

Existence and Attractivity Theorems for Nonlinear Hybrid Fractional Integrodifferential Equations with Anticipation and Retardation

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ABSTRACT

In this paper, we establish the existence and a global attractivity results for a nonlinear mixed quadratic and linearly perturbed hybrid fractional integrodifferential equation of second type involving the Caputo fractional derivative on unbounded intervals of real line with the mixed arguments of anticipations and retardation. The hybrid fixed point theorem of Dhage is used in the analysis of our nonlinear fractional integrodifferential problem. A positivity result is also obtained under certain usual natural conditions. Our hypotheses and claims have also been explained with the help of a natural realization.

RESUMEN

En este artículo, se establecen resultados de existencia y de atractividad global para una ecuación no lineal cuadrática mixta e híbrida fraccionaria integrodiferencial linealmente perturbada de segundo tipo involucrando la derivada fraccional de Caputo en intervalos no acotados de la recta real con argumentos mixtos de anticipación y retardo. El teorema de punto fijo híbrido de Dhage es usado en el análisis de nuestro problema no lineal fraccionario integrodiferencial. También se obtiene un resultado de positividad bajo ciertas condiciones naturales usuales. Nuestras hipótesis y afirmaciones también se explican con la ayuda de una realización natural.

Keywords and Phrases: Hybrid fractional integrodifferential equation, Dhage fixed point theorem, Existence theorem, Attractivity of solutions, Asymptotic stability.

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1 Introduction

Let $t_0 \in \mathbb{R}$ be a fixed real number and let $J_\infty = [t_0, \infty)$ be a closed but unbounded interval in \mathbb{R} . Let $\mathcal{CRB}(J_\infty)$ denote the class of **pulling functions** $a : J_\infty \rightarrow (0, \infty)$ satisfying the following properties:

- (i) a is continuous, and
- (ii) $\lim_{t \rightarrow \infty} a(t) = \infty$.

The notion of the pulling function is introduced in Dhage [15, 17] and Dhage *et al.* [21]. There do exist functions $a : J_\infty \rightarrow (0, \infty)$ satisfying the above two conditions. In fact, if $a_1(t) = |t| + 1$, $a_2(t) = e^{|t|}$, then $a_1, a_2 \in \mathcal{CRB}(J_\infty)$. Again, the class of continuous and strictly monotone functions $a : J_\infty \rightarrow (0, \infty)$ going to ∞ satisfy the above criteria. Note that if $a \in \mathcal{CRB}(J_\infty)$, then the reciprocal function $\bar{a} : J_\infty \rightarrow \mathbb{R}_+$ defined by $\bar{a}(t) = \frac{1}{a(t)}$ is continuous and $\lim_{t \rightarrow \infty} \bar{a}(t) = 0$. It has been shown in Dhage [16, 18, 19, 20] and Dhage *et al.* [21] that the pulling functions are useful in proving different asymptotic characterizations of the solutions of nonlinear differential and integral equations. In this paper we employ the pulling functions for characterizing the solutions of a nonlinear hybrid fractional differential equation when the value of independent variable is large.

It is now well-known that several anomalous real world problems in sciences and engineering are adequately modelled on fractional differential equations (see Hilfer [25] and Kilbas *et al.* [27]). Sometimes one may be interested in the behaviour of the anomalous dynamic system in the long duration of time which depend upon both past history as well as the future data of the process in question. In such cases, we take help of fractional differential equations with retardatory and anticipatory arguments on the unbounded intervals of real line. Motivated by this reason, in this paper we discuss asymptotic behaviour of a nonlinear hybrid fractional integrodifferential equation with retardation and anticipation on the unbounded intervals via hybrid fixed point theory of Dhage [8, 9, 16].

We need the following fundamental definitions from fractional calculus (see Podlubny [28], Kilbas *et al.* [27] and references therein) in what follows.

Definition 1.1. *If $J_\infty = [t_0, \infty)$ be an interval of the real line \mathbb{R} for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, then for any $x \in L^1(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional integral of fractional order $q > 0$ is defined as*

$$I_{t_0}^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t \frac{x(s)}{(t-s)^{1-q}} ds, \quad t \in J_\infty,$$

provided the right hand side is pointwise defined on (t_0, ∞) , where Γ is the Euler's gamma function defined by $\Gamma(q) = \int_0^\infty e^{-t} t^{q-1} dt$.

Definition 1.2. If $x \in C^n(J_\infty, \mathbb{R})$, then the Caputo fractional derivative ${}^C D_{t_0}^q x$ of x of fractional order q is defined as

$${}^C D_{t_0}^q x(t) = \frac{1}{\Gamma(n - q)} \int_{t_0}^t (t - s)^{n-q-1} x^{(n)}(s) ds, \quad t \in J_\infty,$$

where $n - 1 < q \leq n$, $n = [q] + 1$, $[q]$ denotes the integer part of the real number q , and Γ is the Euler's gamma function. Here $C^n(J_\infty, \mathbb{R})$ denotes the space of real valued functions $x(t)$ which are n times continuously differentiable on J_∞ .

Given a pulling function $a \in \mathcal{CRB}(J_\infty) \cap C^1(J_\infty, \mathbb{R})$, we consider the following nonlinear hybrid fractional integrofractional differential equation (in short HFRIGDE) involving the Caputo fractional derivative,

$${}^C D_{t_0}^q \left[\begin{array}{l} a(t)x(t) - \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t))) \\ \frac{\quad}{f(t, x(t), x(\theta(t)))} \end{array} \right] = g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \quad \left. \begin{array}{l} \\ x(t_0) = x_0, \end{array} \right\} \quad (1.1)$$

where ${}^C D_{t_0}^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$, I^{α_j} are the Riemann-Louville fractional integration of fractional order $\alpha_j \geq 0$ for $j = 1, \dots, m$, $f : J_\infty \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$, $h_j : J_\infty \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, $g : J_\infty \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is Carathéodory and $\eta, \theta, \gamma : J_\infty \rightarrow J_\infty$ are the continuous functions such that η and θ are anticipatory and γ is retardatory, that is, $\eta(t) \geq t$, $\theta(t) \geq t$ and $\gamma(t) \leq t$ for all $t \in J_\infty$ with $\eta(t_0) = t_0 = \theta(t_0)$.

Definition 1.3. By a solution for the hybrid fractional differential equation (1.1) we mean a function $x \in C^1(J_\infty, \mathbb{R})$ such that

- (i) the map $(x, y, z) \mapsto \frac{a(t)x - \sum_{j=1}^m I^{\alpha_j} h_j(t, x, z)}{f(t, x, y)}$ is well defined for each $t \in J_\infty$,
- (ii) the map $t \mapsto \frac{a(t)x(t) - \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\theta(t)))}{f(t, x(t), x(\theta(t)))} = z(t)$ is differentiable on J_∞ and $z' \in C(J_\infty, \mathbb{R})$, and
- (iii) x satisfies the equations in (1.1) on J_∞ ,

where $C^1(J_\infty, \mathbb{R})$ is the space of continuous real-valued functions defined on J_∞ whose first derivative x' exists and $x' \in C(J_\infty, \mathbb{R})$.

As the functions θ and γ in the HFRIGDE (1.1) are respectively anticipatory and retardatory, the arguments in the problem (1.1) are deviating over the unbounded interval J_∞ . Therefore, the behaviour of the dynamic system modelled on the HFRIGDE (1.1) depends upon both back

history as well as future data. As a result the existence analysis of the HFRIGDE (1.1) involves both anticipation and retardation information of the state variable. In a nutshell, the HFRIGDE (1.1) is a nonlinear problem with anticipation and retardation.

The HFRIGDE (1.1) is a mixed linear and quadratic perturbation of second type obtained by multiplying the unknown function under Caputo derivative with a scalar function a together with a subtraction of the term containing unknown function and dividing by a nonlinearity f . The classification of the different types of perturbations of a differential equation is given in Dhage [6]. When $h_j \equiv h$ on $J_\infty \times \mathbb{R} \times \mathbb{R}$, the HFRIGDE (1.1) reduces to the nonlinear ordinary quadratic Caputo fractional differential equation,

$$\left. \begin{aligned} {}^C D_{t_0}^q \left[\frac{a(t)x(t) - h(t, x(t), x(\eta(t)))}{f(t, x(t), x(\theta(t)))} \right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned} \right\} \quad (1.2)$$

which again, when $h_j \equiv 0$, includes the class of the nonlinear quadratic Caputo fractional differential equations

$$\left. \begin{aligned} {}^C D_{t_0}^q \left[\frac{a(t)x(t)}{f(t, x(t), x(\theta(t)))} \right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned} \right\} \quad (1.3)$$

as a special case. The HFRIGDE (1.2) is new to the literature whereas the HFRIGDE (1.3) is studied in Dhage [18] for existence and attractivity of the solutions on unbounded interval J_∞ . When $f(t, x, y) = 1$ and $g(t, x, y) = g(t, x)$ for all $(t, x, y) \in J_\infty \times \mathbb{R} \times \mathbb{R}$, we obtain the following Caputo fractional differential equation,

$$\left. \begin{aligned} {}^C D_{t_0}^q [a(t)x(t)] &= g(t, x(t)), \quad t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned} \right\} \quad (1.4)$$

The equation (1.4) is studied in Dhage [17] for existence, uniqueness and asymptotic attractivity and stability of solutions via classical fixed point theory.

We note that when $q = 1$, the hybrid fractional differential equations (in short HFRDEs) (1.2), (1.3) and (1.4) reduce to the ordinary nonlinear hybrid differential equations,

$$\left. \begin{aligned} \frac{d}{dt} \left[\frac{a(t)x(t) - h(t, x(t), x(\eta(t)))}{f(t, x(t), x(\theta(t)))} \right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned} \right\}, \quad (1.5)$$

$$\left. \begin{aligned} \frac{d}{dt} \left[\frac{a(t)x(t)}{f(t, x(t), x(\theta(t)))} \right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned} \right\} \quad (1.6)$$

and

$$\left. \begin{aligned} \frac{d}{dt} [a(t)x(t)] &= g(t, x(t)), \quad t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned} \right\} \quad (1.7)$$

which are discussed in Dhage [17], Dhage [18] and [15] respectively. The hybrid differential equation (1.7) also includes the nonlinear differential equation treated in Burton and Furumochi [4] as the special case. Therefore the existence and attractivity results of this paper include the similar results for the ordinary nonlinear hybrid classical and fractional differential equations (1.2) through (1.7) as special cases.

Now we state a couple of well-known results fractional calculus which are helpful in transforming the Caputo fractional differential equations into Riemann-Louville fractional integral equations and vice versa.

Lemma 1.1 (Kilbas et al. [27]). *Suppose that $x \in C^n(J, \mathbb{R})$ and $q \in (n - 1, n)$, $n \in \mathbb{N}$. Then, the general solution of the fractional differential equation*

$${}^c D_{t_0}^q x(t) = 0$$

is given by

$$x(t) = c_0 + c_1(t - t_0) + c_2(t - t_0)^2 + \dots + c_{n-1}(t - t_0)^{n-1}$$

for all $t \in J$, where c_i , $i = 0, 1, \dots, n - 1$ are constants and $C^n(J, \mathbb{R})$ is the space of n times continuously differentiable real-valued functions defined on $J = [a, b]$.

Lemma 1.2. (Kilbas et al. [27, page 96]) *Let $x \in C^n(J, \mathbb{R})$ and $q > 0$. Then, we have*

$$I_{t_0}^q \left({}^c D_{t_0}^q x(t) \right) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(t_0)}{k!} (t - t_0)^k = x(t) + \sum_{k=0}^{n-1} c_k (t - t_0)^k$$

for all $t \in J = [a, b]$, where $n - 1 < q \leq n$, $n = [q] + 1$ and c_0, \dots, c_{n-1} are constants.

The converse of the above lemma is not true. It is mentioned in Kilbas et al. [27, page 95] that if $q > 0$ and $x \in C(J, \mathbb{R})$, then ${}^c D_{t_0}^q \left(I_{t_0}^q x(t) \right) = x(t)$ for all $t \in J = [a, b]$, however it has been proved recently in Cohen and Salem [1, 2] that it is not true for any continuous function on J .

Remark 1.1. *The conclusion of the above Lemmas 1.1 and 1.2 also remains true if we replace the function spaces $C^n([a, b], \mathbb{R})$ and $C([a, b], \mathbb{R})$ with the function spaces $BC^n(J_\infty, \mathbb{R})$ and $BC(J_\infty, \mathbb{R})$ respectively.*

2 Auxiliary Results

Let X be a non-empty set and let $\mathcal{T} : X \rightarrow X$. An invariant point under \mathcal{T} in X is called a fixed point of \mathcal{T} , that is, the fixed points are the solutions of the functional equation $\mathcal{T}x = x$. Any

statement asserting the existence of fixed point of the mapping \mathcal{T} is called a fixed point theorem for the mapping \mathcal{T} in X . The fixed point theorems are obtained by imposing the conditions on T or on X or on both \mathcal{T} and X . By experience, better the mapping \mathcal{T} or X , we have better fixed point principles. As we go on adding richer structure to the non-empty set X , we derive richer fixed point theorems useful for applications to different areas of mathematics and particularly to nonlinear differential and integral equations. Below we give some fixed point theorems useful in establishing the attractivity and ultimate positivity of the solutions for HFRIGDE (1.1) on unbounded intervals. Before stating these results we give some preliminaries.

Definition 2.1 (Dhage [8, 9, 10]). *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\psi(0) = 0$ is called a **D-function** on \mathbb{R}_+ . Let X be an infinite dimensional Banach space with the norm $\|\cdot\|$. A mapping $\mathcal{T} : X \rightarrow X$ is called **D-Lipschitz** if there is a D-function $\psi_{\mathcal{T}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi_{\mathcal{T}}(\|x - y\|) \quad (2.1)$$

for all $x, y \in X$.

If $\psi_{\mathcal{T}}(r) = kr$, $k > 0$, then \mathcal{T} is called **Lipschitz** with the Lipschitz constant k . In particular, if $k < 1$, then \mathcal{T} is called a **contraction** on X with the contraction constant k . Further, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then \mathcal{T} is called **nonlinear D-contraction** and the function $\psi_{\mathcal{T}}$ is called **D-function** of \mathcal{T} on X . There do exist D-functions and the commonly used D-functions are $\psi_{\mathcal{T}}(r) = kr$ and $\phi(r) = \frac{r}{1+r}$, etc. (see Banas and Dhage [3] and the references therein).

Definition 2.2. *An operator \mathcal{T} on a Banach space X into itself is called totally bounded if for any bounded subset S of X , $\mathcal{T}(S)$ is a relatively compact subset of X . If \mathcal{T} is continuous and totally bounded, then it is called completely continuous on X .*

The operator theoretic technique is a powerful method often times used in the analysis of different types of nonlinear equations. Our essential tool used in the chapter is the following hybrid fixed point theorem of Dhage [9, 16] for a quadratic operator equation involving three operators in a Banach algebra X which uses arguments from analysis and topology. See also Dhage [6, 7, 9, 16] and Dhage and O'Regan [22] for some related results and applications.

Theorem 2.1 (Dhage fixed point theorem [9, 16]). *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $\mathcal{A}, \mathcal{C} : X \rightarrow X$ and $\mathcal{B} : S \rightarrow X$ be three operators such that*

- (a) \mathcal{A} and \mathcal{C} are D-Lipschitz with D-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively,
- (b) \mathcal{B} is completely continuous,

- (c) $M_{\mathcal{B}}\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, where $M_{\mathcal{B}} = \|\mathcal{B}(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\}$, and
- (d) $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x \implies x \in S$ for all $y \in S$.

Then the operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$ has a solution in S .

The above hybrid fixed point theorem of Dhage is a fifth important operator theoretic technique or tool that used in the subject of nonlinear analysis in line with Banach, Schauder, Krasnoselskii and Dhage (see [23],[5]). The nonlinear alternatives related to Dhage fixed point theorem, Theorem 2.1 on the lines of Leray-Schauder and Schafer are also available in the literature (see Dhage [7, 8, 9, 10] and references therein), however the present version is more convenient to apply in the theory of nonlinear hybrid differential equations. A collection of a good number of applicable fixed point theorems may be found in the monographs of Granas and Dugundji [23], Deimling [5], Dhage [16] and the references therein. In the following section we give different types of characterizations of the solutions for nonlinear fractional integrodifferential equations on unbounded intervals of the real line.

3 Characterizations of Solutions

We seek solutions of the HFRIGDE (1.1) in the function space $BC(J_{\infty}, \mathbb{R})$ of continuous and bounded real-valued functions defined on J_{∞} . Define a standard supremum norm $\|\cdot\|$ and a multiplication “ \cdot ” in $BC(J_{\infty}, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J_{\infty}} |x(t)|$$

and

$$(x \cdot y)(t) = (xy)(t) = x(t)y(t), \quad t \in J_{\infty}.$$

Clearly, $BC(J_{\infty}, \mathbb{R})$ becomes a Banach algebra w.r.t. the above norm and the multiplication. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} : BC(J_{\infty}, \mathbb{R}) \rightarrow BC(J_{\infty}, \mathbb{R})$ be three continuous operators and consider the following operator equation in the Banach algebra $BC(J_{\infty}, \mathbb{R})$,

$$\mathcal{A}x(t)\mathcal{B}x(t) + \mathcal{C}x(t) = x(t) \tag{3.1}$$

for all $t \in J_{\infty}$. Below we give different characterizations of the solutions for the operator equation (3.1) in the function space $BC(J_{\infty}, \mathbb{R})$.

Definition 3.1. We say that solutions of the operator equation (3.1) are **locally attractive** if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J_{\infty}, \mathbb{R})$ for some $x_0 \in BC(J_{\infty}, \mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.1) belonging to $\overline{B}_r(x_0)$ we have that

$$\lim_{t \rightarrow \infty} (x(t) - y(t)) = 0. \tag{3.2}$$

In the case when the limit (3.2) is uniform with respect to the set $\overline{B}_r(x_0)$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leq \varepsilon \quad (3.3)$$

for all $x, y \in \overline{B}_r(x_0)$ being solutions of ((3.1) and for $t \geq T$, we will say that solutions of equation (3.1) are **uniformly locally attractive** on J_∞ .

Definition 3.2. A solution $x = x(t)$ of equation (3.1) is said to be **globally attractive** if (3.2) holds for each solution $y = y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$. In other words, we may say that solutions of the equation (3.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$, the condition (3.2) is satisfied. In the case when the condition (3.2) is satisfied uniformly with respect to the space $BC(J_\infty, \mathbb{R})$, i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (3.2) is satisfied for all $x, y \in BC(J_\infty, \mathbb{R})$ being the solutions of (3.1) and for $t \geq T$, we will say that solutions of the equation (3.1) are **uniformly globally attractive** on J_∞ .

Remark 3.1. Let us mention that the details of the global attractivity of solutions may be found in a recent paper of Hu and Yan [26] while the concepts of uniform local and global attractivity (in the above sense) may be found in Banas and Dhage [3], Dhage [10, 12, 13] and references therein.

Now we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (3.1) in the space $BC(J_\infty, \mathbb{R})$.

Definition 3.3 (Dhage [11]). A solution x of the equation (3.1) is called **locally ultimately positive** if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J_\infty, \mathbb{R})$ for some $x_0 \in BC(J_\infty, \mathbb{R})$ such that $x \in \overline{B}_r(0)$ and

$$\lim_{t \rightarrow \infty} [|x(t)| - x(t)] = 0. \quad (3.4)$$

In the case when the limit (3.4) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$||x(t)| - x(t)| \leq \varepsilon \quad (3.5)$$

for all x being solutions of (3.1) in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we will say that solutions of equation (3.1) are **uniformly locally ultimately positive** on J_∞ .

Definition 3.4 (Dhage [13]). A solution $x \in BC(J_\infty, \mathbb{R})$ of the equation (3.1) is called **globally ultimately positive** if (3.4) is satisfied. In the case when the limit (3.5) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (3.5) is satisfied for all x being solutions of (3.1) in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we will say that solutions of equation (3.1) are **uniformly globally ultimately positive** on J_∞ .

Finally, we have the the following characterization of the asymptotic stability of the solution of the equation (3.1) on J_∞ .

Definition 3.5. A solution of the equation (3.1) is called *asymptotically stable* to t -axis or zero if $\lim_{t \rightarrow \infty} x(t) = 0$. Again, x is called *uniformly asymptotically stable* to zero if for $\epsilon > 0$ there exists a real number $T \geq t_0$ such that $|x(t)| \leq \epsilon$ for all $t \geq T$.

Remark 3.2. We note that global attractivity implies the local attractivity and uniform global attractivity implies the uniform local attractivity of the solutions for the operator equation (3.1) on J_∞ . Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation (3.1) on an unbounded interval J_∞ . However, the converse of the above two statements may not be true.

4 Attractivity and Positivity Results

Now, in this section, we discuss the attractivity results for the ordinary hybrid functional fractional integrodifferential equation (1.1) on J_∞ . We need the following definition in the sequel.

Definition 4.1. A function $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is called *Carathéodory* if

- (i) the map $t \mapsto \beta(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$, and
- (ii) the map $(x, y) \mapsto \beta(t, x, y)$ is jointly continuous for each $t \in J_\infty$.

The following lemma is often used in the study of nonlinear differential equations (see Granas *et al.* [24] and references therein).

Lemma 4.1 (Carathéodory). Let $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function. Then the map $(t, x, y) \mapsto \beta(t, x, y)$ is jointly measurable. In particular the map $t \mapsto \beta(t, x(t), y(t))$ is measurable on J_∞ for all $x, y \in C(J_\infty, \mathbb{R})$.

We need the following hypotheses in the sequel.

- (A₁) The function f is continuous and there exists a function $\ell \in BC(J_\infty, \mathbb{R}_+)$ and a constant $K > 0$ such that

$$|f(t, x_1, x_2) - f(t, x_1, y_2)| \leq \frac{\ell(t) \max\{|x_1 - x_2|, |x_2 - y_2|\}}{K + \max\{|x_1 - x_2|, |x_2 - y_2|\}}$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, $\sup_{t \in J_\infty} \ell(t) = L$.

- (A₂) The function $t \mapsto |f(t, 0, 0)|$ is bounded with bound F .

- (B₁) The function g is Carathéodory and bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound M_g .
- (C₁) The functions h_j are continuous and there exist a functions $\ell_j \in BC(J_\infty, \mathbb{R}_+)$ and a constants $K_j > 0$ such that

$$|h_j(t, x_1, x_2) - h_j(t, y_1, y_2)| \leq \frac{\ell_j(t) \max\{|x_1 - x_2|, |x_2 - y_2|\}}{K_j + \max\{|x_1 - x_2|, |x_2 - y_2|\}}$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $j = 1, \dots, m$. Moreover, $\sup_{t \in J_\infty} \ell_j(t) = L_j$.

- (C₂) The function $t \mapsto |h_j(t, 0, 0)|$ is bounded with bound H_j .
- (D₁) The pulling function a satisfies $\lim_{t \rightarrow \infty} \bar{a}(t) t^q = 0 = \lim_{t \rightarrow \infty} \bar{a}(t) t^{\alpha_j}$ for each $j = 1, \dots, m$.

Remark 4.1. If $a \in CRB(J_\infty)$, then $\bar{a} \in BC(J_\infty, \mathbb{R}_+)$ and so the number $\|\bar{a}\| = \sup_{t \in J_\infty} \bar{a}(t)$ exists. Again, since the hypothesis (D₁) holds, the function $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by the expression $w(t) = \bar{a}(t) t^q$ is continuous on J_∞ and satisfies the relation $\lim_{t \rightarrow \infty} w(t) = 0$. So the number $W = \sup_{t \geq t_0} w(t)$ exists. Similarly, the function $w_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by the expression $w_j(t) = \bar{a}(t) t^{\alpha_j}$ is continuous on J_∞ and satisfies the relation $\lim_{t \rightarrow \infty} w_j(t) = 0$ for each $j = 1, \dots, m$. Hence, the number $W_j = \sup_{t \geq t_0} w_j(t)$ exists for each $j = 1, \dots, m$.

The following lemma is useful in the sequel.

Lemma 4.2. If for any function $h \in L^1(J_\infty, \mathbb{R})$, the function $x \in BC(J_\infty, \mathbb{R})$ is a solution of the HFRIGDE

$${}^C D_{t_0}^q \left[\frac{a(t)x(t) - \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t)))}{f(t, x(t), x(\theta(t)))} \right] = h(t), \quad t \in J_\infty, \quad (4.1)$$

and

$$x(0) = x_0, \quad (4.2)$$

then x satisfies the hybrid fractional integral equation (in short HFRIE)

$$\begin{aligned} x(t) = [f(t, x(t), x(\theta(t)))] & \left(c_0 \bar{a}(t) + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} h(s) ds \right) \\ & + \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t))) \end{aligned} \quad (4.3)$$

for all $t \in J_\infty$, where $c_0 = \frac{a(t_0)x_0}{f(t_0, x_0, x_0)}$ and $x_0 \neq 0$.

Proof. Let $h \in L^1(J_\infty, \mathbb{R})$. Assume first that x is a solution of the HFRIGDE (4.1)-(4.2) defined on J_∞ and $x_0 \neq 0$. We apply the Riemann-Liouville fractional integration $I_{t_0}^q$ of fractional order q from t_0 to t on both sides of the HFRIGDE (4.1). Then, by an application Lemma 1.2, the HFRIGDE (4.1)-(4.2) is transformed into the HFRIE (4.3) on J_∞ . ■

Definition 4.2. A solution $x \in BC(J_\infty, \mathbb{R})$ of the FRIE (4.3) is called a **mild solution** of the HFRIGDE (4.1)-(4.2) defined on J_∞ .

In the following we shall deal with the mild solution of the HFRIGDE (1.1) on unbounded interval J_∞ of the real line \mathbb{R} . Our main existence and global attractivity result is as follows.

Theorem 4.1. Assume that the hypotheses $(A_1) - (A_2)$, (B_1) , $(C_1) - (C_2)$ and (D_1) hold. Further, assume that

$$(m + 1) \cdot \max \left\{ L \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}\| + \frac{M_g W}{\Gamma q} \right), \frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)} \right\} \leq \min\{K, K_1, \dots, K_m\}. \tag{4.4}$$

Then the HFRIGDE (1.1) has a mild solution and mild solutions are uniformly globally attractive defined on J_∞ .

Proof. Now, using Lemma 4.2, it can be shown that the mild solution x of the HFRIGDE (1.1) is equivalent to the nonlinear hybrid fractional integral equation (in short HFRIE)

$$x(t) = [f(t, x(t), x(\theta(t)))] \left(c_0 \bar{a}(t) + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right) + \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t))) \tag{4.5}$$

for all $t \in J_\infty$, where $c_0 = \frac{a(t_0)x_0}{f(t_0, x_0, x_0)}$. Set $X = BC(J_\infty, \mathbb{R})$ and define a closed ball $\bar{B}_r(0)$ in X centered at origin of radius r given by

$$r = (L + F) \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}\| + \frac{M_g W}{\Gamma q} \right) + \sum_{j=1}^m \frac{L_j + H_j}{\Gamma(\alpha_j)} W_j.$$

Define three operators \mathcal{A} and \mathcal{C} on X and \mathcal{B} on $\bar{B}_r(0)$ by

$$\mathcal{A}x(t) = f(t, x(t), x(\theta(t))), \quad t \in J_\infty, \tag{4.6}$$

$$\mathcal{B}x(t) = c_0 \bar{a}(t) + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds, \quad t \in J_\infty \tag{4.7}$$

and

$$\mathcal{C}x(t) = \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t))), \quad t \in J_\infty. \tag{4.8}$$

Then the HFRIE (4.5) is transformed into the operator equation as

$$\mathcal{A}x(t) \mathcal{B}x(t) + \mathcal{C}x(t) = x(t), \quad t \in J_\infty. \tag{4.9}$$

We show that the operators \mathcal{A} , \mathcal{B} and \mathcal{C} satisfy all the conditions of Theorem 2.1 on $BC(J_\infty, \mathbb{R})$. First we show that the operators \mathcal{A} , \mathcal{B} and \mathcal{C} define the mappings $\mathcal{A}, \mathcal{C} : X \rightarrow X$ and $\mathcal{B} : \overline{B}_r(0) \rightarrow X$. Let $x \in X$ be arbitrary. Obviously, $\mathcal{A}x$ is a continuous function on J_∞ . We show that $\mathcal{A}x$ is bounded on J_∞ . Thus, if $t \in J_\infty$, then we obtain:

$$\begin{aligned} |\mathcal{A}x(t)| &= |f(t, x(t), x(\theta(t)))| \\ &\leq |f(t, x(t), x(\theta(t))) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq \ell(t) \frac{\max\{|x(t)|, |x(\theta(t))|\}}{K + \max\{|x(t)|, |x(\theta(t))|\}} + F \\ &\leq L + F. \end{aligned}$$

Therefore, taking the supremum over t ,

$$\|\mathcal{A}x\| \leq L + F = N.$$

Thus $\mathcal{A}x$ is continuous and bounded on J_∞ . As a result $\mathcal{A}x \in X$. Again, we have

$$\begin{aligned} |\mathcal{C}x(t)| &\leq \left| \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t))) - \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} h_j(t, 0, 0) \right| \\ &\quad + \left| \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} h_j(t, 0, 0) \right| \\ &\leq \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} |h(t, x(t), x(\eta(t))) - h(t, 0, 0)| + \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} |h_j(t, 0, 0)| \\ &\leq \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} \frac{\ell_j(t) \max\{|x(t)|, |x(\eta(t))|\}}{K_j + \max\{|x(t)|, |x(\eta(t))|\}} + \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} H_j \\ &\leq \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} \frac{\ell_j(t) \|x\|}{K_j + \|x\|} + \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} H_j \\ &\leq \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} L_j + \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} H_j \\ &\leq \sum_{j=1}^m \frac{L_j}{\Gamma(\alpha_j)} W_j + \sum_{j=1}^m \frac{H_j}{\Gamma(\alpha_j)} W_j \\ &\leq \sum_{j=1}^m \frac{L_j + H_j}{\Gamma(\alpha_j)} W_j \end{aligned}$$

for all $t \in t_\infty$. Taking the supremum over t as $t \rightarrow \infty$, we obtain

$$\|\mathcal{C}x\| \leq \sum_{j=1}^m \frac{L_j + H_j}{\Gamma(\alpha_j)} W_j.$$

As a result $(\mathcal{C}x)$ is continuous and bounded on J_∞ . Hence, $\mathcal{C}x \in X$. Similarly, it can be shown that $\mathcal{B}x \in X$ and in particular, $\mathcal{A}, \mathcal{C} : X \rightarrow X$ and $\mathcal{B} : \overline{B}_r(0) \rightarrow X$. We show that \mathcal{A} is a Lipschitz

on X . Let $x, y \in X$ be arbitrary. Then, by hypothesis (A_1) ,

$$\begin{aligned} \|\mathcal{A}x - \mathcal{A}y\| &= \sup_{t \in J_\infty} |\mathcal{A}x(t) - \mathcal{A}y(t)| \\ &\leq \sup_{t \in J_\infty} \ell(t) \frac{\max\{|x(t) - y(t)|, |x(\theta(t)) - y(\theta(t))|\}}{K + \max\{|x(t) - y(t)|, |x(\theta(t)) - y(\theta(t))|\}} \\ &\leq \frac{L\|x - y\|}{K + \|x - y\|} \\ &= \psi_{\mathcal{A}}(\|x - y\|) \end{aligned}$$

for all $x, y \in X$. This shows that \mathcal{A} is a \mathcal{D} -Lipschitz on X with \mathcal{D} -function $\psi_{\mathcal{A}}(r) = \frac{Lr}{K + r}$.

Similarly, by hypothesis (C_1) , we have

$$\begin{aligned} \|\mathcal{C}x - \mathcal{C}y\| &= \sup_{t \in J_\infty} |\mathcal{C}x(t) - \mathcal{C}y(t)| \\ &\leq \sup_{t \in J_\infty} \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} |h_j(t, x(t), x(\eta(t))) - h_j(t, y(t), y(\eta(t)))| \\ &\leq \sup_{t \in J_\infty} \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} \frac{\ell_j(t) \max\{|x(t) - y(t)|, |x(\theta(t)) - y(\theta(t))|\}}{K_j + \max\{|x(t) - y(t)|, |x(\theta(t)) - y(\theta(t))|\}} \\ &\leq \sup_{t \in J_\infty} \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} \frac{L_j \|x - y\|}{K_j + \|x - y\|} \\ &\leq \sum_{j=1}^m \frac{\frac{L_j W_j}{\Gamma(\alpha_j)} \|x - y\|}{K_j + \|x - y\|} \\ &\leq \sum_{j=1}^m \frac{W_j}{\Gamma(\alpha_j)} \cdot \frac{L_j \|x - y\|}{K_j + \|x - y\|} \\ &\leq m \cdot \frac{\max\{\frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)}\} \|x - y\|}{\min\{K_1, \dots, K_m\} + \|x - y\|} \end{aligned}$$

This shows that \mathcal{C} is a \mathcal{D} -Lipschitz on X with \mathcal{D} -function $\psi_{\mathcal{C}}(r)$ given by

$$\psi_{\mathcal{C}}(r) = m \cdot \frac{\max\{\frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)}\} r}{\min\{K_1, \dots, K_m\} + r}.$$

Next, we show that \mathcal{B} is a completely continuous operator on $\overline{B}_r(0)$. First, we show that \mathcal{B} is continuous on $\overline{B}_r(0)$. To do this, let us fix arbitrarily $\epsilon > 0$ and let $\{x_n\}$ be a sequence of points

in $\overline{B}_r(0)$ converging to a point $x \in \overline{B}_r(0)$. Then we get:

$$\begin{aligned}
 & |(\mathcal{B}x_n)(t) - (\mathcal{B}x)(t)| \\
 & \leq \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} |g(s, x_n(s), x_n(\gamma(s))) - g(s, x(s), x(\gamma(s)))| ds \\
 & \leq \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} [|g(s, x_n(s), x_n(\gamma(s)))| + |g(s, x(s), x(\gamma(s)))|] ds \\
 & \leq 2M_g \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} ds \\
 & = \frac{2M_g}{\Gamma q} \cdot w(t),
 \end{aligned}$$

where, $w(t) = \overline{a}(t) t^q$.

Hence, by virtue of hypothesis (D₁), we infer that there exists a $T > 0$ such that $w(t) \leq \epsilon$ for $t \geq T$. Thus, for $t \geq T$, from the estimate (3.3) we derive that

$$|(\mathcal{B}x_n)(t) - (\mathcal{B}x)(t)| \leq \frac{2M_g}{\Gamma q} \epsilon \quad \text{as } n \rightarrow \infty.$$

Furthermore, let us assume that $t \in [t_0, T]$. Then, by dominated convergence theorem, we obtain the estimate:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \mathcal{B}x_n(t) &= \lim_{n \rightarrow \infty} \left[c_0 \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x_n(s), x_n(\gamma(s))) ds \right] \\
 &= c_0 \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} \left[\lim_{n \rightarrow \infty} g(s, x_n(s), x_n(\gamma(s))) \right] ds \\
 &= c_0 \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \\
 &= \mathcal{B}x(t)
 \end{aligned}$$

for all $t \in [t_0, T]$. Moreover, it can be shown as below that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in X . Now, following the arguments similar to that given in Granas *et al.* [23], it is proved that \mathcal{B} is a continuous operator on $\overline{B}_r(0)$.

Next, we show that \mathcal{B} is a compact operator on $\overline{B}_r(0)$. To finish, it is enough to show that every sequence $\{\mathcal{B}x_n\}$ in $\mathcal{B}(\overline{B}_r(0))$ has a Cauchy subsequence. Now, proceeding with the earlier arguments it is proved that

$$\|\mathcal{B}x_n\| \leq |c_0| \|\overline{a}\| + \frac{M_f W}{\Gamma q} = r$$

for all $n \in \mathbb{N}$. This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in $\mathcal{B}(\overline{B}_r(0))$.

Next, we show that $\{\mathcal{B}x_n\}$ is also an equicontinuous sequence in $\mathcal{B}(\overline{B}_r(0))$. Let $\epsilon > 0$ be given. Since $\lim_{t \rightarrow \infty} w(t) = 0$, there is a real number $T_1 > t_0 \geq 0$ such that $|w(t)| < \frac{\epsilon}{8M_f/\Gamma(q)}$ for all

$t \geq T_1$. Similarly, since $\lim_{t \rightarrow \infty} \bar{a}(t) = 0$, for above $\epsilon > 0$, there is a real number $T_2 > t_0 \geq 0$ such that $|\bar{a}(t)| < \frac{\epsilon}{8|c_0|}$ for all $t \geq T_2$. Thus, if $T = \max\{T_1, T_2\}$, then

$$|w(t)| < \frac{\epsilon}{8M_f/\Gamma(q)} \quad \text{and} \quad |\bar{a}(t)| < \frac{\epsilon}{8|c_0|} \tag{4.10}$$

for all $t \geq T$. Let $t, \tau \in J_\infty$ be arbitrary. If $t, \tau \in [t_0, T]$, then we have

$$\begin{aligned} & |\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \\ & \leq |c_0| |\bar{a}(t) - \bar{a}(\tau)| \\ & + \left| \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} f(s, x(s)) ds - \frac{\bar{a}(\tau)}{\Gamma q} \int_{t_0}^\tau (\tau-s)^{q-1} f(s, x(s)) ds \right| \\ & \leq |c_0| |\bar{a}(t) - \bar{a}(\tau)| \\ & + \left| \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} f(s, x(s)) ds - \frac{\bar{a}(\tau)}{\Gamma q} \int_{t_0}^t (\tau-s)^{q-1} f(s, x(s)) ds \right| \\ & + \left| \frac{\bar{a}(\tau)}{\Gamma q} \int_{t_0}^t (\tau-s)^{q-1} f(s, x(s)) ds - \frac{\bar{a}(\tau)}{\Gamma q} \int_{t_0}^\tau (\tau-s)^{q-1} f(s, x(s)) ds \right| \\ & \leq |c_0| |\bar{a}(t) - \bar{a}(\tau)| \\ & + \frac{M_f}{\Gamma q} \int_{t_0}^t |\bar{a}(t)(t-s)^{q-1} - \bar{a}(\tau)(\tau-s)^{q-1}| ds \\ & + \frac{M_f}{\Gamma q} \left| \int_\tau^t |\bar{a}(\tau)(\tau-s)^{q-1}| ds \right| \\ & \leq |c_0| |\bar{a}(t) - \bar{a}(\tau)| \\ & + \frac{M_f}{\Gamma q} \int_{t_0}^T |\bar{a}(t)(t-s)^{q-1} - \bar{a}(\tau)(\tau-s)^{q-1}| ds \\ & + \frac{M_f \|\bar{a}\|}{\Gamma q} |(t-\tau)^q|. \end{aligned}$$

Since the functions $t \mapsto \bar{a}(t)$ and $t \mapsto \bar{a}(t)(t-s)^{q-1}$ are continuous on compact $[t_0, T]$, they are uniformly continuous there. Therefore, by the uniform continuity, for above ϵ we have the real numbers $\delta_1 > 0$ and $\delta_2 > 0$ depending only on ϵ such that

$$|t - \tau| < \delta_1 \implies |\bar{a}(t) - \bar{a}(\tau)| < \frac{\epsilon}{9|c_0|}$$

and

$$|t - \tau| < \delta_2 \implies |\bar{a}(t)(t-s)^{q-1} - \bar{a}(\tau)(\tau-s)^{q-1}| < \frac{\epsilon}{9M_f T/\Gamma q}.$$

Similarly, choose the real number $\delta_3 = \left(\frac{\epsilon}{9M_f \|\bar{a}\|/\Gamma(q)} \right)^{1/q} > 0$ so that

$$|t - \tau| < \delta_3 \implies |(t-\tau)^q| < \frac{\epsilon}{9M_f \|\bar{a}\|/\Gamma(q)}.$$

Let $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$|t - \tau| < \delta_4 \implies |\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| < \frac{\epsilon}{3}$$

for all $n \in \mathbb{N}$. Again, if $t, \tau > T$, then we have a $\delta_5 > 0$ depending only on ϵ such that

$$\begin{aligned} & |\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \\ & \leq |c_0| |a(t) - a(\tau)| + \frac{\bar{a}(t)}{\Gamma q} \left| \int_{t_0}^t (t-s)^{q-1} f(s, x_n(s)) ds \right| \\ & \quad + \frac{\bar{a}(\tau)}{\Gamma q} \left| \int_{t_0}^{\tau} (\tau-s)^{q-1} f(s, x_n(s)) ds \right| \\ & \leq |c_0| [|\bar{a}(t)| + |\bar{a}(\tau)|] + \frac{M_f}{\Gamma(q)} [w(t) + w(\tau)] \\ & < \frac{\epsilon}{2} < \epsilon \end{aligned}$$

for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta_5$. Similarly, if $t, \tau \in \mathbb{R}_+$ with $t < T < \tau$, then we have

$$|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \leq |\mathcal{B}x_n(t) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(\tau)|.$$

Take $\delta = \min\{\delta_4, \delta_5\} > 0$ depending only on ϵ . Therefore, from the above obtained estimates, it follows that

$$|\mathcal{B}x_n(t) - \mathcal{B}x_n(T)| < \frac{\epsilon}{2} \text{ and } |\mathcal{B}x_n(T) - \mathcal{B}x_n(\tau)| < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta$. As a result, $|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| < \epsilon$ for all $t, \tau \in J_\infty$ and for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta$. This shows that $\{\mathcal{B}x_n\}$ is a equicontinuous sequence in X . Now an application of Arzelà-Ascoli theorem yields that $\{\mathcal{B}x_n\}$ has a uniformly convergent subsequence on the compact subset $[t_0, T]$ of J_∞ . Without loss of generality, call the subsequence to be the sequence itself. We show that $\{\mathcal{B}x_n\}$ is Cauchy in X . Now $|\mathcal{B}x_n(t) - \mathcal{B}x(t)| \rightarrow 0$ as $n \rightarrow \infty$ for all $t \in [t_0, T]$. Then for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{t_0 \leq t \leq T} \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} |f(s, x_m(s)) - f(s, x_n(s))| ds < \frac{\epsilon}{2}$$

for all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$, then we have

$$\begin{aligned} & \|\mathcal{B}x_m - \mathcal{B}x_n\| \\ & = \sup_{t_0 \leq t < \infty} \left| \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} |f(s, x_m(s)) - f(s, x_n(s))| ds \right| \\ & \leq \sup_{t_0 \leq t \leq T} \left| \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} |f(s, x_m(s)) - f(s, x_n(s))| ds \right| \\ & \quad + \sup_{t \geq T} \left| \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} [|f(s, x_m(s))| + |f(s, x_n(s))|] ds \right| \\ & < \epsilon. \end{aligned}$$

This shows that $\{\mathcal{B}x_n\} \subset \mathcal{B}(\overline{B}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{\mathcal{B}x_n\}$ converges to a point in X . As $\mathcal{B}(\overline{B}_r(0))$ is closed, we have that $\{\mathcal{B}x_n\}$ converges to a point in $\mathcal{B}(\overline{B}_r(0))$. Hence $\mathcal{B}(\overline{B}_r(0))$ is relatively compact and consequently \mathcal{B} is a continuous and compact operator on $B_r(0)$ into itself.

Next, we estimate the value of the constant $M_{\mathcal{B}}$ of the hypothesis (c) of the Theorem 2.1. By definition of $M_{\mathcal{B}}$, one has

$$\begin{aligned} \|\mathcal{B}(B_r(0))\| &= \sup\{\|\mathcal{B}x\| : x \in \overline{B}_r(0)\} \\ &= \sup\left\{ \sup_{t \in J_\infty} |\mathcal{B}x(t)| : x \in \overline{B}_r(0) \right\} \\ &\leq \sup_{x \in \overline{B}_r(0)} \left\{ \sup_{t \in J_\infty} \left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| |\overline{a}(t)| \right. \\ &\quad \left. + \frac{1}{\Gamma q} \cdot \sup_{t \in J_\infty} |\overline{a}(t)| \int_{t_0}^t (t-s)^{q-1} |g(s, x(s), x(\gamma(s)))| ds \right\} \\ &\leq \left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\overline{a}\| + \frac{M_g}{\Gamma q} \cdot \sup_{t \in J_\infty} \overline{a}(t) \int_{t_0}^t (t-s)^{q-1} ds \\ &\leq \left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\overline{a}\| + \frac{M_g}{\Gamma q} \cdot \sup_{t \in J_\infty} \overline{a}(t) t^q \\ &\leq \left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} = M_{\mathcal{B}}. \end{aligned}$$

Thus,

$$\|\mathcal{B}x\| \leq \left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} = M_{\mathcal{B}}$$

for all $x \in \overline{B}_r(0)$. Hence, we have

$$\begin{aligned} &M_{\mathcal{B}}\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) \\ &\leq \frac{L \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} \right) r}{K + r} \\ &\quad + m \cdot \frac{\max\left\{ \frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)} \right\} r}{\min\{K_1, \dots, K_m\} + r} \\ &\leq (m + 1) \cdot \frac{\max\left\{ L \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} \right), \frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)} \right\} r}{\min\{K, K_1, \dots, K_m\} + r} \\ &< r \end{aligned}$$

for $r > 0$, because

$$\begin{aligned} &(m + 1) \cdot \max\left\{ L \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} \right), \frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)} \right\} \\ &\leq \min\{K, K_1, \dots, K_m\} \end{aligned}$$

Therefore, hypothesis (c) of Theorem 2.1 is satisfied.

Next, let $y \in \overline{B}_r(0)$ be arbitrary and let $x = \mathcal{A}x \mathcal{B}y + \mathcal{C}x$. Then,

$$\begin{aligned}
 |x(t)| &\leq |\mathcal{A}x(t)| |\mathcal{B}y(t)| + |\mathcal{C}x(t)| \\
 &\leq \|\mathcal{A}x\| \|\mathcal{B}y\| + \|\mathcal{C}x\| \\
 &\leq \|\mathcal{A}(X)\| \|\mathcal{B}(\overline{B}_r(0))\| + \|\mathcal{C}(X)\| \\
 &\leq (L + F) M_B + \sum_{j=1}^m \frac{L_j + H_j}{\Gamma(\alpha_j)} W_j \\
 &\leq (L + F) \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}\| + \frac{M_g W}{\Gamma_q} \right) + \sum_{j=1}^m \frac{L_j + H_j}{\Gamma(\alpha_j)} W_j
 \end{aligned}$$

for all $t \in J_\infty$. Therefore, we have:

$$\|x\| \leq (L + F) \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}\| + \frac{M_g W}{\Gamma_q} \right) + \sum_{j=1}^m \frac{L_j + H_j}{\Gamma(\alpha_j)} W_j = r.$$

This shows that $x \in \overline{B}_r(0)$ and hypothesis (c) of Theorem 2.1 is satisfied. Now we apply Theorem 2.1 to the operator equation $\mathcal{A}x \mathcal{B}x + \mathcal{C}x = x$ to yield that the HFRIGDE (1.1) has a mild solution on J_∞ . Moreover, the mild solutions of the HFRIGDE (1.1) are in $\overline{B}_r(0)$. Hence, mild solutions are global in nature.

Finally, let $x, y \in \overline{B}_r(0)$ be any two mild solutions of the HFRIGDE (1.1) on J_∞ . Then, from

(4.5) we obtain

$$\begin{aligned}
 |x(t) - y(t)| &\leq \left| [f(t, x(t), x(\theta(t)))] \times \right. \\
 &\quad \times \left(\frac{a(t_0)x_0\bar{a}(t)}{f(t_0, x_0, x_0)} + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right) \\
 &\quad - [f(t, y(t), y(\theta(t)))] \times \\
 &\quad \times \left(\frac{a(t_0)x_0\bar{a}(t)}{f(t_0, x_0, x_0)} + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, y(s), y(\gamma(s))) ds \right) \left| \right. \\
 &\quad + \sup_{t \in J_\infty} \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} |h_j(t, x(t), x(\eta(t))) - h_j(t, y(t), y(\eta(t)))| \\
 &\leq \left| [f(t, x(t), x(\theta(t))) - f(t, y(t), y(\theta(t)))] \right. \\
 &\quad \times \left(\frac{a(t_0)x_0\bar{a}(t)}{f(t_0, x_0, x_0)} + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right) \left| \right. \\
 &\quad + | f(t, y(t), y(\theta(t))) | \times \\
 &\quad \times \left| \left(\frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds - g(s, y(s), y(\gamma(s))) ds \right) \right| \\
 &\quad + \sup_{t \in J_\infty} \bar{a}(t) \sum_{j=1}^m I^{\alpha_j} \frac{L_j \|x - y\|}{K_j + \|x - y\|} \\
 &\leq |f(t, x(t), x(\theta(t))) - f(t, y(t), y(\theta(t)))| \times \\
 &\quad \times \left| \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}(t)\| + \frac{M_g W}{\Gamma q} w(t) \right) \right| \\
 &\quad + 2[|f(t, x(t), x(\theta(t))) - f(t, 0, 0)| + |f(t, 0, 0)|] \frac{M_g W}{\Gamma q} w(t) \\
 &\quad + \sum_{j=1}^m \frac{w_j(t)}{\Gamma(\alpha_j)} \cdot \frac{L_j \|x - y\|}{K_j + \|x - y\|} \\
 &\leq \ell(t) \frac{|x(t) - y(t)|}{K + |x(t) - y(t)|} \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}\| + \frac{M_g W}{\Gamma q} \right) \\
 &\quad + \frac{2 M_g W}{\Gamma q} \left[\frac{\ell(t) \max\{|x(t)|, |x(\theta(t))|\}}{K + \max\{|x(t)|, |x(\theta(t))|\}} + F \right] w(t) \\
 &\quad + \sum_{j=1}^m \frac{L_j w_j(t)}{\Gamma(\alpha_j)} \\
 &\leq \frac{L \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}\| + \frac{M_g W}{\Gamma q} \right) |x(t) - y(t)|}{K + |x(t) - y(t)|} \\
 &\quad + \frac{2 M_g W}{\Gamma q} (L + F) w(t) + \sum_{j=1}^m \frac{L_j w_j(t)}{\Gamma(\alpha_j)}. \tag{4.11}
 \end{aligned}$$

Taking the limit superior as $t \rightarrow \infty$ in the above inequality (4.11) yields, $\lim_{t \rightarrow \infty} |x(t) - y(t)| = 0$. Therefore, there is a real number $T > 0$ such that $|x(t) - y(t)| < \epsilon$ for all $t \geq T$. Consequently, the mild solutions of HFRIGDE (1.1) are uniformly globally attractive on J_∞ . This completes the proof. ■

Remark 4.2. The conclusion of Theorem 4.1 also remains true under if we replace the hypotheses (A_1) , (A_2) , (C_1) and (C_2) with the following modified conditions:

(A'_1) The function f is continuous and there exists a \mathcal{D} -function $\psi_f \in \mathfrak{D}$ such that

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \psi_f(\max\{|x_1 - y_1|, |x_2 - y_2|\})$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(A'_2) The function f is bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound M_f .

(C'_1) The functions h'_j 's are continuous and there exist \mathcal{D} -functions $\psi_{h_j} \in \mathfrak{D}$ such that

$$|h_j(t, x_1, x_2) - h_j(t, y_1, y_2)| \leq \psi_{h_j}(\max\{|x_1 - y_1|, |x_2 - y_2|\})$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $j = 1, \dots, m$.

(C'_2) The functions h_j are bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound M_{h_j} .

Theorem 4.2. Assume that the hypotheses (A'_1) - (A'_2) , (B_1) , (C'_1) - (C'_2) and (D_1) hold. Furthermore, assume that

$$\left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \|\bar{a}\| + \frac{M_g W}{\Gamma q} \right) \psi_f(r) + \sum_{j=1}^m \frac{W_j}{\Gamma(\alpha_j)} \psi_{h_j}(r) < r, \quad r > 0. \quad (4.12)$$

Then the HFRIGDE (1.1) has a mild solution and mild solutions are uniformly globally attractive defined on J_∞ .

Proof. The proof is similar to Theorem 4.1 and hence we omit the details. ■

Theorem 4.3. Assume that the hypotheses (A_1) - (A_2) , (B_1) , (C_1) - (C_2) and (D_1) hold. Then the HFRIGDE (1.1) has a mild solution and mild solutions are uniformly globally attractive and ultimately positive defined on J_∞ .

Proof. By Theorem 4.1, the HFRIGDE (1.1) has a global mild solution in the closed ball $\bar{B}_r(0)$, where the radius r is given as in the proof of Theorem 4.1, and the mild solutions are uniformly globally attractive on J_∞ . We know that for any $x, y \in \mathbb{R}$, one has the inequality,

$$|x| |y| = |xy| \geq xy,$$

and therefore,

$$||xy| - (xy)| \leq |x| ||y| - y| + |x| - x| |y| \tag{4.13}$$

for all $x, y \in \mathbb{R}$. Now, for any mild solution x of the HFRIGDE (1.1) in $\overline{B}_r(0)$, one has

$$\begin{aligned} & ||x(t)| - x(t)| \\ &= \left| \left| f(t, x(t), x(\theta(t))) \right| \times \right. \\ &\quad \times \left| \left(\frac{a(t_0)x_0\bar{a}(t)}{f(t_0, x_0, x_0)} + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right) \right| \\ &\quad - \left| f(t, x(t), x(\theta(t))) \right| \times \\ &\quad \times \left(\frac{a(t_0)x_0\bar{a}(t)}{f(t_0, x_0, x_0)} + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right) \left| \right. \\ &\quad \left. + \left| \sum_{j=1}^m I^{\alpha_j} |h_j(t, x(t), x(\eta(t)))| - \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t))) \right| \right| \\ &\leq \left| \left| f(t, x(t), x(\theta(t))) \right| \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| - \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right) \bar{a}(t) \right| \\ &\quad + \left| \left| f(t, x(t), x(\theta(t))) \right| \times \right. \\ &\quad \times \left| \left| \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right| \right. \\ &\quad \quad \left. - \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right| \\ &\quad \left. + \left| \left| f(t, x(t), x(\theta(t))) \right| - f(t, x(t), x(\theta(t))) \right| \right. \\ &\quad \times \left| \frac{a(t_0)x_0\bar{a}(t)}{f(t_0, x_0, x_0)} + \frac{\bar{a}(t)}{\Gamma q} \int_{t_0}^t (t-s)^{q-1} g(s, x(s), x(\gamma(s))) ds \right| \\ &\quad \left. + \sum_{j=1}^m I^{\alpha_j} \left| |h_j(t, x(t), x(\eta(t)))| - h_j(t, x(t), x(\eta(t))) \right| \right| \\ &\leq \left[4(L + F) \left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \right] \bar{a}(t) + \left[4(L + F) \frac{M_g}{\Gamma q} \right] w(t) \\ &\quad + 2 \sum_{j=1}^m \frac{H_j}{\Gamma(\alpha_j)} w_j(t) \tag{4.14} \end{aligned}$$

for all $t \in J_\infty$.

Taking the limit superior as $t \rightarrow \infty$ in the above inequality (4.14), we obtain the estimate that $\lim_{t \rightarrow \infty} ||x(t)| - x(t)| = 0$. Therefore, there is a real number $T > 0$ such that $||x(t)| - x(t)| \leq \epsilon$ for all $t \geq T$. Hence, mild solutions of the HFRIGDE (1.1) are uniformly globally attractive as well as ultimately positive defined on J_∞ . This completes the proof. ■

Theorem 4.4. *Assume that the hypotheses (A₁) - (A₂) and (B₁) hold. Then the HFRDE (1.1) has a mild solution and mild solutions are uniformly globally attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on J_∞.*

Proof. By Theorems 4.1 and 4.2, the HFRIGDE (1.1) has a global mild solution in the closed ball $\overline{B}_r(0)$, where the radius r is given as in the proof of Theorem 4.1, and the mild solutions are uniformly globally attractive and uniformly ultimately positive on J_∞ . Now, for any mild solution $x \in \overline{B}_r(0)$, we have from (4.10),

$$|x(t)| \leq (L + F) \left(\left| \frac{a(t_0)x_0}{f(t_0, x_0, x_0)} \right| \overline{a}(t) + \frac{M_g}{\Gamma_q} w(t) \right) + \sum_{j=1}^m \frac{L_j + H_j}{\Gamma(\alpha_j)} w_j(t).$$

Taking the limit superior as $t \rightarrow \infty$ in the above inequality yields that $\lim_{t \rightarrow \infty} |x(t)| = 0$. Therefore, for $\epsilon > 0$ there exists a real number $T \geq t_0$ such that $|x(t)| < \epsilon$ whenever $t \geq T$. Consequently, the mild solution x is a uniformly asymptotically stable to zero defined on J_∞ . This completes the proof. ■

Example 4.1 Let $J_\infty = \mathbb{R}_+ = [0, \infty) \subset \mathbb{R}$. Given a pulling function $a(t) = e^t \in \mathcal{CRB}(\mathbb{R}_+)$, consider the following nonlinear hybrid fractional Caputo differential equation with the mixed arguments of anticipation and retardation,

$$\left. \begin{aligned} {}^C D_0^q \left[\frac{e^t x(t) - \frac{t}{t^2 + 1} I^{3/2} \left(\frac{|x(t)| + |x(3t)|}{4 + |x(t)| + |x(3t)|} \right)}{1 + \frac{1}{t^2 + 1} \left(\frac{|x(t)| + |x(2t)|}{2 + |x(t)| + |x(2t)|} \right)} \right] &= \frac{e^{-t} \log(1 + |x(t)| + |x(t/2)|)}{2 + |x(t)| + |x(t/2)|}, \quad t \in \mathbb{R}_+, \\ x(0) &= 0, \end{aligned} \right\} \tag{4.15}$$

for all $t \in \mathbb{R}_+$, where ${}^C D_0^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$.

Here, $a(t) = e^t$, $\theta(t) = 2t$, $\eta(t) = 3t$, $\gamma(t) = \frac{t}{2}$ for $t \in \mathbb{R}_+$ and hence $\theta(0) = 0 = \eta(0)$. Next, $\alpha = 3/2$ and the functions $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+ \setminus \{0\}$ and $g, h : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are defined by

$$f(t, x, y) = 1 + \frac{1}{t^2 + 1} \left[\frac{|x| + |y|}{2 + |x| + |y|} \right],$$

$$h(t, x, y) = \frac{t}{t^2 + 1} \left[\frac{|x| + |x|}{4 + |x| + |x|} \right]$$

and

$$g(t, x, y) = \frac{e^{-t} \log(|x| + |y|)}{1 + |x| + |y|}.$$

Clearly, the function f is continuous and bounded real function on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$ and in particular, $F = 1$. Now, it can be shown as in Banas and Dhage [3] that the function f satisfies the hypothesis (A₁) with $\ell(t) = \frac{1}{t^2 + 1}$ and $K = 1$. So we have $L = 1$. Furthermore,

the function h is also continuous and bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound $M_h = 1$. Next, the function h satisfies the hypothesis (C_1) with the function $\ell_h(t) = \frac{t}{t^2 + 1}$ so that we have $L_h = \frac{1}{2}$ and $K_h = 4$. Again, the function g is continuous and bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ and therefore, satisfies the hypotheses (B_1) with $M_g = 1$. Next, we have

$$\lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} e^{-t} t^q = 0 = \lim_{t \rightarrow \infty} e^{-t} t^{3/2} = \lim_{t \rightarrow \infty} w_h(t)$$

and so the hypothesis (D_1) is satisfied. Now, $\|\bar{a}\| = \sup_{t \in \mathbb{R}_+} e^{-t} = 1, W = \sup_{t \in \mathbb{R}_+} e^{-t} t^q = 1$ and $W_h = 1$. Finally, it is verified that the the functions a, f, g and h satisfy the condition (4.4) of Theorem 4.1. Consequently, the HFRIGDE (4.15) has a mild solution and mild solutions are globally uniformly attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on \mathbb{R}_+ . In particular, the HFRIGDE

$${}^C D_0^{2/3} \left[\frac{e^t x(t) - \frac{t}{t^2 + 1} I^{3/2} \left(\frac{|x(t)| + |x(3t)|}{4 + |x(t)| + |x(3t)|} \right)}{1 + \frac{1}{t^2 + 1} \left(\frac{|x(t)| + |x(2t)|}{2 + |x(t)| + |x(2t)|} \right)} \right] = \frac{e^{-t} \log(1 + |x(t)| + |x(t/2)|)}{2 + |x(t)| + |x(t/2)|}, t \in \mathbb{R}_+, \left. \vphantom{{}^C D_0^{2/3}} \right\} x(0) = 0,$$

has a mild solution and mild solutions are globally uniformly attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on \mathbb{R}_+ .

Remark 4.3. Finally, we remark that the ideas of this paper may be extended with appropriate modifications to a more general hybrid fractional integrodifferential equation with Caputo fractional derivative,

$${}^C D_{t_0}^q \left[\frac{a(t)x(t) - \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta_1(t)), \dots, x(\eta_m(t)))}{f(t, x(\theta_1(t)), \dots, x(\theta_n(t)))} \right] = g(t, x(\gamma_1(t)), \dots, x(\gamma_n(t))), t \in J_\infty, \left. \vphantom{{}^C D_{t_0}^q} \right\} x(t_0) = x_0 \in \mathbb{R}, \tag{4.16}$$

where ${}^C D_{t_0}^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$, Γ is a Euler's gamma function, $f : J_\infty \times \mathbb{R} \times \dots (n \text{ times}) \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$, $g, h_j : J_\infty \times \mathbb{R} \times \dots (n \text{ times}) \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and $\theta_i, \gamma_i : J_\infty \rightarrow J_\infty$ are continuous functions which are respectively anticipatory and retardatory, that is, $\theta_i(t) \geq t$ and $\gamma_i(t) \leq t$ for all $t \in J_\infty$ with $\theta_i(t_0) = t_0 = \eta_i(t_0)$ for $i = 1, \dots, n$.

Remark 4.4. If g is assumed to be continuous function on $J_\infty \times \mathbb{R} \times \mathbb{R}$, then the attractivity and existence results for the HFRIGDE (1.1) may be obtained via another approach of using measure of noncompactness. In that case we need to construct a handy tool for the measure of noncompactness which is not the case with the present approach in the qualitative study of such nonlinear fractional integrodifferential equations. See the details of this procedure that appears in Banas and Dhage [3], Hu and Yan [26], Dhage [11, 14] and the references therein.

5 The Conclusion

From the foregoing discussion, it is clear that the pulling functions and the hybrid fixed point theorems are very much useful for proving the existence theorems as well as characterizing the mild solutions of different types of nonlinear fractional integrodifferential equations on unbounded intervals of the real line when the nonlinearity is not necessarily continuous. The choices of the pulling function and the fixed point theorem depends upon the situations and the circumstances of the nonlinearities involved in the nonlinear problem. The clever selection of the fixed point theorems yields very powerful existence results as well as different characterizations of the nonlinear fractional differential equations. In this article, we have been able to prove in Theorems 4.1, 4.2, 4.3 and 4.4 the existence as well as global attractivity, ultimate positivity and asymptotic stability of the mild solutions for a quadratic type of nonlinear hybrid fractional differential equation (1.1) on the unbounded interval $J_\infty = [t_0, \infty)$ of right half of the real line \mathbb{R}_+ , however, other nonlinear fractional integrodifferential equations can be treated in the similar way for these and some other characterizations such as monotonic global attractivity, monotonic asymptotic attractivity and monotonic ultimate positivity etc. of the mild solutions on unbounded intervals of the real line. It is known that several real world phenomena in physics and chemistry such as growth and decay of the radioactive elements continue for a very long period of time and the existence results of the type proved in this paper may be applicable for the situation to understand the behavior of the process after a sufficient lapse of time. In a forthcoming paper, it is proposed to discuss the global asymptotic and monotonic attractivity of the mild solutions for nonlinear hybrid fractional integrodifferential equations involving three nonlinearities via classical and applicable hybrid fixed point theory.

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