Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 24, no. 03, pp. 369–392, December 2022
DOI: 10.56754/0719-0646.2403.0369

Dual digraphs of finite semidistributive lattices

Andrew Craig[1, ✉]
Miroslav Haviar[1, 2]
José São João[3, 4]

[1] *Department of Mathematics and Applied Mathematics, University of Johannesburg, PO Box 524, Auckland Park, 2006, South Africa.*
`acraig@uj.ac.za` ✉

[2] *Department of Mathematics, Faculty of Natural Sciences, M. Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia.*
`miroslav.haviar@umb.sk`

[3] *Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden.*
`zealvro98@outlook.com`

[4] *Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden.*

ABSTRACT

Dual digraphs of finite join-semidistributive lattices, meet-semidistributive lattices and semidistributive lattices are characterised. The vertices of the dual digraphs are maximal disjoint filter-ideal pairs of the lattice. The approach used here combines representations of arbitrary lattices due to Urquhart (1978) and Ploščica (1995). The duals of finite lattices are mainly viewed as TiRS digraphs as they were presented and studied in Craig–Gouveia–Haviar (2015 and 2022). When appropriate, Urquhart's two quasi-orders on the vertices of the dual digraph are also employed. Transitive vertices are introduced and their role in the domination theory of the digraphs is studied. In particular, finite lattices with the property that in their dual TiRS digraphs the transitive vertices form a dominating set (respectively, an in-dominating set) are characterised. A characterisation of both finite meet- and join-semidistributive lattices is provided via minimal closure systems on the set of vertices of their dual digraphs.

Accepted: 22 August, 2022
Received: 01 June, 2022

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RESUMEN

Se caracterizan los digrafos duales de reticulados finitos unión-semidistributivos, encuentro-semidistributivos y semidistributivos. Los vértices de los digrafos duales son pares filtro-ideales disjuntos maximales del reticulado. El enfoque usado combina las representaciones de reticulados arbitrarios de Urquhart (1978) and Ploščica (1995). Los duales de reticulados finitos son vistos principalmente como digrafos TiRS como fueron presentados y estudiados en Craig–Gouveia–Haviar (2015 y 2022). Cuando sea apropiado, también se emplean los dos cuasi-órdenes de Urquhart en los vértices del digrafo dual. Se introducen los vértices transitivos y se estudia su rol en la teoría de dominación de digrafos. En particular, se caracterizan los reticulados finitos con la propiedad que en sus digrafos TiRS duales los vértices transitivos forman un conjunto dominante (respectivamente un conjunto dominante interior). Se entrega una caracterización de reticulados encuentro- y unión-semidistributivos a través de sistemas de clausura mínima en el conjunto de vértices de sus digrafos duales.

Keywords and Phrases: semidistributive lattice, TiRS digraph, join-semidistributive lattice, meet-semidistributive lattice, dual digraph, domination.

2020 AMS Mathematics Subject Classification: 06B15, 06A75, 06D75, 05C20, 05C69.

Accepted: 22 August, 2022
Received: 01 June, 2022

1 Introduction

Semidistributivity was first described by Jónsson [16] while he was studying sublattices of a free lattice. He proved [16, Lemma 2.6] that every free lattice is semidistributive.

A lattice is *join-semidistributive* if it satisfies the following quasi-equation for all $x, y, z \in L$:

$$(\mathrm{SD}_\vee) \;\; x \vee y = x \vee z \quad \Longrightarrow \quad x \vee y = x \vee (y \wedge z).$$

Dually, L is *meet-semidistributive* if it satisfies:

$$(\mathrm{SD}_\wedge) \;\; x \wedge y = x \wedge z \quad \Longrightarrow \quad x \wedge y = x \wedge (y \vee z).$$

A lattice is *semidistributive* if it satisfies both $(\mathrm{SD}_\vee)$ and $(\mathrm{SD}_\wedge)$.

For background on semidistributive lattices we refer to the papers by Adaricheva *et al.* [1] and [2], the chapter by Adaricheva and Nation [3], and the paper by Davey *et al.* [10].

The aim of our paper is to investigate dual digraphs of finite semidistributive lattices. Theorem 3.6 provides a representation of finite semidistributive lattices via a certain class of TiRS digraphs (see Definition 2.4). This theorem is a generalisation of Birkhoff's representation of finite distributive lattices via finite ordered sets [6] (see comments in the next paragraph regarding the distributive case). In addition, we study transitive vertices in the dual digraphs and their role in the domination theory of the digraphs, and also explore closure systems on the set of vertices of the dual digraphs.

We employ representations for finite lattices due to Urquhart [20] and Ploščica [17]. In Urquhart's representation the elements of the dual space are maximal disjoint filter-ideal pairs of the lattice. Urquhart considered two quasi-orders $\leqslant_1$ and $\leqslant_2$ on them and studied the dual of the lattice as a certain doubly (quasi-) ordered space. In Ploščica's representation, the dual space of a lattice L is formed by maximal partial homomorphisms from L into the two-element lattice, which correspond to Urquhart's maximal disjoint filter-ideal pairs of L. When L is a distributive lattice, these maximal partial homomorphisms become total homomorphisms from L into the two-element lattice, which form the Priestley dual of L [18]. The close relationship between Ploščica's representation of general lattices and Priestley's representation of distributive lattices lies in the single binary relation E, which Ploščica considered on his dual space. When L is distributive, E becomes exactly Priestley's order on the dual space. Ploščica's dual space of a finite lattice L is therefore a finite digraph where the vertices are the maximal partial homomorphisms from L into the two-element lattice and the binary relation E, which mimics Priestley's order, forms the edge set of the digraph. These dual digraphs of lattices were presented and studied as TiRS digraphs in two papers by Craig, Gouveia and Haviar [7, 8].

In our approach we combine Urquhart's and Ploščica's representations of finite lattices: the vertices of our dual digraphs are maximal disjoint filter-ideal pairs of the lattice in the Urquhart style, but we mainly study them as TiRS digraphs using the Ploščica binary relation E on the vertices. Only in a small part of our investigation do we swap Ploščica's relation E for Urquhart's two quasi-orders on the vertices to present our results in a different yet rather satisfactory way (the end of Section 3).

In Section 2 we give preliminary results that will prove useful in the subsequent three sections of the paper. In Section 3 we provide several characterisations of the dual digraphs of finite meet-semidistributive, finite join-semidistributive, and finite semidistributive lattices. In Section 4 we introduce transitive vertices in the dual digraphs and we study their role in the domination theory of the digraphs. In particular, we are able to characterise finite lattices having the properties that in their dual TiRS digraphs the transitive vertices form a dominating set, respectively an in-dominating set. In Section 5 we characterise both finite meet-semidistributive and finite join-semidistributive lattices via minimal closure systems on the set of vertices of their dual digraphs.

In Section 6 we make some concluding remarks and observations. In particular, we note connections to other representations of finite semidistributive lattices, and we propose several directions for future research in this area.

2 Preliminaries

Ploščica's representation of arbitrary bounded lattices [17] uses the set of maximal partial homomorphisms (MPHs) from a bounded lattice L to the two-element bounded lattice $(\{0,1\},\wedge,\vee,0,1)$ as the underlying set of the dual space of L. We recall that a *partial homomorphism* from a bounded lattice $(L,\wedge,\vee,0,1)$ into the two-element bounded lattice $(\{0,1\},\wedge,\vee,0,1)$ is a partial map $f : L \to \{0,1\}$ such that $\operatorname{dom} f$ is a bounded sublattice of L and the restriction $f{\restriction}_{\operatorname{dom} f}$ is a bounded lattice homomorphism. A *maximal partial homomorphism* is a partial homomorphism with no proper extension. The set of MPHs is then equipped with a binary relation and a topology.

Definition 2.1 ([20, Section 3]). Let L be a lattice. Then $\langle F, I\rangle$ is a *disjoint filter-ideal pair* of L if F is a filter of L and I is an ideal of L such that $F \cap I = \varnothing$. We say that a disjoint filter-ideal pair $\langle F, I\rangle$ is maximal if there is no disjoint filter-ideal pair $\langle G, J\rangle \neq \langle F, I\rangle$ such that $F \subseteq G$ and $I \subseteq J$. A maximal disjoint filter-ideal pair $\langle F, I\rangle$ of L is *total in* L if $F \cup I = L$.

There is a one-to-one correspondence between the set of MPHs from L to $2 = (\{0,1\},\wedge,\vee,0,1)$ and the maximal disjoint filter-ideal pairs (MDFIPs) of L. The latter were used in the dual representation of Urquhart [20]. We will use a combination of the two approaches: for a lattice L, the elements of our dual set X_L will be MDFIPs, but we will equip the set with the binary relation due to Ploščica, and hence will obtain a digraph. (Later, when desirable, we will also equip the set X_L of all MDFIPs of L with Urquhart's two quasi-orders $\leqslant_1$ and $\leqslant_2$.) We do not require the topologies used by Ploščica and Urquhart because we are only working with finite lattices.

Ploščica's binary relation on the set of MPHs is defined as follows for MPHs f and g from L to 2:

(E1) $fEg \quad \iff \quad (\forall x \in \operatorname{dom} f \cap \operatorname{dom} g)(f(x) \leqslant g(x))$.

The digraph dual to a finite bounded lattice L in Ploščica's representation is $G_L = (V_L, E)$ where the set of vertices V_L is formed by all MPHs from L to 2 and the relation E is defined by (E1) above. We will now present this dual digraph as $G_L = (X_L, E)$ where the set of vertices will be X_L, *i.e.* is formed by all MDFIPs of L, and the corresponding Ploščica relation E will be defined below by (E2).

For two MDFIPs $\langle F, I\rangle$ and $\langle G, J\rangle$, Ploščica's relation E is determined as follows:

(E2) $\langle F, I\rangle E\langle G, J\rangle \quad \iff \quad F \cap J = \emptyset$.

For finite lattices every filter is the up-set of a unique element and every ideal is the down-set of a unique element, so we can represent every disjoint filter-ideal pair $\langle F, I\rangle$ by an ordered pair $\langle {\uparrow}x, {\downarrow}y\rangle$ where $x = \bigwedge F$ and $y = \bigvee I$. Hence for finite lattices we have $\langle {\uparrow}x, {\downarrow}y\rangle E\langle {\uparrow}a, {\downarrow}b\rangle$ if and only if $x \nleqslant b$.

In Figure 1 we present a number of examples of finite (non-distributive) lattices and their dual digraphs. To make the labelling more compact, we denote by xy the MDFIP $\langle {\uparrow}x, {\downarrow}y\rangle$. Also, to keep the display simpler, we have not included the loop on each vertex. Notice that the directed edge set is not a transitive relation.

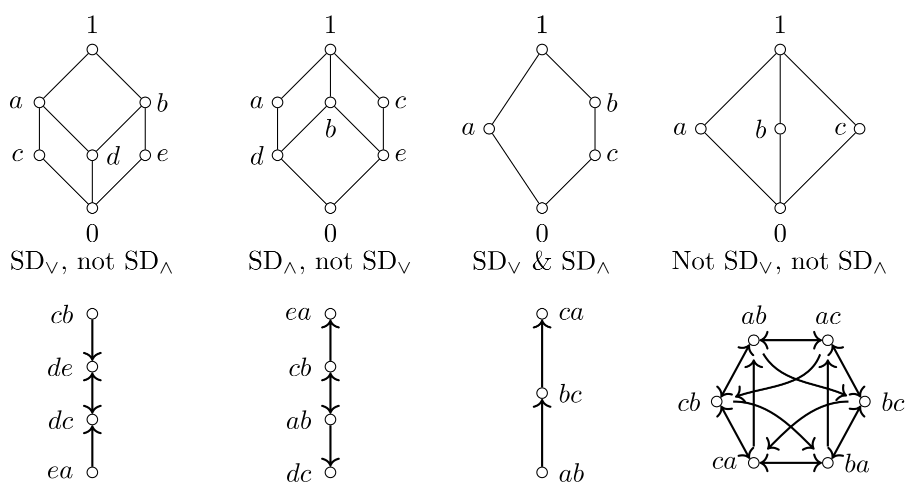

Figure 1: Some finite lattices and their dual digraphs.

The fact below was noted by Urquhart and will be useful later.

Proposition 2.2 ([20, p. 52])**.** *Let L be a finite lattice. If $\langle F, I\rangle$ is a maximal disjoint ideal-filter pair of L then $\bigwedge F$ is join-irreducible and $\bigvee I$ is meet-irreducible.*

Some of what appears in the proposition below can be found in the paper by Gaskill and Nation [13, p. 353]. We will make frequent use of this result and its proof reveals some important features of MDFIPs.

Proposition 2.3. *Let L be a finite lattice and $\langle F, I\rangle$ be a maximal disjoint filter-ideal pair of L. Then the following are equivalent:*

- (i) *$\bigwedge F$ is join-prime;*
- (ii) *$\bigvee I$ is meet-prime;*
- (iii) *$F \cup I = L$;*
- (iv) *F is a prime filter;*
- (v) *I is a prime ideal.*

The equivalences $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ hold even when L is not finite.

Proof. Let L be a finite lattice and let $\langle F, I\rangle$ be a maximal disjoint filter-ideal pair of L. Let $\bigwedge F = x$ and $\bigvee I = y$.

First we show that $(iii) \Rightarrow (i)$. Assume that $F \cup I = L$. Let $a, b \in L$ such that $x \leqslant a \vee b$. We claim that $a \in F$ or $b \in F$. Suppose for a contradiction that $a \notin F$ and $b \notin F$. Then $a, b \in L \setminus F = I$. That implies $a \vee b \in I$, whence $x \in I$, a contradiction.

Now we show that $(i) \Rightarrow (iii)$. Assume that x is join-prime. Let $a \in L$ such that $a \notin F \cup I$. We will consider three cases for the element $a \vee y$ and derive a contradiction for each case.

Case 1: If $a \vee y \in I$ then $a \leqslant a \vee y = y$, thus $a \in I$, a contradiction.

Case 2: If $a \vee y \in F$ then $x \leqslant a \vee y$. Since x is join-prime, $x \leqslant a$ or $x \leqslant y$. If $x \leqslant a$ then $a \in F$, contradicting $a \notin F \cup I$. If $x \leqslant y$ then $x \in I$, contradicting $F \cap I = \varnothing$.

Case 3: Suppose $a \vee y \notin F \cup I$. Since $a \vee y \notin {\uparrow}x$, ${\downarrow}(a \vee y) \cap {\uparrow}x = \varnothing$. From $a \vee y \notin {\downarrow}y$ it follows that ${\downarrow}y \subset {\downarrow}(a \vee y)$. Hence $\langle {\uparrow}x, {\downarrow}(a \vee y)\rangle$ is a disjoint filter-ideal pair properly containing $\langle F, I\rangle$, which contradicts the maximality of $\langle F, I\rangle$.

The equivalence of (ii) and (iii) can be shown analogously.

Now we drop the assumption that L is finite and show that $(iii) \Rightarrow (iv)$. Let $a \vee b \in F$. If $a \notin F$ and $b \notin F$ then we have $a, b \in L \backslash F = I$. Since I is an ideal we would get $a \vee b \in I$, a contradiction. Therefore $a \in F$ or $b \in F$.

To show $(iv) \Rightarrow (iii)$, and the equivalence of (iv) and (v), one uses the fact that a filter $F \subseteq L$ is prime if and only if $L \backslash F$ is a prime ideal. $\square$

The properties of the digraphs dual to bounded lattices were described by Craig, Gouveia and Haviar [7]. There they were called *TiRS graphs*; in this paper we will use the terminology *TiRS digraphs*. We recall the necessary facts. (We note that in the definition below $xE = \{ y \in V \mid (x,y) \in E \}$ and $Ex = \{ y \in V \mid (y,x) \in E \}$.)

Definition 2.4 ([7, Definition 2.2])**.** A TiRS digraph $G = (V,E)$ is a set V and a reflexive relation $E \subseteq V \times V$ such that:

(S) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$.

(R) For all $x, y \in V$, if $xE \subset yE$ then $(x,y) \notin E$, and if $Ey \subset Ex$ then $(x,y) \notin E$.

(Ti) For all $x, y \in V$, if xEy then there exists $z \in V$ such that $zE \subseteq xE$ and $Ez \subseteq Ey$.

We recall that the vertices of the dual digraph G_L of a bounded lattice L are formed by the set X_L of MDFIPs of L and Ploščica's relation E is determined by (E2). Using these facts, the following result can be stated.

Proposition 2.5 ([7, Proposition 2.3])**.** *For any bounded lattice L, its dual digraph $G_L = (X_L, E)$ is a TiRS digraph.*

We recall from [17] a fact concerning general digraphs $G = (X,E)$. Let $\underset{\sim}{2} = (\{0,1\}, \leqslant)$ denote the two-element digraph. A partial map $\varphi \colon X \to \underset{\sim}{2}$ is said to preserve the relation E if $\varphi(x) \leqslant \varphi(y)$ whenever $x, y \in \operatorname{dom}\varphi$ and $(x,y) \in E$. The lattice of maximal partial E-preserving maps from G to $\underset{\sim}{2}$ is denoted by $\mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$.

Lemma 2.6 ([17, Lemma 1.3])**.** *Let $G = (X,E)$ be a digraph and let us consider $\varphi \in \mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$. Then*

(i) $\varphi^{-1}(0) = \{ x \in X \mid$ *there is no* $y \in \varphi^{-1}(1)$ *with* $(y,x) \in E \}$*;*

(ii) $\varphi^{-1}(1) = \{ x \in X \mid$ *there is no* $y \in \varphi^{-1}(0)$ *with* $(x,y) \in E \}$*.*

The above lemma allows us to observe that for a digraph $G = (X,E)$ and $\varphi, \psi \in \mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$ we have

$$\varphi^{-1}(1) \subseteq \psi^{-1}(1) \iff \psi^{-1}(0) \subseteq \varphi^{-1}(0).$$

This implies that the reflexive and transitive binary relation $\leqslant$ defined on $\mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$ by

$$\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

is a partial order. For a digraph $G = (X,E)$ we let $\mathbb{C}(G) = (\mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2}), \leqslant)$.

Theorem 2.7 ([7, Theorem 1.7 and p. 87]). *For any finite bounded lattice L we have that L is isomorphic to $\mathbb{C}(G_L)$ and for any finite TiRS digraph $G = (V, E)$ we have that G is isomorphic to $G_{\mathbb{C}(G)}$.*

In later sections, we will frequently make use of Theorem 2.7 in the following way: given any finite TiRS digraph $G = (V, E)$, we can consider G to be the dual digraph $G_L = (X_L, E)$ for some finite lattice L.

There are a number of different constructions that yield complete lattices isomorphic to the complete lattice $\mathbb{C}(G)$ described above, which is assigned to a digraph $G = (X, E)$ (see [9]). In particular, later we will use the lattice obtained via the polarity $\mathbb{K}(G) = (X, X, E^{\complement})$, which will be described in Section 5.

At the end of this preliminary section we recall from [20] how the set X_L of all MDFIPs of a finite bounded lattice L can be equipped with two quasi-orders $\leqslant_1$ and $\leqslant_2$. Urquhart in [20, p. 47] defined two binary relations $\leqslant_1$ and $\leqslant_2$ on the set set X_L of all MDFIPs of an arbitrary lattice L as follows: for two MDFIPs $\langle F, I\rangle$ and $\langle G, J\rangle$,

($\leqslant_1$) $\langle F, I\rangle \leqslant_1 \langle G, J\rangle \iff F \subseteq G;$

($\leqslant_2$) $\langle F, I\rangle \leqslant_2 \langle G, J\rangle \iff I \subseteq J.$

It is clear that the binary relations $\leqslant_1$ and $\leqslant_2$ are reflexive and transitive on the set X_L, and hence are quasi-orders.

3 Characterisation of dual digraphs

The theorem below will play a crucial role in the proof of our first result. Our presentation is slightly different to [3]; we have re-stated their items to suit our purposes. We use $\mathsf{J}(L)$, respectively $\mathsf{M}(L)$, to denote the join-irreducible, respectively meet-irreducible, elements of L.

Theorem 3.1 ([3, Theorem 3-1.4]). *Let L be a finite lattice. Then the following are equivalent:*

(*i*) *L satisfies $SD_\vee$;*

(*ii*) *For each $x \in \mathsf{M}(L)$, there exists a unique minimal element of the set*

$$S(x) = \{k \in L \mid k \nleqslant x \ \& \ k \leqslant x^*\},$$

where x^* is the unique upper cover of x, and moreover, this minimal element of $S(x)$ is in $\mathsf{J}(L)$.

(*iii*) *There exists a map* $\kappa\colon \mathsf{M}(L) \to \mathsf{J}(L)$ *such that for each* $x \in \mathsf{M}(L)$, $\kappa(x)$ *is the minimal element of the set* $S(x)$.

Using the previous result, in the next theorem we characterise finite join-semidistributive and meet-semidistributive lattices via their MDFIPs.

Theorem 3.2. *Let* L *be a finite lattice.*

(*i*) L *is not join-semidistributive if and only if there exist distinct maximal disjoint filter-ideal pairs of the form* $\langle{\uparrow}y, {\downarrow}x\rangle$ *and* $\langle{\uparrow}z, {\downarrow}x\rangle$ *for some* $x, y, z \in L$.

(*ii*) L *is not meet-semidistributive if and only if there exist distinct maximal disjoint filter-ideal pairs of the form* $\langle{\uparrow}x, {\downarrow}y\rangle$ *and* $\langle{\uparrow}x, {\downarrow}z\rangle$ *for some* $x, y, z \in L$.

Proof. For the necessity, assume L is not join-semidistributive, whence by Theorem 3.1, for some $x \in \mathsf{M}(L)$ there exist two minimal elements y and z of the set $S(x)$. Then ${\uparrow}y \cap {\downarrow}x = \varnothing$ and ${\uparrow}z \cap {\downarrow}x = \varnothing$ so $\langle{\uparrow}y, {\downarrow}x\rangle$ and $\langle{\uparrow}z, {\downarrow}x\rangle$ are disjoint filter-ideal pairs. We claim that $\langle{\uparrow}y, {\downarrow}x\rangle$ and $\langle{\uparrow}z, {\downarrow}x\rangle$ are maximal. Suppose on the contrary that there is a disjoint filter-ideal pair $\langle{\uparrow}a, {\downarrow}b\rangle$ of L such that ${\uparrow}y \subseteq {\uparrow}a$ and ${\downarrow}x \subseteq {\downarrow}b$ but $\langle{\uparrow}a, {\downarrow}b\rangle \neq \langle{\uparrow}y, {\downarrow}x\rangle$. This gives us two possible cases:

Case 1: If $a \neq y$ then since y is minimal in the set $S(x)$ and $a \leqslant y \leqslant x^*$ we have that $a \leqslant x$. But $x \leqslant b$, which implies that $a \leqslant b$, contradicting ${\uparrow}a \cap {\downarrow}b = \varnothing$.

Case 2: If $b \neq x$ then $x^* \leqslant b$ since x^* is the unique upper cover of x. But $a \leqslant y \leqslant x^*$, which implies that $a \leqslant b$, contradicting again ${\uparrow}a \cap {\downarrow}b = \varnothing$.

Thus $\langle{\uparrow}y, {\downarrow}x\rangle$ is maximal and we can use a similar argument to prove that $\langle{\uparrow}z, {\downarrow}x\rangle$ is maximal.

For the sufficiency, assume that there exist distinct maximal disjoint filter-ideal pairs of the form $\langle{\uparrow}y, {\downarrow}x\rangle$ and $\langle{\uparrow}z, {\downarrow}x\rangle$ for some $x, y, z \in L$. We will prove that y and z are both minimal elements of the set $S(x)$. If follows from ${\uparrow}y \cap {\downarrow}x = \varnothing$ and ${\uparrow}z \cap {\downarrow}x = \varnothing$ that $y \nleqslant x$ and $z \nleqslant x$. We will argue $y \leqslant x^*$ by contradiction. Suppose $y \nleqslant x^*$, then ${\uparrow}y \cap {\downarrow}x^* = \varnothing$. Since $x < x^*$ implies that ${\downarrow}x \subset {\downarrow}x^*$, we get that $\langle{\uparrow}y, {\downarrow}x\rangle$ is properly contained in $\langle{\uparrow}y, {\downarrow}x^*\rangle$, which is a contradiction. Therefore $y \leqslant x^*$ and $y \in S(x)$. Using a similar argument, $z \in S(x)$. Now if $a \in S(x)$ and $a < y$, then ${\uparrow}y \subset {\uparrow}a$. Since $a \nleqslant x$, we have ${\uparrow}a \cap {\downarrow}x = \varnothing$. Therefore $\langle{\uparrow}a, {\downarrow}x\rangle$ is a disjoint filter-ideal pair with ${\uparrow}y \subset {\uparrow}a$, contradicting the maximality of $\langle{\uparrow}y, {\downarrow}x\rangle$. Similarly, if $b \in S(x)$ such that $b < z$, then $\langle{\uparrow}b, {\downarrow}x\rangle$ is a disjoint filter-ideal pair properly containing $\langle{\uparrow}z, {\downarrow}x\rangle$, which is a contradiction. Therefore y and z are both minimal elements of $S(x)$.

The proof of (*ii*) follows by an order-dual argument. $\square$

Corollary 3.3. *Let $G = (V, E)$ be a finite TiRS digraph which is the dual digraph of a finite lattice L. If the relation E is antisymmetric, then L is semidistributive.*

Proof. In accordance with our remarks after Theorem 2.7, we can consider G to be G_L and so its vertex set V will be X_L.

Suppose for a contradiction that L is not semidistributive. Then L is not join-semidistributive or L is not meet-semidistributive. If L is not join-semidistributive then by Theorem 3.2 (i) there are maximal disjoint filter-ideal pairs of the form $\langle\uparrow y, \downarrow x\rangle$ and $\langle\uparrow z, \downarrow x\rangle$ for some $x, y, z \in L$. Since G is the dual digraph of L, we have $\langle\uparrow y, \downarrow x\rangle, \langle\uparrow z, \downarrow x\rangle \in V$. Clearly $\langle\uparrow y, \downarrow x\rangle E \langle\uparrow z, \downarrow x\rangle$ and $\langle\uparrow z, \downarrow x\rangle E \langle\uparrow y, \downarrow x\rangle$. This contradicts the antisymmetry of the relation E.

If L is not meet-semidistributive, then the argument is analogous. □

Remark 3.4. The converse to Corollary 3.3 does not hold. We can see it on the lattice in Figure 2.

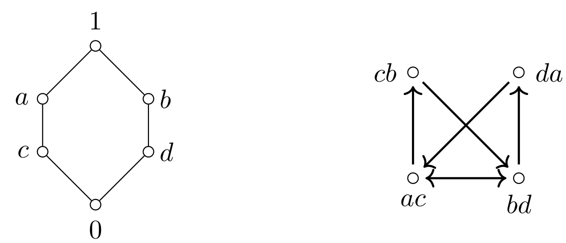

Figure 2: A finite semidistributive lattice and its dual digraph.

The lattice is semidistributive but we see on its dual digraph, which contains a "double arrow" between the elements ac and bd, that the relation E of the digraph is not antisymmetric.

Hence the condition in Corollary 3.3 is sufficient but not necessary for a finite lattice to be semidistributive. An interesting task that is left open is to possibly weaken the given sufficient condition to some form of "weak antisymmetry" of the relation E so that the resulting condition on E is necessary and sufficient for a finite lattice to be semidistributive.

In the statement and the proof of the following result we again use the fact that, by Theorem 2.7, $G = (V, E)$ is isomorphic to the dual digraph $G_L = (X_L, E_L)$ of the lattice L, whose vertex set X_L is the set of all MDFIPs of L.

Lemma 3.5. *Let $G = (V, E)$ be a finite TiRS digraph with dual lattice L. Let $u, v \in V$ be distinct. Then:*

(i) $Eu = Ev$ if and only if u and v are the isomorphic images of $\langle\uparrow x, \downarrow y\rangle$ and $\langle\uparrow z, \downarrow y\rangle$ in X_L for some $x, y, z \in L$;

(*ii*) *$uE = vE$ if and only if u and v are the isomorphic images of $\langle{\uparrow}x, {\downarrow}y\rangle$ and $\langle{\uparrow}x, {\downarrow}z\rangle$ in X_L for some $x, y, z \in L$.*

Proof. Let $u, v \in V$. To show the sufficiency of the condition in (i), let u and v be the isomorphic images of the vertices $\langle{\uparrow}x, {\downarrow}y\rangle$ and $\langle{\uparrow}z, {\downarrow}y\rangle$ in G_L for some $x, y, z \in L$. Since G is isomorphic to G_L, we only need to show that $E_L\langle{\uparrow}x, {\downarrow}y\rangle = E_L\langle{\uparrow}z, {\downarrow}y\rangle$. To this end, let $\langle F, I\rangle \in E_L\langle{\uparrow}x, {\downarrow}y\rangle$, then $F \cap {\downarrow}y = \varnothing$. Thus $\langle F, I\rangle \in E_L\langle{\uparrow}z, {\downarrow}y\rangle$. Similarly, if $\langle F, I\rangle \in E_L\langle{\uparrow}z, {\downarrow}y\rangle$, then $F \cap {\downarrow}y = \varnothing$ and $\langle F, I\rangle \in E_L\langle{\uparrow}x, {\downarrow}y\rangle$. Therefore $E_L\langle{\uparrow}x, {\downarrow}y\rangle = E_L\langle{\uparrow}z, {\downarrow}y\rangle$ and $Eu = Ev$.

For the necessity of the condition in (i), let $\langle{\uparrow}x, {\downarrow}y\rangle$ and $\langle{\uparrow}z, {\downarrow}w\rangle$ be isomorphic images of u and v in X_L and let $Eu = Ev$. We will show ${\downarrow}y = {\downarrow}w$. Let $a \in {\downarrow}y$. For all $\langle F, I\rangle \in E_L\langle{\uparrow}z, {\downarrow}w\rangle$ we have $F \cap {\downarrow}y = \varnothing$ since $E_L\langle{\uparrow}x, {\downarrow}y\rangle = E_L\langle{\uparrow}z, {\downarrow}w\rangle$. For $S = \bigcup\{F \mid \langle F, I\rangle \in E_L\langle{\uparrow}z, {\downarrow}w\rangle\}$ now $a \notin S$ as $a \in {\downarrow}y$. We claim that $a \in {\downarrow}w$. Suppose on the contrary that $a \notin {\downarrow}w$. Then $a \nleq w$ and ${\uparrow}a \cap {\downarrow}w = \varnothing$. This shows $\langle{\uparrow}a, {\downarrow}w\rangle$ is a disjoint filter-ideal pair. Hence there is an MDFIP $\langle H, J\rangle$ such that ${\uparrow}a \subseteq H$ and ${\downarrow}w \subseteq J$. But ${\downarrow}w \subseteq J$ and $H \cap J = \varnothing$ implies that $H \cap {\downarrow}w = \varnothing$. Then $\langle H, J\rangle \in E_L\langle{\uparrow}z, {\downarrow}w\rangle$, so $H \subseteq S$, which means $a \in S$, a contradiction. Thus $a \in {\downarrow}w$. The reverse inclusion can be shown analogously. Therefore ${\downarrow}y = {\downarrow}w$ and the proof of (i) is complete. Part (ii) can be proven analogously. ☐

Theorem 3.6. *Let $G = (V, E)$ be a finite TiRS digraph with $u, v \in V$. Then*

(*i*) *G is the dual digraph of a join-semidistributive lattice if and only if whenever $u \neq v$ then $Eu \neq Ev$.*

(*ii*) *G is the dual digraph of a meet-semidistributive lattice if and only if whenever $u \neq v$ then $uE \neq vE$.*

(*iii*) *G is the dual digraph of a semidistributive lattice if and only if whenever $u \neq v$ then $Eu \neq Ev$ and $uE \neq vE$.*

Proof. Let G be a finite TiRS digraph with dual lattice L. To show the necessity in (i), assume there exist distinct $u, v \in V$ such that $Eu = Ev$. Then by Lemma 3.5 there exist distinct MDFIPs $\langle{\uparrow}x, {\downarrow}y\rangle$ and $\langle{\uparrow}z, {\downarrow}y\rangle$ in L. It then follows from Theorem 3.2(i) that L is not join-semidistributive.

To show the sufficiency in (i), assume that L is not join-semidistributive. Then by Theorem 3.2(i) there exist distinct MDFIPs $\langle{\uparrow}x, {\downarrow}y\rangle$ and $\langle{\uparrow}z, {\downarrow}y\rangle$. By Lemma 3.5 there exist distinct vertices $u, v \in V$ such that $Eu = Ev$.

Part (ii) can be shown analogously, and part (iii) follows directly from (i) and (ii). ☐

We recall that the "separation property" (S) in the definition of TiRS digraphs is defined as follows:

(S) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$.

Hence it should be emphasized that the condition (*iii*) in the theorem above characterising the semidistributivity is clearly strengthening the separation condition (S) by replacing in it the logical connective "or" with "and". Thus it can be considered as a certain "strong separation property":

(sS) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ and $Ex \neq Ey$.

It is interesting to realise that finite semidistributive lattices are exactly those finite lattices whose dual digraphs have the "separation property" (S) strengthened to the "strong separation property" (sS).

A remark of Urquhart [20, Section 7] says that a finite lattice L is meet-semidistributive if and only if the quasi-order $\leqslant_1$ is an order. We state that result (and its dual) below and prove it using the results from earlier in the section.

Theorem 3.7. *Let L be a finite lattice.*

(*i*) *L is join-semidistributive if and only if the quasi-order $\leqslant_2$ on the vertices of the dual digraph is an order.*

(*ii*) *L is meet-semidistributive if and only if the quasi-order $\leqslant_1$ on the vertices of the dual digraph is an order.*

Proof. Assume firstly that the quasi-order $\leqslant_2$ on the vertices of the dual digraph is not an order, that is, the relation $\leqslant_2$ is not antisymmetric. Then there exist distinct vertices x and y such that $x \leqslant_2 y$ and $y \leqslant_2 x$. If we consider the vertices x and y as the MDFIPs $x = \langle F, I \rangle$ and $y = \langle G, J \rangle$, then by definition of $\leqslant_2$ we have $I \subseteq J$ and $J \subseteq I$, hence the MDFIPs x and y have the same ideal part. By Theorem 3.2 it follows that L is not join-semidistributive.

Conversely, if L is not join-semidistributive, then by Theorem 3.2 there exist distinct MDFIPs x and y with the same ideal part, whence $x \leqslant_2 y$ and $y \leqslant_2 x$. It follows that the relation $\leqslant_2$ is not antisymmetric, hence the quasi-order $\leqslant_2$ is not an order. ☐

Now we can rephrase Lemma 3.5 in terms of quasi-orders $\leqslant_1$ and $\leqslant_2$:

Corollary 3.8. *Let $G = (V, E)$ be a finite TiRS digraph with dual lattice L. Let $u, v \in V$ be distinct. Then:*

(*i*) *$Eu = Ev$ if and only if $u \leqslant_2 v$ and $v \leqslant_2 u$;*

(*ii*) *$uE = vE$ if and only if $u \leqslant_1 v$ and $v \leqslant_1 u$.*

We can finally summarise the previous results in the following characterisations of join-semidistributivity, meet-semidistributivity and semidistributivity of finite lattices via the properties of their dual digraphs:

Corollary 3.9. *Let $G = (V, E)$ be a finite TiRS digraph.*

(1) The following are equivalent:

- *(i) G is the dual digraph of a join-semidistributive lattice;*
- *(ii) for all $u, v \in V$, if $u \neq v$ then $Eu \neq Ev$;*
- *(iii) the quasi-order $\leqslant_2$ on V is an order.*

(2) The following are equivalent:

- *(i) G is the dual digraph of a meet-semidistributive lattice;*
- *(ii) for all $u, v \in V$, if $u \neq v$ then $uE \neq vE$;*
- *(iii) the quasi-order $\leqslant_1$ on V is an order.*

(3) The following are equivalent:

- *(i) G is the dual digraph of a semidistributive lattice;*
- *(ii) for all $u, v \in V$, if $u \neq v$ then $Eu \neq Ev$ and $uE \neq vE$;*
- *(iii) both the quasi-orders $\leqslant_1$ and $\leqslant_2$ on V are orders.*

4 Domination in dual digraphs

In the dual digraph of a lattice L, there are certain vertices that play an important role. It turns out that these vertices correspond to MDFIPs where $F \cup I = L$.

Definition 4.1. A vertex v of a digraph $G = (V, E)$ is said to be *transitive* in G if uEv and vEw imply uEw for all $u, w \in V$.

With respect to the illustration of the following result, the reader is reminded to return to Figure 1 for examples.

Theorem 4.2. *Let L be a lattice with dual digraph $G_L = (X_L, E)$. A maximal disjoint filter-ideal pair $\langle F, I \rangle$ is total in L if and only if it is transitive in G_L.*

Proof. Let $\langle F, I \rangle$ be total in L. Assume that $\langle G, J \rangle$ and $\langle H, K \rangle$ are maximal disjoint filter-ideal pairs such that $\langle G, J \rangle E \langle F, I \rangle$ and $\langle F, I \rangle E \langle H, K \rangle$. By the definition of E we have that $G \cap I = \varnothing$ and $F \cap K = \varnothing$. We claim that $G \cap K = \varnothing$. Notice that since $F \cap K = \varnothing$ and $\langle F, I \rangle$ is total, it follows that $K \subseteq L \backslash F = I$. But $G \cap I = \varnothing$ and hence $G \cap K = \varnothing$. By the definition of E we get $\langle G, J \rangle E \langle H, K \rangle$ and therefore $\langle F, I \rangle$ is transitive.

For the converse, assume that $\langle F, I\rangle$ is not total in L. Take $x \in L\backslash(F\cup I)$ and consider the disjoint filter-ideal pairs $\langle{\uparrow}x, I\rangle$ and $\langle F, {\downarrow}x\rangle$. These can be extended to maximal disjoint filter-ideal pairs $\langle G, J\rangle$ (where ${\uparrow}x \subseteq G$ and $I \subseteq J$) and $\langle H, K\rangle$ (with $F \subseteq H$ and ${\downarrow}x \subseteq K$). Since $I \subseteq J$, we have $G \cap I = \varnothing$ and hence $\langle G, J\rangle E\langle F, I\rangle$. Since $F \subseteq H$ we get $F \cap K = \varnothing$ and hence $\langle F, I\rangle E\langle H, K\rangle$. But, since $x \in G \cap K$ we do not have $\langle G, J\rangle E\langle H, K\rangle$ and so $\langle F, I\rangle$ is not transitive. ☐

The following result was first shown in a more restricted context by Gaskill and Nation [13]. This more general statement is folklore.

Proposition 4.3 ([13, Lemma 1])**.** *Let L be a join-semidistributive lattice with greatest element 1. Then L has a prime ideal. Dually, if L is a meet-semidistributive lattice with least element 0, then L has a prime filter.*

Proof. Let I be an ideal that is maximal with respect to not containing 1. Suppose that $y, z \notin I$. Then there is an element $x \in I$ such that $x \vee y = x \vee z = 1$. Since L satisfies $SD_\vee$ we get $x \vee (y \wedge z) = 1$ and hence $y \wedge z \notin I$. ☐

Corollary 4.4. *Let L be a bounded lattice. If the dual digraph $G_L = (X_L, E)$ does not have a transitive vertex then L satisfies neither $SD_\vee$ nor $SD_\wedge$.*

Proof. Assume that G_L does not have a transitive element. Then every MDFIP of L is such that $F \cup I \neq L$. By Proposition 2.3 we have that no filter $F \subseteq L$ can be prime. Since L has both a greatest and least element, by Proposition 4.3, L cannot be join-semidistributive and it cannot be meet-semidistributive. ☐

Notice that the converse of Corollary 4.4 does not hold. The lattice L_3 from [10] satisfies neither $SD_\vee$ nor $SD_\wedge$ but there exists a maximal disjoint filter-ideal pair $\langle F, I\rangle$ with $F \cup I = L$ (or, a total homomorphism from L_3 to 2).

As stated earlier, the transitive elements in a finite TiRS digraph can play a special role. Notice that when a TiRS digraph G is a poset (*i.e.* it is the dual digraph of a finite distributive lattice) then *every* element of G is transitive.

The next lemma captures two familiar facts about finite join-semidistributive and meet-semidistributive lattices.

Lemma 4.5 ([13, Lemma 1])**.** (*i*) *The co-atoms of a finite join-semidistributive lattice are meet-prime.*

(*ii*) *The atoms of a finite meet-semidistributive lattice are join-prime.*

Proof. We prove only (*i*) as the proof of (*ii*) will follow using a dual argument.

Let L be a finite join-semidistributive lattice and let $x \in L$ be a co-atom such that $x \geqslant a \wedge b$ for some $a, b \in L$. Suppose that $x \ngeqslant a$ and $x \ngeqslant b$. We then have $x \vee a > x$ and $x \vee b > x$. Since x is a co-atom, we get $x \vee a = 1 = x \vee b$. However, since L is join-semidistributive, we get $x = x \vee (a \wedge b) = x \vee a = 1$, a contradiction. Thus $x \geqslant a$ or $x \geqslant b$. □

In the definition below we note that the original source uses 'arc' instead of 'edge'.

Definition 4.6 ([15, Definition 2])**.** Given a digraph $D = (V, E)$, with vertex set V and edge set E, a set $S \subseteq V$ is a *dominating set* if for every vertex $v \in V \backslash S$, there is a vertex $u \in S$ such that uEv.

Proposition 4.7. *Let $G = (V, E)$ be a finite TiRS digraph. If G is dual to a finite join-semidistributive lattice L, then the transitive vertices of G form a dominating set.*

Proof. Assume that $G = G_L = (X_L, E)$ for some finite join-semidistributive lattice L. If x is a vertex of G then $x = \langle \uparrow a, \downarrow b \rangle$ for some $a, b \in L$. Since $b \neq 1$ we have that $b \leqslant c$ for some co-atom c. By Lemma 4.5 we have that c is meet-prime and so by Proposition 2.3 we know that $\downarrow c$ is a prime ideal and that there exists $d \in L$ such that $\uparrow d$ is a prime filter with $\uparrow d \cap \downarrow c = \varnothing$ and $\uparrow d \cup \downarrow c = L$. By Theorem 4.2, $y = \langle \uparrow d, \downarrow c \rangle$ is a transitive vertex of G_L. Since $\downarrow b \subseteq \downarrow c$ we have $\uparrow d \cap \downarrow b = \varnothing$ and hence yEx. □

The converse of the above proposition does not hold. Let L' be the diamond M_3 with a new top element t. Then its dual digraph G is the same as the dual digraph of M_3 (see Figure 1) except it has an extra vertex $v = \langle \uparrow t, \downarrow 1 \rangle$, which is transitive since it is total. In G the edges obviously go from the vertex v to every other vertex. Hence the set $\{v\}$ of transitive vertices of G is the dominating set, yet the lattice L' is not join-semidistributive as it contains a sublattice isomorphic to M_3 (*cf.* [10]).

Since transitive elements are connected to join- and meet-prime elements, the previous result is partly related to how the join-primes or meet-primes sit inside the lattice. The next result characterises finite TiRS digraphs G dual to finite lattices, in which the transitive vertices of G form a dominating set.

Theorem 4.8. *Let $G = (V, E)$ be a finite TiRS digraph. Then G is dual to a finite lattice L in which every co-atom is meet-prime if and only if the transitive vertices of G form a dominating set.*

Proof. Let $G = (V, E)$ be the dual digraph G_L for some finite lattice L in which every co-atom is meet-prime. If $x \in V$ then $x = \langle \uparrow a, \downarrow b \rangle$ for some $a, b \in L$. Since $b \neq 1$ we have that $b \leqslant c$ for some co-atom c. By Proposition 2.3 we know that $\downarrow c$ is a prime ideal and that there exists $d \in L$

such that $\uparrow d$ is a prime filter with $\uparrow d \cap \downarrow c = \varnothing$ and $\uparrow d \cup \downarrow c = L$. By Theorem 4.2, $y = \langle \uparrow d, \downarrow c\rangle$ is a transitive vertex of $G_L = G$. Since $\downarrow b \subseteq \downarrow c$ we have $\uparrow d \cap \downarrow b = \varnothing$ and hence yEx.

Next, assume that the transitive vertices of G form a dominating set and let c be a co-atom of L. The pair $\langle \uparrow 1, \downarrow c\rangle$ is a disjoint filter-ideal pair that can be extended to a maximal disjoint filter-ideal pair $\langle \uparrow b, \downarrow c\rangle$. Since the transitive vertices form a dominating set, there exists a transitive vertex $\langle \uparrow x, \downarrow y\rangle$ such that $\langle \uparrow x, \downarrow y\rangle E \langle \uparrow b, \downarrow c\rangle$, *i.e.* $\uparrow x \cap \downarrow c = \emptyset$. Since $\langle \uparrow x, \downarrow y\rangle$ is transitive, we have by Proposition 2.3 and Theorem 4.2 that x is join-prime. Now, we have that $\langle \uparrow x, \downarrow c\rangle$ is a disjoint filter-ideal pair which can be extended to a maximal disjoint filter-ideal pair $\langle \uparrow a, \downarrow c\rangle$ where $a \leqslant x$. Since $a \nleqslant c$ we have $c < a \vee c = 1$. Clearly now $x \leqslant a \vee c$ and hence $x \leqslant a$ or $x \leqslant c$. The latter cannot happen as $\uparrow x \cap \downarrow c = \varnothing$ so $x \leqslant a$ and hence $x = a$. Now $\langle \uparrow x, \downarrow c\rangle$ is a maximal disjoint filter-ideal pair with x join-prime, and hence c is meet-prime. ☐

Remark 4.9. It is well-known (*cf.* [11, Theorem 2.24]; see also [3, Theorem 3-1.4]) that a finite lattice L satisfies $SD_\vee$ if and only if each element in L has a so-called canonical join representation. Using [13, Lemma 1(ii)] we are able to show that the equivalent conditions of Theorem 4.8 hold for the TiRS digraph G dual to a finite lattice L if and only if the top element 1 of L has a canonical join representation. Since canonical join representations are not the focus of this paper, we have decided to present the proof in a separate paper where this will be explored with the proper context and in more depth.

Definition 4.10 ([15, Definition 3])**.** Given a digraph $D = (V, E)$, with vertex set V and edge set E, a set $S \subseteq V$ is an *in-dominating set* if for every vertex $v \in V \backslash S$, there is a vertex $u \in S$ such that vEu.

Theorem 4.11. *Let $G = (V, E)$ be a finite TiRS digraph. Then G is dual to a finite lattice L in which every atom is join-prime if and only if the transitive vertices of G form an in-dominating set.*

Proof. Let $G_L = (X_L, E)$ be the dual digraph of some finite lattice L in which every atom is join-prime. If $x \in V$ then $x = \langle \uparrow a, \downarrow b\rangle$ for some $a, b \in L$. Assume that x is not transitive. Since $a \neq 0$ we have that $c \leqslant a$ for some atom $c \in L$. By Proposition 2.3 we know that $\uparrow c$ is a prime filter and that there exists $d \in L$ such that $\downarrow d$ is a prime ideal with $\uparrow c \cap \downarrow d = \varnothing$ and $\uparrow c \cup \downarrow d = L$. By Theorem 4.2, $y = \langle \uparrow c, \downarrow d\rangle$ is a transitive vertex of G_L. Since $\uparrow a \subseteq \uparrow c$ we have $\uparrow c \cap \downarrow b = \varnothing$ and hence xEy.

Next, assume that the transitive vertices of $G = (V, E)$ form an in-dominating set and let c be an atom of L. The pair $\langle \uparrow c, \downarrow 0\rangle$ is a disjoint filter-ideal pair that can be extended to an MDFIP $\langle \uparrow c, \downarrow b\rangle$. Since the transitive vertices form an in-dominating set, there exists a transitive vertex $\langle \uparrow x, \downarrow y\rangle$ such that $\langle \uparrow c, \downarrow b\rangle E \langle \uparrow x, \downarrow y\rangle$, *i.e.* $\uparrow c \cap \downarrow y = \emptyset$. Since $\langle \uparrow x, \downarrow y\rangle$ is transitive, we have by Proposition 2.3 and Theorem 4.2 that y is meet-prime.

Now, we have that $\langle \uparrow c, \downarrow y\rangle$ is a disjoint filter-ideal pair which can be extended to a maximal disjoint filter-ideal pair $\langle \uparrow c, \downarrow a\rangle$ where $y \leqslant a$. Since $c \nleqslant a$ we have $0 = a \wedge c < c$. Clearly now $a \wedge c < y$ and hence $a \leqslant y$ or $c \leqslant y$. The latter cannot happen as $\uparrow c \cap \downarrow y = \varnothing$ so $a \leqslant y$ and hence $y = a$. Now $\langle \uparrow c, \downarrow y\rangle$ is an MDFIP with y is meet-prime, and hence c is join-prime. □

Corollary 4.12. *Let $G = (V, E)$ be a finite TiRS digraph. If G is dual to a finite meet-semidistributive lattice L, then the transitive vertices of G form an in-dominating set.*

Proof. Let $G = (V, E)$ be a finite TiRS digraph. Assume G is dual to a finite meet-semidistributive lattice L. Then by Lemma 4.5 the atoms of L are join-prime. It then follows from Theorem 4.11 that the transitive elements of L form an in-dominating set. □

We think it is an interesting problem to try and characterise the dual digraphs of finite join-semidistributive lattices within the class of finite TiRS digraphs whose transitive vertices form a dominating set (and dually). We attempted to do so but were unable to identify the required condition.

5 Minimal closure systems from dual digraphs

Closure systems appear in many different areas of mathematics. They were investigated in relation to join-semidistributive lattices by Adaricheva *et al.* [1]. A comprehensive account of the theory can be found in the book chapters by Adaricheva and Nation [4, 5]. The definitions below all follow the notational conventions used in Adaricheva and Nation [4, Section 4-2] although in some cases the reference is to another source.

Definition 5.1 ([14, Definition 30])**.** Let X be a set and $\phi : \wp(X) \to \wp(X)$. Then ϕ is a *closure operator* on X if for all $Y, Z \in \wp(X)$,

(*i*) $Y \subseteq \phi(Y)$,

(*ii*) $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$,

(*iii*) $\phi(\phi(Y)) = \phi(Y)$.

If X is a set and ϕ a closure operator on X then the pair $\langle X, \phi\rangle$ is called a *closure system.*

For $Y \subseteq X$ we say that Y is *closed* if $\phi(Y) = Y$. The closed sets of a closure operator ϕ on X form a complete lattice, denoted by $\mathrm{Cld}(X, \phi)$.

Example 5.2. Let L be a finite lattice. If $a \in L$ let $J_a = \{x \in \mathsf{J}(L) \mid x \leqslant a\}$ and define $\tau\colon \wp(\mathsf{J}(L)) \to \wp(\mathsf{J}(L))$ by $\tau(A) = \bigcap\{J_a \mid a \in L \text{ and } A \subseteq J_a\}$. Then $\langle \mathsf{J}(L), \tau\rangle$ is a closure system. Notice that every finite lattice L is isomorphic to $\mathrm{Cld}(\mathsf{J}(L), \tau)$ via the isomorphism $a \mapsto J_a$.

From any digraph $G = (X, E)$ we get the closure system $\langle X, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement} \rangle$ (see [9, Theorem 3.3]). Here we recall necessary facts from [9, Section 3].

For a digraph $G = (X, E)$ one can consider the triple (called a *context*) $\mathbb{K}(G) := (X, X, E^{\complement})$, where the relation $E^{\complement} \subseteq X \times X$ is the complement of the digraph relation E: $E^{\complement} = (X \times X) \backslash E$. One can then define a Galois connection via so-called *polars* as follows. The maps

$$E_{\rhd}^{\complement} : (\wp(X), \subseteq) \to (\wp(X), \supseteq) \quad \text{and} \quad E_{\lhd}^{\complement} : (\wp(X), \supseteq) \to (\wp(X), \subseteq)$$

are given by

$$E_{\rhd}^{\complement}(Y) = \{ \, x \in X \mid (\forall \, y \in Y)(y, x) \notin E \, \},$$
$$E_{\lhd}^{\complement}(Y) = \{ \, z \in X \mid (\forall \, y \in Y)(z, y) \notin E \, \}.$$

The so-called *concept lattice* $\mathrm{CL}(\mathbb{K}(G))$ of the context $\mathbb{K}(G) = (X, X, E^{\complement})$, given by

$$\mathrm{CL}(\mathbb{K}(G)) = \{ \, Y \subseteq X \mid (E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement})(Y) = Y \, \},$$

is a complete lattice when ordered by inclusion.

The isomorphism in Proposition 5.3 below is different to the original source but is equivalent because of the one-to-one correspondence between the sets V_L and X_L. We recall that the definition of the lattice $\mathbb{C}(G_L)$ is given directly before Theorem 2.7.

Proposition 5.3 ([9, Proposition 3.1 and Corollary 3.2])**.** *If L is a finite lattice and $G_L = (X_L, E)$ is its dual digraph, we have*

$$L \cong \mathbb{C}(G_L) \cong \mathrm{CL}(\mathbb{K}(G_L)).$$

The map $a \mapsto \{ \, \langle F, I \rangle \in X_L \mid a \in F \, \}$ is the isomorphism from L to $\mathrm{CL}(\mathbb{K}(G_L))$.

The definition below is important in understanding the notion of a minimal closure system later on.

Definition 5.4 ([4, Definition 4-2.1])**.** Closure systems $\langle X, \phi \rangle$ and $\langle Y, \psi \rangle$ are called *equivalent* if $\mathrm{Cld}(X, \phi) \cong \mathrm{Cld}(Y, \psi)$. Two equivalent systems are called *isomorphic* if there exists a bijection $\rho : X \to Y$ such that $\rho(\phi(Z)) = \psi(\rho(Z))$ for all $Z \subseteq X$.

The left-most lattice in Figure 1 is referred to as L_4^{∂} in [10]. We use this lattice to provide an illustration of Definition 5.4.

Example 5.5. Let $L = L_4^{\partial}$ and consider its dual digraph $G_L = (X_L, E) = (\{cb, de, dc, ea\}, E)$. From this digraph we get the closure system $\langle X_L, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement} \rangle$ with

$$\mathrm{Cld}(X_L, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement}) = \{\emptyset, \{cb\}, \{ea\}, \{de, dc\}, \{cb, de, dc\}, \{ea, de, dc\}, X_L\}.$$

If we let $Y = \{cb, de, ea\}$ and $\phi_Y(S) = Y \cap (E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement})(S)$ then

$$\mathrm{Cld}(Y, \phi_Y) = \{\emptyset, \{cb\}, \{ea\}, \{de\}, \{cb, de\}, \{ea, de\}, Y\}.$$

It is easy to see that $\langle X_L, E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement}\rangle$ and $\langle Y, \phi_Y\rangle$ are equivalent but not isomorphic.

Proposition 5.6. *Let $\langle X, \phi\rangle$ and $\langle Y, \psi\rangle$ be closure systems and let $f\colon X \to Y$ be a mapping. If $f(A)$ is closed in Y for all closed sets $A \subseteq X$ and $f^{-1}(B)$ is closed in X for all closed sets $B \subseteq Y$ then $f(\phi(A)) = \psi(f(A))$ for all $A \subseteq X$.*

Proof. Let f be such that $f(A)$ is closed in Y for all closed sets $A \subseteq X$ and $f^{-1}(B)$ is closed in X for all closed sets $B \subseteq Y$. Notice that for all $S \subseteq X$ we have that $\phi(S) = \bigcap\{A \subseteq X \mid S \subseteq A$ and A is closed in $X\}$, and similarly for ψ. Let $S \subseteq X$. To show the inclusion $f(\phi(A)) \subseteq \psi(f(A))$, let $B \in \mathrm{Cld}(Y, \psi)$ such that $f(S) \subseteq B$. Then $S \subseteq f^{-1}(B)$. But $f^{-1}(B)$ is closed in X by our assumption. Hence $\phi(S) \subseteq f^{-1}(B) = \phi(f^{-1}(B))$. This implies that $f(\phi(S)) \subseteq B$. Since B was arbitrary, this is true for all closed sets containing $f(S)$. Therefore $f(\phi(S)) \subseteq \psi(f(S)) = \bigcap\{A \subseteq X \mid f(S) \subseteq A$ and A is closed in $X\}$. For the reverse inclusion notice that since $A \subseteq \phi(A)$ we get that $f(A) \subseteq f(\phi(A))$. But $f(\phi(A))$ is closed by our assumption. Thus $\psi(f(A)) \subseteq f(\phi(A))$. □

Further, Adaricheva and Nation [4] posed the following problem: given a closure system $\langle X, \phi\rangle$, can we find a $\subseteq$-minimal subset Y of X and a closure operator ψ on Y such that $\langle Y, \psi\rangle$ is equivalent to $\langle X, \phi\rangle$? Such a closure system is then said to be *minimal* for $\langle X, \phi\rangle$.

Theorem 5.7 ([5, Lemma 4-2.13])**.** *A closure system $\langle X, \phi\rangle$ with lattice of closed sets L is minimal if and only if it is isomorphic to $\langle \mathsf{J}(L), \tau\rangle$.*

Proposition 5.8. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then the mapping $f\colon X \to \mathsf{J}(L)$ defined by $f(\langle F, I\rangle) = \bigwedge F$ is surjective and satisfies $f(E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement}(S)) = \tau(f(S))$ for all $S \subseteq X$.*

Proof. We start by proving the surjectivity of f. Let $x \in \mathsf{J}(L)$ and let $T(x)$ denote the set $\{a \in L \mid x_* \leqslant a$ and $x \nleqslant a\}$ where x_* is the unique lower cover of x. We notice that the set $T(x)$ is non-empty since $x_* \in T(x)$. Let $y \in T(x)$ be a maximal element (which exists since $T(x)$ is a finite ordered set). Then we claim that $\langle{\uparrow}x, {\downarrow}y\rangle$ is an MDFIP. We have that ${\uparrow}x \cap {\downarrow}y = \varnothing$ since $x \nleqslant y$. Now let $\langle{\uparrow}a, {\downarrow}b\rangle$ be an MDFIP such that ${\uparrow}x \subseteq {\uparrow}a$ and ${\downarrow}y \subseteq {\downarrow}b$ and $\langle{\uparrow}a, {\downarrow}b\rangle \neq \langle{\uparrow}x, {\downarrow}y\rangle$. We get two cases from this.

Case 1: If ${\uparrow}x \neq {\uparrow}a$ then $a < x$ so $a \leqslant x_*$. Thus we get that $a \leqslant x_* \leqslant y \leqslant b$, which is a contradiction.

Case 2: If ${\downarrow}y \neq {\downarrow}b$ then $y < b$ and so $x_* \leqslant y < b$. But y is maximal in $T(x)$ so we have that $a \leqslant x \leqslant b$. Again, this is a contradiction.

Thus $\langle {\uparrow}x, {\downarrow}y \rangle$ is an MDFIP and $f(\langle {\uparrow}x, {\downarrow}y \rangle) = x$. Hence f is surjective.

To help us prove that f preserves closure, we define $B_a = \{\langle F, I \rangle \in X_L \mid a \in F\}$ and $J_a = \{x \in \mathsf{J}(L) \mid x \leqslant a\}$ for $a \in L$. Notice that the closed sets from $\langle X_L, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement} \rangle$ are exactly the sets B_a for all $a \in L$ and the closed sets from $\langle \mathsf{J}(L), \tau \rangle$ are exactly the sets J_a for all $a \in L$ (see Proposition 5.3 and Example 5.2). We claim that $f(B_a) = J_a$ and $f^{-1}(J_a) = B_a$ for all $a \in L$.

Let $a \in L$. We prove firstly that $f(B_a) = J_a$. To show the inclusion $f(B_a) \subseteq J_a$, let $x \in f(B_a)$. Then $x = \bigwedge F$ for some $\langle F, I \rangle \in B_a$. Since $\langle F, I \rangle \in B_a$ we have that $a \in F$. This implies that $x \leqslant a$. But $x \in \mathsf{J}(L)$ and thus $x \in J_a$. To show the reverse inclusion $f(B_a) \supseteq J_a$, let $x \in J_a$. Then by the surjectivity there is $y \in L$ such that $\langle {\uparrow}x, {\downarrow}y \rangle \in X_L$. Then since $x \in J_a$, we have that $x \leqslant a$. This implies that $a \in {\uparrow}x$ and that $\langle {\uparrow}x, {\downarrow}y \rangle \in B_a$. Since $\langle {\uparrow}x, {\downarrow}y \rangle \in B_a$, we get that $x \in f(B_a)$. Thus $f(B_a) = J_a$.

Now we prove that $f^{-1}(J_a) = B_a$ for all $a \in L$. To show $f^{-1}(J_a) \subseteq B_a$, let $\langle F, I \rangle \in f^{-1}(J_a)$. Then $f(\langle F, I \rangle) = x \in J_a$. Since $x \in J_a$, we have that $x \leqslant a$ and that $a \in {\uparrow}x = F$. Therefore $\langle F, I \rangle \in B_a$. To show $f^{-1}(J_a) \supseteq B_a$, let $\langle F, I \rangle \in B_a$. Then $a \in F$ and $f(\langle F, I \rangle) = \bigwedge F \leqslant a$. Therefore $f(\langle F, I \rangle) \in J_a$ and $\langle F, I \rangle \in f^{-1}(J_a)$. Thus $f^{-1}(J_a) = B_a$.

By Proposition 5.6 we get $f(E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement}(S)) = \tau(f(S))$ for all $S \subseteq X$. □

The main result of this section is the theorem below. We again refer the reader to Figure 1 for basic illustrative examples, while Example 5.5 provides a demonstration of what can happen when L is not meet-semidistributive.

Theorem 5.9. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then $\langle X_L, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement} \rangle$ is a minimal closure system for itself if and only if L is meet-semidistributive.*

Proof. The necessity will be proved by contraposition. Assume L is not meet-semidistributive. By Proposition 5.8 we have that $|\mathsf{J}(L)| \leq |X_L|$ since f is surjective. But by Theorem 3.2 there exist distinct MDFIPs $\langle {\uparrow}x, {\downarrow}y \rangle$ and $\langle {\uparrow}x, {\downarrow}z \rangle$ where $x \in \mathsf{J}(L)$. This implies that f is not injective and hence $|\mathsf{J}(L)| < |X_L|$. Therefore by [5, Lemma 4-2.13], $\langle X, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement} \rangle$ is not minimal.

For the sufficiency, assume that L is meet-semidistributive. We will show that f defined in Proposition 5.8 is a bijection. We only need to show that f is injective. Let $\langle F, I \rangle, \langle G, J \rangle \in X$ be such that $f(\langle F, I \rangle) = f(\langle G, J \rangle) = x$. Then $F = G = {\uparrow}x$. By Theorem 3.2 we have that $I = J$. Therefore $\langle F, I \rangle = \langle G, J \rangle$ and hence f is injective. Thus it follows from Propositions 5.6 and 5.8 that f is an isomorphism of closure systems. By [5, Lemma 4-2.13] this implies that $\langle X, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement} \rangle$ is minimal. □

Before stating the dual of Theorem 5.9, we need to make some observations. As observed earlier in the section, if L is a finite lattice, with $G_L = (X_L, E)$ its dual digraph, then $L \cong \mathrm{Cld}(X_L, E_{\lhd}^{\complement} \circ E_{\rhd}^{\complement}) \cong$

$\mathrm{Cld}(\mathsf{J}(L), \tau)$. If we reverse the order of the polar maps $E_{\triangleleft}^{\complement}$ and $E_{\triangleright}^{\complement}$, we again get a closure operator, but with $L^{\partial} \cong \mathrm{Cld}(X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement})$. For a finite lattice L, it is easy to show that $g : X_L \to X_{L^{\partial}}$, defined for $\langle \uparrow a, \downarrow b \rangle \in X_L$ by $g(\langle \uparrow a, \downarrow b \rangle) = \langle \uparrow b, \downarrow a \rangle$, is a bijection. From this we get that $\langle X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement} \rangle$ is isomorphic to $\langle X_{L^{\partial}}, E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement} \rangle$.

Theorem 5.10. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then $\langle X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement} \rangle$ is a minimal closure system for itself if and only if L is join-semidistributive.*

Proof. We know that L is join-semidistributive if and only if L^{∂} is meet-semidistributive. We can then apply Theorem 5.9 to the closure system $\langle X_{L^{\partial}}, E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement} \rangle$. □

Corollary 5.11. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then $\langle X_L, E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement} \rangle$ and $\langle X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement} \rangle$ are minimal closure systems for themselves if and only if L is semidistributive.*

6 Conclusion and future research

In this paper we characterised dual digraphs of finite meet-semidistributive, join-semidistributive and semidistributive lattices. We combined Urquhart's and Ploščica's representations of finite lattices in the following sense: the vertices of our dual digraphs were maximal disjoint filter-ideal pairs of the lattice in the Urquhart style, but we mainly viewed the duals as TiRS digraphs using the Ploščica binary relation E on the vertices. We introduced transitive vertices in our digraphs and explored their role in the domination theory. In particular, we characterised the finite lattices with the property that in their dual digraphs the transitive vertices form a dominating set resp. an in-dominating set. Finally, we characterised finite meet-semidistributive and join-semidistributive lattices via minimal closure systems on the set of vertices of their dual digraphs.

We wish to take note of two other settings in which dual representations of finite semidistributive lattices have been developed. The older of these is that of Formal Concept Analysis, where a characterisation of both finite join-semidistributive and meet-semidistributive lattices is available [12, Section 6.3]. There is also a recent paper by Reading, Speyer and Thomas [19] where they give a representation of finite semidistributive lattices via *two-acyclic factorization systems*. They define a two-acyclic factorization system to be a 4-tuple $\langle W, \to, \twoheadrightarrow, \hookrightarrow \rangle$ with a set W and three binary relations $\to, \hookrightarrow, \twoheadrightarrow$ on W. The relations $\twoheadrightarrow$ and $\hookrightarrow$ are required to be partial orders. The representation then comes from defining a factorization system on the set of join-irreducible elements of a semidistributive lattice. The triple $(X, \hookrightarrow, \twoheadrightarrow)$ is isomorphic to Urquhart's dual of the lattice L. We note that, in our representation, join- and meet-semidistributive lattices can be considered separately, but in the setting of factorization systems this separation is not yet possible (see [19, Remark 5.14]).

Lastly, we wish to point to some promising directions for future research. These would build on the representation of finite join- and meet-semidistributive lattices obtained in Section 3. The first of these would be to attempt to study finite sublattices of free lattices via their dual digraphs. This would require first finding a dual description of the well-known Whitman's Condition. The second direction would be the study of finite convex geometries (see [1, 5]) via their dual digraphs. Finite convex geometries are closure systems that are often studied via their lattice of closed sets. These lattices of closed sets are join-semidistributive and lower semimodular. Work is already under way to find a dual characterisation of upper and lower semimodularity.

Acknowledgements

The first author acknowledges the National Research Foundation (NRF) of South Africa (grant 127266). The second author acknowledges his appointment as a Visiting Professor at the University of Johannesburg from 1 June 2020 and the support by Slovak VEGA grants 1/0152/22 and 2/0078/20. The authors would like to express their appreciation for the referee's suggestions, which improved the final version of this paper, in particular for the one that led to Remark 4.9.

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