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ON BASES OF CONSTANT CURVATURE

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Abstract

The aim of this paper is to study the Riemannian manifolds that have bases along which their sectional curvatures are constant.

1 Introduction

Let M^n be a n-dimensional Riemannian manifold with curvature tensor R. Given $p \in M$, let $X, Y \in T_pM$ be two linearly independent vectors. The sectional curvature of M along the plane spanned by X and Y is defined by

$$K(X,Y) = \frac{\langle R(X,Y)X, Y\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}.$$

An orthornomal basis $\beta = \{E_1, E_2, \ldots, E_n\}$ is called a *basis of constant curvature* c if

$$K(E_i, E_j) = c, \ \forall\ 1 \leq i \neq j \leq n.$$

We show in Example 1.2 that this condiction does not imply that M has constant curvature at p.

Example 1.1 *The space forms have bases of constant curvature at all points.*

Example 1.2 *Let $SO(3)$ be the Lie group of the rotations in Euclidean space $\mathbb{R}^3$. We consider $SO(3)$ equipped with the left-invariant metric such that $\{F_1, F_2, F_3\}$ is an orthonormal basis of $T_I SO(3)$ (I is the identity matrix of $SO(3)$), where*

$$F_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{e} \quad F_3 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}..$$

We have

$$[F_1, F_2] = 2F_3, \quad [F_2, F_3] = \frac{1}{2}F_1 \quad \text{and} \quad [F_3, F_1] = \frac{1}{2}F_2.$$

Now, by using Theorem 4.3 of [Miln], we get that $\{F_1, F_2, F_3\}$ diagonalizes the Ricci tensor (see Section 2) of $SO(3)$ and the Ricci curvatures at F_i, $1 \le i \le 3$, are given by

$$\mathrm{Ricc}(F_1) = -\frac{1}{2}, \quad \mathrm{Ricc}(F_2) = -\frac{1}{2} \quad \text{and} \quad \mathrm{Ricc}(F_3) = 1.$$

In particular, the scalar curvature of $SO(3)$ at I is zero. Hence, if X, Y are orthonormal vectors in $T_I SO(3)$, then $K(X,Y) = -\mathrm{Ricc}(X \times Y)$ (see Lemma 2.1), where $\times$ indicates the cross product in $T_I SO(3)$. Thus $SO(3)$ has not constant curvature. Consider the following vectors of the tangent space $T_I SO(3)$:

$$E_1 = \begin{pmatrix} 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{3} & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & -\frac{1}{-3+\sqrt{3}} & \frac{1}{2}\frac{-1+\sqrt{3}}{-3+\sqrt{3}} \\ \frac{1}{-3+\sqrt{3}} & 0 & -\frac{-2+\sqrt{3}}{-3+\sqrt{3}} \\ -\frac{1}{2}\frac{-1+\sqrt{3}}{-3+\sqrt{3}} & \frac{-2+\sqrt{3}}{-3+\sqrt{3}} & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & -\frac{1}{3}\frac{-3+2\sqrt{3}}{-1+\sqrt{3}} & \frac{1}{6}\frac{-3+\sqrt{3}}{-1+\sqrt{3}} \\ \frac{1}{3}\frac{-3+2\sqrt{3}}{-1+\sqrt{3}} & 0 & \frac{1}{3}\frac{\sqrt{3}}{-1+\sqrt{3}} \\ -\frac{1}{6}\frac{-3+\sqrt{3}}{-1+\sqrt{3}} & -\frac{1}{3}\frac{\sqrt{3}}{-1+\sqrt{3}} & 0 \end{pmatrix}.$$

Clearly $\{E_1, E_2, E_3\}$ is an orthonormal basis of curvature zero. Now, if we consider the left invariant vector fields induced by E_1, E_2 and E_3, we obtain a frame field of curvature zero along the whole $SO(3)$.

This example shows that there exist manifolds with bases of constant curvature, but which have not constant curvature. In fact, Example 1.2 is a particular case of the following.

Theorem 1.3 *All tridimensional Riemannian manifold has, at least, a basis of constant curvature at all points.*

We also obtained a very large family of manifolds with bases of constant curvature:

Theorem 1.4 *Let M^n be a conformally flat manifold. Then, given $p \in M$, there exists a basis of constant curvature in T_pM.*

The converse is not true, as the following example shows.

Example 1.5 *Let M be the Riemannian product $SO(3) \times N$, where N is either S^1 or $\mathbb{R}$, and $SO(3)$ is as in Example 1.2. Let $\beta = \{E_1, E_2, E_3, E_4\}$ be an orthonormal basis of $T_{(I,x)}M$, $x \in N$, where $\{E_1, E_2, E_3\}$ is as in Example 1.2 and $E_4 \in T_xN$. Then β is a basis of zero curvature of M. Now, applying the Kulkarni Theorem (see [Kulk]) to the quadruple F_1, F_2, F_3 and E_4, we see that M cannot be conformally flat.*

2 Basic Material

In this section we present the basic definitions and results which will be used in proof of Theorem 1.2 and Theorem 1.4.

Let M a Riemannian manifold with metric $\langle\ ,\ \rangle$ and curvature tensor[1] R. Fix $p \in M$ and let $\{E_1, E_2, \ldots E_n\}$ be a orthonormal basis of T_pM. The *Ricci tensor* of M at p is given by

$$Q(X) = \sum_{i=1}^{n} R_{E_i X} E_i, \quad X \in T_pM.$$

The quadratic form associated to Q will be indicated by Ricc. So

$$\mathrm{Ricc}(X) = \langle Q(X), X \rangle = \langle \sum_{i=1}^{n} R_{E_i X} E_i, X \rangle, \quad X \in T_pM.$$

[1] We are using the following definition for R:

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

where ∇ indicates the Riemannian connection of M.

If $U \in T_pM$ is a unit vector, then $\mathrm{Ricc}(U)$ is called the ***Ricci curvature*** of M at direction U. In particular,

$$\mathrm{Ricc}(E_k) = \sum_{i=1(i\neq k)}^{n} K(E_i, E_k).$$

The real number

$$S(p) = \sum_{i=1}^{n} \mathrm{Ricc}(E_i) = 2\sum_{i<j}^{n} K(E_i, E_j)$$

is the *scalar curvature* of M at p.

When M has dimension 3, K, S and Q are related as follows.

Lemma 2.1 *Let M be a 3-dimensional Riemannian manifold, let $p \in M$ and let U and V be two orthonormal vectors of T_pM. Then*

$$K(U,V) = \frac{S(p)}{2} - \mathrm{Ricc}(U \times V).$$

Proof: Since $\{U, V, U \times V\}$ is an orthonormal basis, we get

$$\begin{aligned} \mathrm{Ricc}(U) &= K(U,V) + K(U, U \times V) \\ \mathrm{Ricc}(V) &= K(U,V) + K(V, U \times V) \\ \mathrm{Ricc}(U \times V) &= K(U, U \times V) + K(V, U \times V). \end{aligned}$$

Hence

$$S(p) = \mathrm{Ricc}(U) + \mathrm{Ricc}(V) + \mathrm{Ricc}(U \times V) = 2K(U,V) + 2\,\mathrm{Ricc}(U \times V),$$

which proves the lemma. □

Now we present a linear algebraic lemma which is essential in the proofs of 1.3 and 1.4.

Lemma 2.2 *Let $\mathbf{V}$ be a n-dimensional vector space equipped with an inner product. Let $B : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{R}$ be a traceless bilinear form. Then there exists a basis $\{W_1, W_2, \ldots, W_n\}$ such that $B(W_i, W_i) = 0$, for $i = 1, \ldots, n$.*

Proof: Let $W_n \in \mathbf{V}$ be an unit vector such that $B(W_n, W_n) = 0$. Let $\widetilde{\mathbf{V}}$ be the subspace of $\mathbf{V}$ orthogonal to W_n, and $\{V_1, V_2, \ldots, V_{n-1}\}$ be an orthonormal basis of $\widetilde{\mathbf{V}}$. Then

$$B(V_1, V_1) + B(V_2, V_2) + \cdots + B(V_{n-1}, V_{n-1}) = 0,$$

since the trace of B is equal to zero. Now, by making induction on n, we obtain an orthonormal basis of $\widetilde{\mathbb{V}}$, say $\{W_1, W_2, \ldots, W_{n-1}\}$, such that $B(W_i, W_i) = 0$, for $i = 1, \ldots, n-1$. Hence $\{W_1, W_2, \ldots, W_{n-1}, W_n\}$ is a orthonormal basis of $\mathbb{V}$ with the desired property. □

Corollary 2.3 *Let $\mathbb{V}$ be a n-dimensional vector space equipped with the inner product $\langle\ ,\ \rangle$. Let $B : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ be a bilinear form. Then there exists a basis $\{W_1, W_2, \ldots, W_n\}$ such that $B(W_i, W_i) = \frac{\operatorname{tr} B}{n}$, for $i = 1, \ldots, n$.*

Proof: Let $\widetilde{B} : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ be the bilinear form defined by

$$\widetilde{B}(X, Y) = B(X, Y) - \frac{\operatorname{tr} B}{n}\langle X, Y\rangle.$$

So $\operatorname{tr} \widetilde{B} = 0$ and then by Lemma 2.2 there is an orthonormal basis

$$\{W_1, W_2, \ldots, W_{n-1}, W_n\}$$

such that $\widetilde{B}(W_i, W_i) = 0$, for $i = 1, \ldots, n$. Hence $B(W_i, W_i) = (\operatorname{tr} B)/n$, for $i = 1, \ldots, n$. □

3 Constructing Constant Curvature Bases

We begin this section with the following proposition, which shows that the scalar curvature of a manifold M determines completely the value of the constant of a basis of constant curvature, if there exists such a basis.

Proposition 3.1 *If $\{E_1, E_2, \ldots, E_n\} \subset T_pM$ is a basis of constant curvature c of a Riemannian manifold M, then $c = S(p)/n(n-1)$, where $S(p)$ is the scalar curvature of M at p.*

Proof: Just observe that $S(p) = 2\sum_{i<j} K(E_i, E_j) = n(n-1)c$. □

Now we can construct a family of Riemannian manifolds which has no bases of constant curvature.

Proposition 3.2 *Let $M = P^2 \times F^k$, where P^2 is a 2-dimensional Riemannian manifold, and F^k, $k \geq 2$, is a k-dimensional flat manifold. If the sectional curvature of P is never zero, then M has no bases of constant curvature.*

Proof: In fact, if $\{V_1, V_2, \ldots, V_n\}$ is such a basis at some $p = (a, b) \in M$, then $K(V_i, V_j) = 2\overline{K}/n(n-1)$, $1 \leq i \neq j \leq n$, $n = k+2$, where $\overline{K}$ is the curvature of P

at a. Now, writing $V_i = X_i + Y_i$, where $X_i \in T_aP$ and $Y_i \in T_bF$, for $1 \leq i \leq n$, we obtain that

$$K(V_i, V_j) = \langle \overline{R}(X_i, X_j)X_i, X_j \rangle = \frac{2\overline{K}}{n(n-1)}, \quad 1 \leq i \neq j \leq n,$$

where $\overline{R}$ is the curvature tensor of P. Hence

$$\|X_i\|^2\|X_j\|^2 - \langle X_i, X_j \rangle^2 = \frac{2}{n(n-1)} > 0, \quad 1 \leq i \neq j \leq n,$$

which implies that the vectors X_i, $1 \leq i \leq n$, are pairwise linearly independent and the parallelograms spanned by the pairs $\{X_i, X_j\}$, $1 \leq i \neq j \leq n$, have the same area. This is not possible, since T_aP has dimension 2 and $n \geq 4$. So M cannot have bases of constant curvature. □

Proof of Theorem 1.3: Let $p \in M$ and let B be the symmetric bilinear form induced by the Ricci tensor of M, Q (see Section 2), that is,

$$B(X, Y) = \langle Q(X), Y \rangle, \quad X, Y \in T_pM.$$

So $S(p) = \operatorname{tr} B$. From Corollary 2.3 it follows that there exists an orthonormal basis $\{E_1, E_2, E_3\}$ of T_pM such that $B(E_i, E_i) = \operatorname{Ricc}(E_i) = (\operatorname{tr} B)/3$, $1 \leq i \leq 3$. But $S(p) = \operatorname{tr} B$. Hence $K(E_i, E_j) = S(p)/6$, by Lemma 2.1. □

Proof of Theorem 1.4: We use the same notation as in the proof of Theorem 1.3. Let $\{E_1, E_2, \ldots, E_n\}$ be an orthonormal basis of M such that

$$B(E_i, E_i) = \operatorname{Ricc}(E_i) = \frac{\operatorname{tr} B}{n} = \frac{S(p)}{n}, \quad 1 \leq i \leq n.$$

Since M is conformally flat, we get from Theorem 3.2 of [Kulk] that

$$K(E_i, E_j) = \frac{\operatorname{Ricc}(E_i) + \operatorname{Ricc}(E_j)}{n-2} - \frac{S(p)}{(n-1)(n-2)}, \quad 1 \leq i \leq n.$$

Hence $K(E_i, E_j) = S(p)/n(n-1)$ and the proof is complete. □

As a application of Theorem 1.4, we obtain the following example.

Example 3.3 *Let M be one of the manifolds listed.*

(•) *a hypersurface of rotation of $\mathbb{R}^n$;*

(•) *a warped product of the type $\mathbb{R} \times_\phi N$, where N is a space form;*

(•) *a Riemannian product $S^m(1/\sqrt{c}) \times H^n(-1/\sqrt{c})$, where $H^n(-1/\sqrt{c})$ is the hyperbolic space of curvature $-c$.*

Then M has bases of constant curvature. In fact, in any case, M is conformally flat.

4 References

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