CUBO, A Mathematical Journal
Vol. 24, no. 01, pp. 105–114, April 2022
DOI: 10.4067/S0719-06462022000100105

Some results on the geometry of warped product CR-submanifolds in quasi-Sasakian manifold

Shamsur Rahman

Department of Mathematics, Maulana Azad National Urdu University Polytechnic Satellite Campus Darbhanga Bihar- 846002, India.
shamsur@rediffmail.com

ABSTRACT

The present paper deals with a study of warped product submanifolds of quasi-Sasakian manifolds and warped product CR-submanifolds of quasi-Sasakian manifolds. We have shown that the warped product of the type $M = D_\perp \times_y D_T$ does not exist, where $D_\perp$ and D_T are invariant and anti-invariant submanifolds of a quasi-Sasakian manifold $\bar{M}$, respectively. Moreover we have obtained characterization results for CR-submanifolds to be locally CR-warped products.

RESUMEN

El presente artículo trata de un estudio de subvariedades producto alabeadas de variedades cuasi-Sasakianas y CR-subvariedades producto alabeadas de variedades cuasi-Sasakianas. Hemos mostrado que el producto alabeado de tipo $M = D_\perp \times_y D_T$ no existe, donde $D_\perp$ y D_T son subvariedades invariantes y anti-invariantes de una variedad cuasi-Sasakiana $\bar{M}$, respectivamente. Más aún, hemos obtenido resultados de caracterización para que CR-subvariedades sean localmente CR-productos alabeados.

Keywords and Phrases: Warped product, CR-submanifolds, quasi Sasakian manifold, canonical structure.

2020 AMS Mathematics Subject Classification: 53C25, 53C40.

Accepted: 18 January, 2022
Received: 02 May, 2021

1 Introduction

If (D, g_D) and (E, g_E) are two semi-Riemannian manifolds with metrics g_D and g_E respectively and y a positive differentiable function on D, then the warped product of D and E is the manifold $D \times_y E = (D \times E, g)$, where $g = g_D + y^2 g_E$. Further, let T be tangent to $M = D \times E$ at (p, q). Then we have

$$\|T\|^2 = \|d\pi_1 T\|^2 + y^2 \|d\pi_2 T\|^2$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $D \times E$ onto D and E.

A warped product manifold $D \times_y E$ is said to be trivial if the warping function y is constant. In a warped product manifold, we have

$$\nabla_U V = \nabla_V U = (U \ln y) V \tag{1.1}$$

for any vector fields U tangent to D and V tangent to E [5].

The idea of a warped product manifold was introduced by Bishop and O'Neill [5] in 1969. Chen [2] has studied the geometry of warped product submanifolds in Kaehler manifolds and showed that the warped product submanifold of the type $D_\perp \times_y D_T$ is trivial where D_T and $D_\perp$ are ϕ-invariant and anti-invariant submanifolds of a Sasakian manifold, respectively. Many research articles appeared exploring the existence or nonexistence of warped product submanifolds in different spaces [1, 10, 6]. The idea of CR-submanifolds of a Kaehlerian manifold was introduced by A. Bejancu [9]. Later, A. Bejancu and N. Papaghiue [11], introduced and studied the notion of semi-invariant submanifolds of a Sasakian manifold. These submanifolds are closely related to CR-submanifolds in a Kaehlerian manifold. However the existence of the structure vector field implies some important changes. Later on, Binh and De [4] studied CR-warped product submanifolds of a quasi-Saskian manifold. The purpose of this paper is to study the notion of a warped product submanifold of quasi-Sasakian manifolds. In the second section we recall some results and formulae for later use. In the third section, we prove that the warped product in the form $M = D_\perp \times_y D_T$ does not exist except for the trivial case, where D_T and $D_\perp$ are invariant and anti-invariant submanifolds of a quasi-Sasakian manifold $\bar{M}$, respectively. Also, we obtain a characterization result of the warped product CR-submanifolds of the type $M = D_\perp \times_y D_T$.

2 Preliminaries

If $\bar{M}$ is a real $(2n+1)$ dimensional differentiable manifold, endowed with an almost contact metric structure (f, ξ, η, g), then

$$f^2 U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad f(\xi) = 0, \quad \eta(fU) = 0, \tag{2.1}$$

$$\eta(U) = g(U,\xi), \quad g(fU,fV) = g(U,V) - \eta(U)\eta(V), \tag{2.2}$$

for any vector fields U, V tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$.

Throughout the paper, all manifolds and maps are differentiable of class C^∞. We denote by $\digamma\bar{M}$ the algebra of the differentiable functions on $\bar{M}$ and by $\Gamma(E)$ the $\digamma\bar{M}$ module of the sections of a vector bundle E over $\bar{M}$.

The Nijenhuis tensor field, denoted by N_f, with respect to the tensor field f, is given by

$$N_f(U,V) = [fU,fV] + f^2[U,V] - f[fU,V] + f[U,fV],$$

and the fundamental 2-form Λ is given by

$$\Lambda(U,V) = g(U,fV), \quad \forall U,V \in \Gamma(T\bar{M}).$$

The curvature tensor field of $\bar{M}$, denoted by $\bar{R}$ with respect to the Levi-Civita connection $\bar{\nabla}$, is defined by

$$\bar{R}(U,V)W = \bar{\nabla}_U\bar{\nabla}_V W - \bar{\nabla}_V\bar{\nabla}_U W - \bar{\nabla}_{[U,V]}W, \quad \forall U,V \in \Gamma(T\bar{M}),$$

Definition 2.1.

(a) An almost contact metric manifold $\bar{M}(f,\xi,\eta,g)$ is called normal if

$$N_f(U,V) + 2d\eta(U,V)\xi = 0, \quad \forall U,V \in \Gamma(T\bar{M}),$$

or equivalently

$$(\bar{\nabla}_{fU}f)V = f(\bar{\nabla}_U f)V - g(\bar{\nabla}_U\xi,V)\xi, \quad \forall U,V \in \Gamma(T\bar{M}).$$

(b) The normal almost contact metric manifold $\bar{M}$ is called cosympletic if $d\Lambda = d\eta = 0$.

If $\bar{M}$ is an almost contact metric manifold, then $\bar{M}$ is a quasi-Sasakian manifold if and only if ξ is a Killing vector field [7] and

$$(\bar{\nabla}_U f)V = g(\bar{\nabla}_{fU}\xi,V)\xi - \eta(V)\bar{\nabla}_{fU}\xi, \quad \forall U,V \in \Gamma(T\bar{M}). \tag{2.3}$$

Next we define a tensor field F of type $(1,1)$ by

$$FU = -\bar{\nabla}_U\xi, \quad \forall U \in \Gamma(T\bar{M}). \tag{2.4}$$

Lemma 2.1. *For a quasi-Sasakian manifold $\bar{M}$, we have*

(i) $(\bar{\nabla}_\xi f)U = 0, \quad \forall U \in \Gamma(T\bar{M})$,

(ii) $f \circ F = F \circ f$,

(iii) $F\xi = 0$,

(iv) $g(FU, V) + g(U, FV) = 0$,

(v) $\eta \circ F = 0$,

(vi) $(\bar{\nabla}_U F)V = \bar{R}(\xi, U)V$,

for all $U, V \in \Gamma(T\bar{M})$.

The tensor field f defines on $\bar{M}$ an f-structure in sense of K. Yano [12], that is

$$f^3 + f = 0.$$

If M is a submanifold of a quasi-Sasakian manifold $\bar{M}$ and denote by N the unit vector field normal to M. Denote by the same symbol g the induced tensor metric on M, by ∇ the induced Levi-Civita connection on M and by $TM^\perp$ the normal vector bundle to M. The Gauss and Weingarten methods are

$$\bar{\nabla}_U V = \nabla_U V + \sigma(U, V), \tag{2.5}$$

$$\bar{\nabla}_U \lambda = -A_\lambda U + \nabla^\perp_U \lambda, \quad \forall U, V \in \Gamma(TM), \tag{2.6}$$

where $\nabla^\perp$ is the induced connection in the normal bundle, σ is the second fundamental form of M and A_λ is the Weingarten endomorphism associated with λ. The second fundamental form σ and the shape operator A are related by

$$g(A_\lambda U, V) = g(h(U, V), \lambda), \tag{2.7}$$

where g denotes the metric on $\bar{M}$ as well as the induced metric on M [7].

For any $U \in TM$, we write

$$fU = rU + sU, \tag{2.8}$$

where rU is the tangential component of fU and sU is the normal component of fU, respectively. Similarly, for any vector field λ normal to M, we put

$$f\lambda = J\lambda + K\lambda \tag{2.9}$$

where $J\lambda$ and $K\lambda$ are the tangential and normal components of $f\lambda$, respectively.

For all $U, V \in \Gamma(TM)$ the covariant derivatives of the tensor fields r and s are defined as

$$(\bar{\nabla}_U r)V = \nabla_U rV - r\nabla_U V, \tag{2.10}$$

$$(\bar{\nabla}_U s)V = \nabla^\perp_U sV - s\nabla_U V. \tag{2.11}$$

3 Warped Product Submanifolds

If D_T and $D_\perp$ are invariant and anti-invariant submanifolds of a quasi-Sasakian manifold $\bar{M}$, then their warped product CR-submanifolds are one of the following forms:

(i) $M = D_\perp \times_y D_T$,

(ii) $M = D_T \times_y D_\perp$.

For case (i), when $\xi \in TD_T$, we have the following theorem.

Theorem 3.1. *There do not exist warped product CR-submanifolds $M = D_\perp \times_y D_T$ in a quasi-Sasakian manifold $\bar{M}$ such that D_T is an invariant submanifold, $D_\perp$ is an anti-invariant submanifold of $\bar{M}$ and ξ is tangent to M.*

Proof. If $M = D_\perp \times_y D_T$ is a warped product CR-submanifold of a quasi-Sasakian manifold $\bar{M}$ such that D_T is an invariant submanifold tangent to ξ and $D_\perp$ is an anti-invariant submanifold of $\bar{M}$, then from (1.1), we have

$$\nabla_U W = \nabla_W U = (W \ln y)U,$$

for any vector fields W and U tangent to $D_\perp$ and D_T, respectively.
In particular,

$$\nabla_W \xi = (W \ln y)\xi, \tag{3.1}$$

using (2.4), (2.5) and ξ is tangent to $D_\perp$, we have

$$\nabla_W \xi = -FW, \quad h(W, \xi) = 0. \tag{3.2}$$

It follows from (3.1) and (3.2) that $W \ln y = 0$, for all $W \in TD_\perp$, *i. e.*, y is constant for all $W \in TD_\perp$. □

Now, the other case, when ξ tangent to $D_\perp$ is dealt in the following two results.

Lemma 3.1. *Let $M = D_\perp \times_y D_T$ be a warped product CR-submanifold of a quasi-Sasakian manifold such that ξ is tangent to $D_\perp$, where $D_\perp$ and D_T are any Riemannian submanifolds of $\bar{M}$. Then*

(i) $\xi \ln y = -F$,

(ii) $g(\sigma(U, fU), sW) = -\{\eta(W)F + (W \ln y)\}\|U\|^2$,

for any $U \in TD_T$ and $W \in TD_\perp$.

Proof. Let $\xi \in TD_\perp$ then for any $U \in TD_T$, we have

$$\nabla_U \xi = (\xi \ln y)U, \tag{3.3}$$

From (2.4) and the fact that ξ is tangent to $D_\perp$, we have $\bar{\nabla}_U \xi = -FU$. With the help of (2.5), we have

$$\nabla_W \xi = -FW, \quad h(W, \xi) = 0. \tag{3.4}$$

From (3.3) and (3.4), we have $\xi \ln y = -F$. Now, for any $U \in TD_T$ and $W \in TD_\perp$, we have $\bar{\nabla}_U fW = (\bar{\nabla}_U f)W + f(\bar{\nabla}_U W)$. Using (2.3), (2.6), (2.8), (2.9) and by the orthogonality of the two distributions, we derive

$$-\eta(W)\bar{\nabla}_{fU}\xi = -A_{sW}U + \nabla^\perp_U sW - r\nabla_U W - s\nabla_U W - Jh(U, W) - Kh(U, W).$$

Equating the tangential components, we get

$$-\eta(W)FfU = A_{sW}U + r\nabla_U W + Jh(U, W).$$

Taking the product with fU and using (2.2) and (2.3), we derive

$$\begin{aligned}-\eta(W)Fg(fU, fU) &= g(A_{sW}U, fU) + (W \ln y)g(rU, fU) + g(Jh(U, W), fU)\\ &= g(h(fU, fU), sW) + (W \ln y)g(fU, fU) + g(fh(U, W), fU).\end{aligned}$$

Using (2.2), we obtain

$$g(\sigma(U, fU), sW) = -\{\eta(W)F + (W \ln y)\}\|U\|^2. \tag{3.5}$$

□

Theorem 3.2. *If $M = D_\perp \times_y D_T$ is a warped product CR-submanifold of a quasi-Sasakian manifold $\bar{M}$ such that ξ is tangent to $D_\perp$ and if $\sigma(U, fU) \in \mu$ the invariant normal subbundle of M, then $W \ln y = -\eta(W)F$, for all $U \in TD_T$ and $Z \in TN_\perp$.*

Proof. The affirmation follows from formula (3.5) by means of the known truth. □

The warped product $M = D_T \times_y D_\perp$, we have the following theorem.

Theorem 3.3. *There do not exist warped product CR-submanifolds $M = D_T \times_y D_\perp$ in a quasi-Sasakian manifold $\bar{M}$ such that ξ is tangent to $D_\perp$.*

Proof. If $\xi \in TN_\perp$, then from (1.1), we have

$$\nabla_U \xi = (U \ln y)\xi, \tag{3.6}$$

for any $U \in TD_T$. While using (2.4), (2.5) and $\xi \in TD_\perp$, we have

$$\nabla_U \xi = -FU, \quad h(U, \xi) = 0. \tag{3.7}$$

From (3.6) and (3.7), it follows that $U \ln y = 0$, for all $U \in TD_T$, and this means that y is constant on N_T. □

The remaining case, when $\xi \in TD_T$ is dealt with the following two theorems.

Theorem 3.4. *Let $M = D_T \times_y D_\perp$ be a warped product CR-submanifold of a quasi-Sasakian manifold $\bar{M}$ such that ξ is tangent to D_T. Then $(\bar{\nabla}_U F)W \in \mu$, for each $U \in TD_T$ and $W \in TD_\perp$, where μ is an invariant normal subbundle of TM.*

Proof. For any $U \in TD_T$ and $W \in TD_\perp$, we have

$$g(f\bar{\nabla}_U W, fW) = g(\bar{\nabla}_U W, W) = g(\nabla_U W, W).$$

Using (1.1), we get

$$g(f\bar{\nabla}_U W, fW) = (U \ln y)\|W\|^2. \tag{3.8}$$

On the other hand, we have

$$\bar{\nabla}_U fW = (\bar{\nabla}_U f)W + f(\bar{\nabla}_U W),$$

for any $U \in TD_T$ and $W \in TD_\perp$. Using (2.3) and the fact that ξ is tangent to D_T, the left-hand side of the above equation is identically zero, that is

$$\bar{\nabla}_U fW = f(\bar{\nabla}_U W). \tag{3.9}$$

Taking the product with fW in (3.9) and making use of formula (2.6), we obtain

$$g(f\bar{\nabla}_U W, fW) = g(\nabla^\perp_U sW, sW).$$

Then from (2.10), we derive $g(f\bar{\nabla}_U W, fW) = g((\bar{\nabla}_U s)W, sW) + g(s\nabla_U W, sW)$.

From (1.1) we have

$$\begin{aligned} g(f\bar{\nabla}_U W, fW) &= (U \ln y)g(sW, sW) + g((\bar{\nabla}_U s)W, sW) \\ &= (U \ln y)g(fW, fW) + g((\bar{\nabla}_U s)W, sW). \end{aligned}$$

Therefore by (2.2), we obtain

$$g(f\bar{\nabla}_U W, fW) = (U \ln y)\|W\|^2 + g((\bar{\nabla}_U s)W, sW). \tag{3.10}$$

Thus (3.8) and (3.9) imply

$$g((\bar{\nabla}_U s)W, sW) = 0. \tag{3.11}$$

Also, as D_T is an invariant submanifold then $fQ \in TD_T$, for any $Q \in TD_T$, thus on using (2.11) and the fact that the product of tangential components with normal is zero, we obtain

$$g((\bar{\nabla}_U s)W, fQ) = 0. \tag{3.12}$$

Hence from (3.11) and (3.12), it follows that $(\bar{\nabla}_U s)W \in \mu$, for all $U \in TD_T$ and $W \in TD_\perp$. $\square$

Theorem 3.5. *A CR-submanifold M of a quasi-Sasakian manifold $(\bar{M}, f, \xi, g)$ is a CR-warped product if and only if the shape operator of M satisfies*

$$A_{fW}U = (fU\mu)W, \quad U \in B \oplus \langle \xi \rangle, \quad W \in B^{\perp}, \tag{3.13}$$

for some function μ on M, fulfilling $C(\mu) = 0$, for each $C \in B^{\perp}$.

Proof. If $M = D_T \times_y D_{\perp}$ is a CR-warped product submanifold of a quasi-Sasakian manifold $\bar{M}$, with $\xi \in TD_T$, then for any $U \in TD_T$ and $W, Q \in TD_{\perp}$, we have

$$\begin{aligned} g(A_{fW}U, Q) &= g(\sigma(U, Q), fW) = g(\bar{\nabla}_Q U, fW) = g(f\bar{\nabla}_Q U, W) \\ &= g(\bar{\nabla}_Q fU, W) - g((\bar{\nabla}_Q f)U, W). \end{aligned}$$

By equations (1.1), (2.3) and the fact that ξ is tangent to D_T, we derive

$$g(A_{fW}U, Q) = (fU \ln y) g(W, Q). \tag{3.14}$$

On the other hand, we have $g(\sigma(U, V), sW) = g(f\bar{\nabla}_U V, W) = -g(fV, \bar{\nabla}_U W)$, for each $U, V \in TD_T$ and $W \in TN_{\perp}$. Using (1.1), we obtain $g(\sigma(U, V), sW) = 0$. Taking into account this fact in (3.14), we obtain (3.13).

Conversely, suppose that M is a proper CR-submanifold of a quasi-Sasakian manifold M satisfying (3.13), then for any $U, V \in B \oplus \langle \xi \rangle$,

$$g(\sigma(U, V), fW) = g(A_{fW}U, V) = 0.$$

This implies that $g(\bar{\nabla}_U fV, W) = 0$, that is, $g(\nabla_U V, W) = 0$. This means $B \oplus \langle \xi \rangle$ is integrable and its leaves are totally geodesic in M. Now, for any $W, Q \in B^{\perp}$ and $U \in B \oplus \langle \xi \rangle$, we have

$$g(\nabla_W Q, fU) = g(\bar{\nabla}_W Q, fU) = g(f\bar{\nabla}_W Q, U) = g(\bar{\nabla}_W fQ, U) - g((f\bar{\nabla}_W f)Q, U).$$

By equations (2.3) and (2.6), it follows that $g(\nabla_W Q, fU) = -g(A_{fQ}W, U)$. Thus from (2.6), we arrive at $g(\nabla_W Q, fU) = -g(\sigma(W, U), fQ)$. Again using (2.7) and (3.13), we obtain

$$g(\nabla_W Q, fU) = -g(A_{fQ}U, W) = -(fU\mu) g(W, Q). \tag{3.15}$$

If $N_{\perp}$ is a leaf of $B^{\perp}$ and $\sigma^{\perp}$ is the second fundamental form of the immersion of $D_{\perp}$ into M, then for any $W, Q \in B^{\perp}$, we have

$$g(\sigma^{\perp}(W, Q), fU) = g(\nabla_W Q, fU). \tag{3.16}$$

Hence, from (3.15) and (3.16), we find that

$$g(\sigma^{\perp}(W, Q), fU) = -(fU\mu) g(W, Q).$$

This means that the integral manifold $D_\perp$ of $B^\perp$ is totally umbilical in M. Since $C(\mu) = 0$ for each $C \in B^\perp$, which implies that the integral manifold of $B^\perp$ is an extrinsic sphere in M, this means that the curvature vector field is nonzero and parallel along $N_\perp$. Hence by virtue of a result in [7], M is locally a warped product $D_T \times_y D_\perp$, where D_T and $N_\perp$ denote the integral manifolds of the distributions $B \oplus \langle \xi \rangle$ and $B^\perp$, respectively and y is the warping function. $\square$

Acknowledgements

The authors grateful the referee(s) for the corrections and comments in the revision of this paper.

References

[1] K. Arslan, R. Ezentas, I. Mihai and C. Murathan, "Contact CR-warped product submanifolds in Kenmotsu space forms", J. Korean Math. Soc., vol. 42, no. 5, pp. 1101–1110, 2005.

[2] A. Bejancu, "CR-submanifold of a Kaehler manifold. I", Proc. Amer. Math. Soc., vol. 69, no. 1, 135–142, 1978.

[3] A. Bejancu and N. Papaghiuc, "Semi-invariant submanifolds of a Sasakian manifold.", An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.), vol 27, no. 1, pp. 163–170, 1981.

[4] T.-Q. Binh and A. De, "On contact CR-warped product submanifolds of a quasi-Sasakian manifold", Publ. Math. Debrecen, vol. 84, no. 1-2, pp. 123–137, 2014.

[5] R. L. Bishop and B. O'Neill, "Manifolds of negative curvature", Trans. Amer. Math. Soc., vol. 145, pp. 1–49, 1969.

[6] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., vol. 509, Berlin-New York: Springer-Verlag, 1976.

[7] C. Calin, "Contributions to geometry of CR-submanifold", PhD Thesis, University of Iași, Iași, Romania, 1998.

[8] B.-Y. Chen, "Geometry of warped product CR-submanifolds in Kaehler manifolds", Monatsh. Math., vol. 133, no. 3, pp. 177–195, 2001.

[9] I. Hasegawa and I. Mihai, "Contact CR-warped product submanifolds in Sasakian manifolds", Geom. Dedicata, vol. 102, pp. 143–150, 2003.

[10] S. Hiepko, "Eine innere Kennzeichnung der verzerrten Produkte", Math. Ann., vol. 241, no. 3, pp. 209–215, 1979.

[11] M.-I. Munteanu, "A note on doubly warped product contact CR-submanifolds in trans-Sasakian manifolds", Acta Math. Hungar., vol 116, no. 1-2, pp. 121–126, 2007.

[12] K. Yano, "On structure defined by a tensor field f of type $(1,1)$ satisfying $f^3 + f = 0$", Tensor (N.S.), vol. 14, pp. 99–109, 1963.

[13] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, vol. 3, Singapore: World Scientific Publishing Co., 1984.