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Characterization of Upper Detour Monophonic Domination Number

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ABSTRACT

This paper introduces the concept of *upper detour monophonic domination number* of a graph. For a connected graph G with vertex set $V(G)$, a set $M \subseteq V(G)$ is called minimal detour monophonic dominating set, if no proper subset of M is a detour monophonic dominating set. The maximum cardinality among all minimal monophonic dominating sets is called *upper detour monophonic domination number* and is denoted by $\gamma_{dm}^{+}(G)$. For any two positive integers p and q with $2 \leq p \leq q$ there is a connected graph G with $\gamma_m(G) = \gamma_{dm}(G) = p$ and $\gamma_{dm}^{+}(G) = q$. For any three positive integers p, q, r with $2 < p < q < r$, there is a connected graph G with $m(G) = p$, $\gamma_{dm}(G) = q$ and $\gamma_{dm}^{+}(G) = r$. Let p and q be two positive integers with $2 < p < q$ such that $\gamma_{dm}(G) = p$ and $\gamma_{dm}^{+}(G) = q$. Then there is a minimal DMD set whose cardinality lies between p and q. Let p, q and r be any three positive integers with $2 \leq p \leq q \leq r$. Then, there exist a connected graph G such that $\gamma_{dm}(G) = p, \gamma_{dm}^{+}(G) = q$ and $|V(G)| = r$.

RESUMEN

Este artículo introduce el concepto de *número de dominación de desvío monofónico superior* de un grafo. Para un grafo conexo G con conjunto de vértices $V(G)$, un conjunto $M \subseteq V(G)$ se llama conjunto dominante de desvío monofónico minimal, si ningún subconjunto propio de M es un conjunto dominante de desvío monofónico. La cardinalidad máxima entre todos los conjuntos dominantes de desvío monofónico minimales se llama *número de dominación de desvío monofónico superior* y se denota por $\gamma_{dm}^{+}(G)$. Para cualquier par de enteros positivos p y q con $2 \leq p \leq q$ existe un grafo conexo G con $\gamma_m(G) = \gamma_{dm}(G) = p$ y $\gamma_{dm}^{+}(G) = q$. Para cualquiera tres enteros positivos p, q, r con $2 < p < q < r$, existe un grafo conexo G con $m(G) = p$, $\gamma_{dm}(G) = q$ y $\gamma_{dm}^{+}(G) = r$. Sean p y q dos enteros positivos con $2 < p < q$ tales que $\gamma_{dm}(G) = p$ y

$\gamma^+_{dm}(G) = q$. Entonces existe un conjunto DMD mínimo cuya cardinalidad se encuentra entre p y q. Sean p, q y r tres enteros positivos cualquiera con $2 \leq p \leq q \leq r$. Entonces existe un grafo conexo G tal que $\gamma_{dm}(G) = p, \gamma^+_{dm}(G) = q$ y $|V(G)| = r$.

Keywords and Phrases: Monophonic number, Domination Number, Detour monophonic number, Detour monophonic domination number, Upper detour monophonic domination number.

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1 Introduction

Consider an undirected connected graph $G(V, E)$ without loops or multiple edges. Let P : $u_1, u_2, ...u_n$ be a path of G. An edge e is said to be a *chord* of P if it is the join of two non adjacent vertices of P. A path is said to be *monophonic path* if there is no chord. If S is a set of vertices of G such that each vertex of G lies on an $u - v$ monophonic path in G for some $u, v \in S$, then S is called *monophonic set*. *Monophonic number* is the minimum cardinality among all the monophonic sets of G. It is denoted by $m(G)$ [1,2].

A vertex v in a graph G dominates itself and all its neighbours. A set T of vertices in a graph G is a *dominating set* if $N[T] = V(G)$. The minimum cardinality among all the dominating sets of G is called *domination number* and is dented by $\gamma(G)$[4]. A set $T \subset V(G)$ is a *monophonic dominating set* of G if T is both monophonic set and dominating set. The *monophonic domination number* is the minimum cardinality among all the monophonic dominating sets of G and is denoted by $\gamma_m(G)$[5,6]. A monophonic set M in a connected graph G is *minimal monophonic set* if no proper subset of M is a monophonic set. The *upper monophonic number* is the maximum cardinality among all minimal monophonic sets and is denoted by $m^+(G)$[9].

The shortest $x - y$ path is called *geodetic path* and longest $x - y$ monophonic path is called *detour monophonic path*. If every vertex of G lies on a $x - y$ detour monophonic path in G for some $x, y \in M \subseteq V(G)$, M could be identified as a *detour monophonic set*. The minimum cardinality among all the detour monophonic set is the *detour monophonic number* and is denoted by $dm(G)$. A *minimal detour monophonic set* D of a connected graph G is a subset of $V(G)$ whose any proper subset is not a detour monophonic set of G. The maximum cardinality among all minimal detour monophonic sets is called *upper detour monophonic set*, denoted by $dm^+(G)$ [10].

If D is both a detour monophonic set and a dominating set, it could be a *detour monophonic dominating set*. The minimum cardinality among all detour monophonic dominating sets of G is the *detour monophonic dominating number*(DMD number) and is denoted by $\gamma_{dm}(G)$[7,8]. A vertex v is an *extreme vertex* if the sub graph induced by its neighbourhood is complete. A vertex u in a connected graph G is a *cut-vertex* of G, if $G - u$ is disconnected. In this article, we consider

G as a connected graph of order $n \geq 2$ if otherwise not stated. For basic notations and terminology refer [3].

Theorem 1.1 (8)**.** *Each extreme vertex of a connected graph G belongs to every detour monophonic dominating set of G.*

Example 1.1. *Consider the graph G given in Figure 1. Here $M_1 = \{v_1, v_4\}$ is a monophonic set. Therefore $m(G) = 2$. M_1 also dominate G. Hence $\gamma(G) = 2$. The set $M_2 = \{v_1, v_2, v_3\}$ is a minimum detour monophonic set. Thus $dm(G) = 3$. M_2 does not dominate G. $M_2 \cup \{v_4\}$ is a minimum DMD set. Therefore $\gamma_{dm}(G) = 4$.*

2 UDMD Number of a Graph

Definition 2.1. *A detour monophonic dominating set M in a connected graph G is called minimal detour monophonic dominating set if no proper subset of M is a detour monophonic dominating set. The maximum cardinality among all minimal detour monophonic dominating sets is called upper detour monophonic domination number and is denoted by $\gamma^+_{dm}(G)$.*

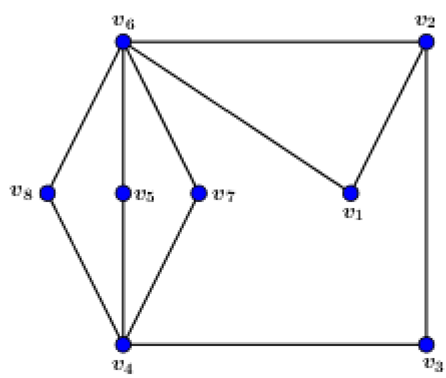

Figure 1: Graph G with UDMD number 5

Example 2.1. *Consider the graph G given in Figure 1. The set $M = \{v_1, v_5, v_6, v_7, v_8\}$ is a minimal DMD set with maximum cardinality. Therefore $\gamma^+_{dm}(G) = 5$.*

Theorem 2.1. *Let G be a connected graph and v an extreme vertex of G. Then v belongs to every minimal detour monophonic dominating set of G.*

Proof. Every minimal detour monophonic dominating set is a minimum detour monophonic set. Since each extreme vertex belongs to every minimum detour monophonic dominating set, the result follows. ∎

Theorem 2.2. *Let v be a cut- vertex of a connected graph G. If M is a minimal DMD set of G, then each component of $G-v$ have an element of M.*

Proof. Suppose let A is a component of $G-v$ having no vertices of M. Let u be any one of the vertex in A. Since M is a minimal DMD set, there exist two vertices p, q in M such that u lies on a $p-q$ detour monophonic path $P : p, u_0, u_1, ..., u, ..., u_m = q$ in G. Consider two sub-paths $P_1 : p-u$ and $P_2 : u-q$ of P. Given v is a cut-vertex of G. Therefore both P_1 and P_2 contain v. Hence P is not a path. This is a contradiction. That is, each component of $G-v$ have an element of every minimal DMD set. ∎

Theorem 2.3. *For a connected graph G of order n, $\gamma_{dm}(G) = n$ if and only if $\gamma^+_{dm}(G) = n$.*

Proof. First, suppose $\gamma^+_{dm}(G) = n$. That is $M = V(G)$ is the unique minimal DMD set of G, so that no proper subset of M is a DMD set. Hence M is the unique DMD set. Therefore $\gamma_{dm}(G) = n$. Conversely, let $\gamma_{dm}(G) = n$. Since every DMD set is a minimal DMD set, $\gamma_{dm}(G) \leq \gamma^+_{dm}(G)$. Therefore $\gamma^+_{dm}(G) \geq n$. Since $V(G)$ is the maximum DMD set, $\gamma^+_{dm}(G) = n$. ∎

3 UDMD Number of Some Standard Graphs

Example 3.1. ***Complete bipartite graph*** $K_{m,n}$

For complete bipartite graph $G = K_{m,n}$,

$$\gamma^+_{dm}(G) = \begin{cases} 2, & if \quad m = n = 1; \\ n, & if \quad n \geq 2, m = 1; \\ 4, & if \quad m = n = 3 \\ max\{m, n\} & if \quad m, n \geq 2, m, n \neq 3 \end{cases}$$

Proof. Case (i): Let $m = n = 1$. Then $K_{m,n} = K_2$. Therefore $\gamma^+_{dm}(G) = 2$.
Case (ii): Let $n \geq 2, m = 1$. This graph is a rooted tree. There are n end vertices. All these are extreme vertices. Therefore they belong to every DMD set and consequently every minimal DMD set.
Case (iii): If $m = n = 3$, then exactly two vertices from both the particians form a minimal DMD set.
Case (iv): Let $m, n \geq 2, m, n \neq 3$. Assume that $m \leq n$. Let $A = \{a_1, a_2, ...a_m\}$ and $B = \{b_1, b_2, ...b_n\}$ be the partitions of G. First, prove $M = B$ is a minimal DMD set. Take a vertex $a_j, 1 \leq j \leq m$, which lies in a detour monophonic path $b_i a_j b_k$ for $k \neq j$ so that M is a detour monophonic set. They also dominate G. Hence M is a DMD set.

Next, let S be any minimal DMD set such that $|S| > n$. Then S contains vertices from both the sets A and B. Since A and B are themselves minimal DMD sets, they do not completely belongs to S. Note that if S contains exactly two vertices from A and B, then it is a minimum DMD set. Thus $\gamma^+_{dm}(G) = n = max\{m, n\}$. ∎

Example 3.2. ***Complete graph*** K_n

For complete graph $G = K_n$, $\gamma^+_{dm}(G) = n$.

Proof. For a complete graph G, every vertex in G is an extreme vertex. By theorem 2.1 they belong to every minimal DMD set. ∎

Example 3.3. ***Cycle graph*** C_n

For Cycle graph $G = C_n$ with n vertices ,

$$\gamma^+_{dm}(G) = \begin{cases} 3, & if \quad n \leq 7, n \neq 4 \\ 2, & if \quad n = 4 \\ 4 + \dfrac{n-7-r}{3}, & if \quad n \geq 8, \quad n-7 \equiv r \ mod(3) \end{cases}$$

Proof. For $n \leq 7$ the results are trivial. For $n \geq 8$, let $C_n : v_1, v_2, v_3, ..., v_n, v_1$ be the cycle with n vertices. Then the set of vertices $\{v_1, v_3, v_{n-1}\}$ is a minimal detour monophonic set but not dominating. This set dominates only seven vertices. There are $n-7$ remaining vertices. If r is the reminder when $n-7$ is divided by 3, then $\dfrac{n-7-r}{3} + 1$ vertices dominate the remaining vertices. Therefore every minimal DMD set contains $4 + \dfrac{n-7-r}{3}$ vertices. ∎

4 Characterization of $\gamma^+_{dm}(G)$

Theorem 4.1. *For any two positive integers p and q with $2 \leq p \leq q$ there is a connected graph G with $\gamma_m(G) = \gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$.*

Proof. Construct a graph G as follows. Let $C_6 : u_1, u_2, u_3, u_4, u_5, u_6, u_1$ be the cycle of order 6. Join $p-1$ disjoint vertices $M_1 = \{x_1, x_2, ..., x_{p-1}\}$ with the vertex u_1. Let $M_2 = \{y_1, y_2, ..., y_{q-p-1}\}$ be a set of $q-p-1$ disjoint vertices. Add each vertex in M_2 with u_4 and u_6. Let x_{p-1} be adjacent with u_2 and u_6. This is the graph G given in Figure 2.

Since all vertices except x_{p-1} in M_1 are extreme, they belong to every minimum monophonic dominating set and DMD set. The set $M = M_1 \cup \{u_4\}$ is a minimum monophonic dominating set. Therefore $\gamma_m(G) = p$. Moreover, the set of all vertices in M form a DMD set and is minimum. That is $\gamma_{dm}(G) = p$.

Next, we prove that $\gamma^+_{dm}(G) = q$. Clearly $N = M_1 \cup M_2 \cup \{u_5, u_6\}$ is a DMD set. N is also a minimal DMD set of G. For the proof, let N' be any proper subset of N. Then there exists at least one vertex $u \in N$ and $u \notin N'$. If $u = y_i$, for $1 \leq i \leq q - p - 1$, then y_i does not lie on any $x - y$ detour monophonic path for some $x, y \in N'$. Similarly if $u \in \{u_5, u_6, x_{p-1}\}$, then that vertex does not lie on any detour monophonic path in N'. Thus N is a minimal DMD set. Therefore $\gamma^+_{dm}(G) \geq q$. ∎

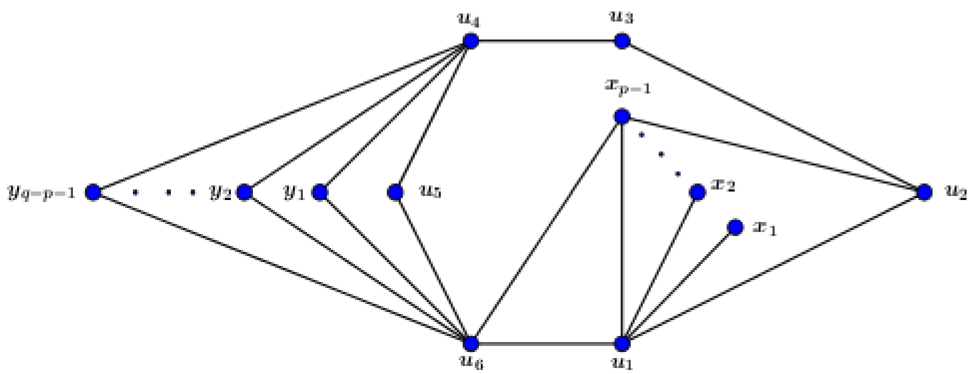

Figure 2: $\gamma_m(G) = \gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$.

Note that N is a minimal DMD set with maximum cardinality. On the contrary, suppose there exists a minimal DMD set, say T, whose cardinality is strictly greater than q. Then there is a vertex $u \in T, u \notin N$. Therefore $u \in \{u_2, u_3, u_4\}$. If $u = u_4$, then $M_1 \cup \{u_4\}$ is a DMD set properly contained in T which is a contradiction. If $u = u_3$, then the set $M_1 \cup \{u_3, u_5\}$ is a DMD set which is a proper subset of T and is a contradiction. If $u = u_2$, then the set $(N - \{u_6\}) \cup \{u_2\}$ is a DMD set properly contained in T and is a contradiction. Thus $\gamma^+_{dm}(G) = q$.

Theorem 4.2. *For any three positive integers p, q, r with $2 < p < q < r$, there is a connected graph G with $m(G) = p$, $\gamma_{dm}(G) = q$ and $\gamma^+_{dm}(G) = r$.*

Proof. Let G be the graph constructed as follows. Take $q - p$ copies of a cycle of order 5 with each cycle C_i has a vertex set $\{d_i, e_i, f_i, g_i, h_i\}$, for $1 \leq i \leq q - p$. Join each e_i with all other vertices in C_i. Also join the vertex f_{i-1} of C_{i-1} with the vertex d_i of C_i. Let $\{u, v\}$ and $\{b_1, b_2, ..., b_{r-q+1}\}$ be two sets of mutually non adjacent vertices. Join each b_i with u and v, for $1 \leq i \leq r - q + 1$. Join another $p - 2$ pendent vertices with u and one pendent vertex with d_1. This is the graph G given in Figure 3.

The set $M_1 = \{a_0, a_1, a_2 ..., a_{p-2}\}$ is the set of all extreme vertices and belongs to every monophonic dominating set and DMD set (Theorem 1.1). Clearly M_1 is not monophonic. But $M_1 \cup \{v\}$ is a monophonic set and is minimum. Therefore $m(G) = p$. Take $M_2 = \{e_1, e_2, ..., e_{q-p}\}$. Then $M_1 \cup M_2 \cup \{v\}$ is a DMD set and is minimum. Therefore $\gamma_{dm}(G) = p - 1 + q - p + 1 = q$. ∎

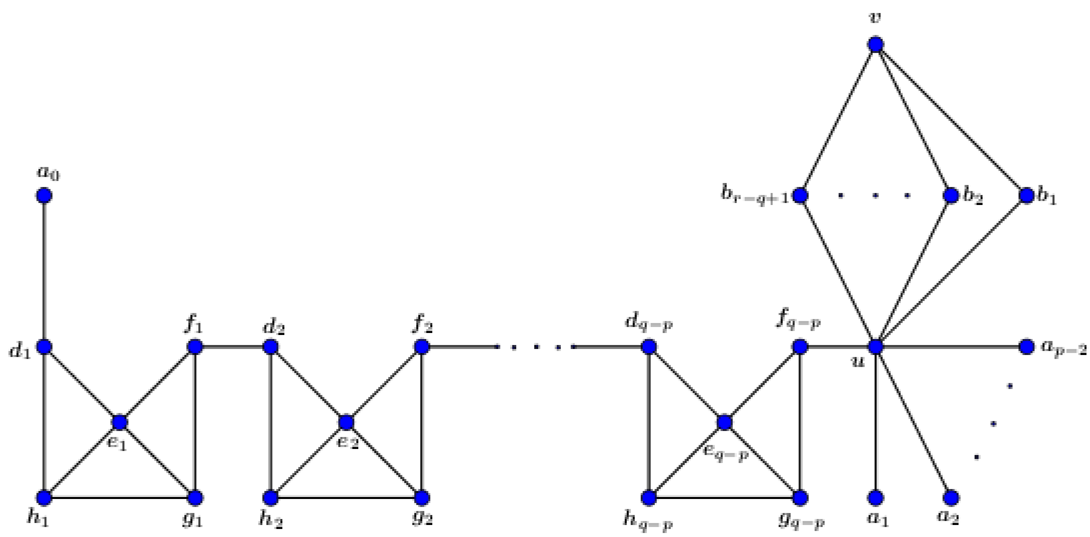

Figure 3: Graph G with $m(G) = p$, $\gamma_{dm}(G) = q$ and $\gamma^+_{dm}(G) = r$.

Let $M_3 = \{b_1, b_2, ..., b_{r-q+1}\}$. Then $M = M_1 \cup M_2 \cup M_3$ is a DMD set. Now M is a minimal DMD set. On the contrary, suppose N is any proper DMD subset of M so that there exists at least one vertex in M which does not belong to N. Let $u \in M$ and $u \notin N$. Clearly $u \notin M_1$ since M_1 is the set of all extreme vertices. If $u = e_i$ for some i, then the vertex e_i does not belong to any detour monophonic path induced by N. Therefore $u \notin M_2$. Similarly $u \notin M_3$. This is a contradiction. Hence M is a minimal DMD set with maximum cardinality. Therefore $\gamma^+_{dm}(G) = |M_1| + |M_2| + |M_3| = (p - 1) + (q - p) + (r - q + 1) = r$.

Theorem 4.3. *Let p and q be two positive integers with $2 < p < q$ such that $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$. Then there is a minimal DMD set whose cardinality lies between p and q.*

Proof. Consider three sets of mutually disjoint vertices $M_1 = \{a_1, a_2, ..., a_{q-n+1}\}$, $M_2 = \{b_1, b_2, ..., b_{n-p+1}\}$ and $M_3 = \{x, y, z\}$. Join each vertex a_i with x and z and each vertex b_j with y and z. Add $p - 2$ pendent vertices $M_4 = \{c_1, c_2, ..., c_{p-2}\}$ with the vertex y. This is the graph G given in Figure 4. Since M_4 is the set of all extreme vertices, it belongs to every DMD set. But M_4 is not a DMD set. The set $M = M_4 \cup \{x, z\}$ is a minimum DMD set. Therefore $\gamma_{dm}(G) = p$.

Consider the set $N = M_1 \cup M_2 \cup M_4$. We claim N is a minimal DMD set with maximum cardinality. On the contrary, suppose there is a set $N' \subset N$ which is a DMD set of G. Then there exists at least one vertex, say u in N which does not belong to N'. Clearly $u \notin M_4$ since it is the set of all extreme vertices. If $u \in M_1$, then $u = a_i$ for some i. Then the vertex a_i does not lie on any detour monophonic path, which is a contradiction. Similarly, if $u \in M_2$, we get a contradiction. Thus N is a minimal DMD set. Therefore $\gamma^+_{dm}(G) \geq q$. ∎

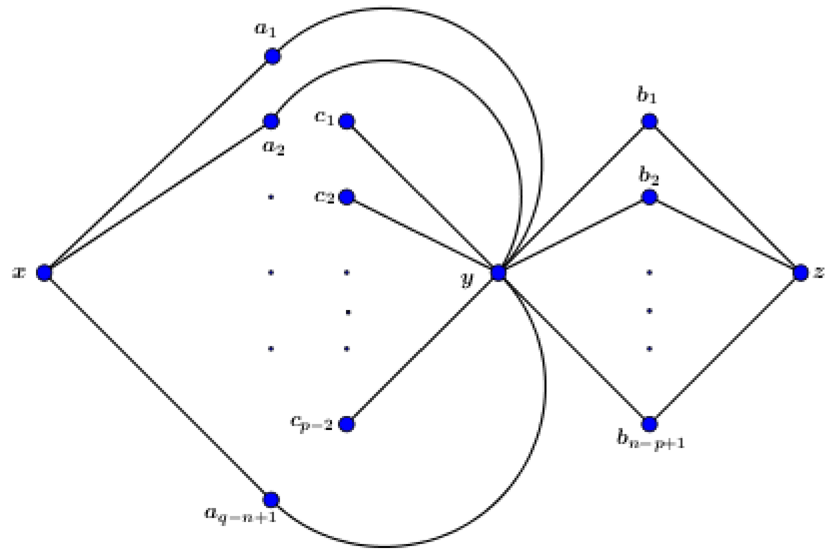

Figure 4: Graph G with $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$

Next, we claim that N has the maximum cardinality of any minimal DMD set. If $\gamma^+_{dm}(G) > q$, there is at least one vertex $v \in V(G), v \notin N$ and belongs to a minimal DMD set. Therefore $v \in M_3$. If $v = x$, then the set $M_2 \cup M_4 \cup \{v\}$ is a minimal DMD set having less than q vertices. Similarly if $v = z$, then the set $M_1 \cup M_4 \cup \{v\}$ is a minimal DMD set. For $v = y$, the set $N \cup \{y\}$ is not a minimal DMD set. Therefore $\gamma^+_{dm}(G) \leq q$.

Let n be any number which lies between p and q. Then there is a minimal DMD set of cardinality n. For the proof, consider the set $T = M_2 \cup M_4 \cup \{x\}$. T is a minimal DMD set. If T is not a minimal DMD set, there is a proper subset T' of T such that T' is a minimal DMD set. Let $u \in T$ and $u \notin T'$. Since each vertex in M_4 is an extreme vertex, $v \notin M_4$. If $u = x$, then the vertex u is not an internal vertex of any detour monophonic path in T'. A similar argument may be made if $u \in M_2$. This leads to a contradiction. Therefore T is a minimal DMD set with cardinality $(n-p+1)+(p-2)+1 = n$.

Theorem 4.4. *Let p , q and r be any three positive integers with $2 \leq p \leq q \leq r$. Then, there exists a connected graph G such that $\gamma_{dm}(G) = p, \gamma^+_{dm}(G) = q$ and $|V(G)| = r$.*

Proof. Let $K_{1,p}$ is a star graph with leaves set $M_1 = \{u_1, u_2, ..., u_p\}$ and let u be the support vertex of $K_{1,p}$. Insert $r-q-1$ vertices $M_2 = \{v_1, v_2, ..., v_{r-q-1}\}$ in the edges uu_i respectively for $1 \leq i \leq r-q-1$. Add $q-p$ vertices $M_3 = \{x_1, x_2, ..., x_{q-p}\}$ with this graph and join each x_i with u and u_1. This is the graph G as shown in Figure 5. Here $|V(G)| = (q-p)+p+(r-q-1)+1 = r$. The length of a detour monophonic path is 4.

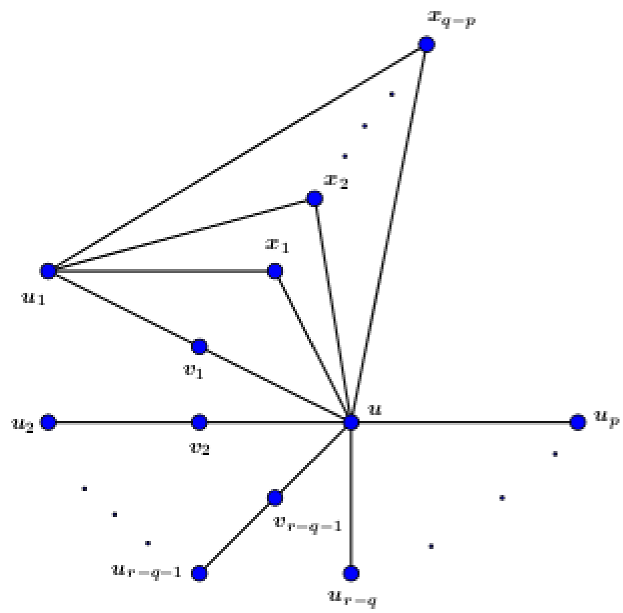

Figure 5: Graph G with $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$

Let $T = M_1 - \{u_1\}$. All the vertices in T are extreme vertices and belong to all DMD sets and minimal DMD sets. Clearly M_1 is a DMD set with minimum cardinality. Therefore $\gamma_{dm}(G) = p$. Let $N = T \cup M_3 \cup \{v_1\}$. Then $|N| = (p-1) + (q-p) + 1 = q$. We claim that N is a minimal DMD set with maximum cardinality.

On the contrary, suppose there is a proper subset N' of N which is a minimal DMD set of G. Then there exists at least one vertex $x \in N, x \notin N'$. Clearly $x \notin T$. If $x \in M_3$, then $x = x_i$ for some i, $1 \leq i \leq q-p$. Then the vertex x_i does not lie on any $u - v$ detour monophonic path for $u, v \in N'$. If $x = v_1$ then v_1 does not lies on any detour monophonic path in N'. Thus no such vertex x exists. This is a contradiction. Therefore $\gamma^+_{dm}(G) \geq q$.

To prove maximum cardinality of N, suppose there exists a minimal DMD set S with $|S| > q$. Since S contains T, the set of all extreme vertices, the vertex x lies on some $u-v$ detour monophonic path for all $x \in \{u, v_2, v_3, .., v_{r-q-1}\}$. Now S is a minimal DMD set having more than q vertices and $u, v_2, v_3, ..., v_{r-q-1} \notin S$. Therefore $S = \{v_1\} \cup M_3 \cup \{u_1\} \cup T$. Then N is properly contained in S. This is a contradiction. Therefore $\gamma^+_{dm}(G) = q$. Hence the proof. ∎

References

[1] P. A. P. Sudhahar, M. M. A. Khayyoom and A. Sadiquali, "Edge Monophonic Domination Number of Graphs". J.Adv.in Mathematics, vol. 11, no. 10, pp. 5781–5785, 2016.

[2] P. A. P. Sudhahar, M. M. A. Khayyoom and A. Sadiquali, "The Connected Edge Monophonic Domination Number of Graphs". Int. J Comp.Applications, vol. 145, no. 12, pp. 18–21, 2016.

[3] G. Chartrand and P. Zhang, *Introduction to Graph Theory*. MacGraw Hill, 2005.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundementals of Domination in Graphs*. 208, Marcel Dekker Inc, New York, 1998.

[5] J. Jhon and P. A. P. Sudhahar, "On The Edge Monophonic Number of a Graph Filomat", vol. 26, no. 6, pp. 1081–1089, 2012.

[6] J.Jhon and P.Arul Paul Sudhahar, "The Monophonic Domination Number of a Graph, Proceedings of the International Conference on Mathematics and Business Managment", pp. 142–145, 2012.

[7] M. M. A. Khayyoom and P. A. P. Sudhahar. "Edge Detour Monophonic Domination Number of a Graph. International Journal of Pure and Applied Mathematics", vol. 120, no. 7, pp. 195–203, 2018.

[8] M. M. A. Khayyoom and P. A. P. Sudhahar, "Connected Detour Monophonic Domination Number of a Graph". Global Journal of Pure and Applied Mathematics, vol. 13, no. 5, pp. 241–249, 2017.

[9] S. R. Chellathurai, and S. Padma Vijaya, "Upper Geodetic Domination Number of a Graph" Int. Journal of Cont. Math Sci., vol. 10, no. 1, pp. 23–36, 2015.

[10] P. Titus, A. P. Santhakumaran, K. Ganesamoorthy, "Upper Detour Monophonic Number of a Graph", Electronic Note in Discrete Mathematics, vol. 53, pp. 331–342, 2016.