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Extension of Exton's hypergeometric function K_{16}

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ABSTRACT

The purpose of this article is to introduce an extension of Exton's hypergeometric function K_{16} by using the extended beta function given by Özergin *et al.* [11]. Some integral representations, generating functions, recurrence relations, transformation formulas, derivative formula and summation formulas are obtained for this extended function. Some special cases of the main results of this paper are also considered.

RESUMEN

El propósito de este artículo es introducir una extensión de la función hipergeométrica de Exton K_{16} usando la función beta extendida dada por Özergin *et al.* [11]. Se obtienen algunas representaciones integrales, funciones generatrices, relaciones de recurrencia, fórmulas de transformación, fórmulas de derivadas y fórmulas de sumación para esta función extendida. Se consideran también algunos casos especiales de los resultados principales de este artículo.

Keywords and Phrases: Extended beta function, Extended Exton's function, Integral representations, Generating functions, Recurrence relation, Transformation formula, Derivative formula, Summation formula.

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1 Introduction

In recent years, some extensions of beta function and Gauss hypergeometric function have been considered by several authors (see [3, 5, 6, 11]). The following extended beta function and extended Gauss hypergeometric function are introduced by Özergin *et al.* [11]:

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{1.1}$$

$$(\Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(p) \geq 0,\ \Re(x) > 0,\ \Re(y) > 0)$$

and

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.2}$$

$$(\Re(c) > \Re(b) > 0,\ |z| < 1).$$

They [11] presented the following integral representation:

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{1.3}$$

$$\Re(p) > 0;\ p = 0 \quad \text{and} \quad |arg(1-z)| < \pi;\ \Re(c) > \Re(b) > 0.$$

Clearly, we have

$$B_0^{(\alpha,\beta)}(x,y) = B(x,y)$$

and

$$F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z),$$

where $B(x,y)$ and ${}_2F_1(z,b;c;z)$ are the classical beta function and Gauss hypergeometric function defined by (see [13])

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \Re(x) > 0, \quad \Re(y) > 0 \tag{1.4}$$

and

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \quad c \neq 0,-1,-2,\ldots, \tag{1.5}$$

where $(\lambda)_n$ $(n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\})$ denotes the Pochhammer's symbol defined by [13]

$$(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.6}$$

Many authors have considered certain interesting extensions of some hypergeometric functions of two and three variables (see [1, 2, 8, 10]). By using the extended beta function in (1.1), Liu [8] defined the extended Appell's function F_1 as follows:

$$F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n,d-a)(b)_m(c)_n}{B(a,d-a)}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.7}$$

and obtained the following integral representation:

$$F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y) = \frac{1}{B(a,d-a)} \times \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt. \tag{1.8}$$

Observe that

$$F_{1,0}^{(\alpha,\beta)}(a,b,c;d;x,y) = F_1(a,b,c;d;x,y),$$

where $F_1(a,b,c;d;x,y)$ is Appell's hypergeometric function [13]

$$F_1(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{1.9}$$

The Exton's hypergeometric function K_{16} is defined by [7] as follows:

$$K_{16}(a_1,a_2,a_3,a_4;b;x,y,z,t) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+p}(a_3)_{n+q}(a_4)_{p+q}}{(b)_{m+n+p+q}} \frac{x^m\ y^n\ z^p\ t^q}{m!\ n!\ p!\ q!}. \tag{1.10}$$

In this paper, we use the extended beta function given in (1.1) to define extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ as follows:

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(d)_{r+s}\ x^m y^n z^r u^s}{B(a,e-a)(e-a)_{r+s}\ \ m!\ n!\ r!\ s!}. \tag{1.11}$$

The extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ given in (1.11) can be written as follows:

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \sum_{r,s=0}^{\infty} \frac{(d)_{r+s}(b)_r(c)_s}{(e)_{r+s}} F_{1,p}^{(\alpha,\beta)}(a,b+r,c+s;e+r+s;x,y)\frac{z^r\ u^s}{r!\ s!}. \tag{1.12}$$

Observe that:

- The special case $d = e-a$ of (1.11) yields the following extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}$:

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}\ x^m\ y^n\ z^r\ u^s}{B(a,e-a)\ m!\ n!\ r!\ s!}. \tag{1.13}$$

- The special case $p = 0$ of (1.11) yields the Exton's hypergeometric function K_{16}

$$K_{16,0}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = K_{16}(a,b,c,d;e;x,y,z,u). \tag{1.14}$$

2 Integral representations

In this section, we present some integral representations for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ in (1.11).

Theorem 2.1. *The integral representations (2.1), (2.4), (2.5) of $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ hold for $\Re(p)>0$, $\Re(e)>\Re(a)>0$; $|x|+|z|<1$, $|y|+|u|<1$ and the others hold for $\Re(p)>0$, $\Re(e)>\Re(a)>\Re(d)>0$; $|x|+|z|<1$, $|y|+|u|<1$:*

$$\begin{aligned} K_{16,p}^{(\alpha,\beta)}&(a,b,c,d;e;x,y,z,u)\\ &=\frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}\\ &\times F_1\left(d,b,c;e-a;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt \end{aligned} \tag{2.1}$$

$$\begin{aligned} K_{16,p}^{(\alpha,\beta)}&(a,b,c,d;e;x,y,z,u)\\ &=\frac{1}{B(a,e-a)}\frac{1}{B(d,e-a-d)}\int_0^1\int_0^1 t^{a-1}s^{d-1}(1-t)^{e-a-1}(1-s)^{e-a-d-1}\\ &\times(1-xt-zs(1-t))^{-b}(1-yt-us(1-t))^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dtds \end{aligned} \tag{2.2}$$

$$\begin{aligned} K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)&=\frac{1}{B(a,e-a)\ B(d,e-a-d)}\\ \times\int_0^1\int_0^1 & t^{a-1}s^{d-1}(1-t)^{e-a-1}(1-s)^{e-a-d-1}(1-zs)^{-b}(1-us)^{-c}\\ \times\left(1-\left(\frac{x-zs}{1-zs}\right)t\right)^{-b}&\left(1-\left(\frac{y-us}{1-us}\right)t\right)^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dtds \end{aligned} \tag{2.3}$$

$$\begin{aligned} K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)&=\frac{2}{B(a,e-a)}\\ \times\int_0^{\frac{\pi}{2}}&\sin^{2a-1}\theta\cos^{2e-2a-1}\theta(1-x\sin^2\theta)^{-b}(1-y\sin^2\theta)^{-c}\\ \times F_1\left(d,b,c;e-a;\frac{z\cos^2\theta}{1-x\sin^2\theta},\frac{u\cos^2\theta}{1-y\sin^2\theta}\right)&{}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right)d\theta \end{aligned} \tag{2.4}$$

$$\begin{aligned} K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)&=\frac{1}{B(a,e-a)}\\ \times\int_0^{\infty}&\xi^{a-1}(1+\xi)^{c+b-e}(1+(1-x)\xi)^{-b}(1+(1-y)\xi)^{-c}\\ \times F_1\left(d,b,c;e-a;\frac{z}{1+(1-x)\xi},\frac{u}{1+(1-y)\xi}\right)&{}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right)d\xi. \end{aligned} \tag{2.5}$$

Proof of (2.1). Using (1.1) in (1.11) and interchanging the order of summation and integration, we have

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \frac{1}{B(a,e-a)}\\
&\times \int_0^1 t^{a-1}(1-t)^{e-a-1} \sum_{r,s=0}^{\infty} \frac{(d)_{r+s}(b)_r(c)_s(z(1-t))^r(u(1-t))^s}{(e-a)_{r+s}\ r!\ s!}\\
&\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)\left(\sum_{m=0}^{\infty}\frac{(b+r)_m(xt)^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{(c+s)_n(yt)^n}{n!}\right)dt\\
&= \frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)\left\{\sum_{r,s=0}^{\infty}\frac{(d)_{r+s}(b)_r(c)_s}{(e-a)_{r+s}\ r!\ s!}\left(\frac{z(1-t)}{1-xt}\right)^r\left(\frac{u(1-t)}{1-yt}\right)^s\right\}dt,
\end{aligned}$$

which by applying the definition of Appell hypergeometric function F_1 (1.9), we have the desired result (2.1). The integral representation (2.2) can be obtained easily from (2.1) by using the following integral representation of F_1 [12]:

$$F_1(a,b,c;d;x,y) = \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}dt. \tag{2.6}$$

Also the integral representation (2.3) can be obtained directly from (2.2) if we use the following relation:

$$(1-xt-z(1-t))^{-a} = (1-z)^{-a}\left(1-\frac{(x-z)t}{1-z}\right)^{-a}. \tag{2.7}$$

Finally, the integral representations (2.4) and (2.5) can be easily obtained by taking the transformations $t=\sin^2\theta$ and $t=\frac{\xi}{1+\xi}$ in (2.1), respectively. This completes the proof of theorem 2.1. □

The special case $d=e-a$ of (2.1), (2.4) and (2.5), yields the following results:

Corollary 2.2.

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&\quad= \frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}\\
&\quad\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = \frac{2}{B(a,e-a)}\\
&\times \int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\cos^{2e-2a-1}\theta(1-x\sin^2\theta-z\cos^2\theta)^{-b}(1-y\sin^2\theta-u\cos^2\theta)^{-c}\\
&\times {}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right)d\theta
\end{aligned} \tag{2.9}$$

and

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = \frac{1}{B(a,e-a)} \times \int_0^\infty \xi^{a-1}(1+\xi)^{c+b-e}(1+(1-x)\xi-z)^{-b}(1+(1-y)\xi-u)^{-c} \times {}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right)d\xi. \tag{2.10}$$

3 Generating functions

In this section, we derive certain generating functions for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ in (1.11).

Theorem 3.1. *The following generating functions holds true:*

$$\sum_{k=0}^\infty \frac{(b)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a,b+k,c,d;e;x,y,z,u) = (1-t)^{-b} K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;\frac{x}{1-t},y,\frac{z}{1-t},u\right) \tag{3.1}$$

$$\sum_{k=0}^\infty \frac{(c)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a,b,c+k,d;e;x,y,z,u) = (1-t)^{-c} K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;x,\frac{y}{1-t},z,\frac{u}{1-t}\right) \tag{3.2}$$

$$\sum_{k=0}^\infty \frac{(d)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a,b,c,d+k;e;x,y,z,u) = (1-t)^{-d} K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;x,y,\frac{z}{1-t},\frac{u}{1-t}\right). \tag{3.3}$$

Proof of (3.1). Using (1.11) in the L.H.S. of equation (3.1), we get

$$\sum_{k=0}^\infty \frac{(b)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a,b+k,c,d;e;x,y,z,u)$$
$$= \sum_{m,n,r,s,k=0}^\infty \frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r+k}(c)_{n+s}(d)_{r+s}\; x^m y^n z^r u^s t^k}{B(a,e-a)(e-a)_{r+s}\; m!\; n!\; r!\; s!\; k!}$$
$$= \sum_{m,n,r,s=0}^\infty \frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(d)_{r+s}\; x^m y^n z^r u^s}{B(a,e-a)(e-a)_{r+s}\; m!\; n!\; r!\; s!} \sum_{k=0}^\infty \frac{(b+m+r)_k t^k}{k!}$$
$$= (1-t)^{-b} K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;\frac{x}{1-t},y,\frac{z}{1-t},u\right).$$

This completes the proof of (3.1). The generating functions (3.2) and (3.3) can be proved by a similar method as in the proof of (3.1). □

Setting $p=0$ in (3.1), (3.2) and (3.3), we get known results [4].

Theorem 3.2. *The following generating functions holds true:*

$$\sum_{k=0}^\infty \frac{(\lambda)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a,b,c,-k;e;x,y,z,u) = (1-t)^{-\lambda} K_{16,p}^{(\alpha,\beta)}\left(a,b,c,\lambda;e;x,y,\frac{-zt}{1-t},\frac{-ut}{1-t}\right) \tag{3.4}$$

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a, b, -k, d; e; x, y, z, u) = (1-t)^{-\lambda} K_{16,p}^{(\alpha,\beta)}\left(a, b, \lambda, d; e; x, \frac{-yt}{1-t}, z, \frac{-ut}{1-t}\right) \quad (3.5)$$

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a, -k, c, d; e; x, y, z, u) = (1-t)^{-\lambda} K_{16,p}^{(\alpha,\beta)}\left(a, \lambda, c, d; e; \frac{-xt}{1-t}, y, \frac{-zt}{1-t}, u\right). \quad (3.6)$$

Proof of (3.4). Using (1.11) in the L.H.S. of equation (3.4) and using the result [13]

$$(-k)_r = \frac{(-1)^r k!}{(k-r)!}, \quad 0 \le r \le k, \quad (3.7)$$

we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} K_{16,p}^{(\alpha,\beta)}(a, b, c, -k; e; x, y, z, u) \\
&= \sum_{m,n,k=0}^{\infty} \sum_{r=0}^{k} \sum_{s=0}^{k-r} \frac{B_p^{(\alpha,\beta)}(a+m+n, e-a+r+s)(b)_{m+r}(c)_{n+s}(\lambda)_k \; x^m y^n (-z)^r (-u)^s t^k}{B(a, e-a)(e-a)_{r+s} \; m! \; n! \; r! \; s! \; (k-r-s)!} \\
&= \sum_{m,n,r,s,k=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n, e-a+r+s)(b)_{m+r}(c)_{n+s}(\lambda)_{k+r+s} \; x^m y^n (-zt)^r (-ut)^s t^k}{B(a, e-a)(e-a)_{r+s} \; m! \; n! \; r! \; s! \; k!} \\
&= \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n, e-a+r+s)(b)_{m+r}(c)_{n+s}(\lambda)_{r+s} \; x^m y^n (-zt)^r (-ut)^s}{B(a, e-a)(e-a)_{r+s} \; m! \; n! \; r! \; s!} \sum_{k=0}^{\infty} \frac{(\lambda+r+s)_k t^k}{k!} \\
&= (1-t)^{-\lambda} K_{16,p}^{(\alpha,\beta)}\left(a, b, c, \lambda; e; x, y, \frac{-zt}{1-t}, \frac{-ut}{1-t}\right).
\end{aligned}$$

This completes the proof of (3.4). The generating functions (3.5) and (3.6) can be proved by a similar method as in the proof of (3.4). □

4 Recurrence relations

In this section, we deduce some recurrence relations for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a, b, c, d; e; x, y, z, u)$ in (1.11) by using the recurrence relations of the confluent function ${}_1F_1$ and Appell's function F_1.

Theorem 4.1. *The following recurrence relation holds true:*

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a, b, c, d+1; e; x, y, z, u) - K_{16,p}^{(\alpha,\beta)}(a, b, c, d; e; x, y, z, u) \\
&- \frac{bz}{e} K_{16,p}^{(\alpha,\beta)}(a, b+1, c, d+1; e+1; x, y, z, u) - \frac{cu}{e} K_{16,p}^{(\alpha,\beta)}(a, b, c+1, d+1; e+1; x, y, z, u) = 0
\end{aligned} \quad (4.1)$$

Proof. To prove Theorem 4.1, we consider the following recurrence relation of Appell's function F_1 [14]:

$$\begin{aligned}
F_1(\alpha+1, \beta_1, \beta_2; \gamma; x, y) - F_1(\alpha, \beta_1, \beta_2; \gamma; x, y) - \frac{x\beta_1}{\gamma} F_1(\alpha+1, \beta_1+1, \beta_2; \gamma+1; x, y) \\
- \frac{y\beta_2}{\gamma} F_1(\alpha+1, \beta_1, \beta_2+1; \gamma+1; x, y) = 0
\end{aligned} \quad (4.2)$$

In (4.2) replacing α, β_1, β_2, γ, x, y by d, b, c, $e-a$, $\frac{z(1-t)}{1-xt}$, $\frac{u(1-t)}{1-yt}$ respectively, multiplying both sides by $\frac{1}{B(a,e-a)}t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)$ and integrating the resulting equation with respect to t between the limits 0 to 1, we get

$$\begin{aligned}
&\frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\times F_1\left(d+1,b,c;e-a;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\
&-\frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\times F_1\left(d,b,c;e-a;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\
&-\frac{bz}{(e-a)B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a}(1-xt)^{-b-1}(1-yt)^{-c}\\
&\times F_1\left(d+1,b+1,c;e-a+1;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\
&-\frac{cu}{(e-a)B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a}(1-xt)^{-b}(1-yt)^{-c-1}\\
&\times F_1\left(d+1,b,c+1;e-a+1;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt=0.
\end{aligned}$$

Finally, using the integral representation (2.1), we get the desired result (4.1). $\square$

Theorem 4.2. *The following recurrence relations hold true:*

(i)

$$\begin{aligned}
&(\beta-\alpha)K_{16,p}^{(\alpha-1,\beta)}(a,b,c,e-a;e;x,y,z,u)-\alpha K_{16,p}^{(\alpha+1,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&+(2\alpha-\beta)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&+\frac{pB(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)=0
\end{aligned}\tag{4.3}$$

(ii)

$$\begin{aligned}
&\beta(\beta-1)K_{16,p}^{(\alpha,\beta-1)}(a,b,c,e-a;e;x,y,z,u)-\beta(\beta-1)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&-\frac{\beta pB(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)\\
&+\frac{p(\alpha-\beta)B(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta+1)}(a-1,b,c,e-a-1;e-2;x,y,z,u)=0
\end{aligned}\tag{4.4}$$

(iii)

$$\begin{aligned}
&\alpha\beta K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)-\alpha\beta K_{16,p}^{(\alpha+1,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&+\frac{p\beta B(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)\\
&-\frac{p(\beta-\alpha)B(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta+1)}(a-1,b,c,e-a-1;e-2;x,y,z,u)=0
\end{aligned}\tag{4.5}$$

(iv)

$$\beta K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) - \beta K_{16,p}^{(\alpha-1,\beta)}(a,b,c,e-a;e;x,y,z,u) + \frac{pB(a-1,e-a-1)}{B(a,e-a)} K_{16,p}^{(\alpha,\beta+1)}(a-1,b,c,e-a-1;e-2;x,y,z,u) = 0 \tag{4.6}$$

(v)

$$(\beta-\alpha-1)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) + \alpha K_{16,p}^{(\alpha+1,\beta)}(a,b,c,e-a;e;x,y,z,u) - (\beta-1)K_{16,p}^{(\alpha,\beta-1)}(a,b,c,e-a;e;x,y,z,u) = 0 \tag{4.7}$$

(vi)

$$\begin{aligned}&(\alpha-1)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)\\ &+ \frac{pB(a-1,e-a-1)}{B(a,e-a)} K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)\\ &+ (\beta-\alpha)K_{16,p}^{(\alpha-1,\beta)}(a,b,c,e-a;e;x,y,z,u)\\ &- (\beta-1)K_{16,p}^{(\alpha,\beta-1)}(a,b,c,e-a;e;x,y,z,u) = 0.\end{aligned} \tag{4.8}$$

Proof. To prove our results in Theorem 4.2, we require the following recurrence relations of the confluent function ${}_1F_1$ [9]:

$$(\beta-\alpha)\,{}_1F_1(\alpha-1;\beta;z) - \alpha\ {}_1F_1(\alpha+1;\beta;z) + (2\alpha-\beta+z)\,{}_1F_1(\alpha;\beta;z) = 0 \tag{4.9}$$

$$\beta(\beta-1)\,{}_1F_1(\alpha;\beta-1;z) - \beta(\beta-1+z)\,{}_1F_1(\alpha;\beta;z) + (\beta-\alpha)z\,{}_1F_1(\alpha;\beta+1;z) = 0 \tag{4.10}$$

$$\beta(\alpha+z)\,{}_1F_1(\alpha;\beta;z) - \alpha\ \beta\ {}_1F_1(\alpha+1;\beta;z) - (\beta-\alpha)z\ {}_1F_1(\alpha;\beta+1;z) = 0 \tag{4.11}$$

$$\beta\ {}_1F_1(\alpha;\beta;z) - \beta\ {}_1F_1(\alpha-1;\beta;z) - z\ {}_1F_1(\alpha;\beta+1;z) = 0 \tag{4.12}$$

$$(\beta-\alpha-1)\,{}_1F_1(\alpha;\beta;z) + \alpha\ {}_1F_1(\alpha+1;\beta;z) - (\beta-1)\,{}_1F_1(\alpha;\beta-1;z) = 0 \tag{4.13}$$

$$(\alpha+z-1)\,{}_1F_1(\alpha;\beta;z) + (\beta-\alpha)\,{}_1F_1(\alpha-1;\beta;z) - (\beta-1)\,{}_1F_1(\alpha;\beta-1;z) = 0. \tag{4.14}$$

Proof of (4.3). In (4.9) replacing z by $-\frac{p}{t(1-t)}$, multiplying both sides by $t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}/B(a,e-a)$ and integrating the resulting equation with respect to t between the limits 0 to 1, we get

$$\begin{aligned}&\frac{\beta-\alpha}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}\,{}_1F_1\left(\alpha-1;\beta;-\frac{p}{t(1-t)}\right)dt\\ &-\frac{\alpha}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\ &+\frac{2\alpha-\beta}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\ &+\frac{p}{B(a,e-a)}\int_0^1 t^{a-2}(1-t)^{e-a-2}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt = 0\end{aligned}$$

Finally, using the integral representation (2.8), we get the desired result (4.3). ∎

The results (4.4)-(4.8) can be proved by a similar method as in the proof of (4.3) and we use here the recurrence relations (4.10)-(4.14). □

5 Transformation, differentiation and summation formulas

In this section, we derive certain transformation, derivative and summation formulas for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ in (1.11).

Theorem 5.1. *The following transformation formula of $K_{16,p}^{\alpha,\beta}(a,b,c,d;e;x,y,z,u)$ holds true:*

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = (1-z)^{-b}(1-u)^{-c}F_{1,p}^{(\alpha,\beta)}\left(a,b,c;e;\frac{x-z}{1-z},\frac{y-u}{1-u}\right). \tag{5.1}$$

Proof. Using (2.7) in (2.8), we have

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = \frac{(1-z)^{-b}(1-u)^{-c}}{B(a,e-a)}$$
$$\times \int_0^1 t^{a-1}(1-t)^{e-a-1}\left(1-\left(\frac{x-z}{1-z}\right)t\right)^{-b}\left(1-\left(\frac{y-u}{1-u}\right)t\right)^{-c} {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,$$

which by using (1.8), we get the desired result (5.1). □

Setting $p=0$ in (5.1), we get a known result [7]

$$K_{16}(a,b,c,e-a;e;x,y,z,u) = (1-z)^{-b}(1-u)^{-c}F_1\left(a,b,c;e;\frac{x-z}{1-z},\frac{y-u}{1-u}\right). \tag{5.2}$$

Theorem 5.2. *The following derivative formula holds true:*

$$\frac{d^{m+n+r+s}}{dx^m\ dy^n\ dz^r\ du^s}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\} = \frac{(a)_{m+n}(b)_{m+r}(c)_{n+s}(d)_{r+s}}{(e)_{m+n+r+s}}$$
$$\times K_{16,p}^{(\alpha,\beta)}(a+m+n,b+m+r,c+n+s,d+r+s;e+m+n+r+s;x,y,z,u). \tag{5.3}$$

Proof. Differentiating (1.11) with respect to x,y,z and u, we have

$$\frac{d}{dx\ dy\ dz\ du}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\}$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{B_p(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(d)_{r+s}x^{m-1}y^{n-1}z^{r-1}u^{s-1}}{B(a,e-a)(e-a)_{r+s}(m-1)!(n-1)!(r-1)!(s-1)!}$$

setting $m\to m+1,\ n\to n+1,\ r\to r+1,\ s\to s+1$ and using the following identities:

$$B(a,e-a) = \frac{e}{a}B(a+1,e-a) = \frac{e(e+1)}{a(a+1)}B(a+2,e-a),$$
$$(a)_{p+q+2} = a(a+1)(a+2)_{p+q},$$

we obtain

$$\frac{d}{dx\ dy\ dz\ du}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\} = \frac{(a)_2(b)_2(c)_2(d)_2}{(e)_2(e-a)_2}$$
$$\times \sum_{m,n,r,s=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+2,e-a+r+s+2)(b+2)_{m+r}(c+2)_{n+s}(d+2)_{r+s}x^m y^n z^r u^s}{B(a+2,e-a)(e-a+2)_{r+s}\ m!\ n!\ r!\ s!}$$

Now using

$$B(a+2,e-a)=\frac{(e)_4}{(e)_2(e-a)_2}B(a+2,e-a+2),$$

we have

$$\frac{d}{dx\,dy\,dz\,du}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\}=\frac{(a)_2(b)_2(c)_2(d)_2}{(e)_4}$$
$$\times K_{16,p}^{(\alpha,\beta)}(a+2,b+2,c+2,d+2;e+4;x,y,z,u).$$

Thus by repeatedly differentiating, we find that the result (5.3) can be derived by induction. □

Theorem 5.3. *The following summation formulas hold true:*

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;1,1,1,1)=\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(a)\Gamma(e-a-d)\Gamma(e-a-b-c)}B_p^{(\alpha,\beta)}(a,e-a-b-c) \tag{5.4}$$

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;1+a+b+d-c;1,1,1,-1)$$
$$=\frac{\Gamma(1-c)\Gamma(1+\frac{1}{2}d)\Gamma(1+a+b+d-c)}{\Gamma(a)\Gamma(1+d)\Gamma(1+b-c)\Gamma(1+\frac{1}{2}d-c)}B_p^{(\alpha,\beta)}(a,d-2c+1). \tag{5.5}$$

Proof. Setting $x=y=z=u=1$ in (2.1) and using the following formula:

$$F_1(a,b,c;d;1,1)=\frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-a)\Gamma(d-b-c)}, \tag{5.6}$$

we get

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;1,1,1,1)=\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(a)\Gamma(e-a-d)\Gamma(e-a-b-c)}$$
$$\times\int_0^1 t^{a-1}(1-t)^{e-a-b-c-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt \tag{5.7}$$

Now, by using (1.1) in (5.7), we obtain the desired result (5.4). The summation formula (5.5) can be obtained easily by putting $e=1+a+b+d-c,\ x=y=z=1,\ u=-1$ in (2.1) and using the formula

$$F_1(a,b,c;1+a+b-c;1,-1)=\frac{\Gamma(1-c)\Gamma(1+\frac{1}{2}a)\Gamma(1+a+b-c)}{\Gamma(1+a)\Gamma(1+b-c)\Gamma(1+\frac{1}{2}a-c)}. \tag{5.8}$$

This completes the proof of the theorem (5.3). □

Setting $p=0$ in (5.4) and (5.5), we get respectively the following summation formulas of Exton's hypergeometric function K_{16}:

$$K_{16}(a,b,c,d;e;1,1,1,1)=\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(e-a-d)\Gamma(e-b-c)} \tag{5.9}$$

and

$$K_{16}(a,b,c,d;1+a+b+d-c;1,1,1,-1)=\frac{\Gamma(1-c)\Gamma(1+\frac{1}{2}d)\Gamma(1+a+b+d-c)\Gamma(d-2c+1)}{\Gamma(1+d)\Gamma(1+b-c)\Gamma(1+\frac{1}{2}d-c)\Gamma(a+d-2c+1)}. \tag{5.10}$$

6 Conclusion

In this paper, we have introduced the extended Exton's hypergeometric function $K_{16,p}^{\alpha,\beta}(a, b, c, d; e; x, y, z, u)$ by using the extended beta function $B_p^{\alpha,\beta}(x, y)$ given by Özergin *et al.* [11]. For this function we have presented some integral representations, generating functions, recurrence relations, transformation formulas, derivative formula and summation formulas. We have also established some a known and new generating functions, transformation formulas, and summation formulas for the classical Exton's hypergeometric function $K_{16}(a, b, c, d; e; x, y, z, u)$.

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