CUBO A Mathematical Journal
Vol.21, Nº 03, (29–38). December 2019
http://dx.doi.org/10.4067/S0719-06462019000300029

Ostrowski-Sugeno fuzzy inequalities

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ABSTRACT

We present Ostrowski-Sugeno fuzzy type inequalities. These are Ostrowski-like inequalities in the context of Sugeno fuzzy integral and its special properties are investigated. Tight upper bounds to the deviation of a function from its Sugeno-fuzzy averages are given. This work is greatly inspired by [3] and [1].

RESUMEN

Presentamos desigualdades de Ostrowski-Sugeno de tipo fuzzy. Estas son desigualdades de tipo Ostrowski en el contexto de integrales fuzzy de Sugeno y se investigan sus propiedades especiales. Se entregan cotas superiores ajustadas para la desviación de una función de sus promedios fuzzy de Sugeno. Este trabajo está inspirado principalmente por [3] y [1].

Keywords and Phrases: Sugeno fuzzy, integral, function fuzzy average, deviation from fuzzy mean, fuzzy Ostrowski inequality.

2010 AMS Mathematics Subject Classification: Primary: 26D07, 26D10, 26D15, 41A44, Secondary: 26A24, 26D20, 28A25.

1 Introduction

The famous Ostrowski ([3]) inequality motivates this work and has as follows:

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\left\|f'\right\|_\infty,$$

where $f \in C'([a,b])$, $x \in [a,b]$, and it is a sharp inequality. One can easily notice that

$$\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a) = \frac{(x-a)^2+(b-x)^2}{2(b-a)}.$$

Another motivation is author's article [1].

First we give a survey about Sugeno fuzzy integral and its basic properties. Then we derive a series of Ostrowski-like inequalities to all directions in the context of Sugeno integral and its basic important particular properties. We also give applications to special cases of our problem we deal with.

2 Background

In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented.

Definition 2.1. *(Fuzzy measure [5, 7]) Let* Σ *be a* σ*-algebra of subsets of* X*, and let* $\mu : \Sigma \to [0,+\infty]$ *be a non-negative extended real-valued set function. We say that* μ *is a fuzzy measure iff:*

(1) $\mu(\varnothing) = 0$,

(2) $E, F \in \Sigma : E \subseteq F$ *imply* $\mu(E) \leq \mu(F)$ *(monotonicity),*

(3) $E_n \in \Sigma$ *(*$n \in \mathbb{N}$*),* $E_1 \subset E_2 \subset \ldots$*, imply* $\lim_{n\to\infty}\mu(E_n) = \mu\left(\cup_{n=1}^{\infty}E_n\right)$ *(continuity from below);*

(4) $E_n \in \Sigma$ *(*$n \in \mathbb{N}$*),* $E_1 \supset E_2 \supset \ldots$*,* $\mu(E_1) < \infty$*, imply* $\lim_{n\to\infty}\mu(E_n) = \mu\left(\cap_{n=1}^{\infty}E_n\right)$ *(continuity from above).*

Let (X,Σ,μ) be a fuzzy measure space and f be a non-negative real-valued function on X. We denote by $\mathcal{F}_+$ the set of all non-negative real valued measurable functions, and by $L_\alpha f$ the set: $L_\alpha f := \{x \in X : f(x) \geq \alpha\}$, the α-level of f for $\alpha \geq 0$.

Definition 2.2. *Let* (X,Σ,μ) *be a fuzzy measure space. If* $f \in \mathcal{F}_+$ *and* $A \in \Sigma$*, then the Sugeno integral (fuzzy integral) [6] of* f *on* A *with respect to the fuzzy measure* μ *is defined by*

$$(S)\int_A f\,d\mu := \vee_{\alpha\geq 0}\left(\alpha \wedge \mu(A\cap L_\alpha f)\right), \tag{1}$$

where $\vee$ *and* $\wedge$ *denote the sup and inf on* $[0,\infty]$*, respectively.*

The basic properties of Sugeno integral follow:

Theorem 2.3. *([4, 7]) Let (X, Σ, μ) be a fuzzy measure space with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. Then*

1) $(S)\int_A f d\mu \leq \mu(A)$;

2) $(S)\int_A k d\mu = k \wedge \mu(A)$ *for a non-negative constant* k*;*

3) if $f \leq g$ *on* A*, then* $(S)\int_A f d\mu \leq (S)\int_A g d\mu$;

4) if $A \subset B$*, then* $(S)\int_A f d\mu \leq (S)\int_B f d\mu$;

5) $\mu(A \cap L_\alpha f) \leq \alpha \Rightarrow (S)\int_A f d\mu \leq \alpha$;

6) if $\mu(A) < \infty$*, then* $\mu(A \cap L_\alpha f) \geq \alpha \Leftrightarrow (S)\int_A f d\mu \geq \alpha$;

7) when $A = X$*,* $(S)\int_A f d\mu = \vee_{\alpha \geq 0} (\alpha \wedge \mu(L_\alpha f))$;

8) if $\alpha \leq \beta$*, then* $L_\beta f \subseteq L_\alpha f$;

9) $(S)\int_A f d\mu \geq 0$.

Theorem 2.4. *([7, p. 135]) Let* $f \in \mathcal{F}_+$*, the class of all finite nonnegative measurable functions on* (X, Σ, μ)*. Then*

1) if $\mu(A) = 0$*, then* $(S)\int_A f d\mu = 0$*, for any* $f \in \mathcal{F}_+$*;*

2) if $(S)\int_A f d\mu = 0$*, then* $\mu(A \cap \{x | f(x) > 0\}) = 0$;

3) $(S)\int_A f d\mu = (S)\int_A f \cdot \chi_A d\mu$*, where* χ_A *is the characteristic function of* A*;*

4) $(S)\int_A (f + a) d\mu \leq (S)\int_A f d\mu + (S)\int_A a d\mu$*, for any constant* $a \in [0, \infty)$.

Corollary 2.5. *([7, p. 136]) Let* $f, f_1, f_2 \in \mathcal{F}_+$*. Then*

1) $(S)\int_A (f_1 \vee f_2) d\mu \geq (S)\int_A f_1 d\mu \vee (S)\int_A f_2 d\mu$;

2) $(S)\int_A (f_1 \wedge f_2) d\mu \leq (S)\int_A f_1 d\mu \wedge (S)\int_A f_2 d\mu$;

3) $(S)\int_{A \cup B} f d\mu \geq (S)\int_A f d\mu \vee (S)\int_B f d\mu$;

4) $(S)\int_{A \cap B} f d\mu \leq (S)\int_A f d\mu \wedge (S)\int_B f d\mu$.

In general we have

$$(S)\int_A (f_1 + f_2) d\mu \neq (S)\int_A f_1 d\mu + (S)\int_A f_2 d\mu,$$

and

$$(S)\int_A a f d\mu \neq a (S)\int_A f d\mu, \text{ where } a \in \mathbb{R},$$

see [7, p. 137].

Lemma 2.6. *([7, p. 138])* $(S)\int_A f d\mu = \infty$ *if and only if* $\mu(A \cap L_\alpha f) = \infty$ *for any* $\alpha \in [0, \infty)$.

We need

Definition 2.7. *([2]) A fuzzy measure* μ *is subadditive iff* $\mu(A \cup B) \leq \mu(A) + \mu(B)$*, for all* $A, B \in \Sigma$.

We mention the following result

Theorem 2.8. *([2]) If* μ *is subadditive, then*

$$(S)\int_X (f+g)\, d\mu \leq (S)\int_X f d\mu + (S)\int_X g d\mu, \tag{2}$$

for all measurable functions $f, g : X \to [0, \infty)$.

Moreover, if (2) holds for all measurable functions $f, g : X \to [0, \infty)$ *and* $\mu(X) < \infty$*, then* μ *is subadditive.*

Notice here in (1) we have that $\alpha \in [0, \infty)$.

We have the following corollary.

Corollary 2.9. *If* μ *is aubadditive,* $n \in \mathbb{N}$*, and* $f : X \to [0, \infty)$ *is a measurable function, then*

$$(S)\int_X n f d\mu \leq n\,(S)\int_X f d\mu, \tag{3}$$

in particular it holds

$$(S)\int_A n f d\mu \leq n\,(S)\int_A f d\mu, \tag{4}$$

for any $A \in \Sigma$.

Proof. By inequality (2). ☐

A very important property of Sugeno integral follows.

Theorem 2.10. *If* μ *is subadditive measure, and* $f : X \to [0, \infty)$ *is a measurable function, and* $c > 0$*, then*

$$(S)\int_A c f d\mu \leq (c+1)\,(S)\int_A f d\mu, \tag{5}$$

for any $A \in \Sigma$.

Proof. Let the ceiling $\lceil c \rceil = m \in \mathbb{N}$, then by Theorem 2.3 (3) and (4) we get

$$(S)\int_A c f d\mu \leq (S)\int_A m f d\mu \leq m\,(S)\int_A f d\mu \leq (c+1)\,(S)\int_A f d\mu,$$

proving (5). ☐

3 Main Results

From now on in this article we work on the fuzzy measure space $([a,b], \mathcal{B}, \mu)$, where $[a,b] \subset \mathbb{R}$, $\mathcal{B}$ is the Borel σ-algebra on $[a,b]$, and μ is a finite fuzzy measure on $\mathcal{B}$. Typically we take it to be subadditive.

The functions f we deal with here are continuous from $[a,b]$ into $\mathbb{R}_+$.

We make the following remark

Remark 3.1. *Let* $f \in C^1([a,b], \mathbb{R}_+)$, *and* μ *is a subadditive fuzzy measure such that* $\mu([a,b]) > 0$, $x \in [a,b]$. *We will estimate*

$$E := \left| (S)\int_{[a,b]} f(x)\, d\mu(t) - \mu([a,b]) \wedge f(x) \right| \tag{6}$$

(by Theorem 2.3 (2))

$$= \left| (S)\int_{[a,b]} f(t)\, d\mu(t) - (S)\int_{[a,b]} f(x)\, d\mu(t) \right|.$$

We notice that

$$f(t) = f(t) - f(x) + f(x) \leq |f(t) - f(x)| + f(x),$$

then (by Theorem 2.3 (3) and Theorem 2.4 (4))

$$(S)\int_{[a,b]} f(t)\, d\mu(t) \leq (S)\int_{[a,b]} |f(t) - f(x)|\, d\mu(t) + (S)\int_{[a,b]} f(x)\, d\mu(t), \tag{7}$$

that is

$$(S)\int_{[a,b]} f(t)\, d\mu(t) - (S)\int_{[a,b]} f(x)\, d\mu(t) \leq (S)\int_{[a,b]} |f(t) - f(x)|\, d\mu(t). \tag{8}$$

Similarly, we have

$$f(x) = f(x) - f(t) + f(t) \leq |f(t) - f(x)| + f(t),$$

then (by Theorem 2.3 (3) and Theorem 2.8)

$$(S)\int_{[a,b]} f(x)\, d\mu(t) \leq (S)\int_{[a,b]} |f(t) - f(x)|\, d\mu(t) + (S)\int_{[a,b]} f(t)\, d\mu(t),$$

that is

$$(S)\int_{[a,b]} f(x)\, d\mu(t) - (S)\int_{[a,b]} f(t)\, d\mu(t) \leq (S)\int_{[a,b]} |f(t) - f(x)|\, d\mu(t). \tag{9}$$

By (8) and (9) we derive that

$$\left| (S)\int_{[a,b]} f(t)\, d\mu(t) - (S)\int_{[a,b]} f(x)\, d\mu(t) \right| \leq (S)\int_{[a,b]} |f(t) - f(x)|\, d\mu(t). \tag{10}$$

Consequently it holds

$$E \overset{\text{(by (6), (10))}}{\leq} (S)\int_{[a,b]} |f(t)-f(x)|\, d\mu(t)$$

(and by $|f(t)-f(x)| \leq \|f'\|_{\infty} |t-x|$*)*

$$\leq (S)\int_{[a,b]} \|f'\|_{\infty} |t-x|\, d\mu(t) \overset{\text{(by (5))}}{\leq} (\|f'\|_{\infty}+1)\,(S)\int_{[a,b]} |t-x|\, d\mu(t). \tag{11}$$

We have proved the following Ostrowski-like inequality

$$\left|\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \frac{\mu([a,b]\wedge f(x))}{\mu([a,b])}\right| \leq \tag{12}$$

$$\frac{(\|f'\|_{\infty}+1)}{\mu([a,b])}(S)\int_{[a,b]} |t-x|\, d\mu(t).$$

The last inequality can be better written as follows:

$$\left|\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1\wedge \frac{f(x)}{\mu([a,b])}\right)\right| \leq$$

$$\frac{(\|f'\|_{\infty}+1)}{\mu([a,b])}(S)\int_{[a,b]} |t-x|\, d\mu(t). \tag{13}$$

Notice here that $\left(1\wedge \frac{f(x)}{\mu([a,b])}\right) \leq 1$*, and* $\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) \leq \frac{\mu([a,b])}{\mu([a,b])} = 1$*, where* $(S)\int_{[a,b]} f(t)\, d\mu(t) \geq 0$.

I.e. If $f : [a,b] \to \mathbb{R}_+$ *is a Lipschitz function of order* $0 < \alpha \leq 1$*, i.e.* $|f(x)-f(y)| \leq K|x-y|^{\alpha}$*,* $\forall\, x,y \in [a,b]$*, where* $K > 0$*, denoted by* $f \in \mathrm{Lip}_{\alpha,K}([a,b],\mathbb{R}_+)$*, then we get similarly the following Ostrowski-like inequality:*

$$\left|\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1\wedge \frac{f(x)}{\mu([a,b])}\right)\right| \leq$$

$$\frac{(K+1)}{\mu([a,b])}(S)\int_{[a,b]} |t-x|^{\alpha}\, d\mu(t). \tag{14}$$

We have proved the following Ostrowski-Sugeno inequalities:

Theorem 3.2. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$*,* $x \in [a,b]$.

1) Let $f \in C^1([a,b],\mathbb{R}_+)$*, then*

$$\left|\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1\wedge \frac{f(x)}{\mu([a,b])}\right)\right| \leq$$

$$\frac{(\|f'\|_{\infty}+1)}{\mu([a,b])}(S)\int_{[a,b]} |t-x|\, d\mu(t). \tag{15}$$

2) Let $f \in Lip_{\alpha,K}([a,b], \mathbb{R}_+)$, $0 < \alpha \leq 1$, *then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{(K+1)}{\mu([a,b])} (S) \int_{[a,b]} |t-x|^{\alpha}\, d\mu(t). \tag{16}$$

We make the following remark

Remark 3.3. *Let* $f \in C^1([a,b], \mathbb{R}_+)$ *and* $g \in C^1([a,b])$, *by Cauchy's mean value theorem we get that*

$$(f(t) - f(x))\, g'(c) = (g(t) - g(x))\, f'(c),$$

for some c *between* t *and* x*; for any* $t, x \in [a,b]$.

If $g'(c) \neq 0$, *we have*

$$(f(t) - f(x)) = \left(\frac{f'(c)}{g'(c)}\right)(g(t) - g(x)).$$

Here we assume that $g'(t) \neq 0$, $\forall\ t \in [a,b]$. *Hence it holds*

$$|f(t) - f(x)| \leq \left\| \frac{f'}{g'} \right\|_{\infty} |g(t) - g(x)|, \tag{17}$$

for all $t, x \in [a,b]$.

We have again as before (see (11))

$$E \leq (S) \int_{[a,b]} |f(t) - f(x)|\, d\mu(t) \overset{\text{(by (17))}}{\leq}$$

$$(S) \int_{[a,b]} \left\| \frac{f'}{g'} \right\|_{\infty} |g(t) - g(x)|\, d\mu(t) \overset{\text{(by (5))}}{\leq}$$

$$\left(\left\| \frac{f'}{g'} \right\|_{\infty} + 1 \right) (S) \int_{[a,b]} |g(t) - g(x)|\, d\mu(t). \tag{18}$$

We have established the following general Ostrowski-Sugeno inequality:

Theorem 3.4. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$, $x \in [a,b]$. *Let* $f \in C^1([a,b], \mathbb{R}_+)$ *and* $g \in C^1([a,b])$ *with* $g'(t) \neq 0$, $\forall\ t \in [a,b]$. *Then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{\left(\left\| \frac{f'}{g'} \right\|_{\infty} + 1 \right)}{\mu([a,b])} (S) \int_{[a,b]} |g(t) - g(x)|\, d\mu(t). \tag{19}$$

We give for $g(t) = e^t$ the next result

Corollary 3.5. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$, $x \in [a,b]$. *Let* $f \in C^1([a,b], \mathbb{R}_+)$, *then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{\left(\left\| \frac{f'}{e^t} \right\|_\infty + 1\right)}{\mu([a,b])} (S) \int_{[a,b]} \left| e^t - e^x \right| d\mu(t). \tag{20}$$

When $g(t) = \ln t$ we get the following corollary.

Corollary 3.6. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$, $x \in [a,b]$ *and* $a > 0$. *Let* $f \in C^1([a,b], \mathbb{R}_+)$. *Then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{\left(\|tf'(t)\|_\infty + 1\right)}{\mu([a,b])} (S) \int_{[a,b]} \left| \ln \frac{t}{x} \right| d\mu(t). \tag{21}$$

Many other applications of Theorem 3.4 could follow but we stop it here.

We make the following remark.

Remark 3.7. *Let* $f \in \left[C([a,b], \mathbb{R}_+) \cap C^{n+1}([a,b])\right]$, $n \in \mathbb{N}$, $x \in [a,b]$. *Then by Taylor's theorem we get*

$$f(y) - f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} (y-x)^k + R_n(x,y), \tag{22}$$

where the remainder

$$R_n(x,y) := \int_x^y \left(f^{(n)}(t) - f^{(n)}(x)\right) \frac{(y-t)^{n-1}}{(n-1)!} dt; \tag{23}$$

here y *can be* $\geq x$ *or* $\leq x$.

By [1] we get that

$$|R_n(x,y)| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!} |y-x|^{n+1}, \quad \textit{for all } x,y \in [a,b]. \tag{24}$$

Here we assume $f^{(k)}(x) = 0$, *for all* $k = 1, ..., n$.

Therefore it holds

$$|f(t) - f(x)| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!} |t-x|^{n+1}, \ \textit{for all } t,x \in [a,b]. \tag{25}$$

Here we have again

$$E \le (S)\int_{[a,b]} |f(t)-f(x)|\, d\mu(t) \overset{\text{(by Theorem 2.3 (3) and (25))}}{\le}$$

$$(S)\int_{[a,b]} \frac{\left\|f^{(n+1)}\right\|_{\infty}}{(n+1)!}|t-x|^{n+1}\, d\mu(t) \overset{\text{(by (5))}}{\le}$$

$$\left(\frac{\left\|f^{(n+1)}\right\|_{\infty}}{(n+1)!}+1\right)(S)\int_{[a,b]} |t-x|^{n+1}\, d\mu(t). \tag{26}$$

We have derived the following high order Ostrowski-Sugeno inequality:

Theorem 3.8. *Let* $f \in \left[C([a,b],\mathbb{R}_+) \cap C^{n+1}([a,b])\right]$, $n \in \mathbb{N}$, $x \in [a,b]$. *We assume that* $f^{(k)}(x) = 0$, *all* $k = 1,...,n$. *Here* μ *is subadditive with* $\mu([a,b]) > 0$. *Then*

$$\left|\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right)\right| \le$$

$$\frac{\left(\frac{\left\|f^{(n+1)}\right\|_{\infty}}{(n+1)!}+1\right)}{\mu([a,b])}(S)\int_{[a,b]} |t-x|^{n+1}\, d\mu(t), \tag{27}$$

which generalizes (15).

When $x = \frac{a+b}{2}$ we get the following corollary

Corollary 3.9. *Let* $f \in \left[C([a,b],\mathbb{R}_+) \cap C^{n+1}([a,b])\right]$, $n \in \mathbb{N}$. *Assume that* $f^{(k)}\left(\frac{a+b}{2}\right) = 0$, $k = 1,...,n$. *Here* μ *is subadditive with* $\mu([a,b]) > 0$. *Then*

$$\left|\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f\left(\frac{a+b}{2}\right)}{\mu([a,b])}\right)\right| \le$$

$$\frac{\left(\frac{\left\|f^{(n+1)}\right\|_{\infty}}{(n+1)!}+1\right)}{\mu([a,b])}(S)\int_{[a,b]} \left|t-\frac{a+b}{2}\right|^{n+1} d\mu(t). \tag{28}$$

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