CUBO, A Mathematical Journal
Vol.22, N°03, (361–377). December 2020
http://dx.doi.org/10.4067/S0719-06462020000300361
Received: 29 May, 2020 | Accepted: 10 November, 2020

Mild solutions of a class of semilinear fractional integro-differential equations subjected to noncompact nonlocal initial conditions

Abdeldjalil Aouane[1,3], Smaïl Djebali[2,3] and Mohamed Aziz Taoudi[4]

[1] *Département de Sciences Exactes et Informatique,*
École Normale Supérieure,
Constantine, Algeria.

[2] *Department of Mathematics, Faculty of Sciences,*
Imam Mohammad Ibn Saud Islamic University (IMSIU),
PB 90950. Riyadh 11623, Saudi Arabia.

[3] *Laboratoire Théorie du Point Fixe et Applications*
ENS, BP 92 Kouba,
Algiers, 16006. Algeria.

[4] *Cadi Ayyad University,*
National School of Applied Science
Marrakesh, Morocco.

abdeldjalilens@hotmail.com, djebali@hotmail.com, a.taoudi@uca.ma

ABSTRACT

In this paper, we prove the existence of mild solutions of a class of fractional semilinear integro-differential equations of order $\beta \in (1, 2]$ subjected to noncompact initial nonlocal conditions. We assume that the linear part generates an arbitrarily strongly continuous β-order fractional cosine family, while the nonlinear forcing term is of Carathéodory type and satisfies some fairly general growth conditions. Our approach combines the Monch fixed point theorem with some recent results regarding the measure of noncompactness of integral operators. Our conclusions improve and generalize many earlier related works. An example is provided to illustrate the main results.

RESUMEN

En este artículo, probamos la existencia de soluciones leves de una clase de ecuaciones integro-diferenciales fraccionales semilineales de orden $\beta \in (1,2]$ con condiciones no-compactas iniciales no-locales. Asumimos que la parte lineal genera una familia coseno de orden fraccional β arbitrariamente fuertemente continua , mientras que el término no-lineal de forzamiento es de tipo Carathéodory y satisface algunas condiciones de crecimiento bastante generales. Nuestro enfoque combina el teorema de punto fijo de Monch con algunos resultados recientes sobre la medida de no-compacidad de operadores integrales. Nuestras conclusiones mejoran y generalizan muchos trabajos anteriores relacionados . Se provee un ejemplo para ilustrar los resultados principales.

Keywords and Phrases: Cosine operator, fractional integro-differential operator, abstract differential equation, noncompact nonlocal condition.

2020 AMS Mathematics Subject Classification: 34A08, 34G20, 35F25, 47D09, 47D60, 47H08, 47H10, 47G20.

1 Introduction

In recent years, the investigation of fractional differential equations in Banach spaces has attracted many research works due to its applications in various areas of engineering, physics, bio-engineering, and other applied sciences. Notable contributions have been made to both theory and applications of fractional differential equations; we refer, e.g., to [1, 6, 13, 14, 15, 16, 18, 19, 25] and the references therein. Actually, it has been found that differential equations involving fractional derivatives in time are more realistic to describe many phenomena in practical situations than those of integer order. The most significant advantage of fractional derivatives compared with integer derivatives is that it can be used to describe the property of memory and heredity of various materials and processes [5, 8, 22]. For more details about fractional calculus and fractional differential equations, we refer the reader to [2, 4, 10].

In this paper, we are concerned with the existence of mild solutions of the following class of fractional semilinear integro-differential equations:

$$\begin{cases} {}^cD_t^\beta u(t) &=& Au(t) + f(t, u(t), Gu(t), Su(t)), \quad t \in [0,a], \\ u(0) &=& u_0 + q(u), \\ u'(0) &=& v_0 + p(u), \end{cases} \tag{1.1}$$

where $\beta \in (1,2]$ and ${}^cD_t^\beta$ is the standard Caputo fractional derivative of order β. The operator A is the infinitesimal generator of a strongly continuous β-order fractional cosine family $\{C_\beta(t) : t \geq 0\}$

in a Banach space E, f, q, p are suitably defined functions satisfying certain conditions to be specified later, x_0, y_0 are given elements of E and G, S are two linear operators defined by

$$Gu(t) = \int_0^t K(t,s)u(s)ds \text{ and } Su(t) = \int_0^a H(t,s)u(s)ds, \quad t \in [0,a], \tag{1.2}$$

where $H \in C\left[[0,a] \times [0,a], \mathbb{R}^+\right]$, $K \in C\left[U, \mathbb{R}^+\right]$, and

$$U = \left\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq a\right\}.$$

Here $\mathbb{R}^+$ refers to the set of nonnegative real numbers. The problem of the existence of mild solutions to (1.1) has been addressed by many investigators in the case where $\beta \in (0,1]$. We quote for instance the contributions by Shu and Wang [21], Qin *et al.* [20], and the pioneering works of Travis and Webb [23, 24]. However, only a few papers have been up to now devoted to the case $\beta \in (1,2]$. We quote the paper [25], where the authors proved the existence of mild solutions to (1.1) with $\beta \in (1,2]$ when p and q are compact. In many applications, nonlocal conditions are not compact. Specifically, periodic $p(u) = u(a)$, anti-periodic $p(u) = -u(a)$, or multipoint discrete nonlocal conditions $p(u) = \sum_{i=1}^m c_i u(t_i)$, $0 < t_1 < \cdots < t_m$ are not compact.

As a matter of fact, the first and major aim of this paper is to address the problem of existence of mild solutions to (1.1) in the case where p and q are not necessarily compact. Moreover, we merely assume that the operator A generates an arbitrarily strongly continuous β-order fractional cosine family, which is an extra interesting feature. Our approach combines the Monch fixed point theorem with some recent results concerning the measure of noncompactness of integral operators.

The outline of the paper is as follows: In Section 2, we present the main technical tools which will be used in this work. In Section 3, we investigate the existence of mild solution to problem (1.1) by means of a fixed point method. Finally, in Section 4, we include an example to illustrate our results.

2 Preliminaries and auxiliary results

In this section, we recall some background and collect several useful results which are crucial for our further work. To do this, let $(E, \|\cdot\|)$ be a Banach space and $C([0,a],E)$ be the space of all continuous functions defined on $[0,a]$ with values in E, equipped with the standard sup-norm. Let $\mathcal{L}(E)$ denote the space of all bounded linear operators on E endowed with the classical operator norm. We first list some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. *[4] For $0 < \gamma < 1$, consider the function of Wright type defined by*

$$\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)} = \frac{1}{2\pi i} \int_\Gamma \mu^{\gamma-1} \exp\left(\mu - z\mu^\gamma\right) d\mu, \tag{2.1}$$

where Γ *is a contour which starts and ends at* $-\infty$ *and encircles the origin once counterclockwise.* $\Phi_\gamma(t)$ *is a probability density function:*

$$\Phi_\gamma(t) \geq 0 \;\; for \;\; t > 0 \;\; and \;\; \int_0^\infty \Phi_\gamma(t)dt = 1. \tag{2.2}$$

Definition 2.2. *[4] The Riemann-Liouville fractional integral of order* $\beta > 0$ *of a function* $f \in L^1([0,a];E)$ *is defined by*

$$I_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds, \quad t > 0, \tag{2.3}$$

where $\Gamma(\cdot)$ *stands for the Gamma function.*

Definition 2.3. *[4] The Riemann-Liouville fractional derivative of order* $1 < \beta \leq 2$ *is defined by*

$$D_t^\beta f(t) = \frac{d^2}{dt^2} I_t^{2-\beta} f(t), \tag{2.4}$$

where $f \in L^1([0,a];E)$ *and* $D_t^\beta f \in L^1([0,a];E)$.

Definition 2.4. *[4] The Caputo fractional derivative of order* $\beta \in (1,2]$ *is defined by*

$${}^c D_t^\beta f(t) = D_t^\beta \left(f(t) - f(0) - f'(0)t \right), \tag{2.5}$$

where $f \in L^1([0,a];E) \cap C^1([0,a];E)$ *and* $D_t^\beta f \in L^1([0,a];E)$.

Consider the following problem

$${}^c D_t^\beta x(t) = Ax(t), \qquad x(0) = \eta, \qquad x'(0) = 0, \tag{2.6}$$

where $\beta \in (1,2]$, $A : D(A) \subset E \to E$ is a closed densely defined linear operator in Banach space E.

Definition 2.5. *[4] Let* $\beta \in (1,2]$. *A family* $\{C_\beta\}_{\beta \geq 0} \subset \mathcal{L}(E)$ *is called a solution operator (or a strongly continuous* β*-order fractional cosine family) for the problem* (2.6) *if the following conditions are satisfied:*

(a) $C_\beta(t)$ *is strongly continuous for* $t \geq 0$ *and* $C_\beta(0) = I$,

(b) $C_\beta(t)D(A) \subset D(A)$ *and* $AC_\beta(t)\eta = C_\beta(t)A\eta$ *for all* $\eta \in D(A)$, $t \geq 0$,

(c) $C_\beta(t)\eta$ *is a solution of* $x(t) = \eta + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Ax(s)ds$ *for all* $\eta \in D(A)$, $t \geq 0$.

In this case, A is called the infinitesimal generator of $C_\beta(t)$.

Definition 2.6. *[15] The fractional sine family* $S_\beta : \mathbb{R}^+ \to \mathcal{L}(E)$ *associated with* C_β *is defined by*

$$S_\beta(t) = \int_0^t C_\beta(s)ds, \qquad t \geq 0. \tag{2.7}$$

Definition 2.7. *[15] The fractional Riemann-Liouville family $P_\beta : \mathbb{R}^+ \to \mathcal{L}(E)$ associated with C_β is defined by*

$$P_\beta(t) = I_t^{\beta-1} C_\beta(t) = \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} C_\beta(s) ds, \qquad t \geq 0. \tag{2.8}$$

Definition 2.8. *[4] The strongly continuous β-order fractional cosine family $C_\beta(t)$ is called exponentially bounded if there are constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|C_\beta(t)\| \leq M e^{\omega t}, \qquad t \geq 0. \tag{2.9}$$

An operator A is said to belong to $C^\beta(M, \omega)$, if the problem (2.6) *has a strongly continuous β-order fractional cosine family $C_\beta(t)$ satisfying* (2.9)*. Denote $C^\beta(\omega) = \bigcup \{C^\beta(M, \omega); M \geq 1\}$.*

Theorem 2.1. *[4, Theorem 3.1] Let $0 < \beta' < \beta \leq 2$, $\gamma = \frac{\beta'}{\beta}$, $\omega \geq 0$. If $A \in C^\beta(\omega)$ then $A \in C^{\beta'}(\omega^{\frac{1}{\gamma}})$ and the following representation holds*

$$C_{\beta'}(t) = \int_0^\infty \varphi_{t,\gamma}(s) C_\beta(s) ds, \quad t > 0, \tag{2.10}$$

where $\varphi_{t,\gamma}(s) := t^{-\gamma} \Phi_\gamma(st^{-\gamma})$ and $\Phi_\gamma(z)$ is defined by (2.1)*.*

For more details regarding β-order fractional cosine families, we refer the reader to [4].

Definition 2.9. *A function ψ defined on the set of all bounded subsets of a Banach space E with values in $\mathbb{R}^+$ is called a measure of noncompactness (MNC in short) on E if for any bounded subset M of E we have $\psi(\overline{co}M) = \psi(M)$, where $\overline{co}M$ stands for the closed convex hull of M. An MNC is said to be*

(i) Full: $\psi(M) = 0$ if and only if M is a relatively compact set.

(ii) Monotone: for all bounded subsets M_1 and M_2 of E, we have

$$M_1 \subset M_2 \Longrightarrow \psi(M_1) \leq \psi(M_2).$$

(iii) Nonsingular: $\psi(M \cup \{x\}) = \psi(M)$, for every bounded subset M of E and for all $x \in E$.

One of most important measures of noncompactness is the Hausdorff measure of noncompactness defined by

$$\chi(M) = \inf\{r > 0; M \text{ can be covered by finitely many balls with radii } \leq r\},$$

for each bounded subset M of E. The Hausdorff measure of noncompactness is *full, monotone* and *nonsingular*. Moreover, it enjoys the following additional properties.

Lemma 2.1. *[3]*

(i) $\chi(M_1 + M_2) \leq \chi(M_1) + \chi(M_2)$.

(ii) $\chi(\lambda M) = |\lambda|\chi(M)$, *for all* $\lambda \in \mathbb{R}$.

(iii) $\chi(\overline{co}(M)) = \chi(M)$.

(iv) $\chi(A + x) = \chi(A),\ \forall x \in E$.

(v) if $B\colon E \longrightarrow E$ *is a Lipschitz continuous map with constant* k, *then* $\chi(B(M)) \leq k\chi(M)$ *for all bounded subset* M *of* E.

Lemma 2.2. *[17, 9] If* $\{u_n\}_{n\in\mathbb{N}} \subset L^1([0, a]; E)$ *is uniformly integrable, then the function* $t \mapsto \chi(\{u_n(t)\}_{n\in\mathbb{N}})$ *for* $t \in [0, a]$ *is measurable and*

$$\chi\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leq \int_0^t \chi\big(\{u_n(s)\}_{n=1}^{\infty}\big)ds.$$

In the sequel, we use a measure of noncompactness in the space $C(I; E)$ which was investigated in [11, 12]. In order to define this measure, let us fix a nonempty bounded subset Ω of the space $C(I; E)$. Let

$$\operatorname{mod}_C(\Omega) = \sup\left\{\operatorname{mod}_C(\Omega(t)) : t \in I\right\},$$

where

$$\operatorname{mod}_C(\Omega(t)) = \lim_{\delta\to 0} \sup_{x\in\Omega} \left\{\sup\left\{|x(t_2) - x(t_1)| : t_1, t_2 \in (t - \delta, t + \delta)\right\}\right\},$$

and

$$\chi_\infty(\Omega) = \sup\left\{\chi(\Omega(t)) : t \in I\right\},$$

where χ denotes the Hausdorff measure of noncompactness in E. It is worth noticing that χ_∞ and mod_C are monotone nonsingular MNCs on $C(I; E)$ (see [3, 12]). From an application view point, one of the main disadvantages of these MNCs is the lack of fullness. To overcome this problem, we can define the function ψ_C on the family of bounded subsets in $C(I; E)$ by taking

$$\psi_C(\Omega) = \chi_\infty(\Omega) + \operatorname{mod}_C(\Omega)$$

Lemma 2.3. *[11, Lemma 3.1]* ψ_C *is a full monotone and nonsingular MNC on the space* $C(I; E)$.

Finally, we will make use of Monch's fixed point theorem.

Theorem 2.2. *[17] Let* C *be a closed, convex subset of a Banach space* E *with* $x_0 \in C$. *Suppose there is a continuous map* $T : C \to C$ *with the following property:*

$$\begin{cases} D \subseteq C \text{ countable and } D \subseteq \operatorname{co}\left(\{x_0\} \cup T(D)\right) \\ \text{imply that } D \text{ is relatively compact.} \end{cases}$$

Then, T *has at least one fixed point in* C.

Let $\mathcal{F}$ be a function from $[0,+\infty)$ into $\mathcal{L}(E)$. Suppose that $\mathcal{F}$ is continuous for the strong operator topology, namely

$$\text{The mapping } [0,+\infty) \ni t \to \mathcal{F}(t)x \in E \text{ is continuous for every } x \in E. \tag{2.11}$$

Notice that from the uniform boundedness principle, we know that $\mathcal{F}$ is uniformly bounded on any interval $[0,a]$, i.e., $M_a := \sup_{t\in[0,a]} \|\mathcal{F}(t)\|_{\mathcal{L}(E)} < +\infty$. For later use, let us define the quantity

$$\omega(\mathcal{F}(t)) = \lim_{\delta\to 0} \sup_{\|x\|\le 1} \left\{ \left\| \mathcal{F}(t_2)\, x - \mathcal{F}(t_1)\, x \right\|_E : t_1, t_2 \in (t-\delta, t+\delta) \right\}.$$

Recall that a family $(\mathcal{F}(t))_{t\ge 0}$ is said to be equicontinuous if $\{\mathcal{F}(\cdot)x : x \in \Omega\}$ is equicontinuous at any $t > 0$ for any bounded subset $\Omega \subset X$. It is easily seen that a family $(\mathcal{F}(t))_{t\ge 0}$ is equicontinuous if and only if $\omega(\mathcal{F}(t)) = 0$ for any $t > 0$.

Theorem 2.3. *[7] Let $\mathcal{F}$ be a function from $[0,+\infty)$ into $\mathcal{L}(E)$. Suppose that $\mathcal{F}$ is continuous for the strong operator topology. Then, for any bounded set $\Omega \subset E$ and for any $t \ge 0$, we have*

$$\operatorname{mod}_C(\mathcal{F}(t)\Omega) \le \omega(\mathcal{F}(t))\chi(\Omega).$$

In particular, for any $t \in [0,a]$ we have

$$\operatorname{mod}_C(\mathcal{F}(t)\Omega) \le 2M_a\chi(\Omega).$$

Now, we present two crucial results concerning the integral operator:

$$(\mathcal{S}_0 f)(t) = \int_0^t \mathcal{F}(t-s) f(s) \mathrm{d}s \quad \text{ for } t \in [0,a]$$

where $f \in L^1([0,a];E)$ and $\mathcal{F} : [0,+\infty) \to \mathcal{L}(E)$ verifies (2.11).

Theorem 2.4. *[7] Let $\{f_n\}_{n=1}^{\infty} \subset L^1([0,a];E)$ be integrably bounded, that is,*

$$\|f_n(t)\| \le \nu(t) \text{ for all } n = 1,2,\cdots \text{ and a.e. } t \in [0,a], \tag{2.12}$$

where $\nu \in L^1([0,a])$. Assume that

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \le q(t) \tag{2.13}$$

for a.e. $t \in [0,a]$ where $q \in L^1([0,a])$. Then, for every $t \in [0,a]$ we have:

$$\operatorname{mod}_C\left(\{\mathcal{S}_0 f_n(t)\}_{n=1}^{\infty}\right) \le 4M_a \int_0^t q(s) \mathrm{d}s. \tag{2.14}$$

Theorem 2.5. *[7] Let $\{f_n\}_{n=1}^{\infty} \subset L^1([0,a];E)$ be as in (2.12) Assume that (2.13) holds. Then*

$$\chi\left(\{\mathcal{S}_0 f_n(t)\}_{n=1}^{\infty}\right) \le 2M_a \int_0^t q(s) \mathrm{d}s, \quad \text{for all } t \in [0,a]$$

3 Existence results

In this section, we discuss the existence of mild solutions to the semilinear fractional integro-differential equation (1.1). Before doing so, it is appropriate to clarify the definition of solution we will consider.

Definition 3.1. *Assume $A \in C^{\beta}(M,\omega)$ and $C_\beta(t)$ is the solution operator. We say that $u \in C[I,E]$ is a mild solution of (1.1) if u satisfies*

$$\begin{aligned} u(t) &= C_\beta(t)\,(u_0+q(u)) + S_\beta(t)\,(v_0+p(u)) \\ &\quad + \int_0^t P_\beta(t-s) f(s,u(s),Gu(s),Su(s))ds, \quad t\in I. \end{aligned} \tag{3.1}$$

To allow the abstract formulation of our problem, we define the operator $T : C([0,a];E) \to C([0,a];E)$ by

$$\begin{aligned} Tu(t) &= C_\beta(t)\,(u_0+q(u)) + S_\beta(t)\,(v_0+p(u)) \\ &\quad + \int_0^t P_\beta(t-s) f(s,u(s),Gu(s),Su(s))ds, \quad t\in [0,a] \end{aligned} \tag{3.2}$$

for all $t \in [0,a]$. It is clear that u is a mild solution of (1.1) if and only if it is a fixed point of T.

Our problem will be investigated under the following assumptions:

(C_1) $p, q : C([0,a];E) \to E$ are continuous functions and there exist nonnegative constants k_p and k_q, such that for all bounded subset $D \subset C([0,a];E)$, we have

$$M_a\chi(q(D)) + aM_a\chi(p(D)) \le (M_a k_q + aM_a k_p)\chi_\infty(D),$$

where $M_a = \sup_{t\in[0,a]} \|C_\beta(t)\|_{\mathcal{L}(E)}$.

(C_2) There exist nondecreasing continuous functions $\sigma_1, \sigma_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{cases} \|q(u)\|_E \le \sigma_1\left(\|u\|_\infty\right), & \text{for all } u \in C([0,a];E), \\ \|p(u)\|_E \le \sigma_2\left(\|u\|_\infty\right), & \text{for all } u \in C([0,a];E). \end{cases}$$

(C_3)

$$\begin{cases} f : [0,a]\times E\times E\times E \to E \ \text{ is a Carathéodory function, i.e.,} \\ (i) \text{ the map } t \mapsto f(t,u_1,u_2,u_3) \ \text{ is measurable for all} \\ (u_1,u_2,u_3) \in E\times E\times E, \\ (ii) \text{ the functions } u_1 \mapsto f(t,u_1,u_2,u_3),\ u_2 \mapsto f(t,u_1,u_2,u_3) \ \text{ and} \\ u_3 \mapsto f(t,u_1,u_2,u_3) \ \text{ are continuous for almost } t\in[0,a], \end{cases}$$

(C_4) There exist functions $\rho_1, \rho_2, \rho_3 \in L^1((0,a);\mathbb{R}^+)$ and nondecreasing continuous functions $\Omega_1, \Omega_2, \Omega_3 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,u_1,u_2,u_3)\|_E \le \sum_{i=1}^{3} \rho_i(t)\Omega_i(\|u_i\|_E), \quad \text{for all } t\in[0,a] \text{ and } u_i \in E.$$

(C_5) There exist functions $m_1, m_2, m_3 \in L^1([0,a];\mathbb{R}^+)$ such that for all bounded subset $D_1, D_2, D_3 \subset E$

$$\chi(f(t, D_1, D_2, D_3)) \leq \sum_{i=1}^{3} m_i(t)\chi(D_i), \quad \text{for almost every } t \in [0,a].$$

(C_6) $M_a k_q + aM_a k_p + 2\dfrac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1 < 1,$

where

$$m(s) = m_1(s) + ak_0 m_2(s) + ah_0 m_3(s),\ k_0 = \sup\{K(t,s);\ (t,s) \in U\},$$
$$h_0 = \sup\{H(t,s);\ (t,s) \in U\}, \text{ and } U = \left\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq a\right\}.$$

Remark 3.1. *It is easy to prove that for every $t \geq 0$, we have*

$$\sup_{t\in[0,a]} \|S_\beta(t)\|_{\mathcal{L}(E)} \leq aM_a \quad \text{and} \quad \sup_{t\in[0,a]} \|P_\beta(t)\|_{\mathcal{L}(E)} \leq \frac{M_a a^{\beta-1}}{\Gamma(\beta)}. \tag{3.3}$$

In light of this, we shall show that operator T fulfills all conditions of Theorem 2.2. This will be done in a series of lemmas.

Lemma 3.1. *$T : C([0,a];E) \to C([0,a];E)$ is continuous.*

Proof. Let $(u_n) \subset C([0,a];E)$ be a sequence which converges to $u \in C([0,a];E)$. Then

$$\begin{array}{rcl}\|Tu_n - Tu\|_\infty & \leq & M_a\|q(u_n) - q(u)\|_E + aM_a\|p(u_n) - p(u)\|_E \\ & & +\frac{M_a a^{\beta-1}}{\Gamma(\beta)} \int_0^a \|f(s, u_n(s), Gu_n(s), Su_n(s)) \\ & & -f(s, u(s), Gu(s), Su(s))\|_E ds.\end{array}$$

With assumptions (C_1) and (C_3) in mind, the continuity of G and S entails

$$\lim_{n\to\infty} f(s, u_n(s), Gu_n(s), Su_n(s)) = f(s, u(s), Gu(s), Su(s)).$$

Since (u_n) is convergent then there exists $r > 0$ such that $\|u_n\|_\infty \leq r$, for all $n \in \mathbb{N}$ and $\|u\|_\infty \leq r$. So by (C_4) we have

$$\begin{aligned}&\|f(s, u_n(s), Gu_n(s), Su_n(s)) - f(s, u(s), Gu(s), Su(s))\|_\infty \\ &\qquad \leq 2\left(\rho_1(s)\Omega_1(r) + \rho_2(s)\Omega_2(ak_0 r) + \rho_3(s)\Omega_3(ah_0 r)\right).\end{aligned}$$

Using the dominated convergence theorem, we deduce that T is continuous. $\square$

Lemma 3.2. *Assume that*

$$M_a \liminf_{r\to\infty}\left(\frac{\sigma(r)}{r} + \frac{a^{\beta-1}}{\Gamma(\beta)}\frac{\Omega(r)}{r}\right) < 1, \tag{3.4}$$

where $\sigma(r) = \sigma_1(r) + a\sigma_2(r)$ and

$$\Omega(r) = \Omega_1(r)\|\rho_1\|_{L^1} + \Omega_2(ak_0 r)\|\rho_2\|_{L^1} + \Omega_3(ah_0 r)\|\rho_3\|_{L^1}.$$

Then, there is a $r_0 > 0$ such that T selfmaps the closed ball

$$B_{r_0} = \{u \in C([0,a];E) : \|u\|_\infty \leq r_0\}.$$

Proof. For $u \in B_r$ and $t \in [0,a]$, we have

$$\begin{array}{rcl} \|(Tu)(t)\|_E & \leq & \|C_\beta(t)\,(u_0+q(u))\|_E + \|S_\beta(t)\,(v_0+p(u))\|_E \\ & & + \left\| \int_0^t P_\beta(t-s) f(s,u(s),Gu(s),Su(s))ds \right\|_E \\ & \leq & M_a\left(\|u_0\|_E + \sigma_1(r)\right) + aM_a\left(\|v_0\|_E + \sigma_2(r)\right) \\ & & + \frac{M_a a^{\beta-1}}{\Gamma(\beta)} \int_0^a \Omega_1(r)\rho_1(s) \\ & & + \Omega_2(ak_0r)\rho_2(s) + \Omega_3(ah_0r)\rho_3(s)\mathrm{d}s. \end{array}$$

We claim that there exists $r_0>0$ such that $Tu \in B_{r_0}$ whenever $u \in B_{r_0}$. If is not the case, then for each $r>0$ there exists $u \in B_r$ such that $Tu \notin B_r$, that is

$$r < \|Tu\|_\infty \leq M_a\left(\|u_0\|_E + \sigma_1(r)\right) + aM_a\left(\|v_0\|_E + \sigma_2(r)\right) + \frac{M_a a^{\beta-1}}{\Gamma(\beta)}\Omega(r),$$

which implies when dividing by r that

$$1 < \frac{M_a\,\|u_0\|_E + aM_a\|v_0\|_E}{r} + M_a\frac{\sigma(r)}{r} + \frac{M_a a^{\beta-1}}{\Gamma(\beta)}\frac{\Omega(r)}{r}.$$

Taking the lim inf as $r \to \infty$, we obtain

$$1 \leq M_a \liminf_{r\to\infty}\left(\frac{\sigma(r)}{r} + \frac{a^{\beta-1}}{\Gamma(\beta)}\frac{\Omega(r)}{r}\right),$$

which contradicts the assumption (3.4) Therefore, there exists $r_0>0$ such that

$$\|Tu\|_\infty \leq r_0, \text{ for all } \|u\| \leq r_0.$$

Thus, $Tu \in B_{r_0}$ for all $u \in B_{r_0}$. $\square$

Lemma 3.3. *Let r_0 be as in Lemma 3.2 and let $x_0 \in B_{r_0}$. Let D be a countable subset of B_{r_0}. Then $D \subseteq \overline{\mathrm{co}}\left(\{x_0\} \cup T(D)\right)$ implies that D is relatively compact.*

Proof. Let $D=\{u_n\}_{n=1}^\infty$ be any countable subset of B_{r_0} such that

$$D \subseteq \overline{\mathrm{co}}\left(\{x_0\} \cup T(D)\right). \tag{3.5}$$

We show that D is relatively compact. Notice first that for each $t \in [0,a]$, we have

$$\begin{aligned} \chi(T(D)(t)) &\leq \chi(C_\beta(t)(u_0+q(D))) + \chi(S_\beta(t)(v_0+p(D))) \\ &\qquad + \chi\left(\left\{\int_0^t P_\beta(t-s)f(s,u_n(s),Gu_n(s),Su_n(s))ds\right\}_{n=1}^\infty\right). \end{aligned}$$

Since

$$\|f(s,u_n(s),Gu_n(s),Su_n(s))\|_E \leq \Omega_1(r_0)\rho_1(s) + \Omega_2(ak_0r_0)\rho_2(s) + \Omega_3(ah_0r_0)\rho_3(s)$$

and $\rho_1, \rho_2, \rho_3 \in L^1([0,a]; R+)$, then, in view of Theorem 2.5 and Lemma 2.2, we obtain the following estimates:

$$\begin{array}{ll} & \chi(T(D)(t)) \\ \leq & M_a\chi(q(D)) + aM_a\chi(p(D)) \\ & +2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\int_0^t m_1(s)\chi(D(s)) + m_2(s)\chi(G(D(s))) + m_3(s)\chi(S(D(s)))ds \\ \leq & (M_a k_q + aM_a k_p)\chi_\infty(D) \\ & +2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\int_0^t m_1(s)\chi(D(s)) + ak_0 m_2(s)\chi(D(s)) + ah_0 m_3(s)\chi(D(s))ds \\ \leq & \left(M_a k_q + aM_a k_p + 2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\right)\chi_\infty(D). \end{array}$$

Thus,

$$\chi_\infty(T(D)) \leq \left[M_a k_q + aM_a k_p + 2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\right]\chi_\infty(D). \tag{3.6}$$

On the other hand, referring to Theorem 2.3, Theorem 2.4, and Lemma 2.2, we can see that

$$\begin{array}{lll} \mathrm{mod}_C(T(D)(t)) & \leq & \mathrm{mod}_C(C_\beta(t)q(D)) + \mathrm{mod}_C(S_\beta(t)p(D)) \\ & & +4\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\int_0^t m(s)\chi(D(s))\mathrm{d}s \\ & \leq & 2(M_a k_q + aM_a k_p)\chi_\infty(D) + 4\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\chi_\infty(D). \end{array}$$

Thus,

$$\mathrm{mod}\,_C(T(D)) \leq \left[2(M_a k_q + aM_a k_p) + 4\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\right]\chi_\infty(D). \tag{3.7}$$

Combining (3.5) and (3.6), we arrive at $\chi_\infty(TD) = \chi_\infty(D) = 0$. By (3.7) we get $\mathrm{mod}\,_C(T(D)) = 0$ and therefore $T(D)$ is equicontinuous. Going back to (3.5) we deduce that D is equicontinuous and so relatively compact in $C([0,a]; E)$. This achieves the proof. □

Theorem 3.1. *Assume that* $(C_1) - -(C_6)$ *hold. Then, the nonlocal problem* (1.1) *has at least one mild solution in* $C([0,a]; E)$, *provided that* (3.4) *holds.*

Proof. Invoking Theorem 2.2 together with Lemmas 3.1, 3.2, and 3.3, we infer that T has at least one fixed point in B_{r_0} which is, in turn, a mild solution of (1.1). □

4 Application

To illustrate the application of the theoretical results of this work, we consider the following integro-differential equation:

$$\begin{cases} {}^cD_t^\beta w(t,x) = \dfrac{\partial^2 w(t,x)}{\partial^2 x} + \rho_1(t) f_1(w(t,x)) + \rho_2(t) f_2\left(\int_0^t \frac{ts}{2} w(s,x)ds\right) \\ +\rho_3(t) f_3\left(\int_0^1 \frac{t^2s^2}{2} w(s,x)ds\right), \quad t \in I = [0,1],\ x \in [0,\pi], \\ w(t,0) = w(t,\pi) = 0,\ t \in I, \\ w(0,x) = w_0(x) + \sum\limits_{i=1}^m c_i w(s_i,x), \quad x \in [0,\pi], \\ s_1 < s_2 < \ldots < s_m,\ t_i \in I,\ c_i \in \mathbb{R}, \\ \dfrac{\partial w(t,x)}{\partial t}\Big|_{t=0} = y_0(x) + \sum\limits_{i=1}^n d_i w(t_i,x),\ x \in [0,\pi], \\ t_1 < t_2 < \ldots < t_n,\ t_i \in I,\ d_i \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $\beta \in (1,2]$, the functions $\rho_i : I \to \mathbb{R}$ and $f_i : E \to E$ for $i \in \{1,2,3\}$ satisfy appropriate conditions which are specified later.

To allow the abstract formulation of (4.1), let $E = L^2([0,\pi];\mathbb{R})$ be the Banach space of square integrable functions from $[0,\pi]$ into $\mathbb{R}$. Define the operator $A : D(A) \subset E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous }, w'' \in E, w(0) = w(\pi) = 0\}.$$

It is well known that A is the generator of strongly continuous cosine functions $\{C(t) : t \in \mathbb{R}\}$ on E. Moreover A has a discrete spectrum whose eigenvalues are $-n^2$, $n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) = \sqrt{\frac{2}{\pi}} \sin(n\tau),$$

and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $z \in E$, then $Az = -\sum_{n=1}^\infty n^2 < z,\ z_n > z_n$.

(c) For $z \in E$, $C(t)z = \sum_{n=1}^\infty \cos(nt) < z, \ z_n > z_n$, and the associated sine family is $S(t)z = \sum_{n=1}^\infty \frac{\sin(nt)}{n} < z, z_n > z_n$. $S(t)$ is compact for every $t \in I$ and $\|C(t)\|_{\mathcal{L}(E)} = \|S(t)\|_{\mathcal{L}(E)} \leq 1$, for every $t \in \mathbb{R}$.

For $\beta \in (1,2]$, since A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, from the subordinate principle (Theorem 2.1), it follows that A is the infinitesimal generator of a strongly continuous exponentially bounded fractional cosine family $C_\beta(t)$.

With $u(t)=w(t,\cdot)$, Equation (4.1) may be written in the abstract form:

$$\begin{cases} {}^cD_t^{\beta}u(t)=Au(t)+f(t,u(t),Gu(t),Su(t)),\ t\in I,\\ u(0)=u_0+q(u),\\ u'(0)=v_0+p(u), \end{cases} \tag{4.2}$$

where the function $f: I\times E\times E\times E\to E$ is given by

$$f(t,x,y,z)=\rho_1(t)f_1(x)+\rho_2(t)f_2(y)+\rho_3(t)f_3(z).$$

Here $\rho_i: I\to\mathbb{R}$ is integrable on I, $f_i: E\to E$ is a Lipschitz continuous function with a Lipschitz constant L_i, the functions $p,q: C(I,E)\to E$ are given by

$$q(u)=\sum_{i=1}^{m}c_iu(s_i),\quad 0<s_1<s_2<\cdots<s_m\leq 1,$$

and

$$p(u)=\sum_{i=1}^{n}d_iu(t_i),\quad 0<t_1<t_2<\cdots<t_n\leq 1,$$

and the functions $G,S: C(I,E)\to C(I,E)$ are defined by

$$Gu(t)=\int_0^t\frac{ts}{2}u(s)ds,\quad Su(t)=\int_0^1\frac{t^2s^2}{2}u(s)ds,$$

where $h_0=k_0=\dfrac{1}{2}$.

In order to obtain a mild solution, our strategy is to apply Theorem 3.1. First, by (c) we have $\|C(t)\|_{\mathcal{L}(E)}\leq 1$, for every $t\in\mathbb{R}^+$. In view of Theorem 2.1 and (2.2) we see that there exists a real number $M_a=1>0$ such that $\|C_\beta(t)\|_{\mathcal{L}(E)}\leq M_a$ for $t\geq 0$. Observe further that the function $f: I\times E\times E\times E\to E$ is given by

$$f(t,x,y,z)=\rho_1(t)f_1(x)+\rho_2(t)f_2(y)+\rho_3(t)f_3(z),$$

where $\rho_i: I\to\mathbb{R}$ is integrable on I and $f_i: E\to E$ is a Lipschitz continuous function with a Lipschitz constant L_i $(i=1,2,3)$. This shows that (C_3) is satisfied. On one hand,

$$\|q(u)\|_E\leq\left(\sum_{i=1}^{m}|c_i|\right)\|u\|_\infty=\sigma_1(\|u\|_\infty) \tag{4.3}$$

and

$$\|p(u)\|_E\leq\left(\sum_{i=1}^{n}|d_i|\right)\|u\|_\infty=\sigma_2(\|u\|_\infty), \tag{4.4}$$

where $\sigma_1(r)=\left(\sum_{i=1}^{m}|c_i|\right)r$ and $\sigma_2(r)=\left(\sum_{i=1}^{n}|d_i|\right)r$. In addition, it is easily seen that for any bounded subset D of $C([0,1],E)$ we have

$$\chi(q(D))\leq\sum_{i=1}^{m}|c_i|\,\chi\left(D\left(s_i\right)\right)\leq\left(\sum_{i=1}^{m}|c_i|\right)\chi_\infty(D)=k_q\chi_\infty(D) \tag{4.5}$$

and

$$\chi(p(D)) \leq \sum_{i=1}^{n} |d_i|\, \chi\left(D\left(t_i\right)\right) \leq \left(\sum_{i=1}^{n} |d_i|\right) \chi_\infty(D) = k_p \chi_\infty(D). \tag{4.6}$$

Thus

$$M_a \chi(q(D)) + a M_a \chi(p(D)) \leq (M_a k_q + a M_a k_p)\chi_\infty(D), \tag{4.7}$$

for any bounded subset D of $C([0,1];E)$. This shows that (C_1) and (C_2) are satisfied. Moreover the function f satisfies

$$\begin{aligned}
\|f(t,u_1,u_2,u_3)\|_E &\leq |\rho_1(t)|\|f_1(u_1)\|_E + |\rho_2(t)|\|f_2(u_2)\|_E + |\rho_3(t)|\|f_3(u_3)\|_E \\
&\leq |\rho_1(t)|(\|f_1(0)\|_E + L_1\|u_1\|_E) + |\rho_2(t)|(\|f_2(0)\|_E + L_2\|u_2\|_E) \\
&\qquad + |\rho_3(t)|(\|f_3(0)\|_E + L_3\|u_3\|_E) \\
&\leq |\rho_1(t)|\Omega_1(\|u_1\|_E) + |\rho_2(t)|\Omega_2(\|u_2\|_E) + |\rho_3(t)|\Omega_3(\|u_3\|_E) \\
&\leq \sum_{i=1}^{3} |\rho_i(t)|\Omega_i(\|u_i\|_E),
\end{aligned}$$

where $\Omega_i(\|u_i\|_E) = \|f_i(0)\|_E + L_i\|u_i\|_E$. By virtue of Lemma 2.1, (v) we have

$$\begin{aligned}
\chi(f(t,D_1,D_2,D_3)) &\leq |\rho_1(t)|\chi(f_1(D_1)) + |\rho_2(t)|\chi(f_2(D_2)) + |\rho_3(t)|\chi(f_3(D_3)) \\
&\leq |\rho_1(t)|L_1\chi(D_1) + |\rho_2(t)|L_2\chi(D_2) + |\rho_3(t)|L_3\chi(D_3) \\
&\leq \sum_{i=1}^{3} m_i(t)\chi(D_i),
\end{aligned}$$

for any $t \in [0,a]$ and for any bounded subsets D_1, D_2, D_3 of E. Thus, (C_4) and (C_5) are satisfied. Now the condition (C_6) is given by taking

$$2\frac{a^{\beta-1}M_a}{\Gamma(\beta)}\left(L_1\|\rho_1\|_{L^1} + \frac{1}{2}L_2\|\rho_2\|_{L^1} + \frac{1}{2}L_3\|\rho_3\|_{L^1}\right) + (M_a k_q + a M_a k_p) < 1,$$

because, we have

$$\begin{aligned}
m(s) &= m_1(s) + ak_0 m_2(s) + ah_0 m_3(s) \\
&= L_1|\rho_1(s)| + \frac{1}{2}L_2|\rho_2(s)| + \frac{1}{2}L_3|\rho_3(s)|.
\end{aligned}$$

Then

$$\|m\|_1 = L_1\|\rho_1\|_{L^1} + \frac{1}{2}L_2\|\rho_2\|_{L^1} + \frac{1}{2}L_3\|\rho_3\|_{L^1}.$$

Finally, for

$$\begin{aligned}
\Omega(r) &= \Omega_1(r)\|\rho_1\|_{L^1} + \Omega_2(ak_0 r)\|\rho_2\|_{L^1} + \Omega_3(ah_0 r)\|\rho_3\|_{L^1} \\
&= \Omega_1(r)\|\rho_1\|_{L^1} + \Omega_2(\tfrac{1}{2}r)\|\rho_2\|_{L^1} + \Omega_3(\tfrac{1}{2}r)\|\rho_3\|_{L^1},
\end{aligned}$$

we have

$$\lim_{r\to\infty}\frac{\Omega(r)}{r} = L_1\|\rho_1\|_{L^1} + \frac{1}{2}L_2\|\rho_2\|_{L^1} + \frac{1}{2}L_3\|\rho_3\|_{L^1},$$

and for $\sigma(r) = \sigma_1(r) + a\sigma_2(r) = (k_q + ak_p)r$, notice that

$$\lim_{r\to\infty} \frac{\sigma(r)}{r} = k_q + ak_p.$$

Then

$$\begin{aligned}
& M_a \liminf_{r\to\infty} \left(\frac{\sigma(r)}{r} + \frac{a^{\beta-1}}{\Gamma(\beta)} \frac{\Omega(r)}{r} \right) \\
= \; & \frac{a^{\beta-1} M_a}{\Gamma(\beta)} \left(L_1 \|\rho_1\|_{L^1} + \tfrac{1}{2} L_2 \|\rho_2\|_{L^1} + \tfrac{1}{2} L_3 \|\rho_3\|_{L^1} \right) + (M_a k_q + aM_a k_p) \\
\leq \; & 2\frac{a^{\beta-1} M_a}{\Gamma(\beta)} \left(L_1 \|\rho_1\|_{L^1} + \tfrac{1}{2} L_2 \|\rho_2\|_{L^1} + \tfrac{1}{2} L_3 \|\rho_3\|_{L^1} \right) + (M_a k_q + aM_a k_p) \\
< \; & 1.
\end{aligned}$$

Thus, all conditions of Theorem 3.1 are fulfilled. Therefore Equation (4.1) has a mild solution.

Acknowledgment

The authors thank the three referees for their helpful remarks. The first two authors are grateful to the Direction Générale de la Recherche Scientifique et de Développement Technologique in Algeria for supporting this work.

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