begin{cases} e_{11} = -y_3 \dfrac{\partial}{\partial y_1} + y_4 \dfrac{\partial}{\partial y_2} + y_1 \dfrac{\partial}{\partial y_3} - y_2 \dfrac{\partial}{\partial y_4} \\ e_{12} = -y_4 \dfrac{\partial}{\partial y_1} - y_3 \dfrac{\partial}{\partial y_2} + y_2 \dfrac{\partial}{\partial y_3} + y_1 \dfrac{\partial}{\partial y_4} \\ e_{21} = -y_2 \dfrac{\partial}{\partial y_1} + y_1 \dfrac{\partial}{\partial y_2} - y_4 \dfrac{\partial}{\partial y_3} + y_3 \dfrac{\partial}{\partial y_4} \quad . \end{cases} \tag{8}$$

3. For the third possibility $K < 0$, we consider the manifold

$$Q_r^3 = \{(y_1, y_2, y_3, y_4) \in \mathbf{R}^4 : y_1^2 + y_2^2 - y_3^2 - y_4^2 = r^2\}$$

and the vector fields tangent to $Q_r^3$ determined by

$$\left\{\begin{aligned} e_{11} &= y_3\frac{\partial}{\partial y_1} - y_4\frac{\partial}{\partial y_2} + y_1\frac{\partial}{\partial y_3} - y_2\frac{\partial}{\partial y_4} \\ e_{12} &= y_4\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial y_3} + y_1\frac{\partial}{\partial y_4} \\ e_{21} &= y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_4} \end{aligned}\right. \tag{9}$$

which generate the associated Lie algebra

$$\begin{aligned} [e_{11}, e_{12}] &= -2e_{21} \\ [e_{12}, e_{21}] &= 2e_{11} \\ [e_{21}, e_{11}] &= 2e_{12}. \end{aligned}$$

## 3.2 The Case $\lambda > 0$, $W_1 = W_2 = 0$.

Let us assume that $\lambda > 0$ and $W_1 = W_2 = 0$. In this case, $\Gamma_1 = \Gamma_2 = 0$ and $\Gamma = -\frac{K}{2}$ in (4). Therefore, we get

$$\begin{aligned} [e_{11}, e_{12}] &= -2e_{21} \\ [e_{12}, e_{21}] &= -\frac{K}{2}e_{11} + \lambda e_{12} \\ [e_{21}, e_{11}] &= \lambda e_{11} - \frac{K}{2}e_{12}. \end{aligned} \tag{10}$$

In the sequel we will describe the different possibilities depending on the parameters $K$ and $\lambda$ and its relative positions as real numbers. It turns out that five classical Lie groups appears. In each case, we mention the Lie group $G$ and a basis $(f_1, f_2, f_3)$ for the corresponding Lie algebra $\mathfrak{g}$. To obtain the structure equations (10), we explicity introduce the adapted vector fields $(e_{11}, e_{12}, e_{21})$. On the other hand, in order to write down the associated ordinary differential equations system for the sub-Riemannian geodesics, we use a local coordinate system for the Lie group at the identity element.

(1) Case $K > 2\lambda$.

It turns out that the group is

$$G \cong SU(2) = \{A \in GL(2, \mathbf{C}) : A\overline{A}^t = I \quad \text{and} \quad \det A = 1\}.$$

Here we consider,

$$\begin{array}{lll} f_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} & , & e_{11} = \frac{1}{2}\left(\sqrt{\frac{K}{2}+\lambda}\; f_2 - \sqrt{\frac{K}{2}-\lambda}\; f_1\right) \\ f_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & , & e_{12} = \frac{1}{2}\left(\sqrt{\frac{K}{2}+\lambda}\; f_2 + \sqrt{\frac{K}{2}-\lambda}\; f_1\right) \\ f_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} & , & e_{21} = \frac{1}{4}\left(\sqrt{K^2-4\lambda^2}\right)\; f_3 \end{array} \tag{11}$$

where

$$\begin{array}{rcl} [f_1, f_2] & = & 2f_3 \\ [f_2, f_3] & = & 2f_1 \\ [f_3, f_1] & = & 2f_2 \quad . \end{array} \tag{12}$$

(2) Case $K = 2\lambda$.

The group is $G = \widetilde{E}(2)$, i. e., the universal covering of the Lie group $E(2)$ defined by

$$E(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x_1 & \cos x_3 & -\sin x_3 \\ x_2 & \sin x_3 & \cos x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbf{R} \right\}. \tag{13}$$

A basis of the corresponding Lie algebra and its adapted vector fields are as follows:

$$\begin{array}{lll} f_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & , & e_{11} = \sqrt{2\lambda} f_2 - f_1 \\ f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} & , & e_{12} = \sqrt{2\lambda} f_2 + f_1 \\ f_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & , & e_{21} = \sqrt{2\lambda} f_3 \end{array} \tag{14}$$

(3) Case $-2\lambda < K < 2\lambda$.

Here, the Lie group is $G \cong \widetilde{SU}(1,1)$, the universal covering of $SU(1,1)$ defined by

$$SU(1,1) = \{A \in GL(2,\mathbf{C}) : AJ\overline{A}^t = J \quad \text{and} \quad \det A = 1\}, \tag{15}$$

where $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Analagously

$$\begin{array}{llll} f_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} & , & e_{11} = \frac{1}{2}\left(\sqrt{\lambda + \frac{K}{2}}\ f_2 - \sqrt{\lambda - \frac{K}{2}}\ f_1\right) \\ f_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} & , & e_{12} = \frac{1}{2}\left(\sqrt{\lambda + \frac{K}{2}}\ f_2 + \sqrt{\lambda - \frac{K}{2}}\ f_1\right) \\ f_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & , & e_{21} = \frac{1}{4}\left(\sqrt{4\lambda^2 - K^2}\right)\ f_3 \end{array} \tag{16}$$

(4) $K = -2\lambda$

In this case the group $G$ is given by

$$E(1,1) = \left\{ \begin{bmatrix} 1 & 0 \ \ 0 \\ x_1 & \\ x_2 & A(x_3) \end{bmatrix} : x_1, x_2, x_3 \in \mathbf{R} \quad \text{and} \quad A(x_3) = \begin{bmatrix} \cosh x_3 & \sinh x_3 \\ \sinh x_3 & \cosh x_t \end{bmatrix} \right\} \tag{17}$$

and the Lie algebra basis and its adapted vector field are:

$$\begin{array}{lll} f_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & , & e_{11} = f_2 - \sqrt{2\lambda} f_1 \\ f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & , & e_{12} = f_2 - \sqrt{2\lambda} f_1 \\ f_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & , & e_{21} = \sqrt{2\lambda} f_3 \end{array} \tag{18}$$

(5) Case $K < -2\lambda$.

In this particular situation, $G \cong \widetilde{SU}(1,1)$, $(f_1, f_2, f_3)$ are as defined in (16) and

$$\begin{array}{lll} e_{11} & = & \frac{1}{2}\left(\sqrt{-(\lambda + \frac{K}{2})}\ f_1 - \sqrt{\lambda - \frac{K}{2}}\ f_3\right) \\ e_{12} & = & \frac{1}{2}\left(\sqrt{-(\lambda + \frac{K}{2})}\ f_1 + \sqrt{\lambda - \frac{K}{2}}\ f_3\right) \\ e_{21} & = & \frac{1}{4}\sqrt{K^2 - 4\lambda^2} f_2. \end{array} \tag{19}$$

### 3.3 Case $W_1 = \pm W_2 \neq 0$

Assume $W_2 = W_1 > 0$. Thus, $\Gamma = -\lambda$ and $K = 2\lambda - 2\left(\frac{W_1}{2\lambda}\right)^2$. In particular, $K < 2\lambda$ and $\frac{W_1}{2\lambda} = \sqrt{\lambda - \frac{K}{2}}$. So, the structure equations are

$$\begin{aligned} [e_{11}, e_{12}] &= -\sqrt{\lambda - \frac{K}{2}} e_{11} + \sqrt{\lambda - \frac{K}{2}} e_{12} - 2e_{21} \\ [e_{12}, e_{21}] &= -\lambda e_{11} + \lambda e_{12} \\ [e_{21}, e_{11}] &= \lambda e_{11} - \lambda e_{12}. \end{aligned} \tag{20}$$

We introduce the notation

$$g_1 = e_{12} - e_{11}, \quad g_2 = -e_{12}, \quad g_3 = e_{21},$$

to get

$$\begin{aligned} [g_1, g_2] &= \sqrt{\lambda - \frac{K}{2}} g_1 - 2g_3 \\ [g_2, g_3] &= -\lambda g_1 \\ [g_3, g_1] &= 0. \end{aligned} \tag{21}$$

It follows that $h = \text{Span}\{g_1, g_3\}$ is an Abelian Lie algebra of some Abelian Lie subgroup $H$ of $G$. If $H$ is simply connected, we get $H \cong \mathbf{R}^2$. Let $S \cong \mathbf{R}$ be the corresponding simply connected Lie subgroup associated to the Lie subalgebra generated by $g_2$ and denote by $\mathfrak{s}$ its Lie algebra. Under these condition the map

$$\sigma : s \to gl(\mathfrak{h}),$$

defined by

$$\sigma(X)(Y) = [Y, X],$$

$X \in \mathfrak{s}$ and $Y \in h$ is well defined. Since $\mathbf{R}$ is a simply connected Lie group, associated with the Lie algebra homomorphism $\sigma$ there exists an unique Lie group homomorphism $\widetilde{\sigma} : S \cong \mathbf{R} \to GL(H)$ such that the diagram below commutes, [8]

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\sigma} & gl(\mathbf{R}^2) \\ \exp\downarrow & & \downarrow\exp \\ \mathbf{R} & \xrightarrow{\widetilde{\sigma}} & GL(\mathbf{R}^2) \end{array} .$$

Since $\exp : \mathbf{R} \to \mathbf{R}$ is the identity then $\widetilde{\sigma}$ is explicity determined by the equation

$$\widetilde{\sigma}(t) = \exp\sigma(tg_2)) = \exp(t \begin{bmatrix} \sqrt{\lambda - K/2} & \lambda \\ -2 & 0 \end{bmatrix}).$$

On the other hand, the group $G$ is given by the semi-direct product of $\mathbf{R} \times \mathbf{R}^2$, relative to $\widetilde{\sigma}$.

We must determine $e^A$, where $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, with $a$, $b$, $c$ non zero complex numbers. The characteristic polynomial roots are

$$\lambda_1 = \frac{a+\sqrt{\Delta}}{2}, \qquad \lambda_2 = \frac{a-\sqrt{\Delta}}{2}$$

where $\Delta = a^2 + 4bc$. In particular, $\Delta \neq 0$ implies that $A$ é diagonalizable over $\mathbf{C}$. Then

$$A = T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} T^{-1}$$

where

$$T = \begin{bmatrix} -b & -b \\ \lambda_2 & \lambda_1 \end{bmatrix}.$$

Thus

$$e^A = \exp(T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} T^{-1}) = T \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} T^{-1}$$

therefore

$$e^A = e^{\frac{a}{2}} \left\{ \frac{\sinh(\frac{\sqrt{\Delta}}{2})}{(\frac{\sqrt{\Delta}}{2})} \begin{bmatrix} a/2 & b \\ c & -a/2 \end{bmatrix} + \cosh \frac{\sqrt{\Delta}}{2} I \right\}.$$

If $\Delta < 0$, we obtain

$$\frac{\sinh(\frac{\sqrt{\Delta}}{2})}{(\frac{\sqrt{\Delta}}{2})} = \frac{\sin(\frac{\sqrt{-\Delta}}{2})}{(\frac{\sqrt{-\Delta}}{2})}$$

$$\cosh(\frac{\sqrt{\Delta}}{2}) = \cos(\frac{\sqrt{-\Delta}}{2}).$$

In the case $\Delta = 0$, if we take $T = \begin{bmatrix} -b & -b \\ 0 & \frac{a}{2} \end{bmatrix}$, then $A = TDT^{-1}$, where $D = \frac{a}{2} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. So

$$e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!} = e^{a/2} \begin{bmatrix} 1 & 0 \\ \frac{a}{2} & 1 \end{bmatrix}$$

and

$$\begin{aligned} e^A &= Te^DT^{-1} \\ &= e^{a/2} \left\{ \begin{bmatrix} \frac{a}{2} & b \\ c & -\frac{a}{2} \end{bmatrix} + I \right\}. \end{aligned}$$

In resume, we have

$$e^A = \begin{cases} e^{\frac{a}{2}} \left[ \frac{2}{\sqrt{\Delta}} \sinh(\frac{\sqrt{\Delta}}{2}) B + \cosh(\frac{\sqrt{\Delta}}{2}) I \right] & ,\text{if } \Delta > 0 \\ e^{\frac{a}{2}} [B + I] & ,\text{if } \Delta = 0 \\ e^{\frac{a}{2}} \left[ \frac{2}{\sqrt{-\Delta}} \sin(\frac{\sqrt{-\Delta}}{2}) B + \cos(\frac{\sqrt{-\Delta}}{2}) I \right] & ,\text{if } \Delta < 0 \end{cases}$$

where $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, $B = \begin{bmatrix} a/2 & b \\ c & -a/2 \end{bmatrix}$ and $\Delta = a^2 + 4bc$.

To obtain $\widetilde{\sigma}$ we do $a = t\sqrt{\lambda - \frac{K}{2}}$, $b = \lambda t$ and $c = -2t$, to get

$$\widetilde{\sigma}(t) = \begin{cases} e^{\alpha t}\left[\frac{\sinh(\beta t)}{\beta}B + \cosh(\beta t)I\right] & ,\text{if } K < -14\lambda, \\ \\ e^{\alpha t}[tB + I] & ,\text{if } K = -14\lambda, \\ \\ e^{\alpha t}\left[\frac{\sin(\beta' t)}{\beta'}B + \cos(\beta' t)I\right] & ,\text{if } -14\lambda < K < 2\lambda, \end{cases}$$

where $\alpha = \frac{1}{2}\sqrt{\lambda - \frac{K}{2}}$, $\beta = \frac{1}{2\sqrt{2}}\sqrt{-K - 14\lambda}$, $\beta' = \frac{1}{2\sqrt{2}}\sqrt{K + 14\lambda}$ and $B = \begin{bmatrix} \alpha & \lambda \\ -2 & -\alpha \end{bmatrix}$.

It is an easy verificaton that $\widetilde{\sigma}$ is one to one, so we get the following representation:

$$G_1 \cong \left\{ \begin{bmatrix} 0 & 0 \quad 0 \\ x & \multirow{2}{*}{$\widetilde{\sigma}(t)$} \\ y & \end{bmatrix} : x, y, t \in \mathbf{R} \right\}.$$

In any case,

$$g_1 = -\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_2 = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\alpha & \lambda \\ 0 & -2 & 0 \end{bmatrix}, \quad g_3 = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a basis of the Lie algebra of $G_1$ that satisfies (21). Therefore,

$$e_{11} = -(g_1 + g_2), \quad e_{12} = -g_2, \quad e_{21} = g_3$$

is the basis that satisfies (20).

The case $-W_2 = W_1 > 0$ is similar, so $K < -2\lambda$ and we otain

$$G_2 \cong \left\{ \begin{bmatrix} 1 & 0 \quad 0 \\ x & \multirow{2}{*}{$\varphi(t)$} \\ y & \end{bmatrix} : x, y, t \in \mathbf{R} \right\},$$

where

$$\varphi(t) = e^{\alpha t}\left[\frac{\sinh \beta t}{\beta}B + \cosh \alpha t . I\right], \ \alpha = \frac{1}{2}\sqrt{-\lambda - \frac{K}{2}}, \ \beta = \frac{1}{2\sqrt{2}}\sqrt{14\lambda - K} \text{ and } B = \begin{bmatrix} \alpha & \lambda \\ 2 & -\alpha \end{bmatrix}.$$

A basis of the Lie algebra as in (20) is given by

$$e_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -a & -\lambda \\ 0 & -2 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & \lambda \\ 0 & 2 & 0 \end{bmatrix}, \quad e_{21} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We can resume our classification as:

| Group | $\lambda$ | $K$ | $W_1$ | $W_2$ |
|---|---|---|---|---|
| $H^3$ | $0$ | $0$ | $0$ | $0$ |
| $S^3_r$ | $0$ | $K = 4/r^2$ | $0$ | $0$ |
| $\widetilde{Q}^3_r$ | $0$ | $K = -4/r^2$ | $0$ | $0$ |
| $SU(2)$ | $\lambda > 0$ | $K > 2\lambda$ | $0$ | $0$ |
| $\widetilde{E}(2)$ | $\lambda > 0$ | $K = 2\lambda$ | $0$ | $0$ |
| $\widetilde{SU}(1,1)$ | $\lambda > 0$ | $-2\lambda < K < 2\lambda$ | $0$ | $0$ |
| $E(1,1)$ | $\lambda > 0$ | $K = -2\lambda$ | $0$ | $0$ |
| $\widetilde{SU}(1,1)$ | $\lambda > 0$ | $K < -2\lambda$ | $0$ | $0$ |
| $G_1$ | $\lambda > 0$ | $K = 2\lambda - \frac{W_1^2}{2\lambda^2}$ | $W_1 > 0$ | $W_2 = W_1$ |
| $G_2$ | $\lambda > 0$ | $K = -2\lambda - \frac{W_1^2}{2\lambda^2}$ | $W_1 > 0$ | $W_2 - W_1$ |

Observe that $S^3_r \cong SU(2)$ and $Q^3_r \cong SU(1,1)$.

## 4 Singularities of the Exponential Map

In order to show some details of the graphs of wfs and the singularities at the first conjugate points, we use a specific computational programme (see Apendice A).

### 4.1 Heinseberg group $H^3$

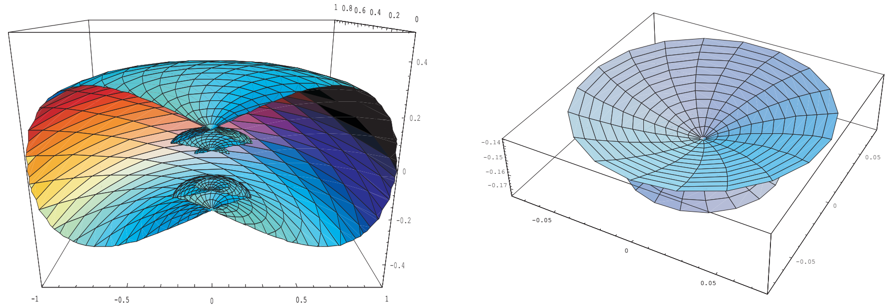
From (5) and (6) we know that the no null terms in the geodesics equations (2) are

$$E_{11}^1 = 1, E_{11}^3 = -x^2, E_{12}^2 = 1, E_{12}^3 = x^1, a_{(11)(12)}^{(21)} = -2,$$

then we get the differential system

$$\begin{cases} \dot{x}^1 = \lambda_{11} \\ \dot{x}^2 = \lambda_{12} \\ \dot{x}^3 = -x^2\lambda_{11} + x^1\lambda_{12} \\ \dot{\lambda}_{11} + 2\lambda_{12}\,\lambda_{21} \ = 0 \\ \dot{\lambda}_{12} - 2\lambda_{11}\,\lambda_{21} \ = 0 \\ \dot{\lambda}_{21} = 0. \end{cases} \tag{22}$$

In Appendix A, we have a programme which allows to show the graphics of the wave front set and some details of the singularity at the first conjugate point. In particular, for the Heinsenberg model we get Figure 1 .



(a) Half of wave front set with $\epsilon = 1$ (b) Singularity at the first conjugate point

Figure 1: Type with circular symetry: Heisenberg group.

## 4.2 $S^3$

In (8) we showed the associated vector fields on $S^3 \subset \mathbf{R}^4$. To write down the differential equations for geodesics, we write these vector fields in the coordinate system

$$\begin{aligned} y_1 &= \cos x_1 \cos x_2 \cos x_3 \\ y_2 &= \cos x_1 \cos x_2 \sin x_3 \\ y_3 &= \cos x_1 \sin x_2 \\ y_4 &= \sin x_1 \end{aligned} \tag{23}$$

to obtain the correspondent vector fields $\tilde{e}_{11}, \tilde{e}_{12}, \tilde{e}_{21}$ on $\mathbf{R}^3$. The components of these vector fields in the coordinate basis are

$$\begin{cases} E_{11}^1 = & -\cos(x^2)\sin(x^3) \\ E_{11}^2 = & \cos(x^3) - \sin(x^2)\sin(x^3)\tan(x^1) \\ E_{11}^3 = & \sec(x^2)\sec(x^3)\tan(x^1) + \\ & \quad + \sin(x^3)(\tan(x^2) - \cos(x^2)\tan(x^1)\tan(x^3) - \sin(x^2)\tan(x^1)\tan(x^2)\tan(x^3)) \\ E_{12}^1 = & \cos(x^2)\cos(x^3) \\ E_{12}^2 = & \sin(x^3) + \cos(x^3)\sin(x^2)\tan(x^1) \\ E_{12}^3 = & \cos(x^2)\sin(x^3)\tan(x^1) + \tan(x^2)(-\sec(x^3) + \sin(x^3)(\sin(x^2)\tan(x^1) + \tan(x^3))) \quad . \end{cases}$$

From equations (7) we know that

$$a_{(11)(12)}^{21} = a_{(12)(21)}^{11} = a_{(21)(11)}^{12} = -2,$$

and all others coefficients are 0. So replacing these terms in (2) we obtain the differential equations for the geodesics. Applying the commands in Appendix A we get Figure 3 that shows half of the wave front set with the singularity at the first conjugate point, which is similar to the Heisenberg case.

## 4.3 $\widetilde{Q^3}$

In (9) we determined the vector fields on $Q^3 \subset \mathbf{R}^4$ generating the associated Lie algebra (10). By using the coordinate system

$$\begin{aligned} y_1 &= \cosh x_1 \cosh x_2 \cos x_3 \\ y_2 &= \cosh x_1 \cosh x_2 \sin x_3 \\ y_3 &= \cosh x_1 \sinh x_2 \\ y_4 &= \sinh x_1 \end{aligned} \tag{24}$$

on $Q^3$ we obtain the correspondent vector fields $\tilde{e}_{11}, \tilde{e}_{12}, \tilde{e}_{21}$ on $\mathbf{R}^3$ whose components in the coordinate basis are

$$\begin{cases} E^1_{11} = & -\cosh(x^2)\sin(x^3) \\ E^2_{11} = & \cos(x^3) + \sinh(x^2)\sin(x^3)\tanh(x^1) \\ E^3_{11} = & -\operatorname{sech}(x^2)\cos(x^3)\tanh(x^1) - \sin(x^3)\tanh(x^2) \\ E^1_{12} = & \cosh(x^2)\cos(x^3) \\ E^2_{12} = & \sin(x^3) - \cos(x^3)\sinh(x^2)\tanh(x^1) \\ E^3_{12} = & -\cosh(x^2)\sin(x^3)\tanh(x^1) + \tanh(x^2)(\cos(x^3) + \sin(x^3)\sinh(x^2)\tanh(x^1)) \quad . \end{cases}$$

From equations 3.1(3) we know that

$$-a^{21}_{(11)(12)} = a^{11}_{(12)(21)} = a^{12}_{(21)(11)} = -2,$$

being 0 all others coefficents. Replacing these coeficients in the geodesic equations (2) we obtain the differential equations of geodesics. As above, we obtain Figure 4 that shows the half of the wave front set with the singularity at the first conjugate point, similar to the Heisenberg case.

## 4.4 $SU(2)$

The case $K > 2\lambda$ corresponds to $G \cong SU(2) = \{A \in GL(2, \mathbf{C}) | A\bar{A}^t = I, \det A = 1\}$. Then

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbf{C}, a\bar{a} + b\bar{b} = 1 \right\}$$

If we write $a = y_1 + iy_2$ and $b = y_3 + y_4$, then $SU(2)$ can be represented as $S^3 \subset R^4$ by the equation $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1$. Let us take (23) as coordinates of $S^3$. A basis of the Lie algebra that verifies

(12) is

$$
\begin{cases}
f_1 = & -\sin x_2 \dfrac{\partial}{\partial x_1} + \cos x_2 \tan x_1 \dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_3} \\
f_2 = & \cos x_2 \sin x_3 \dfrac{\partial}{\partial x_1} + (\cos x_3 + \sin x_2 \sin x_3 \tan x_1) \dfrac{\partial}{\partial x_2} + (-\sec x_2 \sec x_3 \tan x_1 + \\
& \qquad + \sin x_3(\tan x_2 + \cos x_2 \tan x_1 \tan x_3 + \sin x_2 \tan x_1 \tan x_2 \tan x_3)) \dfrac{\partial}{\partial x_3} \\
f_3 = & \cos x_2 \cos x_3 \dfrac{\partial}{\partial x_1} + (-\sin x_3 + \cos x_3 \sin x_2 \tan x_1) \dfrac{\partial}{\partial x_2} + (\cos x_2 \sin x_3 \tan x_1 + \\
& \qquad + \tan x_2(\sec x_3 + \sin x_3(\sin x_2 \tan x_1 - \tan x_3)) \dfrac{\partial}{\partial x_3} + x_3 \dfrac{\partial}{\partial x_4} \quad .
\end{cases}
$$

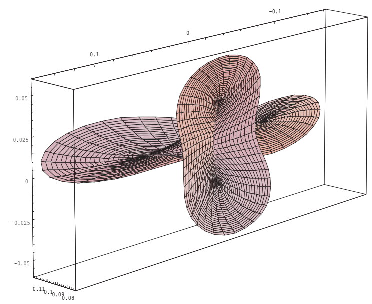
It follows from 11 that

$$
\begin{cases}
E^1_{11} = \beta \sin x_2 + \alpha \cos x_2 \sin x_3 \\
E^2_{11} = -\beta \cos x_2 \tan x_1 + \alpha(\cos x_3 + \sin x_2 \sin x_3 \tan x_1)) \\
E^3_{11} = -\beta + \alpha(-\sec x_2 \sec x_3 \tan x_1 + \\
\qquad + \sin x_3(\tan x_2 + \cos x_2 \tan x_1 \tan x_3 + \sin x_2 \tan x_1 \tan x_2 \tan x_3) \\
E^1_{12} = -\beta \sin x_2 + \alpha \cos x_2 \sin x_3 \\
E^2_{12} = \beta \cos x_2 \tan x_1 + \alpha(\cos x_3 + \sin x_2 \sin x_3 \tan x_1) \\
E^3_{12} = \beta + \alpha(-\sec x_2 \sec x_3 \tan x_1 + \\
\qquad + \sin x_3(\tan x_2 + \cos x_2 \tan x_1 \tan x_3 + \sin x_2 \tan x_1 \tan x_2 \tan x_3)) \quad ,
\end{cases}
$$

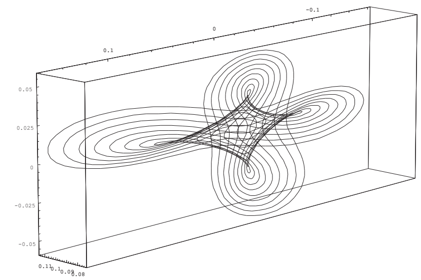
with

$$
\alpha = \frac{1}{2}\sqrt{\frac{K}{2} + \lambda}, \quad \beta = \frac{1}{2}\sqrt{\frac{K}{2} - \lambda} \quad \text{and} \quad \gamma = \frac{1}{4}\sqrt{K^2 - 4\lambda^2}.
$$

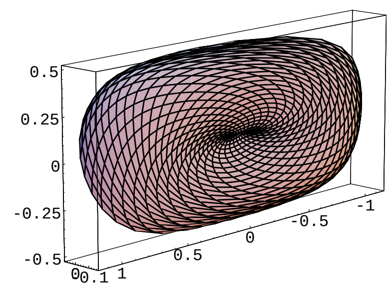
Replacing these coefficients in (2) and taking account the relation on (10) we obtain the differential equations of geodesics. Processing the program in Appendix A with the data above, we get Figure 2 which shows that there are two first conjugate points at each “side” of the wave front set. It also shows that the auto intersection of wfs happens along a segment of line connecting both conjugate points. Some details of the singularity at the first conjugate points, are showed too.



(a) Detail of singularity



(b) Detail showing parallels




(c) Wave front set

Figure 2: Wave front set without circular symmetry: $SU(2)$, $K > 2\lambda$ ($K = 3$, $\lambda = 1$).

## 4.5 $\widetilde{E}(2)$

The case $K = 2\lambda$ corresponds to $G = \widetilde{E}(2)$, where $E(2)$ is defined in (13). In the coordinate system $x_1, x_2, x_3$ a basis for the Lie algebras (14) is

$$\begin{cases} f_1 = \cos x_3 \dfrac{\partial}{\partial x_1} + \sin x_3 \dfrac{\partial}{\partial x_2} \\ \\ f_2 \dfrac{\partial}{\partial x_3} \\ \\ f_3 = -\sin x_3 \dfrac{\partial}{\partial x_1} + \cos x_3 \dfrac{\partial}{\partial x_2} \end{cases}$$

where

$$\begin{array}{l} [f_1, f_2] = -f_3 \\ [f_2, f_3] = -f_1 \\ [f_3, f_1] = 0, \end{array}$$

and the coefficients of generators $(e_{11}, e_{12}, e_{21})$ to replace in equations (2) are

$$E^1_{11} = -\cos x_3,\ E^2_{11} = -\sin x_3,\ E^3_{11} = -\sqrt{2K},\ E^1_{12} = \cos x_3,\ E^2_{12} = \sin x_3,\ E^3_{12} = -\sqrt{2K}.$$

Applying the computational program in Appendix A to the above data, we get Figure 5 with half of the wave front set and the singularity at the first conjugate points. The singularity is the same as $SU(2)$ in subsection 4.4.

## 4.6 $\widetilde{SU}(1,1)$

Let us consider $-2\lambda < K < 2\lambda$. In this case $G \cong \widetilde{SU}(1,1)$, as in (15) where

$$SU(1,1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : a, b \in C, a\bar{a} - b\bar{b} = 1 \right\}.$$

By the identification $a = y_1 + iy_2$ and $b = y_3 + y_4$, we have that $SU(1,1)$ is isomorphic to $Q^3 \subset R^4$. It turns out that $a\bar{a} - b\bar{b} = 1$ transforms into $y_1^2 + y_2^2 - y_3^2 - y_4^2 = 1$. We take the coordinate system (24) on $Q^3$ to have the basis (16) of the Lie algebra generated by

$$\begin{cases} f_1 = \cos x_3 \cosh x_2 \dfrac{\partial}{\partial x_1} - (\sin x_3 + \cos x_3 \sinh x_2 \tanh x_1) \dfrac{\partial}{\partial x_2} + \\ \qquad + (-\cosh x_2 \sin x_3 \tanh x_1 + (-\cos x_3 + \sin x_3 \sinh x_2 \tanh x_1) \tanh x_2) \dfrac{\partial}{\partial x_3} \\ f_2 = -\sinh x_2 \dfrac{\partial}{\partial x_1} + \cosh x_2 \tanh x_1 \dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_3} \\ f_3 = \cosh x_2 \sin x_3 \dfrac{\partial}{\partial x_1} + (\cos x_3 - \sin x_3 \sinh x_2 \tanh x_1) \dfrac{\partial}{\partial x_2} + \\ \qquad + (\cos x_3 \operatorname{sech} x_2 \tanh x_1 - \sin x_3 \tanh x_2) \dfrac{\partial}{\partial x_3} \quad , \end{cases} \tag{25}$$

where

$$\begin{array}{l} [f_1, f_2] = 2f_3 \\ [f_2, f_3] = 2f_1 \\ [f_3, f_1] = -2f_2 \quad . \end{array}$$

It follows also from (16) that the coefficientes to substitute in (2) are

$$\begin{cases} E_{11}^1 = -\beta \cos x_3 \cosh x_2 - \alpha \sinh x_2 \\ E_{11}^2 = \alpha \cosh x_2 \tanh x_1 + \beta(\sin x_3 + \cos x_3 \sinh x_2 \tanh x_1) \\ E_{11}^3 = \alpha - \beta(-\cosh x_2 \sin x_3 \tanh x_1 + (-\cos x_3 + \sin x_3 \sinh x_2 \tanh x_1) \tanh x_2) \\ E_{12}^1 = \beta \cos x_3 \cosh x_2 - \alpha \sinh x_2 \\ E_{12}^2 = \alpha \cosh x_2 \tanh x_1 + \beta(-\sin x_3 - \cos x_3 \sinh x_2 \tanh x_1) \\ E_{12}^3 = \alpha + \beta(-\cosh x_2 \sin x_3 \tanh x_1 + (-\cos x_3 + \sin x_3 \sinh x_2 \tanh x_1) \tanh x_2) \quad , \end{cases}$$

with

$$\alpha = \frac{1}{2}\sqrt{\frac{K}{2} + \lambda}, \beta = \frac{1}{2}\sqrt{\lambda - \frac{K}{2}} \quad \text{and} \quad \gamma = \frac{1}{4}\sqrt{4\lambda^2 - K^2}.$$

We obtain the graphic of the wave front set in Figure 6, which shows the same type of singularity as in the subsections 4.5 and 4.6 before.

## 4.7 $E(1,1)$

For the case $K=-2\lambda$ the associated Lie group is $G=E(1,1)$, see (17). In the coordinate system $x_1,x_2,x_3$ a basis for the Lie algebra is

$$\begin{cases} f_1=\dfrac{\partial}{\partial x_3} \\ \\ f_2=\cosh x_3\dfrac{\partial}{\partial x_1}+\sinh x_3\dfrac{\partial}{\partial x_2} \\ \\ f_3=\sinh x_3\dfrac{\partial}{\partial x_1}+\cosh x_3\dfrac{\partial}{\partial x_2} \end{cases}$$

and

$$\begin{array}{l}[f_1,f_2]=f_3\\ [f_2,f_3]=0\\ [f_1,f_3]=f_2.\end{array}$$

It follows from the adapted basis (18) that the coefficients to be substituted in (2) are

$$E_{11}^1=\cosh x_3,\ E_{11}^2=\sinh x_3,\ E_{11}^3=\sqrt{2\lambda},\ E_{12}^1=\cosh x_3,\ E_{12}^2=\sinh x_3,\ E_{12}^3=-\sqrt{2\lambda}\ ,$$

The graphic of the wave front set in this case is in Figure 7. Again, the graphic shows the same behaviour as the subsections 4.5, 4.6 and 4.7 above.

## 4.8 $\widetilde{SU}(1,1)$

Let's now examine the case $K<-2\lambda$. The Lie group $G$ is isomorphic to the universal covering $\widetilde{SU}(1,1)$ of $SU(1,1)$. We consider the basis (25). The coefficients of the adapted basis (19) are

$$\begin{cases} E_{11}^1=\alpha\cos x_3\cosh x_2-\beta\cosh x_3\sin x_3\\ E_{11}^2=\alpha\cosh x_2\tanh x_1+\beta(\sin x_3+\cos x_3\sinh x_2\tanh x_1)\\ E_{11}^3=\ -\beta(\cos x_3\sec x_2\tanh x_1-\sin x_3\tanh x_2)+\\ \qquad +\alpha(-\cosh x_2\sin x_3\tanh x_1+(-\cos x_3+\sin x_3\sinh x_2\tanh x_1)\tanh x_2)\\ E_{12}^1=\alpha\cos x_3\cosh x_2-+\beta\cosh x_2\sinh x_3\\ E_{12}^2=\alpha(-\sin x_3-\cos x_3-sinx_3\sinh x_2\tanh x_1)\\ E_{12}^3=\beta(\cos x_3\mathrm{sech}x_2\tanh x_1-\sin x_3\tanh x_2)+\\ \qquad \alpha(-\cosh x_2\sin x_3\tanh x_1+(-cosx_3+\sin x_3\sinh x_2\tanh x_1)\tanh x_2)\quad , \end{cases}$$

with

$$\alpha=\frac{1}{2}\sqrt{-\left(\frac{K}{2}+\lambda\right)},\beta=\frac{1}{2}\sqrt{\lambda-\frac{K}{2}}\quad\text{and}\quad\gamma=\frac{1}{4}\sqrt{K^2-4\lambda^2}.$$

Replacing these coefficients in (2) we obtain the graphic of the wave front set in Figure 8. Again, the graphic shows the same form as the three subsections above.

## 4.9 $G_1$

This case corresponds to $W_2 = W_1 > 0$. The structure equations are

$$\begin{aligned}
[e_{11}, e_{12}] &= -2\alpha e_{11} + 2\alpha e_{12} - 2e_{21} \\
[e_{21}, e_{12}] &= \lambda(e_{11} - e_{12}) \\
[e_{21}, e_{11}] &= \lambda(e_{11} - e_{12})
\end{aligned}$$

where $\alpha = \frac{1}{2}\sqrt{\lambda - \frac{K}{2}}$. The associated group is

$$G_1 = \left\{ \begin{bmatrix} 1 & 0 \quad 0 \\ x_1 & \\ x_2 & \tilde{\sigma}(x_3) \end{bmatrix} : x_1, x_2, x_3 \in \mathbf{R} \right\}$$

where

$$\tilde{\sigma}(x_3) \begin{cases} e^{\alpha x_3}\left(\dfrac{\sinh(\beta x_3)}{\beta} B + \cosh(\beta x_3) I\right) & , \text{if} \quad K < -14\lambda \\ e^{\alpha x_3}\left(x_3 B + I\right) & , \text{if} \quad K = -14\lambda \\ e^{\alpha x_3}\left(\dfrac{\sin(\beta' x_3)}{\beta'} B + \cos(\beta' x_3) I\right) & , \text{if} \quad -14\lambda < K < 2\lambda \end{cases}$$

and

$$\alpha = \frac{1}{2}\sqrt{\lambda - \frac{K}{2}}, \quad \beta = \frac{1}{2\sqrt{2}}\sqrt{-K - 14\lambda}, \quad \beta' = \frac{1}{2\sqrt{2}}\sqrt{K + 14\lambda} \;\; e \;\; B = \begin{bmatrix} \alpha & \lambda \\ -2 & -\alpha \end{bmatrix}.$$

In this particular situation, we distinguish three cases, given by conditions $K$ and $\lambda$ on $\tilde{\sigma}$:

**Case $K < -14\lambda$.** A basis for the Lie algebra in a system of canonical coordinates is:

$$\begin{aligned}
g_1 &= -\sigma_{11}\tfrac{\partial}{\partial x_1} - \sigma_{21}\tfrac{\partial}{\partial x_2} \\
g_2 &= \tfrac{\partial}{\partial x_3} \\
g_3 &= -\sigma_{12}\tfrac{\partial}{\partial x_1} - \sigma_{22}\tfrac{\partial}{\partial x_2}
\end{aligned}$$

where

$$\begin{aligned}
[g_2, g_1] &= -2\alpha g_1 + 2g_3 \\
[g_2, g_3] &= -\lambda g_1 \\
[g_3, g_1] &= 0
\end{aligned}$$

and

$$\begin{cases} \sigma_{11} = e^{\alpha x_3}\left(\dfrac{\alpha}{\beta}\sinh(\beta x_3) + \cosh(\beta x_3)\right) \\ \\ \sigma_{21} = e^{\alpha x_3}\left(\dfrac{-2}{\beta}\sinh(\beta x_3)\right) \\ \\ \sigma_{12} = e^{\alpha x_3}\left(\dfrac{\lambda}{\beta}\sinh(\beta x_3)\right) \\ \\ \sigma_{22} = e^{\alpha x_3}\left(-\dfrac{\alpha}{\beta}\sinh(\beta x_3) + \cosh(\beta x_3)\right). \end{cases}$$

The vector fields satisfying the structure equations are

$$\begin{array}{l} e_{11} = -(g_1 + g_2) \\ e_{12} = -g_2 \\ e_{21} = -\sigma_{12}\frac{\partial}{\partial x_1} - \sigma_{22}\frac{\partial}{\partial x_2} \end{array}$$

then

$$E_{11}^1 = \sigma_{11},\ E_{11}^2 = \sigma_{21},\ E_{11}^3 = 1,\ E_{12}^1 = 0,\ E_{12}^2 = 0,\ E_{12}^3 = 1.$$

We show the graphic of singularities at wave front in Figure 9.

**Case $K = -14\lambda$.** In this case, the basis of the Lie algebra is

$$\begin{array}{l} g_1 = -\sigma_{11}\frac{\partial}{\partial x_1} - \sigma_{21}\frac{\partial}{\partial x_2} \\ g_2 = -\frac{\partial}{\partial x_3} \\ g_3 = -\sigma_{12}\frac{\partial}{\partial x_1} - \sigma_{22}\frac{\partial}{\partial x_2} \end{array}$$

where

$$\begin{array}{l} [g_2, g_1] = -2\alpha g_1 + 2g_3 \\ [g_2, g_3] = -\lambda g_1 \\ [g_3, g_1] = 0 \end{array}$$

and

$$\begin{cases} \sigma_{11} = e^{\alpha x_3}(1 + \alpha x_3) \\ \\ \sigma_{21} = e^{\alpha x_3}(-2x_3) \\ \\ \sigma_{12} = 0 \\ \\ \sigma_{22} = 0. \end{cases}$$

The vector fields satisfying the structure equations are

$$\begin{array}{l} e_{11} = -(g_1 + g_2) \\ e_{12} = -g_2 \\ e_{21} = -\sigma_{12}\frac{\partial}{\partial x_1} - \sigma_{22}\frac{\partial}{\partial x_2} \end{array}$$

thus

$$E_{11}^1 = \sigma_{11},\ E_{11}^2 = \sigma_{21},\ E_{11}^3 = 1,\ E_{12}^1 = 0,\ E_{12}^2 = 0,\ E_{12}^3 = 1.$$

The graphics of the singularities at wave front set are in Figure 10.

**Case $-14\lambda < K < 2\lambda$.** Here a basis for the Lie algebra in a system of canonical coordinates is given by:

$$\begin{aligned} g_1 &= -\sigma_{11}\tfrac{\partial}{\partial x_1} - \sigma_{21}\tfrac{\partial}{\partial x_2} \\ g_2 &= -\tfrac{\partial}{\partial x_3} \\ g_3 &= -\sigma_{12}\tfrac{\partial}{\partial x_1} - \sigma_{22}\tfrac{\partial}{\partial x_2} \end{aligned}$$

where

$$\begin{aligned} [g_2, g_1] &= -2\alpha g_1 + 2g_3 \\ [g_2, g_3] &= -\lambda g_1 \\ [g_3, g_1] &= 0 \end{aligned}$$

and

$$\begin{cases} \sigma_{11} = e^{\alpha x_3}\left(\dfrac{\alpha}{\beta'}\sin(\beta' x_3) + \cos(\beta' x_3)\right) \\ \\ \sigma_{21} = e^{\alpha x_3}\left(\dfrac{-2}{\beta'}\sin(\beta' x_3)\right) \\ \\ \sigma_{12} = e^{\alpha x_3}\left(\dfrac{\lambda}{\beta'}\sin(\beta' x_3)\right) \\ \\ \sigma_{22} = e^{\alpha x_3}\left(-\dfrac{\alpha}{\beta'}\sin(\beta' x_3) + \cos(\beta' x_3)\right). \end{cases}$$

The vector fields satisfying the structure equations are

$$\begin{aligned} e_{11} &= -(g_1 + g_2) \\ e_{12} &= -g_2 \\ e_{21} &= -\sigma_{12}\tfrac{\partial}{\partial x_1} - \sigma_{22}\tfrac{\partial}{\partial x_2} \end{aligned}$$

then

$$E_{11}^1 = \sigma_{11},\ E_{11}^2 = \sigma_{21},\ E_{11}^3 = 1,\ E_{12}^1 = 0,\ E_{12}^2 = 0,\ E_{12}^3 = 1 \quad .$$

The graphics of singularity at wave front set are in Figure 11.

## 4.10 $G_2$

This last case corresponds to $-W_2 = W_1 > 0$. The structure equations are

$$\begin{aligned}
[e_{11}, e_{12}] &= -2\alpha e_{11} - 2\alpha e_{12} - 2e_{21} \\
[e_{21}, e_{12}] &= -\lambda(e_{11} + e_{12}) \\
[e_{21}, e_{11}] &= \lambda(e_{11} + e_{12})
\end{aligned}$$

where $\alpha = \frac{1}{2}\sqrt{-\lambda - \frac{K}{2}}$. The group is

$$G_2 = \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ x_1 & \multicolumn{2}{c}{\multirow{2}{*}{$\sigma(x_3)$}} \\ x_2 & & \end{array} \right] : x_1, x_2, x_3 \in \mathbf{R} \right\}.$$

Here

$$\sigma(x_3) = e^{\alpha x_3} \left( \frac{\sinh(\beta x_3)}{\beta} B + \cosh(\beta x_3) I \right),$$

and

$$\beta = \frac{1}{2\sqrt{2}}\sqrt{14\lambda - K}, \quad e \quad B = \left[ \begin{array}{cc} \alpha & \lambda \\ 2 & -\alpha \end{array} \right].$$

Thus

$$\left\{ \begin{array}{l}
E_{11}^1 = -e^{\alpha x_3} \left( \dfrac{\alpha}{\beta} \sinh(\beta x_3) + \cosh(\beta x_3) \right) \\
E_{11}^2 = e^{\alpha x_3} \left( \dfrac{-2}{\beta} \sinh(\beta x_3) \right) \\
E_{11}^3 = 1 \\
E_{12}^1 = 0 \\
E_{12}^2 = 0 \\
E_{12}^3 = 1 \quad .
\end{array} \right.$$

The graphics of singularity at wave front set are in Figure 12.

# 5 Appendix A: Commands in Mathematica to Generate the Wavefront Sets

In this appendix we stablish a list of commands in the MATHEMATICA Programme. By using the values of the functions $E^i_{1,j}$ for the corresponding `Eij` and $a^{\alpha i}_{(\beta j)(\gamma k)}$ in the equations for $y'^i$, these commands builded all the pictures showed in this work,

```
In[1]:=  E11[x1_,x2_,x3_]:=1
In[2]:=  E21[x1_,x2_,x3_]:=0
In[3]:=  E31[x1_,x2_,x3_]:=-x2
In[4]:=  E12[x1_,x2_,x3_]:=0
In[5]:=  E22[x1_,x2_,x3_]:=1
In[6]:=  E32[x1_,x2_,x3_]:=x1
In[7]:=  geo[c1_,c2_,a_,r_]:=Evaluate[{x1[t],x2[t],x3[t]}]/.Flatten[NDSolve[{
         x1'[t]==E11[x1[t],x2[t],x3[t]]*y1[t]+E12[x1[t],x2[t],x3[t]]*y2[t],
         x2'[t]==E21[x1[t],x2[t],x3[t]]*y1[t]+E22[x1[t],x2[t],x3[t]]*y2[t],
         x3'[t]==E31[x1[t],x2[t],x3[t]]*y1[t]+E32[x1[t],x2[t],x3[t]]*y2[t],
         y1'[t]==2*d[t]*y2[t], y2'[t]==-2*d[t]*y1[t], d'[t]==0,
         x1[0]==0,x2[0]==0,x3[0]==0,y1[0]==c1,y2[0]==c2,d[0]==a},
         {x1,x2,x3,y1,y2,d},{t,0,r}]]
In[8]:=  WaveFront[u_,v_,r_]:=geo[Cos[u],Sin[u],v,r]/.t->r
In[9]:=  ParametricPlot3D[WaveFront[u,v,1],{u,0,2 Pi},{v,3,3.4}]
In[10]:= ParametricPlot3D[WaveFront[u,v,1],{u,0,2 Pi},{v,-7,7},PlotPoints->
         {20,100},PlotRange->{{-1,1},{0,1},{-0.5,0.5}},ViewPoint->{0,-1,0.3}]
```

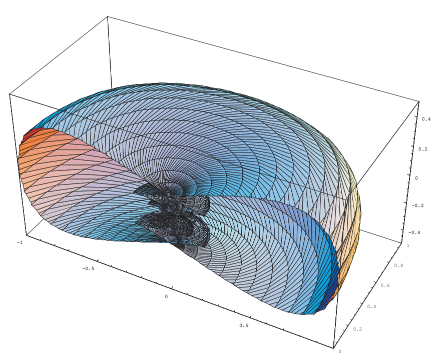


Figure 3: $S^3$ ball section

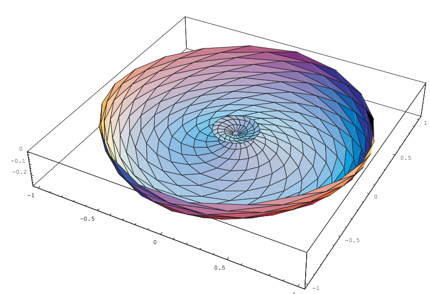


Figure 4: $Q^3$ ball section.

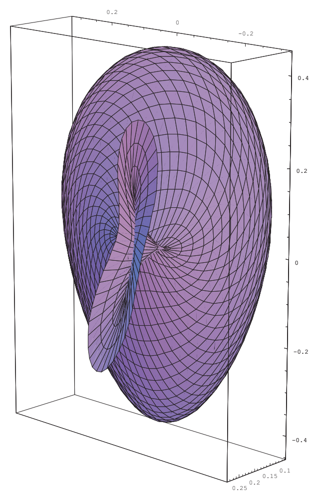



Figure 5: $E(2)$ $(K = 2\lambda = 1)$.

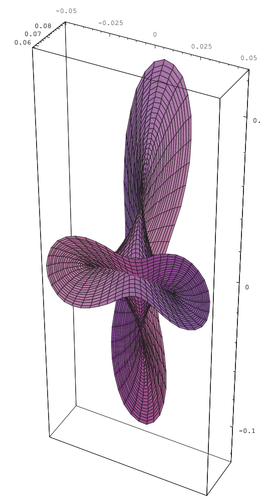



Figure 6: $SU(1,1)$, $-2\lambda < K < 2\lambda$ $(\lambda = 1,\, K = 1)$.

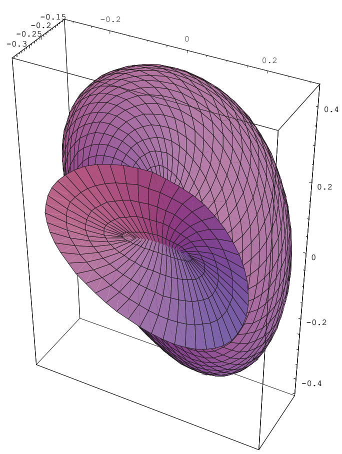



Figure 7: $E(1,1)$ $(\lambda = 1)$.

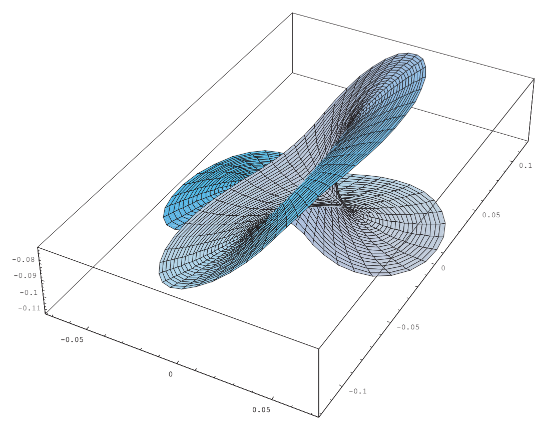



Figure 8: $SU(1,1)$, $K < -2\lambda$ $(\lambda = 1,\, K = -3)$.

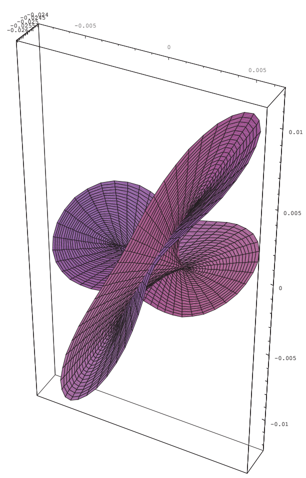


Figure 9: $G_1$, $K < -14\lambda$ ($\lambda = 1$, $K = -15$).

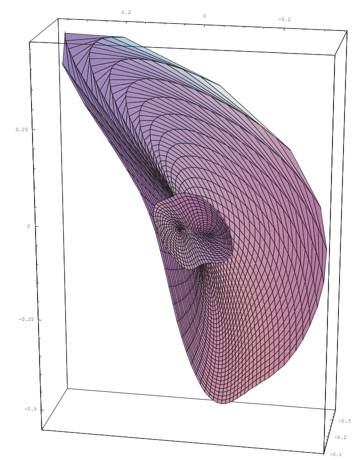


Figure 10: $G_1$, $K = -14\lambda$ ($\lambda = 1$).

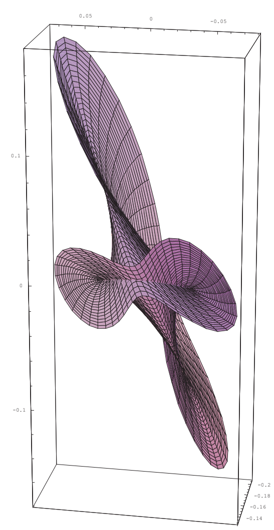


Figure 11: $G_1$, $-14\lambda < K < 2\lambda$ ($\lambda = 1$, $K = 0$).

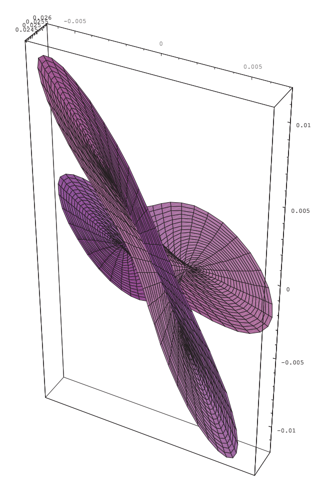


Figure 12: $G_2$ ($\lambda = 1$, $K = -3$).

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