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Existence results for a class of local and nonlocal nonlinear elliptic problems

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ABSTRACT

In this paper, we study the p-Laplacian problems in the case where p depends on the solution itself. We consider two situations, when p is a local and nonlocal quantity. By using a singular perturbation technique, we prove the existence of weak solutions for the problem associated to the following equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2}\nabla u) + |u|^{p(u)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and also for its nonlocal version. The main goal of this paper is to extend the results established by M. Chipot and H. B. de Oliveira (Math. Ann., 2019, 375, 283-306).

RESUMEN

En este artículo, estudiamos los problemas p-Laplacianos en el caso donde p depende de la solución misma. Consideramos dos situaciones, cuando p es una cantidad local y no-local. Usando una técnica de perturbación singular, demostramos la existencia de soluciones débiles para el problema asociado a la siguiente ecuación

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2}\nabla u) + |u|^{p(u)-2}u = f & \text{en } \Omega \\ u = 0 & \text{sobre } \partial\Omega, \end{cases}$$

y también para su versión no-local. El principal objetivo de este artículo es extender los resultados establecidos por M. Chipot y H. B. de Oliveira (Math. Ann., 2019, 375, 283-306).

Keywords and Phrases: $p(u)$-Laplacian; elliptic problems; variable nonlinearity; generalised Sobolev spaces.

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1 Introduction

The study of partial differential equations involving the p-Laplacian generalised several types of problems not only in physics, but also in biophysics, plasma physics, and in the study of chemical reactions. These problems appear, for example, in a general reaction-diffusion system:

$$u_t = -\operatorname{div}\left(a|\nabla u|^{p(\cdot)-2}\nabla u\right) + |u|^{p(\cdot)-2}u,$$

where $a \in \mathbb{R}^+$ is a positive constant, the function u generally describes the concentration, the term $\operatorname{div}\left(a|\nabla u|^{p(\cdot)-2}\nabla u\right)$ corresponds to the diffusion with coefficient $D(u) = a|\nabla u|^{p(\cdot)-2}$, and $|u|^{p(\cdot)-2}u$ is the reaction term related to source and loss processes. In general, the reaction term has a polynomial form with respect to the concentration u.

Because of the importance of this kind of problems, many authors have investigated the existence and uniqueness of their different types of solutions [1, 4, 10].

Our main interest in this work is to extend these results to the case when p may depend both on the space variable x and on the unknown solution u. We first consider the case where the dependency of p on u is a local quantity. Namely, we study the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) + |u|^{p(u)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, f is a given data and p is the nonlinear exponent function $p : \mathbb{R} \to [1, +\infty)$ such that

$$p \text{ is continuous and } \quad 1 < r \leq p \leq s < \infty \quad \text{for some constants } r, s. \tag{1.2}$$

In the second part of this work, we consider also the following nonlocal problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(b(u))-2}\nabla u) + |u|^{p(b(u))-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $p : \mathbb{R} \longrightarrow [1, +\infty)$ and $b : W_0^{1,r}(\Omega) \longrightarrow \mathbb{R}$ are the functions involved in the exponent of nonlinearity, for some constant exponent r such that $1 < r < \infty$.

The fact that in reality physical measurements of certain quantities are not made in a punctual way but through local averages is always the motivation to study non-local problems. This kind of problems appear in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision [5, 6, 17]. J. Türola in [17] presented several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u. A numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$ is presented in [5]. To our best knowledge, there are only a few important contributions concerning the well-posedness of the solutions of this $p(u)$-Laplacian problems. The study of these problems was recently developed by Andreianov *et al.* [3]. They established the partial existence and uniqueness result to the weak solution in the cases of homogeneous Dirichlet boundary condition for the following problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2}\nabla u) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

S. Ouaro and N. Sawadogo in [14] and [15] considered the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \mathrm{div}\, a(x, u, \nabla u) = f & \text{in } \Omega \\ a(x, u, \nabla u) \cdot \eta + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness results of entropy and weak solutions are established by an approximation method and convergent sequences in terms of Young measure.

We were inspired by the work of M. Chipot and H. B. de Oliveira in [7], where the authors have proved the existence of the $p(u)$-problem (1.1) without the second term in the left-hand side, the existence proofs of [2] and [7] are based on the Schauder fixed-point theorem. They considered for the first time in the literature the nonlocal exponent of nonlinearity p as we consider here.

This paper is organized as follows. In Section 2 we introduce the basic assumptions and we recall some definitions and basic properties of generalised Sobolev spaces. Section 3 is devoted to show the existence of a solution to the local problem (1.1) using a singular perturbation technique. In Section 4, we prove the existence of weak solutions to the nonlocal problem (1.3) by using the Minty trick together with the technique of Zhikov (see [18]) for passing to the limit in our sequence of $p(u_n)$-Laplacian problems .

2 Preliminaries

The fact that the function p depends on the solution u and therefore it depends on the space variable x, allows us to look for the weak solutions in a Sobolev space with variable exponents.

Let Ω be a bounded domain of $\mathbb{R}^N$ with $\partial\Omega$ Lipschitz-continuous, we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous in Ω (for more details, see [9]) if

$$\exists C > 0 : \ |p(x) - p(y)| \leq \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \quad \forall x, y \in \Omega, \quad |x - y| < \frac{1}{2}. \tag{2.1}$$

For any Lebesgue-measurable function $p : \Omega \to [1, \infty)$, we define

$$p_- := \operatorname{ess}\inf_{x\in\Omega} p(x) \ , \ p_+ := \operatorname{ess}\sup_{x\in\Omega} p(x), \tag{2.2}$$

and we introduce the variable exponent Lebesgue space by:

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ / \ \rho_{p(\cdot)}(u) \ := \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}. \tag{2.3}$$

Equipped with the Luxembourg norm

$$\|u\|_{p(\cdot)} \ := \inf \left\{ \lambda > 0 \ : \ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}, \tag{2.4}$$

$L^{p(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < p_- \leq p_+ < \infty, \tag{2.5}$$

$L^{p(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where $p'(x)$ is the generalised Hölder conjugate of $p(x)$,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

The next proposition shows that there is a gap between the modular and the norm in $L^{p(\cdot)}(\Omega)$.

Proposition 2.1 (See [11]). *If* (2.5) *holds, for $u \in L^{p(x)}(\Omega)$, then the following assertions hold*

$$\min\left\{ \|u\|_{p(\cdot)}^{p_-}, \ \|u\|_{p(\cdot)}^{p_+} \right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{ \|u\|_{p(\cdot)}^{p_-}, \ \|u\|_{p(\cdot)}^{p_+} \right\},$$

$$\min\left\{ \rho_{p(\cdot)}(u)^{\frac{1}{p_-}} \ , \ \rho_{p(\cdot)}(u)^{\frac{1}{p_+}} \right\} \leq \|u\|_{p(\cdot)} \leq \max\left\{ \rho_{p(\cdot)}(u)^{\frac{1}{p_-}} \ , \ \rho_{p(\cdot)}(u)^{\frac{1}{p_+}} \right\}, \tag{2.6}$$

$$\|u\|_{p(\cdot)}^{p_-} - 1 \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p_+} + 1. \tag{2.7}$$

Proposition 2.2 (Generalised Hölder's inequality. See [13])**.**

- *For any functions* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$*, we have:*

$$\int_\Omega uv\,dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

- *For all* p *satisfying* (2.5)*, we have the following continuous embedding,*

$$L^{p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ whenever } p(x) \geq r(x) \text{ for a.e. } x \in \Omega. \tag{2.8}$$

In generalised Lebesgue spaces, there holds a version of Young's inequality,

$$|uv| \leq \delta\frac{|u|^{p(x)}}{p(x)} + C(\delta)\frac{|v|^{p'(x)}}{p(x)},$$

for some positive constant $C(\delta)$ and any $\delta > 0$.

We define also the generalised Sobolev space by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}. \tag{2.9}$$

The space $W^{1,p(\cdot)}(\Omega)$ is separable and is reflexive when (2.5) is satisfied. We also have

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega) \text{ whenever } p(x) \geq r(x) \text{ for a.e. } x \in \Omega. \tag{2.10}$$

Now, we introduce the following function space

$$W_0^{1,p(\cdot)}(\Omega) := \{u \in \mathrm{W}_0^{1,1}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega)\},$$

endowed with the following norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} := \|u\|_1 + \|\nabla u\|_{p(\cdot)}. \tag{2.11}$$

If $p \in C(\overline{\Omega})$, then the norm in $W_0^{1,p(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{p(\cdot)}$. When p is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.

If p is a measurable function in Ω satisfying $1 \leq p_- \leq p_+ < N$ and the log-Hölder continuity property (2.1), then

$$\|u\|_{p^*(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega),$$

for some positive constant C, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

On the other hand, if p satisfies (2.1) and $p_- > N$, then

$$\|u\|_\infty \leq C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega) \ ,$$

where C is another positive constant.

Lemma 2.3 ([7]). *Assume that*

$$\begin{gathered} 1 < r \leq p_n(x) \leq s < \infty \quad \forall n \in \mathbb{N}, \\ \text{for a.e. } x \in \Omega, \text{ for some constants } r \text{ and } s, \end{gathered} \tag{2.12}$$

$$p_n \to p \ \text{ a.e. in } \Omega, \text{ as } n \to \infty, \tag{2.13}$$

$$\nabla u_n \to \nabla u \text{ in } L^1(\Omega)^N, \text{ as } n \to \infty, \tag{2.14}$$

$$\||\nabla u_n|^{p_n(x)}\|_1 \leq C, \text{ for some positive constant } C \text{ not depending on } n. \tag{2.15}$$

Then $\nabla u \in L^{p(\cdot)}(\Omega)^N$ *and*

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n(x)} dx \geq \int_\Omega |\nabla u|^{p(x)} dx. \tag{2.16}$$

Lemma 2.4 ([8, 12]). *For all* ξ, $\eta \in \mathbb{R}^N$, *the following assertions hold true:*

$$2 \leq p < \infty \Rightarrow \frac{1}{2^{p-1}}|\xi - \eta|^p \leq \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta), \tag{2.17}$$

$$1 < p < 2 \Rightarrow (p-1)|\xi - \eta|^2 \leq \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \left(|\xi|^p + |\eta|^p\right)^{\frac{2-p}{p}}. \tag{2.18}$$

3 Existence for the local problem

In this section, we prove the existence of weak solutions for the local problem (1.1). Firstly, we define the following space:

$$W_0^{1,p(u)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(u)} dx < \infty \right\} \text{ such that } 1 < p(u) < \infty \text{ for all } u \in \mathbb{R}.$$

It is a Banach space for the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ defined at (2.11) which is equivalent to $\|\nabla u\|_{p(u)}$ when $p(u) \in C(\overline{\Omega})$. Since p is continuous then from the fact that $1 < r \leq p$, $W_0^{1,p(u)}(\Omega)$ is separable and reflexive.

Definition 3.1. *Assume that p verifies* (1.2) *and*

$$f \in W^{-1,r'}(\Omega). \tag{3.1}$$

A function $u \in W_0^{1,p(u)}(\Omega)$ is said to be a weak solution to the problem (1.1)*, if*

$$\int_\Omega |\nabla u|^{p(u)-2} \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p(u)-2} uv \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,p(u)}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(W_0^{1,p(u)}(\Omega))'$ and $W_0^{1,p(u)}(\Omega)$.

Theorem 3.2. *Assume that* (1.2) *and* (3.1) *hold together with*

$$N < r \leq p(u) \leq s < +\infty \tag{3.2}$$

and

$$p : \mathbb{R} \longrightarrow [1, +\infty) \text{ is a Lipschitz-continuous function.} \tag{3.3}$$

Then there exists at least one weak solution to problem (1.1) *in the sense of Definition 3.1.*

The proof of Theorem 3.2 is divided into several steps.

Step 1: Approximate problems

For each $\varepsilon > 0$, we consider the following auxiliary problem (namely, the regularized problem)

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2} \nabla u) + |u|^{p(u)-2} u + \varepsilon \left(|u|^{s-2} u - \mathrm{div}(|\nabla u|^{s-2} \nabla u) \right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where

$$N < r \leq p(u) \leq s < \infty \quad \forall u \in \mathbb{R}.$$

Proposition 3.3. *For each $\varepsilon > 0$, the problem* (3.4) *admits a weak solution u_ε.*

Proof. Let $w \in L^2(\Omega)$, then

$$N < r \leq p(w) \leq s < \infty \quad \text{for a.e.} \quad x \in \Omega. \tag{3.5}$$

Recalling that $f \in W^{-1,r'}(\Omega) \subset W^{-1,s'}(\Omega)$. Now, we focus on the operator $T_\varepsilon : W_0^{1,s}(\Omega) \to W^{-1,s'}(\Omega)$ defined by

$$\langle T_\varepsilon(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(w)-2}\nabla u \cdot \nabla v dx + |u|^{p(w)-2}uv\right) dx + \varepsilon \left[\int_\Omega \left(|\nabla u|^{s-2}\nabla u \cdot \nabla v dx + |u|^{s-2}uv\right) dx\right],$$

for all $u, v \in W_0^{1,s}(\Omega)$. We can establish that:

(i) T_ε is continuous, bounded;

(ii) T_ε is strictly monotone (the strict monotonicity follows by Lemma 2.4);

(iii) T_ε is coercive.

According to (i), (ii) and (iii), the operator T_ε is continuous, strictly monotone (hence, maximal monotone too), and coercive. It follows that T_ε is a strictly monotone surjective operator (see Corollary 2.8.7, p. 135, [16]). Therefore, there exists a unique solution $u_w \in W_0^{1,s}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega |\nabla u_w|^{p(w)-2}\nabla u_w \cdot \nabla v dx + \int_\Omega |u_w|^{p(w)-2}u_w v dx + \\ &\varepsilon\left(\int_\Omega |\nabla u_w|^{s-2}\nabla u_w \cdot \nabla v dx + \int_\Omega |u_w|^{s-2}u_w v dx\right) = \langle f, v\rangle \quad \forall v \in W_0^{1,s}(\Omega).\end{aligned} \tag{3.6}$$

We take $v = u_w$ in (3.6) to derive that

$$\int_\Omega |\nabla u_w|^{p(w)}dx + \int_\Omega |u_w|^{p(w)}dx + \varepsilon\left(\int_\Omega |u_w|^s dx + \int_\Omega |\nabla u_w|^s dx\right) \leq \|f\|_{-1,r'}\|\nabla u_w\|_r \leq C\|\nabla u_w\|_s,$$

where $C = C(r, s, \Omega, f)$, and $\|\cdot\|_{-1,r'}$ is the operator norm associated to the norm $\|\nabla\cdot\|_r$. Therefore

$$\varepsilon\|u_w\|_{1,s}^s \leq C\|\nabla u_w\|_s \leq C\|u_w\|_{1,s}.$$

Hence

$$\|u_w\|_{1,s} \leq C, \tag{3.7}$$

where $C = C(r, s, \Omega, \varepsilon, f)$ is a positive constant without w-dependence. From the fact that $s > N \geq 2$, we can deduce that

$$\|u_w\|_{L^2(\Omega)} \leq C. \tag{3.8}$$

Next, we introduce the self-map $T : B \to B$ defined by $T(w) = u_w$, over the set

$$B := \left\{ v \in L^2(\Omega) : \|v\|_{L^2(\Omega)} \leq C \right\}.$$

The compact embedding $W_0^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$ implies that $T(B)$ is relatively compact in B. Appealing to the Schauder fixed-point theorem, we know that the continuity of T is required in obtaining a fixed point of T.

With the assumption that we work on a sequence $\{w_n\}$ in $L^2(\Omega)$ satisfying

$$w_n \to w \text{ in } L^2(\Omega) \quad \text{as} \quad n \to \infty, \tag{3.9}$$

we denote by u_n, for all $n \in \mathbb{N}$, the solution of (3.6) related to $w := w_n$. Therefore, the inequality in (3.7) leads to

$$\|u_n\|_{1,s} \leq C, \quad \text{for some positive constant (without } n\text{-dependence)}\,.$$

Passing to a subsequence if necessary (namely again $\{u_n\}$), for a certain $u \in W_0^{1,s}(\Omega)$ we get

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,s}(\Omega), \quad \text{as} \quad n \to \infty, \tag{3.10}$$

$$u_n \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty. \tag{3.11}$$

We return to (3.6), so that considering (u_n, w_n) instead of (u, w), we get

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(w_n)-2} \nabla u_n + \varepsilon |\nabla u_n|^{s-2} \nabla u_n \right) \cdot \nabla v \, dx + \\ &\int_\Omega \left(|u_n|^{p(w_n)-2} u_n + \varepsilon |u_n|^{s-2} u_n \right) v \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,s}(\Omega). \end{aligned} \tag{3.12}$$

Since the operator on the left-hand side of (3.12) is monotone, then

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(w_n)-2} \nabla u_n + \varepsilon |\nabla u_n|^{s-2} \nabla u_n \right) \cdot \nabla (u_n - v) dx + \\ &\int_\Omega \left(|u_n|^{p(w_n)-2} u_n + \varepsilon |u_n|^{s-2} u_n \right) (u_n - v) dx - \end{aligned} \tag{3.13}$$

$$\int_\Omega \left(|\nabla v|^{p(w_n)-2}\nabla v + \varepsilon|\nabla v|^{s-2}\nabla v\right)\cdot\nabla(u_n - v)dx -$$
$$\int_\Omega \left(|v|^{p(w_n)-2}v + \varepsilon|v|^{s-2}v\right)(u_n - v)dx \geq 0 \quad \forall v \in W_0^{1,s}(\Omega).$$

Considering (3.12) with $v = u_n - v$ as a test function, we use (3.13) to get

$$\langle f, u_n - v\rangle - \int_\Omega \left(|\nabla v|^{p(w_n)-2}\nabla v + \varepsilon|\nabla v|^{s-2}\nabla v\right)\cdot\nabla(u_n - v)dx - \tag{3.14}$$
$$\int_\Omega \left(|v|^{p(w_n)-2}v + \varepsilon|v|^{s-2}v\right)(u_n - v)dx \geq 0 \quad \forall v \in W_0^{1,s}(\Omega).$$

The convergence in (3.9) implies

$$w_n \to w \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty.$$

Since p is a continuous function, we can apply Lebesgue's theorem (in $L^{s'}(\Omega)^N$), therefore

$$|\nabla v|^{p(w_n)-2}\nabla v \to |\nabla v|^{p(w)-2}\nabla v \quad \text{strongly in} \quad L^{s'}(\Omega)^d, \quad \text{as} \quad n \to \infty \tag{3.15}$$

and

$$|v|^{p(w_n)-2}v \to |v|^{p(w)-2}v \quad \text{strongly in} \quad L^s(\Omega), \quad \text{as} \quad n \to \infty, \tag{3.16}$$

for all $v \in W_0^{1,s}(\Omega)$. Finally, by the weak convergence in (3.10) and using (3.15) and (3.16) we can pass to the limit in (3.14) to obtain

$$\langle f, u - v\rangle - \int_\Omega \left(|\nabla v|^{p(w)-2}\nabla v + \varepsilon|\nabla v|^{s-2}\nabla v\right)\cdot\nabla(u - v)dx -$$
$$\int_\Omega \left(|v|^{p(w)-2}v + \varepsilon|v|^{s-2}v\right)(u - v)dx \geq 0 \quad \forall v \in W_0^{1,s}(\Omega). \tag{3.17}$$

Next, choosing $v = u \pm \delta\varphi$, where $\varphi \in W_0^{1,s}(\Omega)$ and $\delta > 0$, we get

$$\pm\Big[\langle f, \varphi\rangle - \int_\Omega \big(|\nabla(u \pm \delta\varphi)|^{p(w)-2}\nabla(u \pm \delta\varphi) + \varepsilon|\nabla(u \pm \delta\varphi)|^{s-2}\nabla(u \pm \delta\varphi)\big)\cdot\nabla\varphi\, dx -$$
$$\int_\Omega \left(|u \pm \delta\varphi|^{p(w)-2}(u \pm \delta\varphi) + \varepsilon|v|^{s-2}(u \pm \delta\varphi)\right)\varphi\, dx\Big] \geq 0. \tag{3.18}$$

We pass to the limit as δ goes to zero in (3.18), and deduce that

$$\int_\Omega \left(|\nabla(u)|^{p(w)-2}\nabla u + \varepsilon|\nabla u|^{s-2}\nabla u\right)\cdot\nabla\varphi dx + \int_\Omega \left(|u|^{p(w)-2}u + \varepsilon|v|^{s-2}u\right)\varphi dx = \langle f, \varphi\rangle \quad \forall \varphi \in W_0^{1,s}(\Omega).$$

Consequently $u = u_w$. In view of (3.11) and by the strong convergence in (3.11), we conclude that

$$u_{w_n} \to u_w \quad \text{strongly in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty,$$

It follows that T is continuous, and this establishes the existence of the fixed point which is the exact weak solution to (3.4). $\square$

Step 2: Passage to the limit as $\varepsilon \to 0$

From Proposition 3.3, for each $\varepsilon > 0$ there exists $u_\varepsilon \in W_0^{1,s}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega |\nabla(u_\varepsilon)|^{p(u_\varepsilon)-2}\nabla u_\varepsilon \nabla v dx + \int_\Omega |u_\varepsilon|^{p(u_\varepsilon)-2} u_\varepsilon v dx + \\ &\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^{s-2}\nabla u_\varepsilon \cdot \nabla v dx + \int_\Omega |u_\varepsilon|^{s-2} u_\varepsilon v dx\right) = \langle f, v\rangle \quad \forall v \in W_0^{1,s}(\Omega)\end{aligned} \tag{3.19}$$

and

$$N < r \le p(u_\varepsilon(x)) \le s < \infty \quad \forall \varepsilon > 0, \quad \text{for a.e.} \quad x \in \Omega.$$

Next, we choose $v = u_\varepsilon$ as a test function in (3.19) to obtain

$$\int_\Omega \left(|\nabla u_\varepsilon|^{p(u_\varepsilon)} + |u_\varepsilon|^{p(u_\varepsilon)}\right) dx + \varepsilon\left(\|\nabla u_\varepsilon\|_s^s + \|u_\varepsilon\|_s^s\right) = \langle f, u_\varepsilon\rangle. \tag{3.20}$$

From (2.7), we deduce that

$$\|u\|_{q(\cdot)} \le (\rho_{q(\cdot)}(u) + 1)^{\frac{1}{q_-}} = \left(\int_\Omega |\nabla u|^{q(x)} dx + 1\right)^{\frac{1}{q_-}}.$$

By using the Hölder inequality, we get

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^r dx &\le C\||\nabla u_\varepsilon|^r\|_{\frac{p(u_\varepsilon)}{r}} \le C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)} dx + 1\right)^{\frac{1}{\left(\frac{p(u_\varepsilon)}{r}\right)_-}} \\ &\le C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)} dx + 1\right),\end{aligned} \tag{3.21}$$

where $C = C(r, s, \Omega)$. Therefore

$$\langle f, u_\varepsilon\rangle \le \|f\|_{-1,r'}||\nabla u_\varepsilon||_r \le C\|f\|_{-1,r'}\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)} dx + 1\right)^{\frac{1}{r}}. \tag{3.22}$$

From (3.20), (3.22) and by using Young's inequality, we obtain

$$\int_\Omega \left(|\nabla u_\varepsilon|^{p(u_\varepsilon)} + |u_\varepsilon|^{p(u_\varepsilon)}\right) dx + \varepsilon\left(\|\nabla u_\varepsilon\|_s^s + \|u_\varepsilon\|_s^s\right) \le C. \tag{3.23}$$

Using (3.21) and (3.22), we can deduce the estimate

$$||u_\varepsilon||_{1,r} \le C, \tag{3.24}$$

where C is a positive constant without ε-dependence.

Now we consider a sequence $\{\varepsilon_n\}$ of positive real numbers. For every $n \in \mathbb{N}$, let u_{ε_n} be the solution to the problem (3.4) associated to ε_n. Since $W_0^{1,r}(\Omega) \hookrightarrow L^2(\Omega)$ compactly, then after passing to a subsequence if needed, for some $u \in W_0^{1,r}(\Omega)$ we have

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{in} \quad W_0^{1,r}(\Omega), \quad \text{as} \quad n \to \infty \tag{3.25}$$

$$\nabla u_{\varepsilon_n} \rightharpoonup \nabla u \quad \text{in} \quad L^r(\Omega)^N, \quad \text{as} \quad n \to \infty \tag{3.26}$$

$$u_{\varepsilon_n} \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty$$

$$u_{\varepsilon_n} \to \quad u \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty. \tag{3.27}$$

The constraints on the exponent range in (3.2) imply that u is Hölder-continuous, then from the condition (3.3), the same conclusion holds for $p(u)$. From (3.27), we deduce that

$$p(u_{\varepsilon_n}) \to p(u) \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty. \tag{3.28}$$

Clearly, the following chain of inequalities is satisfied

$$N < r \leq p(u_{\varepsilon_n}) \leq s < \infty \quad \forall n \in \mathbb{N}, \quad \text{for a.e.} \quad x \in \Omega. \tag{3.29}$$

Using (3.23) written for u_{ε_n}, together with (3.26), (3.28) and (3.29), we conclude that (by Lemma 2.3)

$$u \in W_0^{1,p(u)}(\Omega). \tag{3.30}$$

From the theory of monotone operators, we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2} \nabla u_{\varepsilon_n} + \varepsilon_n |\nabla u_{\varepsilon_n}|^{s-2} \nabla u_{\varepsilon_n} \right) \cdot \nabla(u_{\varepsilon_n} - v) dx + \\
&\int_\Omega \left(|u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2} u_{\varepsilon_n} + \varepsilon_n |u_{\varepsilon_n}|^{s-2} u \right) (u_{\varepsilon_n} - v) dx - \\
&\Big(\int_\Omega \left(|\nabla v|^{p(u_{\varepsilon_n})-2} \nabla v + \varepsilon_n |\nabla v|^{s-2} \nabla v \right) \cdot \nabla(u_{\varepsilon_n} - v) dx + \\
&\int_\Omega \left(|v|^{p(u_{\varepsilon_n})-2} v + \varepsilon_n |v|^{s-2} v \right) (u_{\varepsilon_n} - v) dx \Big) \geq 0 \quad \forall v \in W_0^{1,s}(\Omega).
\end{aligned} \tag{3.31}$$

By replacing u_ε with u_{ε_n} and choosing $u_{\varepsilon_n} - v$ as a test function in (3.19), we can reduce (3.31) to the form

$$\langle f, u_{\varepsilon_n} - v\rangle - \Big(\int_\Omega \Big(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + \varepsilon |\nabla v|^{s-2}\nabla v\Big)\cdot \nabla(u_{\varepsilon_n} - v)dx + \int_\Omega \Big(|v|^{p(u_{\varepsilon_n})-2}v + \varepsilon|v|^{s-2}v\Big)(u_{\varepsilon_n} - v)dx\Big) \geq 0, \tag{3.32}$$

for all $v \in C_0^\infty(\Omega)$. By using (3.28) and the Lebesgue theorem, we have

$$|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v \to |\nabla v|^{p(u)-2}\nabla v \quad \text{in} \quad L^{r'}(\Omega)^d, \quad \text{as} \quad n\to\infty \tag{3.33}$$

and

$$|v|^{p(u_{\varepsilon_n})-2}v \to |v|^{p(u)-2}v \quad \text{in} \quad L^{r}(\Omega), \quad \text{as} \quad n\to\infty. \tag{3.34}$$

We take the limit as n goes to infinity in (3.32), and use (3.24), (3.25), (3.33) and (3.34), therefore

$$\langle f, u - v\rangle - \left(\int_\Omega |\nabla v|^{p(u)-2}\nabla v\cdot\nabla(u-v)dx + \int_\Omega |v|^{p(u)-2}v(u-v)dx\right) \geq 0 \quad \forall v \in C_0^\infty(\Omega). \tag{3.35}$$

From the assumptions (3.2) and (3.3), the functions $p(u)$ is Hölder-continuous which implies that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(u)}(\Omega)$. Thus, (3.34) holds true also for all $v \in W_0^{1,p(u)}(\Omega)$.

So we can take $v = u \pm \delta\varphi$, where $\varphi \in W_0^{1,p(u)}(\Omega)$ and $\delta > 0$, as a test function in (3.34) we get

$$\pm\left(\langle f,\varphi\rangle - \Big(\int_\Omega |\nabla u|^{p(u)-2}\nabla u\cdot\nabla\varphi\, dx + \int_\Omega |u|^{p(u)-2}u\varphi\, dx\Big)\right) \geq 0. \tag{3.36}$$

This implies that,

$$\int_\Omega |\nabla u|^{p(u)-2}\nabla u\cdot\nabla\varphi\, dx + \int_\Omega |u|^{p(u)-2}u\varphi\, dx = \langle f,\varphi\rangle \quad \forall\varphi \in W_0^{1,p(u)}(\Omega). \tag{3.37}$$

Finally, we arrived to a solution for our local problem (1.1) (See Definition 3.1).

4 Nonlocal problems

Along with problem (1.1), we consider in this section its nonlocal version. Firstly, we assume that the function p satisfies the conditions in (1.2). We denote by b a mapping from $W_0^{1,r}(\Omega)$ into $\mathbb{R}$ such that

$$b \text{ is continuous, } \ b \text{ is bounded.} \tag{4.1}$$

The next theorem needs the following revised definition of a weak solution.

Definition 4.1. *A function u is said to be a weak solution to the problem* (1.3) *if*

$$\begin{cases} u \in W_0^{1,p(b(u))}(\Omega)\ , \\ \displaystyle\int_\Omega |\nabla u|^{p(b(u))-2}\nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p(b(u))-2} uv \, dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p(b(u))}(\Omega), \end{cases} \tag{4.2}$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $\left(W_0^{1,p(b(u))}(\Omega)\right)'$ and $W_0^{1,p(b(u))}(\Omega)$.

Since $p(b(u))$ is here a real number and not a function, thus the Sobolev spaces involved are the classical ones.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a bounded domain and assume that* (1.2) *and* (4.1) *hold together with*

$$f \in W^{-1,q'}(\Omega) \quad \text{for} \quad q < r.$$

Then there exists at least one weak solution to the problem (1.3) *in the sense of Definition 4.1.*

To prove Theorem 4.2, we need the following Lemma.

Lemma 4.3. *For $n \in \mathbb{N}$, let u_n be the solution to the problem*

$$\begin{cases} u_n \in W_0^{1,p_n}(\Omega), \\ \displaystyle\int_\Omega |\nabla u_n|^{p_n-2}\nabla u_n \cdot \nabla v \, dx + \int_\Omega |u_n|^{p_n-2} u_n v \, dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p_n}(\Omega), \end{cases} \tag{4.3}$$

where $\langle \cdot,\cdot\rangle$ denotes here the duality pairing between $\left(W_0^{1,p_n}(\Omega)\right)'$ and $W_0^{1,p_n}(\Omega)$.

Assume that

$$p_n \to p, \quad \text{as} \quad n \to \infty, \quad \text{where} \quad p \in (1,\ \infty), \tag{4.4}$$

$$f \in W^{-1,q'}(\Omega) \quad \text{for some} \quad q < p. \tag{4.5}$$

Then

$$u_n \to u \quad \text{in} \quad W_0^{1,q}(\Omega), \quad \text{as} \quad n \to \infty, \tag{4.6}$$

where u is the solution to the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} uv \, dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{4.7}$$

Proof of Lemma 4.3. The proof of Lemma 4.3 is divided into two steps.

Step 1: Weak convergence

In view of $p_n \to p$, as $n \to \infty$, and $q < p$, we may suppose that

$$p+1 > p_n > q \quad \forall n \in \mathbb{N}. \tag{4.8}$$

We choose $v = u_n$ as a test function in (4.3) to obtain

$$\int_\Omega |\nabla u_n|^{p_n} dx + \int_\Omega |u_n|^{p_n} dx \leq \|f\|_{-1,q'} \|\nabla u_n\|_q. \tag{4.9}$$

From (4.8) and Hölder's inequality, we deduce that

$$\|\nabla u_n\|_q \leq C\|\nabla u_n\|_{p_n} \leq C\|u_n\|_{1,p_n}, \tag{4.10}$$

where $C = C(p, q, \Omega)$ is a positive constant. Therefore

$$\|u_n\|_{1,p_n} \leq C, \tag{4.11}$$

where $C = C(p, q, \Omega, f)$ is a positive constant. Combining (4.10) with (4.11), we get

$$\|\nabla u_n\|_q \leq C, \tag{4.12}$$

where C is a positive constant without n-dependence. Passing to a subsequence if necessary still denoted by u_n, for a certain $u \in W_0^{1,q}(\Omega)$ we get

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in} \quad L^q(\Omega), \quad \text{as} \quad n \to \infty. \tag{4.13}$$

On this basis, the convergences in (4.4), (4.8), (4.11) and (4.13) lead to the conclusion that (Lemma 2.3)

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n} dx \geq \int_\Omega |\nabla u|^p dx,$$

and hence

$$u \in W_0^{1,p}(\Omega). \tag{4.14}$$

We observe that, the second line in (4.3) is equivalent to

$$\int_\Omega |\nabla u_n|^{p_n-2}\nabla u_n \cdot \nabla(v-u_n)dx + \int_\Omega |u_n|^{p_n-2}u_n(v-u_n)dx \geq \langle f, v-u_n\rangle \quad \forall v \in W_0^{1,p_n}(\Omega),$$

using the Minty lemma, we have

$$\int_\Omega |\nabla v|^{p_n-2}\nabla v\cdot\nabla(v-u_n)dx+\int_\Omega |v|^{p_n-2}v(v-u_n)dx\geq\langle f,v-u_n\rangle\quad\forall v\in W_0^{1,p_n}(\Omega).\tag{4.15}$$

We choose $v\in C_0^\infty(\Omega)$, then we can take the limit as n goes to infinity in (4.15), and use (4.4) and (4.13), hence we obtain

$$\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla(v-u)dx+\int_\Omega |v|^{p-2}v(v-u)dx\geq\langle f,v-u\rangle\quad\forall v\in C_0^\infty(\Omega).\tag{4.16}$$

Since $C_0^\infty(\Omega)$ dense in $W_0^{1,p}(\Omega)$, we have (4.16) also holds for all $v\in W_0^{1,p}(\Omega)$. Now, choosing $v=u\pm\delta\varphi$, where $\varphi\in W_0^{1,p}(\Omega)$ and $\delta>0$, by passing to the limit as δ goes to zero, we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx+\int_\Omega |u|^{p-2}u\varphi dx=\langle f,\varphi\rangle\quad\forall\varphi\in W_0^{1,p}(\Omega),$$

Finally, it is sufficient to recall that $u\in W_0^{1,p}(\Omega)$ to conclude that we arrived to a solution for the problem (4.7).

Step 2: Strong convergence

In this step we will show that the convergence (4.13) is strong. Firstly, we take $v=u_n$ in (4.3) and using (4.13) to pass to the limit, we get

$$\int_\Omega |\nabla u_n|^{p_n}dx+\int_\Omega |u_n|^{p_n}dx=\langle f,v\rangle\to\int_\Omega |\nabla u|^{p}dx+\int_\Omega |u|^{p}dx=\langle f,v\rangle\quad\text{as}\quad n\to\infty.\tag{4.17}$$

Firstly, we consider the case when

$$p_n\geq p\quad\forall n\in\mathbb{N}.$$

By using Hölder's inequality, we have

$$\int_\Omega |\nabla u_n|^p dx\leq\left(\int_\Omega |\nabla u_n|^{p_n}dx\right)^{\frac{p}{p_n}}|\Omega|^{1-\frac{p}{p_n}}.$$

Thus by (4.17), we deduce that

$$\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^p dx\leq\int_\Omega |\nabla u|^p dx\leq\liminf_{n\to\infty}\int_\Omega |\nabla u_n|^p dx,$$

which implies (from the fact that $\|\nabla u_n\|_p\to\|\nabla u\|_p$, as $n\to\infty$)

$$u_n\to u\quad\text{strongly}\quad\text{in}\quad W_0^{1,p}(\Omega),\quad\text{as}\quad n\to\infty.\tag{4.18}$$

From the fact that $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$, we conclude that

$$u_n \to u \quad \text{in} \quad W_0^{1,q}(\Omega), \qquad \text{as} \quad n \to \infty.$$

Now, we consider the case when

$$q < p_n < p \quad \forall n \in \mathbb{N}, \tag{4.19}$$

we set

$$\begin{aligned} A_n := &\int_\Omega \left(|\nabla u_n|^{p_n-2}\nabla u_n - |\nabla u|^{p_n-2}\nabla u\right) \cdot (\nabla u_n - \nabla u)dx + \\ &\int_\Omega \left(|u_n|^{p_n-2}u_n - |u|^{p_n-2}u\right) \cdot (u_n - u)dx. \end{aligned} \tag{4.20}$$

By the theory of monotone operators, we have $A_n \geq 0$, (4.3) imply that (4.20) reduces to the form

$$A_n = \langle f,\ u_n - u\rangle - \int_\Omega |\nabla u|^{p_n-2}\nabla u \cdot \nabla(u_n - u)dx - \int_\Omega |u|^{p_n-2}u(u_n - u)dx.$$

Due to (4.5) and the convergence in (4.13), we have

$$\langle f, u_n - u\rangle \to 0, \quad \text{as} \quad n \to \infty. \tag{4.21}$$

From the fact that $u \in W_0^{1,p}(\Omega)$ we get

$$||\nabla u|^{p_n-2}\nabla u| \leq \max\{1,\ |\nabla u|^{p-1}\} \in L^{p'}(\Omega), \tag{4.22}$$

$$||u|^{p_n-2}u| \leq \max\{1,\ |u|^{p-1}\} \in L^{p}(\Omega). \tag{4.23}$$

On this basis, we can conclude that

$$A_n \to 0, \quad \text{as} \quad n \to \infty. \tag{4.24}$$

We first consider the case when $p_n \geq 2$. By applying the Lemma 2.4 in (4.20), we get

$$A_n \geq \frac{1}{2^{p_n-1}}\left(\int_\Omega |\nabla(u_n - u)|^{p_n}dx + \int_\Omega |u_n - u|^{p_n}dx\right). \tag{4.25}$$

Since $p_n > q$, we can apply Hölder's inequality to obtain

$$\int_\Omega |\nabla(u_n - u)|^q dx + \int_\Omega |u_n - u|^q dx \leq \left[\left(\int_\Omega |\nabla(u_n - u)|^{p_n}dx\right)^{\frac{q}{p_n}} + \left(\int_\Omega |u_n - u|^{p_n}dx\right)^{\frac{q}{p_n}}\right] |\Omega|^{1-\frac{q}{p_n}}.$$

Hence, from (4.24) and (4.25) we get

$$\int_{\Omega}|\nabla(u_n-u)|^q dx+\int_{\Omega}|u_n-u|^q dx\longrightarrow 0,\quad \text{as}\quad n\to\infty.$$

Therefore,

$$u_n\to u\quad \text{in}\quad W_0^{1,q}(\Omega),\quad \text{as}\quad n\to\infty.$$

Now, we assume that $p_n<2$:

By using the Hölder's inequality we obtain

$$\begin{aligned}
&\int_{\Omega}|\nabla(u_n-u)|^{p_n}dx+\int_{\Omega}|u_n-u|^{p_n}dx\\
&=\int_{\Omega}|\nabla(u_n-u)|^{p_n}\left(|\nabla u_n|+|\nabla u|\right)^{\frac{(p_n-2)p_n}{2}}\left(|\nabla u_n|+|\nabla u|\right)^{\frac{(2-p_n)p_n}{2}}dx\\
&+\int_{\Omega}|u_n-u|^{p_n}\left(|u_n|+|u|\right)^{\frac{(p_n-2)p_n}{2}}\left(|u_n|+|u|\right)^{\frac{(2-p_n)p_n}{2}}dx\\
&\leq\left[\int_{\Omega}|\nabla(u_n-u)|^2\left(|\nabla u_n|+|\nabla u|\right)^{p_n-2}dx\right]^{\frac{p_n}{2}}\left[\int_{\Omega}\left(|\nabla u_n|+|\nabla u|\right)^{p_n}dx\right]^{1-\frac{p_n}{2}}\\
&+\left[\int_{\Omega}|u_n-u|^2\left(|u_n|+|u|\right)^{p_n-2}dx\right]^{\frac{p_n}{2}}\left[\int_{\Omega}\left(|u_n|+|u|\right)^{p_n}dx\right]^{1-\frac{p_n}{2}}.
\end{aligned}\tag{4.26}$$

From Lemma 2.4, one could deduce that

$$A_n\geq C(p_n)\left(\int_{\Omega}|\nabla(u_n-u)|^2\left(|\nabla u_n|+|\nabla u|\right)^{p_n-2}dx+\int_{\Omega}|u_n-u|^2\left(|u_n|+|u|\right)^{p_n-2}dx\right).\tag{4.27}$$

Since $\|u_n\|_{1,p_n}\leq C$, then from (4.24), (4.26) and (4.27) we get

$$\int_{\Omega}|\nabla(u_n-u)|^{p_n}dx+\int_{\Omega}|u_n-u|^{p_n}dx\to 0,\quad \text{as}\quad n\to\infty.$$

Therefore,

$$u_n\to u\quad \text{in}\quad W_0^{1,q}(\Omega),\quad \text{as}\quad n\to\infty.\qquad\square$$

Proof of Theorem 4.2. For any $s>q$ we have, $f\in(W_0^{1,s}(\Omega))'\subset\left(W_0^{1,q}(\Omega)\right)'$. Therefore, for each $\lambda\in\mathbb{R}$, the following $p(\lambda)$-Laplacian problem admits a unique solution u_λ,

$$\begin{cases} u\in W_0^{1,p(\lambda)}(\Omega)\ ,\\ \displaystyle\int_{\Omega}|\nabla u|^{p(\lambda)-2}\nabla u\cdot\nabla v\,dx+\int_{\Omega}|u|^{p(\lambda)-2}uv\,dx=\langle f,v\rangle\quad\forall v\in W_0^{1,p(\lambda)}(\Omega)\ .\end{cases}\tag{4.28}$$

The choice of test function u_λ in (4.28) implies that

$$\int_\Omega |\nabla u_\lambda|^{p(\lambda)}dx + \int_\Omega |u_\lambda|^{p(\lambda)}dx \le \|f\|_{-1,r'}\|\nabla u_\lambda\|_r. \tag{4.29}$$

Now using the Hölder's inequality, one obtains

$$\|u_\lambda\|_{1,r} \le \|u_\lambda\|_{1,p(\lambda)}|\Omega|^{\frac{1}{r}-\frac{1}{p(\lambda)}}. \tag{4.30}$$

From (4.29), it follows that

$$\|u_\lambda\|_{1,p(\lambda)}^{p(\lambda)-1} \le \|f\|_{-1,r'}|\Omega|^{\frac{1}{r}-\frac{1}{p(\lambda)}}. \tag{4.31}$$

Combining (4.30) and (4.31), and using (1.2) to get

$$\|u_\lambda\|_{1,r} \le \|f\|_{-1,r'}^{\frac{1}{p(\lambda)-1}}|z|^{\left(\frac{1}{r}-\frac{1}{p(\lambda)}\right)\frac{p(\lambda)}{p(\lambda)-1}} \le \max_{p\in[r,s]}\|f\|_{1,r'}^{\frac{1}{p-1}}|\Omega|^{\left(\frac{1}{r}-\frac{1}{p}\right)\frac{p}{p-1}}. \tag{4.32}$$

Therefore

$$\|u_\lambda\|_{1,r} \le C. \tag{4.33}$$

The inequality (4.33) and the fact that b is a bounded mapping, imply that there exists $K \in \mathbb{R}$ such that

$$b(u_\lambda) \in [-K,\ K] \quad \forall \lambda \in \mathbb{R}.$$

Next, we introduce the self-map $H : [-K,\ K] \to [-K,\ K]$ defined by $H(\lambda) = b(u_\lambda)$. We know that the continuity of H is required in obtaining a fixed point of H.

Assume that $\lambda_n \to \lambda$ as $n \to \infty$, because p is continuous, $p(\lambda_n) \to p(\lambda)$. Next, we apply Lemma 4.3, so that considering $p(\lambda_n)$ instead of p_n, we deduce that

$$u_{\lambda_n} \longrightarrow u_\lambda \quad \text{in} \quad W_0^{1,r}(\Omega), \quad \text{as} \quad n \to \infty.$$

We use the fact that b is continuous to deduce that $b(u_\lambda) \to b(u_\lambda)$, as n goes to infinity, which implies that the map H is continuous. This establishes the existence of the fixed point λ_0 and a weak solution u_{λ_0} for the problem (4.2). $\square$

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