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Subclasses of λ-bi-pseudo-starlike functions with respect to symmetric points based on shell-like curves

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ABSTRACT

In this paper we define the subclass $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha,\tilde{p}(z))$ of the class Σ of bi-univalent functions defined in the unit disk, called λ-bi-pseudo-starlike, with respect to symmetric points, related to shell-like curves connected with Fibonacci numbers. We determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions $f \in \mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha,\tilde{p}(z))$. Further we determine the Fekete-Szegö result for the function class $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha,\tilde{p}(z))$ and for the special cases $\alpha = 0$, $\alpha = 1$ and $\tau = -0.618$ we state corollaries improving the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

RESUMEN

En este artículo definimos la subclase $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha,\tilde{p}(z))$ de la clase Σ de funciones bi-univalentes definidas en el disco unitario, llamadas λ-bi-pseudo-estrelladas, con respecto a puntos simétricos, relacionadas a curvas espirales en conexión con números de Fibonacci. Determinamos los coeficientes iniciales de Taylor-Maclaurin $|a_2|$ y $|a_3|$ para funciones $f \in \mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha,\tilde{p}(z))$. Más aún determinamos el resultado de Fekete-Szegö para la clase de funciones $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha,\tilde{p}(z))$ y para los casos especiales $\alpha = 0$, $\alpha = 1$ y $\tau = -0.618$ enunciamos corolarios mejorando los coeficientes iniciales de Taylor-Maclaurin $|a_2|$ y $|a_3|$.

Keywords and Phrases: Analytic functions, bi-univalent, shell-like curve, Fibonacci numbers, starlike functions.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The Koebe one quarter theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w \qquad (|w| < r_0(f), \ r_0(f) \geq \tfrac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since $f \in \Sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{1.2}$$

We notice that the class Σ is not empty. For example, the functions z, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of Σ. However, the Koebe function is not a member of Σ. In fact, Srivastava *et al.* [15] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by (see [2, 3, 9, 15, 16, 17]).

An analytic function f is subordinate to an analytic function F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, provided there is an analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F(\omega(z))$. It follows from Schwarz Lemma that

$$f(z) \prec F(z) \quad \Longleftrightarrow \quad f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}), \, z \in \mathbb{U}$$

(for details see [4, 8]). We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Recently, in [14], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition:

Definition 1.1. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$.

It should be observed $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^*$.

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \le r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n \tau + u_{n-1}.$$

In [11] Raina and Sokół showed that

$$\begin{aligned} \tilde{p}(z) &= \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} = \left(t+\frac{1}{t}\right)\frac{t}{1-t-t^2} \\ &= \frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t} - \frac{1}{1-\tau t}\right) \\ &= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n, \end{aligned} \tag{1.3}$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1-\sqrt{5}}{2}, \quad t = \tau z \qquad (n = 1, 2, \ldots). \tag{1.4}$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \cdots$. And they got

$$\begin{aligned} \tilde{p}(z) &= 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n \\ &= 1 + (u_0+u_2)\tau z + (u_1+u_3)\tau^2 z^2 + \sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau^n z^n \\ &= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots . \end{aligned} \tag{1.5}$$

Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $Re\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1.2. *[6] The function* $\tilde{p}(z) = \dfrac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$ *belongs to the class* $\mathcal{P}(\beta)$ *with* $\beta = \sqrt{5}/10 \approx 0.2236$.

Now we give the following lemma which will use in proving.

Lemma 1.3. *[10] Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *then*

$$|c_n| \leq 2, \qquad for \qquad n \geq 1. \tag{1.6}$$

2 Bi-Univalent function class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$

In this section, we introduce a new subclass of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class by subordination.

Firstly, let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}$. Then there exists an analytic function u such that $|u(z)| < 1$ in $\mathbb{U}$ and $p(z) = \tilde{p}(u(z))$. Therefore, the function

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.1}$$

is in the class $\mathcal{P}$. It follows that

$$u(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \tag{2.2}$$

and

$$\begin{aligned} \tilde{p}(u(z)) \quad = \quad & 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_1 + \frac{c_1^2}{4}\tilde{p}_2\right\} z^2 \\ & + \left\{\frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_2 + \frac{c_1^3}{8}\tilde{p}_3\right\} z^3 + \cdots . \end{aligned} \tag{2.3}$$

And similarly, there exists an analytic function v such that $|v(w)| < 1$ in $\mathbb{U}$ and $p(w) = \tilde{p}(v(w))$. Therefore, the function

$$k(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + \cdots \tag{2.4}$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w) = \frac{d_1 w}{2} + \left(d_2 - \frac{d_1^2}{2}\right)\frac{w^2}{2} + \left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\frac{w^3}{2} + \cdots \tag{2.5}$$

and

$$\begin{aligned} \tilde{p}(v(w)) \quad = \quad & 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left\{\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_1 + \frac{d_1^2}{4}\tilde{p}_2\right\} w^2 \\ & + \left\{\frac{1}{2}\left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_2 + \frac{d_1^3}{8}\tilde{p}_3\right\} w^3 + \cdots . \end{aligned} \tag{2.6}$$

The class $\mathcal{L}_\lambda(\alpha)$ of λ-pseudo-starlike functions of order α $(0 \leq \alpha < 1)$ were introduced and investigated by Babalola [1] whose geometric conditions satisfy

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \alpha, \quad \lambda > 0.$$

He showed that all pseudo-starlike functions are Bazilevič of type $\left(1-\frac{1}{\lambda}\right)$ order $\alpha^{\frac{1}{\lambda}}$ and univalent in open unit disk $\mathbb{U}$. If $\lambda = 1$, we have the class of starlike functions of order α, which in this context, are 1-pseudo-starlike functions of order α. A function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\mathbb{U}$ if for every r close to 1, $r < 1$ and every z_0 on $|z| = r$ the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. This class was introduced and studied by Sakaguchi [13] presented the class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points. This class consists of functions $f(z) \in \mathcal{S}$ satisfying the condition

$$\Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > 0, \quad z \in \mathbb{U}.$$

Motivated by $\mathcal{S}_s^*$, Wang *et al.* [18] introduced the class $\mathcal{K}_s$ of functions convex with respect to symmetric points, which consists of functions $f(z) \in \mathcal{S}$ satisfying the condition

$$\Re\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right) > 0, \quad z \in \mathbb{U}.$$

It is clear that, if $f(z) \in \mathcal{K}_s$, then $zf'(z) \in \mathcal{S}_s^*$. For such a function ϕ, Ravichandran [12] presented the following subclasses: A function $f \in A$ is in the class $\mathcal{S}_s^*(\phi)$ if

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \phi(z), \quad z \in \mathbb{U},$$

and in the class $\mathcal{K}_s(\phi)$ if

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \phi(z) \quad z \in \mathbb{U}.$$

Motivated by aforementioned works [1, 13, 12, 18] and recent study of Sokól [14] (also see [11]), in this paper we define the following new subclass $f \in \mathcal{PSL}_{s,\Sigma}^\lambda(\tilde{p}(z))$ of Σ named as λ-bi-pseudo-starlike functions with respect to symmetric points, related to shell-like curves connected with Fibonacci numbers, and determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Further we determine the Fekete-Szegö result for the function class $\mathcal{PSL}_{s,\Sigma}^\lambda(\tilde{p}(z))$ and the special cases are stated as corollaries which are new and have not been studied so far.

Definition 2.1. *For $0 \leq \alpha \leq 1; \lambda > 0; \lambda \neq \frac{1}{3}$,a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}_{s,\Sigma}^\lambda(\alpha, \tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2z(f'(z))^\lambda}{f(z)-f(-z)}\right)^\alpha \left(\frac{2[(z(f'(z)))']^\lambda}{[f(z)-f(-z)]'}\right)^{1-\alpha} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{2.7}$$

and

$$\left(\frac{2w(g'(w))^\lambda}{g(w)-g(-w)}\right)^\alpha \left(\frac{2[(w(g'(w)))']^\lambda}{[g(w)-g(-w)]'}\right)^{1-\alpha} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \tag{2.8}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Specializing the parameter $\lambda = 1$ we have the following definitions, respectively:

Definition 2.2. *For $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(\alpha, \tilde{p}(z)) \equiv \mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(z(f'(z)))'}{[f(z)-f(-z)]'}\right)^{1-\alpha} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{2.9}$$

and

$$\left(\frac{2wg'(w)}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2(w(g'(w)))'}{[g(w)-g(-w)]'}\right)^{1-\alpha} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{2.10}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Further by specializing the parameter $\alpha = 1$ and $\alpha = 0$ we state the following new classes $\mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ respectively.

Definition 2.3. *A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(1, \tilde{p}(z)) \equiv \mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{2.11}$$

and

$$\frac{2wg'(w)}{g(w)-g(-w)} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{2.12}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Definition 2.4. *A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(0, \tilde{p}(z)) \equiv \mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\frac{2(z(f'(z)))'}{[f(z)-f(-z)]'} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{2.13}$$

and

$$\frac{2(w(g'(w)))'}{[g(w)-g(-w)]'} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{2.14}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Definition 2.5. *For $\lambda > 0; \lambda \neq \frac{1}{3}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right) \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{2.15}$$

and

$$\left(\frac{2w(g'(w))^{\lambda}}{g(w)-g(-w)}\right) \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{2.16}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Definition 2.6. *For $\lambda > 0; \lambda \neq \frac{1}{3}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{GSL}_{s,\Sigma}^{\lambda}(\tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2[(z(f'(z)))']^{\lambda}}{[f(z) - f(-z)]'}\right) \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \tag{2.17}$$

and

$$\left(\frac{2[(w(g'(w)))']^{\lambda}}{[g(w) - g(-w)]'}\right) \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2} \tag{2.18}$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases.

Theorem 2.7. *Let f given by* (1.1) *be in the class $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau}}. \tag{2.19}$$

and

$$|a_3| \leq \frac{2\lambda|\tau|\left[2\lambda(\alpha-2)^2 - \{5\lambda(\alpha-2)^2 + 4 - 3\alpha\}\tau\right]}{(3\lambda-1)(3-2\alpha)\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau\right]} \tag{2.20}$$

where $0 \leq \alpha \leq 1; \lambda > 0$ and $\lambda \neq \frac{1}{3}$.

Proof. Let $f \in \mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$ and $g = f^{-1}$. Considering (2.7) and (2.8), we have

$$\left(\frac{2z(f'(z))^{\lambda}}{f(z) - f(-z)}\right)^{\alpha} \left(\frac{2[(z(f'(z)))']^{\lambda}}{[f(z) - f(-z)]'}\right)^{1-\alpha} = \tilde{p}(u(z)) \tag{2.21}$$

and

$$\left(\frac{2w(g'(w))^{\lambda}}{g(w) - g(-w)}\right)^{\alpha} \left(\frac{2[(w(g'(w)))']^{\lambda}}{[g(w) - g(-w)]'}\right)^{1-\alpha} = \tilde{p}(v(w)) \tag{2.22}$$

for some Schwarz functions u and v where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2). Since

$$\left(\frac{2z[f'(z)]^{\lambda}}{f(z) - f(-z)}\right)^{\alpha} \left(\frac{2[(z(f'(z)))']^{\lambda}}{[f(z) - f(-z)]'}\right)^{1-\alpha}$$
$$= 1 - 2\lambda(\alpha-2)a_2 z + \{[2\lambda^2(\alpha-2)^2 + 2\lambda(3\alpha-4)]a_2^2 + (3\lambda-1)(3-2\alpha)a_3\}z^2 + \cdots$$

and

$$\left(\frac{2w(g'(w))^{\lambda}}{g(w) - g(-w)}\right)^{\alpha} \left(\frac{2[(w(g'(w)))']^{\lambda}}{[g(w) - g(-w)]'}\right)^{1-\alpha}$$
$$= 1 + 2\lambda(\alpha-2)a_2 w + \{[2\lambda^2(\alpha-2)^2 + 2\lambda(5-3\alpha) + 2(2\alpha-3)]a_2^2 + (3\lambda-1)(2\alpha-3)a_3\}w^2 + \cdots.$$

Thus we have

$$\begin{aligned}
&1-2\lambda(\alpha-2)a_2z+\{[2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)]a_2^2+(3\lambda-1)(3-2\alpha)a_3\}z^2+\cdots\\
=\ &1+\frac{\tilde{p}_1c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right]z^2\\
&+\left[\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right]z^3+\cdots. \qquad (2.23)
\end{aligned}$$

and

$$\begin{aligned}
&1+2\lambda(\alpha-2)a_2w+\{[2\lambda^2(\alpha-2)^2+2\lambda(5-3\alpha)+2(2\alpha-3)]a_2^2+(3\lambda-1)(2\alpha-3)a_3\}w^2\\
=\ &1+\frac{\tilde{p}_1d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right]w^2\\
&+\left[\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right]w^3+\cdots. \qquad (2.24)
\end{aligned}$$

It follows from (1.5), (2.23) and (2.24) that

$$-2\lambda(\alpha-2)a_2=\frac{c_1\tau}{2}, \qquad (2.25)$$

$$[2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)]a_2^2+(3\lambda-1)(3-2\alpha)a_3=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{3}{4}c_1^2\tau^2, \qquad (2.26)$$

and

$$2\lambda(\alpha-2)a_2=\frac{d_1\tau}{2}, \qquad (2.27)$$

$$[2\lambda^2(\alpha-2)^2+2\lambda(5-3\alpha)+2(2\alpha-3)]a_2^2+(3\lambda-1)(2\alpha-3)a_3=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{3}{4}d_1^2\tau^2. \qquad (2.28)$$

From (2.25) and (2.27), we have

$$c_1=-d_1, \qquad (2.29)$$

and

$$a_2^2=\frac{(c_1^2+d_1^2)}{32\lambda^2(\alpha-2)^2}\tau^2. \qquad (2.30)$$

Now, by summing (2.26) and (2.28), we obtain

$$\left[4\lambda^2(\alpha-2)^2+2(\lambda+2\alpha-3)\right]a_2^2=\frac{1}{2}(c_2+d_2)\tau-\frac{1}{4}(c_1^2+d_1^2)\tau+\frac{3}{4}(c_1^2+d_1^2)\tau^2. \qquad (2.31)$$

By putting (2.30) in (2.31), we have

$$2\left[8\lambda^2(\alpha-2)^2-\{20\lambda^2(\alpha-2)^2-2(\lambda+2\alpha-3)\}\tau\right]a_2^2=(c_2+d_2)\tau^2. \qquad (2.32)$$

Therefore, using Lemma 1.3 we obtain

$$|a_2|\leq\frac{|\tau|}{\sqrt{4\lambda^2(\alpha-2)^2-\{10\lambda^2(\alpha-2)^2-\lambda-2\alpha+3\}\tau}}. \qquad (2.33)$$

Now, so as to find the bound on $|a_3|$, let's subtract from (2.26) and (2.28). So, we find

$$2(3\lambda-1)(3-2\alpha)a_3-2(3\lambda-1)(3-2\alpha)a_2^2=\frac{1}{2}\left(c_2-d_2\right)\tau. \qquad (2.34)$$

Hence, we get

$$2(3\lambda-1)(3-2\alpha)|a_3| \leq 2|\tau| + 2(3\lambda-1)(3-2\alpha)|a_2|^2. \tag{2.35}$$

Then, in view of (2.33), we obtain

$$|a_3| \leq \frac{2\lambda|\tau|\left[2\lambda(\alpha-2)^2 - \{5\lambda(\alpha-2)^2+4-3\alpha\}\tau\right]}{(3\lambda-1)(3-2\alpha)\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2-\lambda-2\alpha+3\}\tau\right]}. \tag{2.36}$$

□

If we can take the parameter $\lambda = 1$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$.

Corollary 2.8. *Let f given by (1.1) be in the class $\mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4(\alpha-2)^2 - 2(5\alpha^2-21\alpha+21)\tau}} \tag{2.37}$$

and

$$|a_3| \leq \frac{|\tau|\left[2(\alpha-2)^2 - \{5\alpha^2-23\alpha+24\}\tau\right]}{(3-2\alpha)\left[4(\alpha-2)^2 - \{10\alpha^2-42\alpha+42\}\tau\right]}. \tag{2.38}$$

Further by taking $\alpha = 1$ and $\alpha = 0$ and $\tau = -0.618$ in Corollary 2.8, we have the following improved initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ respectively.

Corollary 2.9. *Let f given by (1.1) be in the class $\mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-10\tau}} \simeq 0.19369 \tag{2.39}$$

and

$$|a_3| \leq \frac{|\tau|(1-3\tau)}{2-5\tau} \simeq 0.3465. \tag{2.40}$$

Corollary 2.10. *Let f given by (1.1) be in the class $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{16-42\tau}} \simeq 0.0954 \tag{2.41}$$

and

$$|a_3| \leq \frac{4|\tau|(1-3\tau)}{3(8-21\tau)} \simeq 0.17647. \tag{2.42}$$

Corollary 2.11. *Let f given by (1.1) be in the class $\mathcal{PSL}^\lambda_{s,\Sigma}(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4\lambda^2 - \{10\lambda^2-\lambda+1\}\tau}} \tag{2.43}$$

and

$$|a_3| \leq \frac{2\lambda|\tau|\left[2\lambda - \{5\lambda+1\}\tau\right]}{(3\lambda-1)\left[4\lambda^2 - \{10\lambda^2-\lambda+1\}\tau\right]} \tag{2.44}$$

where $\lambda > 0$ and $\lambda \neq \frac{1}{3}$.

Corollary 2.12. *Let f given by* (1.1) *be in the class* $\mathcal{GSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{16\lambda^2 - \{40\lambda^2 - \lambda + 3\}\tau}} \tag{2.45}$$

and

$$|a_3| \leq \frac{2\lambda|\tau|\left[8\lambda - \{20\lambda + 4\}\tau\right]}{3(3\lambda - 1)\left[16\lambda^2 - \{40\lambda^2 - \lambda + 3\}\tau\right]} \tag{2.46}$$

where $\lambda > 0$ *and* $\lambda \neq \frac{1}{3}$.

3 Fekete-Szegö inequality for the function class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$

Fekete and Szegö [7] introduced the generalized functional $|a_3 - \mu a_2^2|$, where μ is some real number. Due to Zaprawa [19], in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$.

Theorem 3.1. *Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let f given by (1.1) be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)} \end{cases}$$

where

$$h(\mu) = \frac{(1 - \mu)\tau^2}{4\left[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}\tau\right]}. \tag{3.1}$$

Proof. From (2.32) and (2.34) we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1 - \mu)(c_2 + d_2)\tau^2}{4\left[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}\tau\right]} + \frac{\tau(c_2 - d_2)}{4(3\lambda - 1)(3 - 2\alpha)} \\ &= \left(\frac{(1 - \mu)\tau^2}{4\left[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}\tau\right]} + \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)}\right) c_2 \\ &+ \left(\frac{(1 - \mu)\tau^2}{4\left[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}\tau\right]} - \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)}\right) d_2. \end{aligned}$$

So we have

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)}\right) c_2 + \left(h(\mu) - \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)}\right) d_2 \tag{3.2}$$

where

$$h(\mu) = \frac{(1 - \mu)\tau^2}{4\left[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}\tau\right]}.$$

Then, by taking modulus of (3.2), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)}. \end{cases}$$ □

Taking $\mu = 1$, we have the following corollary.

Corollary 3.2. *If* $f \in \mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$, *then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{4(3\lambda-1)(3-2\alpha)}. \tag{3.3}$$

If we can take the parameter $\lambda = 1$ in Theorem 3.1, we can state the following:

Corollary 3.3. *Let f given by (1.1) be in the class $\mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{8(3-2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{8(3-2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{8(3-2\alpha)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4(\alpha-2)^2 - \{10(\alpha-2)^2 - 2\alpha + 2\}\tau\right]}.$$

Further by fixing $\lambda = 1$ taking $\alpha = 1$ and $\alpha = 0$ in the above corollary, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}^{*}_{s,\Sigma}(\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$, respectively.

Corollary 3.4. *Let f given by (1.1) be in the class $\mathcal{SL}^{*}_{s,\Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{24}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{24} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{24} \end{cases}$$

where $h(\mu) = \dfrac{(1-\mu)\tau^2}{8\left[2 - 5\tau\right]}$.

Corollary 3.5. *Let f given by (1.1) be in the class $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{8}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{8} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{8} \end{cases}$$

where $h(\mu) = \dfrac{(1-\mu)\tau^2}{8\left[8 - 21\tau\right]}$.

By assuming $\lambda \in \mathbb{R}; \lambda > \frac{1}{3}$ and taking $\alpha = 1$ and $\alpha = 0$ we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ and $\mathcal{GSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$, respectively.

Corollary 3.6. *Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let f given by (1.1) be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{4(3\lambda-1)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda-1)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda-1)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4\lambda^2 - \{10\lambda^2 - \lambda + 1\}\tau\right]}.$$

Corollary 3.7. *Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let f given by (1.1) be in the class $\mathcal{GSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{12(3\lambda-1)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{12(3\lambda-1)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{12(3\lambda-1)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[16\lambda^2 - \{40\lambda^2 - \lambda + 3\}\tau\right]}.$$

Conclusions

Our motivation is to get many interesting and fruitful usages of a wide variety of Fibonacci numbers in Geometric Function Theory. By defining a subclass λ-bi-pseudo-starlike functions with respect to symmetric points of Σ related to shell-like curves connected with Fibonacci numbers we were able to unify and extend the various classes of analytic bi-univalent function, and new extensions were discussed in detail. Further, by specializing $\alpha = 0$ and $\alpha = 1$ and $\tau = -0.618$ we have attempted at the discretization of some of the new and well-known results. Our main results are new and better improvement to initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

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References

[1] K. O. Babalola, "On λ-pseudo-starlike functions", J. Class. Anal., vol. 3, no. 2, pp. 137–147, 2013.

[2] D. A. Brannan, J. Clunie and W. E. Kirwan, "Coefficient estimates for a class of star-like functions", Canad. J. Math., vol. 22, no. 3, pp. 476-485, 1970.

[3] D. A. Brannan and T. S. Taha, "On some classes of bi-univalent functions", Studia Univ. Babes-Bolyai Math., vol. 31, no. 2, pp. 70-77, 1986.

[4] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, New York, Berlin, Heidelberg and Tokyo: Springer-Verlag, 1983.

[5] S. Joshi, S. Joshi and H. Pawar, "On some subclasses of bi-univalent functions associated with pseudo-starlike function", J. Egyptian Math. Soc., vol. 24, no. 4, pp. 522-525, 2016.

[6] J. Dziok, R. K. Raina and J. Sokół, "On $\alpha-$convex functions related to a shell-like curve connected with Fibonacci numbers", Appl. Math. Comput., vol. 218, no. 3, pp. 996–1002, 2011.

[7] M. Fekete and G. Szegö, "Eine Bemerkung über ungerade Schlichte Funktionen", J. London Math. Soc., vol. 8, no. 2, pp. 85-89, 1933.

[8] S. S. Miller and P. T. Mocanu *Differential Subordinations Theory and Applications,* Series of Monographs and Text Books in Pure and Applied Mathematics, vol. 225, New York: Marcel Dekker, 2000.

[9] M. Lewin, "On a coefficient problem for bi-univalent functions", Proc. Amer. Math. Soc., vol. 18, pp. 63-68, 1967.

[10] Ch. Pommerenke, *Univalent Functions*, Math. Math, Lehrbucher, Vandenhoeck and Ruprecht, Göttingen, 1975.

[11] R. K. Raina and J. Sokół, "Fekete-Szegö problem for some starlike functions related to shell-like curves", Math. Slovaca, vol. 66, no. 1, pp. 135-140, 2016.

[12] V. Ravichandran, "Starlike and convex functions with respect to conjugate points", Acta Math. Acad. Paedagog. Nyházi. (N.S.), vol. 20, no. 1, pp. 31-37, 2004.

[13] K. Sakaguchi, "On a certain univalent mapping", J. Math. Soc. Japan, vol. 11, no. 1, pp. 72-75, 1959.

[14] J. Sokół, "On starlike functions connected with Fibonacci numbers", Zeszyty Nauk. Politech. Rzeszowskiej Mat, vol. 23, pp. 111-116, 1999.

[15] H. M. Srivastava, A. K. Mishra and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions", Appl. Math. Lett., vol. 23, no. 10, pp. 1188-1192, 2010.

[16] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, "Coefficient estimates for a certain subclass of analytic and bi-univalent functions", Appl. Math. Lett., vol. 25, no. 6, pp. 990-994, 2012.

[17] X.-F. Li and A.-P. Wang, "Two new subclasses of bi-univalent functions", Int. Math. Forum, vol. 7, no. 30, pp. 1495-1504, 2012.

[18] G. Wang, C. Y. Gao and S. M. Yuan, "On certain subclasses of close-to-convex and quasi-convex functions with respect to $k-$symmetric points", J. Math. Anal. Appl., vol. 322, no. 1, pp. 97–106, 2006.

[19] P. Zaprawa, "On the Fekete-Szegö problem for classes of bi-univalent functions", Bull. Belg. Math. Soc. Simon Stevin, vol. 21, no. 1, pp. 169-178, 2014.