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Boundedness and stability in nonlinear systems of differential equations using a modified variation of parameters formula

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ABSTRACT

In this research we introduce a new variation of parameters for systems of linear and nonlinear ordinary differential equations. We use known mathematical methods and techniques including Gronwall's inequality and fixed point theory to obtain boundedness on all solutions and stability results on the zero solution.

RESUMEN

En esta investigación, introducimos un nuevo método de variación de parámetros para sistemas de ecuaciones diferenciales ordinarias lineales y no lineales. Usamos métodos y técnicas matemáticas conocidas incluyendo la desigualdad de Gronwall y teoría de punto fijo para obtener el acotamiento de todas las soluciones y resultados de estabilidad de la solución cero.

Keywords and Phrases: System, Ordinary differential equations, Linear, Nonlinear, Fundamental matrix, Boundedness, Stability, New variation of parameters.

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1 Introduction

A general approach to solving inhomogeneous linear ordinary differential equations in mathematics is variation of parameters, often known as variation of constants. In this paper we introduce new variation of parameters formula for systems of linear and nonlinear ordinary differential equations. Once the inversion is done, we apply known results such as Gronwall's inequality and the contraction mapping principle to obtain boundedness on all solutions and stability results on the zero solution. It is common practice to linearize around the equilibrium solution for nonlinear systems before drawing conclusions about the stability of the equilibrium solution for the original system using the signs of the linear system's eigenvalues. We demonstrate in our cases how this approach does not work. Utilizing Liapunov functions and functionals to analyze solutions is an additional well-liked technique. However, the method by which such Liapunov functions/functionals are created remains a mystery, and the type of Liapunov functions/functionals determines whether or not the conclusions reached are valid. In general, obtaining a variation of parameters formula relies on heuristics that require guessing and are not applicable to all inhomogeneous linear differential equations, it is typically possible to find solutions to first-order inhomogeneous linear differential equations using integrating factors or undetermined coefficients with a great deal less effort.

Hence, in this research our main intention is to be able to write totally nonlinear systems of the form

$$x' = f(t, x(t)),$$

into an integral system of equations, from which we obtain results concerning the behavior of solutions using fixed point theory. The absence of a linear term in $x' = f(t, x(t))$ is the sole cause for not being able to invert the system and obtain a variation of parameters formula for the solutions. For such systems, usually researchers borrow a linear term for the sake of inversion, and as a result, the resulting integral equation may not satisfy a contraction property.

In [18] the first author used Lyapunov functionals and studied the exponential stability of the zero solution of finite delay Volterra Integro-differential equation

$$x'(t) = Px(t) + \int_{t-\tau}^{t} C(t,s)g(x(s))\, ds. \tag{1.1}$$

Recently, in [5, 6], Burton used the notion of fixed point theory to alleviate some of the difficulties that arise from the use of Liapunov functionals and obtained results concerning the stability and asymptotic stability of the zero solution of (1.1) when it is scalar. We remark that the results of [5, 6, 18] were made possible due to the existence of the linear term Px.

To ease the reader into the main parts of this research, we begin with by considering the scalar differential equation

$$x'(t) = ax(t), \ \ x(0) = x_0 \tag{1.2}$$

with the known solution

$$x(t) = x_0 e^{at}.$$

We notice that if

$$a < 0, \quad \text{then} \quad x(t) = x_0 e^{at} \to 0, \quad \text{as} \quad t \to \infty.$$

To further introduce our topic, we assume $a : \mathbb{R} \to \mathbb{R}$ is continuous and consider

$$x'(t) = a(t)x(t), \ \ x(0) = x_0, \tag{1.3}$$

which has the solution

$$x(t) = x_0 e^{\int_0^t a(s)ds} \to 0, \quad \text{as} \quad t \to \infty$$

provided that

$$\int_0^t a(s)\, ds \to -\infty. \tag{1.4}$$

Condition (1.4) implies that the function $a(t)$ can be positive or oscillates for short time. Now assume the existence of a continuous function

$$v : [0, \infty) \to \mathbb{R}.$$

Multiply both sides of (1.2) by

$$e^{\int_0^t v(s)\, ds},$$

and then integrate from 0 to any $t \in [0, T)$. That is

$$\int_0^t e^{\int_0^u v(s)\, ds}\, x'(u)\, du = \int_0^t ax(u) e^{\int_0^u v(s)\, ds}\, du.$$

Perform an integration by parts on the left side and simplify to get

$$x(t) = x_0 e^{-\int_0^t v(s)ds} + \int_0^t x(u)(v(u) + a) e^{-\int_u^t v(s)\, ds}\, du. \tag{1.5}$$

Expression (1.5) is a *new variation of parameters formula for* (1.2) and of Volterra type integral equation. Note that if

$$v(t) = -a,$$

then (1.5) become the regular solution $x(t) = x_0 e^{at}$ of (1.2). In a similar fashion

$$x'(t) = a(t)x(t), \ \ x(0) = x_0,$$

has the solution

$$x(t) = x_0 e^{-\int_0^t v(s)\,ds} + \int_0^t x(u)(v(u)+a(u))e^{-\int_u^t v(s)\,ds}\,du. \tag{1.6}$$

Again, if we let

$$v(t) = -a(t),$$

then we get the regular known solution $x(t) = x_0 e^{\int_0^t a(s)\,ds}$, and $x(t) \to 0,$ as $t \to \infty$ provided that

$$\int_0^t a(s)\,ds \to -\infty.$$

Again (1.6) is a new variation of parameters formula that can be used to deduce qualitative properties about the solutions. In the mean time for (1.6), by setting up the proper spaces and using the contraction mapping principle, we can show that $x(t) \to 0$ as $t \to \infty$, provided that

$$\int_0^t |v(u)+a(u)|e^{-\int_u^t v(s)\,ds}\,du \leq \alpha, \quad 0<\alpha<1,$$

and

$$\int_0^t v(s)\,ds \to \infty.$$

Suppose $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and consider the nonlinear differential equation

$$x'(t) = f(t, x(t)),\ x(0) = x_0 \quad \text{for a given constant} \quad x_0. \tag{1.7}$$

Then, multiplying by a function $\int_0^t v(s)ds$ then the solution of (1.7) is given by

$$x(t) = x_0 e^{-\int_0^t v(s)\,ds} + \int_0^t \big(x(u)v(u) + f(u, x(u))\big)e^{-\int_u^t v(s)\,ds}\,du. \tag{1.8}$$

Similarly, by setting up the proper space and assuming the right conditions on the function f one can obtain results regarding boundedness of solutions and the stability of the zero solution in the case $f(t, 0) = 0$. In [13] the authors studied obtained a new variation of parameters for the finite delay nonlinear differential equation

$$x'(t) = f(t, x(t-\tau))$$

and arrived at stability and periodicity results. For more on the use of the regular variation of parameters we refer to [5, 6, 7].

2 Homogeneous linear systems

Consider the time-varying homogeneous system

$$x'(t) = A(t)x(t), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{2.1}$$

where $A(t)$ is an $n \times n$ matrix of coefficients $a_{ij}(t)$ that are assumed to be continuous on an interval I. Recall that a solution $x(t)$ of (2.1) is an n-tuple of C^1 functions $x_i : I \to \mathbb{R}$. We adopt the notation that

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

The solution x maybe considered as a C^1 vector-valued functions $x : I \to \mathbb{R}^n$. Such space of functions is denoted by $C^1(I, \mathbb{R}^n)$. If $\mathcal{S}$ is the *solution space* of (2.1), then $\mathcal{S} \subset C^1(I, \mathbb{R}^n)$. We state the following definition regarding the fundamental matrix of (2.1).

Definition 2.1. *A set of n solutions of the linear differential system* (2.1) *all defined on the same open interval I, is called a fundamental set of solutions on I if the solutions are linearly independent functions on I.*

Now we state the following familiar theorem. For its proof we may refer to [9, 10, 11, 12].

Theorem 2.2. *If $\Phi(t)$ is a fundamental matrix of* (2.1) *on an interval I, then $\Phi(t)c$, with $c = \Phi^{-1}(t_0)x_0$ is a solution of* (2.1) *with $x(t_0) = x_0$. That is, the unique solution of* (2.1) *is given by*

$$x(t) = \Phi^{-1}(t)\Phi(t_0)x_0. \tag{2.2}$$

The literature is vast concerning the study of systems of differential equations using variation of parameters or Liapunov functionals. For emphasis, using the regular variation of parameters requires the presence of linear term in the form of $A(t)x$. For comprehensive work on such studies we refer to [1, 2, 3, 4, 5]. For results on comprehensive treatment of Liapunov functions/functionals, we refer to [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. It is worth noting that, in [16], the author constructed what we call today, the Resolvent matrix and used it in the form of variation of parameters to analyze solutions of linear Volterra integro-differential equations. Later on, Burton, in [5, 6, 7] generalized the notion of resolvent to nonlinear systems by borrowing linear terms and obtained results concerning boundedness, stability and periodicity. For various results concerning systems of differential equations, we refer to [13, 14, 15, 16]. As we have previously stated, there is a substantial body of scholarship on parameter variation in books, but not in refereed publications.

In [17], the authors considered considered the nonlinear matrix Lyapunov system

$$T^{(n)} = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} A^{n-r} T(t) B^r,$$

where A and B are constant $n \times n$ matrices. They assumed the existence of the fundamental matrix of $T' = AT$ in order to obtain a variation of parameters formula for all solutions. Our work here does not require the existence of a linear term for the inversion. In addition, the authors in [8] consider different kinds of scalar linear and nonlinear first order differential equations and use Liapunov functions and fixed point theory to get results about the boundedness of solutions, the existence of periodic solutions, and the stability of the zero solution. Although they borrow a linear component in order to be able to invert nonlinear equations, this complicates the formula for the resulting variation of parameters and causes it to immediately encounter problems. Our purpose is to obtain a different variation of parameters that solves (2.1) and hopefully its characteristics are different from those of (2.2). We begin with the following lemma.

Lemma 2.3. *Let $\varphi(t)$ be an $n \times n$ differentiable matrix with continuous entries on the interval I. Assume $\varphi^{-1}(t)$ exists for all $t \in I$. Then $x(t)$ is a solution of (2.1) if and only if*

$$x(t) = \varphi^{-1}(t)\varphi(t_0)x_0 + \int_{t_0}^{t} \varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]x(s)\,ds, \quad t \geq t_0. \tag{2.3}$$

Proof. Multiply both sides of (2.1) from the left with the matrix $\varphi(t)$ and then integrate the resulting equation from t to t_0 and obtain

$$\int_{t_0}^{t} \varphi(s)x'(s)\,ds = \int_{t_0}^{t} \varphi(s)A(s)x(s)\,ds.$$

Integrating the left side by parts by letting

$$u = \varphi(s), dv = x'(s)ds,$$

we arrive at

$$\varphi(t)x(t) - \varphi(t_0)x_0 = \int_{t_0}^{t} \big[\varphi'(s) + \varphi(s)A(s)\big]x(s)\,ds.$$

Multiply from the left by $\varphi^{-1}(t)$ gives the desired result. Since every step is reversible, we have completed the proof. $\square$

Remark 2.4. *We note that if*

$$\varphi'(t) = -\varphi(t)A(t), \quad \textit{for all} \quad t \in I,$$

then equation (2.3) *implies that*

$$x(t) = \varphi^{-1}(t)\varphi(t_0)x_0 \tag{2.4}$$

is a solution of (2.1). *To see this, set* $\varphi'(t) = -\varphi(t)A(t)$, *in* (2.3). *Then* (2.3) *reduces to* $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$ *with* $x(t_0) = x_0$. *Differentiating with respect to* t *we arrive at*

$$\begin{aligned} x'(t) &= \big(\varphi^{-1}(t)\big)'\varphi(t_0)x_0 = -\varphi^{-1}(t)\varphi'(t)\varphi^{-1}(t)\varphi(t_0)x_0 = -\varphi^{-1}(t)\big(-\varphi(t)A(t)\big)\varphi^{-1}(t)\varphi(t_0)x_0 \\ &= A(t)\varphi^{-1}(t)\varphi(t_0)x_0 = A(t)x(t). \end{aligned}$$

Note that a quick comparison of (2.2) *with* (2.4) *we see that*

$$\varphi(t) = \Phi^{-1}(t),$$

a result that is parallel to the scalar equations.

Thus one of the main advantages of using (2.3) with $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$, is that it enables us to find the desired matrix and hence a solution for a time-varying system. Usually finding the fundamental matrix solution of time-varying system (2.1) requires additional conditions that are hard to meet. For the rest of this work we consider system (2.1) over the interval $I = [0, \infty)$. We also assume $\|\cdot\|$ is a suitable matrix norm. Next we consider (2.1) such that $f(t, 0) = 0$.

Definition 2.5. *The zero solution* $(x = 0)$ *of* (2.1)*;*

(a) is stable (S) if for each $\epsilon > 0$ *and* $t_0 \geq 0$, *there is a* $\delta = \delta(t_0, \epsilon) > 0$ *such that* $|x(t_0)| < \delta$ *implies* $|x(t, t_0, x_0)| < \varepsilon$,

(b) is uniformly stable (US) if δ *independent of* t_0,

(c) is unstable if it is not stable,

(d) is asymptotically stable (AS) if it is stable and $\lim_{t\to\infty} |x(t, t_0, x_0)| = 0$.

We have the following theorem regarding boundedness of solutions and stability of the zero solution.

Theorem 2.6. *Assume the existence of a positive constant* K *such that*

$$\|\varphi^{-1}(t)\varphi(s)\| \leq K. \tag{2.5}$$

In addition, if there is a positive constant E *such that*

$$\int_0^\infty \|\varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]\|\, ds \leq E \tag{2.6}$$

then all solutions of (2.1) *are bounded and its zero solution is uniformly stable.*

Proof. Let $x(t)$ be given by (2.3) for all $t \geq t_0 \geq 0$. Since the constant K is independent of the initial time $t_0 \geq 0$ we have from (2.3) that

$$\begin{aligned}
|x(t)| &= \|\varphi^{-1}(t)\varphi(t_0)\||x_0| + \int_{t_0}^{t} \|\varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]\| \, |x(s)| \, ds \\
&\leq K|x_0| + \int_{t_0}^{t} \|\varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]\| \, |x(s)| \, ds \\
&\leq K|x_0| e^{\int_{t_0}^{t} \|\varphi^{-1}(t)\left[\varphi'(s)+\varphi(s)A(s)\right]\| \, ds} \quad \text{(by Gronwall's inequality)} \\
&\leq K|x_0|e^{E} \qquad (2.7)
\end{aligned}$$

Hence inequality (2.7) implies all solutions are bounded. For the uniform stability of the zero solution, we let $\delta = \dfrac{\varepsilon}{Ke^{E}}$ so that for any $\varepsilon > 0$ we have from (2.7) for $|x_0| < \delta$, that $|x(t)| < \varepsilon$. This completes the proof. □

For the next theorems we assume that set $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$, so that the solution of (2.1) is given by $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$ as was indicated by Remark 2.4.

Theorem 2.7. *Let $\varphi(t)$ be as defined in Lemma 2.3 such that $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$. Then the zero solution of* (2.1) *is*

(a) stable if and only if there exists a positive constant M such that

$$\|\varphi^{-1}(t)\| \leq M, \quad t \geq 0,$$

(b) asymptotically stable if and only if

$$\|\varphi^{-1}(t)\| \to 0, \quad \text{as} \quad t \to \infty.$$

Proof. (a) ($\Leftarrow$) Let $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$. Then by the Remark 2.4, we have that $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$ is a solution of (2.1). Let $\epsilon > 0$ and set $\delta = \dfrac{\epsilon}{\|\varphi(t_0)\|M}$ such that for $|x_0| < \delta$ we have that

$$|x(t)| = |\varphi^{-1}(t)\varphi(t_0)x_0| \leq \|\varphi^{-1}(t)\|\|\varphi(t_0)\||x_0| \leq M\|\varphi(t_0)\|\delta = \epsilon.$$

($\Rightarrow$) Set $\epsilon = 1$ from the stability proof. Then

$$|x(t)| = |\varphi^{-1}(t)\varphi(t_0)x_0| < 1, \quad \text{for} \quad t \geq t_0 \quad \text{if} \quad |x_0| < \delta(1, t_0),$$

which implies that

$$\|\varphi^{-1}(t)\varphi(t_0)\| < \frac{1}{\delta(1, t_0)}.$$

Therefore,

$$\|\varphi(t)\| = \|\varphi^{-1}(t)\varphi(t_0)\varphi^{-1}(t_0)\| \leq \|\varphi^{-1}(t)\varphi^{-1}(t_0)\|\|\varphi(t_0)\| \leq \frac{1}{\delta(1,t_0)}|\varphi(t_0)| := M.$$

This completes the proof of (a).

Next we prove (b). We already know the zero solution is stable. Now,

$$|x(t)| = \|\varphi^{-1}(t)\varphi(t_0)x_0\| \to 0, \quad \text{as} \quad t \to \infty$$

if and only if

$$\|\varphi^{-1}(t)\| \to 0, \quad \text{as} \quad t \to \infty.$$

This completes the proof. □

Before we provide an example, we will have the following discussion. We integrate (2.1) from t_0 to t and get $x(t) = e^{\int_{t_0}^t A(s)\,ds}$. Now we let

$$\Phi(t) = e^{\int_{t_0}^t A(s)\,ds}. \tag{2.8}$$

For $\Phi(t)$ to be fundamental matrix solution, we must have

$$A(t)\Big(\int_{t_0}^t A(s)\,ds\Big) = \Big(\int_{t_0}^t A(s)\,ds\Big)A(t). \tag{2.9}$$

Let us see why. Let $J = \int_{t_0}^t A(s)\,ds$. Then

$$e^J = I + J + \frac{1}{2!}J^2 + \frac{1}{3!}J^3 + \cdots + \frac{1}{k!}J^k + \cdots .$$

and

$$\begin{aligned}
\frac{d}{dt}e^{\int_{t_0}^t A(s)\,ds} &= \frac{d}{dt}\Big(I + \int_{t_0}^t A(s)\,ds + \frac{1}{2!}\Big(\int_{t_0}^t A(s)\,ds\Big)^2 \\
&+ \cdots + \frac{1}{(k-1)!}\Big(\int_{t_0}^t A(s)\,ds\Big)^{k-1}A(t) + \frac{1}{k!}\Big(\int_{t_0}^t A(s)\,ds\Big)^k + \cdots\Big) \\
&= A(t) + \int_{t_0}^t A(s)\,ds\ A(t) + \frac{1}{2!}\Big(\int_{t_0}^t A(s)\,ds\Big)^2 A(t) \\
&+ \cdots + \frac{1}{(k-1)!}\Big(\int_{t_0}^t A(s)\,ds\Big)^{k-1}A(t) + \frac{1}{k!}\Big(\int_{t_0}^t A(s)\,ds\Big)^k A(t) + \cdots \\
&= \big[I + \int_{t_0}^t A(s)\,ds + \frac{1}{2!}\Big(\int_{t_0}^t A(s)\,ds\Big)^2 + \cdots \\
&+ \frac{1}{(k-1)!}\Big(\int_{t_0}^t A(s)\,ds\Big)^{k-1}A(t) + \frac{1}{k!}\Big(\int_{t_0}^t A(s)\,ds\Big)^k + \cdots\big]A(t)
\end{aligned}$$

$$
\begin{aligned}
&\neq A(t)\big[I + \int_{t_0}^{t} A(s)\,ds + \frac{1}{2!}\Big(\int_{t_0}^{t} A(s)\,ds\Big)^2 \\
&+ \cdots + \frac{1}{(k-1)!}\Big(\int_{t_0}^{t} A(s)\,ds\Big)^{k-1} A(t) + \frac{1}{k!}\Big(\int_{t_0}^{t} A(s)\,ds\Big)^{k} + \cdots\big] \\
&= A(t)\Phi(t).
\end{aligned}
$$

Thus, if (2.9) holds, then $\Phi'(t) = A(t)\Phi(t)$.

We have the following example.

Example 1. *For $t \geq 0$ we consider the linear system*

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x(0) = x_0. \tag{2.10}
$$

If we let

$$
A(t) = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}
$$

then it is clear that (2.9) *does not hold. Let*

$$
\varphi(t) = \begin{pmatrix} 1+t^2 & -t \\ -t & 1 \end{pmatrix}.
$$

Then one may easily verify that that the matrix φ satisfies $\varphi'(t) = -\varphi(t)A(t)$, for all $t \geq 0$, and hence every solution of (2.10) *satisfies $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$. In addition,*

$$
\varphi^{-1}(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}.
$$

Applying Theorem 2.7 we conclude solutions of (2.10) *are unbounded and its zero solution is unstable.*

In Example 1, it would have been difficult to find the fundamental matrix using the argument of eigenvalues and corresponding eigenfunctions since (2.9) does not hold. Moreover, the method of regular linearization does not work for time-varying systems. We are left with the notion of finding a suitable Liapunov function to prove the unboundedness of solutions and consequently the instability of the zero solution. This author could not find one that would do the job. In conclusion, the above discussion cements the usefulness of our method. A final note: the system may be solved using the Laplace transform. This can be done by writing the system as

$$
x' = -tx_1 + x_2, \quad x_2' = (1-t^2)x_1 + tx_2,
$$

subject to the initial conditions $x_1(0) = x_{01}, x_2(0) = x_{02}$. Looking forward, Laplace transforms can not be used in our next examples.

3 Nonlinear systems

We consider the general nonlinear system of ordinary differential equations

$$\begin{aligned} x_1' &= f_1(t, x_1, \dots, x_n) \\ x_2' &= f_2(t, x_1, \dots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \dots, x_n). \end{aligned}$$

Using the vector notations

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$f(t, x) = \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \\ \vdots \\ f_n(t, x) \end{pmatrix}$$

the above system can be written in the vector form

$$x' = f(t, x) \tag{3.1}$$

and assume $f \in C^1([0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$, is continuous in t and x. Let $\varphi(t)$ be an $n \times n$ matrix with continuous entries on $[0, \infty)$. Assume $\varphi^{-1}(t)$ exists for all $t \geq 0$. We multiply both sides of (3.1) with $\varphi(t)$. By similar work as before, we have

$$x(t) = \varphi^{-1}(t)\varphi(t_0)x_0 + \varphi^{-1}(t) \int_{t_0}^{t} \left[\varphi'(s)x(s) + \varphi(s)f(s, x(s))\right] ds, \quad t \geq t_0. \tag{3.2}$$

Now the advantage of our method is that (3.2) can be used on proper spaces to analyze the solutions of (3.1). In this work it is more convenient to use the following norms for a matrix and a vector.

For $x = (x_1, x_2, \ldots, x_n)$ we consider the norm

$$|x| = \sum_{i=1}^{n} |x_i|.$$

Similarly, we define the norm of a matrix B by

$$|B| = \sum_{i,j=1}^{n} |b_{ij}|,$$

for an $n \times n$ matrix $B = [b_{ij}]$. Under these two norms we have

$$|B(t)x| \le |B(t)||x|,$$

and for any two $n \times n$ matrices B and K we have that

$$|B(t)K(t)| \le |B(t)||K(t)|.$$

We have the following theorem regarding boundedness of solutions and stability of the zero solution of system (3.1).

Theorem 3.1. *Suppose there is a positive constant K and a continuous function $\lambda : [0,\infty) \to [0,\infty)$ such that*

$$|\varphi^{-1}(t)\varphi(t_0)| \le K, \tag{3.3}$$

and

$$|f(t,x)| \le \lambda(t)|x|. \tag{3.4}$$

In addition, if there is a positive constant E such that

$$\int_0^\infty \big[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]\,ds \le E \tag{3.5}$$

then all solutions of (3.1) *are bounded and its zero solution is uniformly stable.*

Proof. Let $x(t)$ be given by (3.2) for all $t \ge t_0 \ge 0$. Since the constant K is independent of the initial time $t_0 \ge 0$ we have from (2.3) that

$$\begin{aligned}
|x(t)| &\le |\varphi^{-1}(t)\varphi(t_0)||x_0| + \int_{t_0}^{t} |\varphi^{-1}(t)\big[\varphi'(s)x(s) + \varphi(s)f(s,x(s))\big]|\,ds \\
&\le K|x_0| + \int_0^\infty \big[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]|x(s)|\,ds \\
&\le K|x_0| e^{\int_{t_0}^{t}\left[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\right]ds} \quad \text{(by Gronwall's inequality)} \\
&\le K|x_0|e^{E}.
\end{aligned} \tag{3.6}$$

Hence inequality (3.6) implies all solutions are bounded. For the uniform stability of the zero solution, we let $\delta = \dfrac{\varepsilon}{Ke^E}$ so that for any $\varepsilon > 0$ we have from (3.6) for $|x_0| < \delta$, that $|x(t)| < \varepsilon$. This completes the proof. $\square$

We provide the following example.

Example 2. *For $t \geq 0$ we consider the nonlinear system*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} \dfrac{x_1 \cos(x_2)\sin(t)}{(1+t)(x_1^2+1)} \\ \\ \dfrac{x_2 \sin(x_1)\cos(t)}{(1+t)(x_2^2+1)} \end{pmatrix}, \quad x(0) = x_0. \tag{3.7}$$

Note that

$$|f(t,x)| = \sum_{i=1}^{2} |f_i(t,x)| = \left|\frac{x_1 \sin(t)}{(1+t)(x_1^2+1)}\right| + \left|\frac{x_2 \cos(t)}{(1+t)(x_2^2+1)}\right| \leq \frac{1}{1+t}\,[|x_1| + |x_2|] = \frac{1}{1+t}|x|.$$

Hence

$$\lambda(t) = \frac{1}{1+t}.$$

To verify the rest of the conditions of Theorem 3.3, we let

$$\varphi(t) = \begin{pmatrix} \sqrt{1+t} & 0 \\ 0 & \sqrt{1+t} \end{pmatrix}.$$

Then

$$\varphi^{-1}(t) = \begin{pmatrix} \dfrac{1}{\sqrt{1+t}} & 0 \\ 0 & \dfrac{1}{\sqrt{1+t}} \end{pmatrix}.$$

One can easily compute that

$$\varphi^{-1}(t)\varphi(s) = \begin{pmatrix} \dfrac{\sqrt{1+s}}{\sqrt{1+t}} & 0 \\ 0 & \dfrac{\sqrt{1+s}}{\sqrt{1+t}} \end{pmatrix},$$

and

$$\varphi^{-1}(t)\varphi'(s) = \begin{pmatrix} \dfrac{1}{2\sqrt{1+s}\sqrt{1+t}} & 0 \\ 0 & \dfrac{1}{2\sqrt{1+s}\sqrt{1+t}} \end{pmatrix}.$$

Thus,

$$|\varphi^{-1}(t)\varphi(0)| \leq \frac{2}{\sqrt{1+t}} \leq 2 =: K, \quad \textit{for all} \quad t \geq 0.$$

Moreover,

$$\int_0^t |\varphi^{-1}(t)\varphi'(s)|\, ds \le \frac{1}{\sqrt{1+t}} \int_0^t \frac{1}{\sqrt{1+s}}\, ds = 2 - \frac{2}{\sqrt{1+t}} \le 2, \quad \textit{for all} \quad t \ge 0.$$

Similarly,

$$\int_0^t |\varphi^{-1}(t)\varphi(s)|\lambda(s)\, ds \le \frac{2}{\sqrt{1+t}} \int_0^t \frac{1}{\sqrt{1+s}}\, ds = 4 - \frac{4}{\sqrt{1+t}} \le 4, \quad \textit{for all} \quad t \ge 0.$$

Finally,

$$\int_0^\infty \left[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\right]\, ds \le 6 =: E.$$

Thus, all conditions of Theorem 3.3 are satisfied which implies that all solutions of (3.7) *are bounded and its zero solutions is uniformly stable.*

Next, we use the contraction principle to show the solution is unique. Let $\mathcal{C}$ be the set of all real-valued continuous functions. Define the space

$$\mathcal{S} = \{\Phi : [0,\infty) \to \mathbb{R}^n \mid \Phi \in \mathcal{C}, |\Phi(t)| \le M\},$$

for positive constant M. Then

$$(\mathcal{S}, |\cdot|)$$

is complete.

Theorem 3.2. *We assume the function f is locally Lipschitz on the set $\mathcal{S}$. That is, for any Φ_1 and $\Phi_2 \in \mathcal{S}$, we have*

$$|f(t,\Phi_1) - f(t,\Phi_2)| \le \Lambda(t)|\Phi_1 - \Phi_2|, \tag{3.8}$$

for continuous $\Lambda : [0,\infty) \to (0,\infty)$. Suppose there is a positive constant $\alpha \in (0,1)$ such that

$$\int_0^\infty \left[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\Lambda(s)\right]\, ds \le \alpha, \tag{3.9}$$

then (3.1) *has a unique solution. In addition if* (3.3) *holds then the unique solution is bounded and the zero solution of* (3.1) *is uniformly stable.*

Proof. For $\Phi \in \mathcal{S}$, define the mapping $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$, by

$$(\mathfrak{P}\Phi)(t) = \varphi^{-1}(t)\varphi(t_0)x_0 + \varphi^{-1}(t)\int_{t_0}^t [\varphi'(s)\Phi(s) + \varphi(s)f(s,\Phi(s))]\, ds, \quad t \ge t_0. \tag{3.10}$$

It is clear that $(\mathfrak{P}\Phi)(0) = x_0$ and $\mathfrak{P}$ is continuous in Φ. Let Φ_1 and $\Phi_2 \in \mathcal{S}$. Then

$$|(\mathfrak{P}\Phi_1)(t) - (\mathfrak{P}\Phi_2)(t)| \le \int_0^\infty [|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\Lambda(s)]ds\, |\Phi_1 - \Phi_2| \le \alpha|\Phi_1 - \Phi_2|.$$

This shows that $\mathfrak{P}$ is a contraction. By Banach's contraction mapping principle, $\mathfrak{P}$ has a unique fixed point $x \in \mathcal{S}$ which is a continuous function. The boundedness of the solution and the uniform stability of the zero solution follow from Theorem 3.2. This completes the proof. $\square$

We will need the following clarifications for the next example. Let $f : D \to \mathbb{R}^n$ where D is a subset of $[0, \infty) \times \mathbb{R}^n$. To check if a function $f : D \to \mathbb{R}^n$ is Lipschitz continuous on some subset D of $[0, \infty) \times \mathbb{R}^n$, it suffices to check that the component functions $f_i : D \to \mathbb{R}^n$ are Lipschitz continuous. This is due to the fact that

$$|f_i(t, z) - f_i(t, w)| \leq L_i|z - w| \text{ for } i = 1, \ldots, n$$

implies under our norm that

$$|f(t, z) - f(t, w)| = \sum_{i=1}^{n} |f_i(t, z) - f_i(t, w)| \leq \sum_{i=1}^{n} L_i|z - w|,$$

which shows that

$$|f(t, z) - f(t, w)| \leq L|z - w| \text{ with } L = \sum_{i=1}^{n} L_i.$$

Example 3. *For $t \geq 0$ we consider the nonlinear system*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} \dfrac{\cos(t)}{20(1+t)} \left[x_2 + \dfrac{x_1}{x_1^2 + 1}\right] \\ \\ \dfrac{\sin(t)}{20(1+t)} \left[x_1 + \dfrac{x_2}{x_2^2 + 1}\right] \end{pmatrix}, \quad x(0) = x_0. \tag{3.11}$$

Note that by a similar argument as in Example 2 we arrive at

$$|f(t, x)| \leq \frac{2}{20(1+t)} \Big[|x_1| + |x_2|\Big] = \frac{1}{10(1+t)}|x|.$$

Hence

$$\lambda(t) = \frac{1}{10(1+t)}.$$

Next we show f is Lipschitz continuous. Let $z = (z_1, z_2), w = (w_1, w_2) \in \mathcal{S}$ with $n = 2$. Then

$$|f_1(t, z) - f_1(t, w)| \leq \frac{1}{20(1+t)} \left[|z_2 - w_2| + \left|\frac{z_1}{z_1^2 + 1} - \frac{w_1}{w_1^2 + 1}\right|\right].$$

$$\left|\frac{z_1}{z_1^2 + 1} - \frac{w_1}{w_1^2 + 1}\right| = \frac{z_1 - w_1 + z_1 w_1(w_1 - z_1)}{1 + z_1^2 w_1^2 + z_1^2 + w_1^2} \leq \frac{1 + |z_1 w_1|}{1 + z_1^2 w_1^2 + z_1^2 + w_1^2}|z_1 - w_1|.$$

We note that

$$1+|z_1w_1| \le (1+|z_1w_1|)^2 = 1+|z_1w_1|^2+2|z_1w_1| \le 1+|z_1w_1|^2+z_1^2+w_1^2.$$

Hence

$$\frac{1+|z_1w_1|}{1+z_1^2w_1^2+z_1^2+w_1^2} \le 1$$

and it follows that

$$\left|\frac{z_1}{z_1^2+1}-\frac{w_1}{w_1^2+1}\right| \le |z_1-w_1|.$$

This implies that

$$|f_1(t,z)-f_1(t,w)| \le \frac{1}{20(1+t)}[|z_2-w_2|+|z_1-w_1|].$$

In a symmetrical argument one can easily shows that

$$|f_2(t,z)-f_2(t,w)| \le \frac{1}{20(1+t)}\big[|z_1-w_1|+|z_2-w_2|\big].$$

Thus from the above discussion we arrive at

$$|f(t,z)-f(t,w)| = \sum_{i=1}^{2}|f_i(t,z)-f_i(t,w)| \le \sum_{i=1}^{2} L_i|z-w| = \frac{2}{20(1+t)}|z-w|.$$

Thus, $\Lambda(t)=\dfrac{1}{10(1+t)}$. *To verify the rest of the conditions of Theorem 3.2 we let*

$$\varphi(t)=\begin{pmatrix} e^{\frac{1}{10(1+t)}} & 0 \\ 0 & e^{\frac{1}{10(1+t)}} \end{pmatrix}.$$

Then

$$\varphi^{-1}(t)=e^{-\frac{1}{5(1+t)}}\begin{pmatrix} e^{\frac{1}{10(1+t)}} & 0 \\ 0 & e^{\frac{1}{10(1+t)}} \end{pmatrix}$$

and

$$\varphi'(t)=-\frac{1}{10(1+t)^2}\begin{pmatrix} e^{\frac{1}{10(1+t)}} & 0 \\ 0 & e^{\frac{1}{10(1+t)}} \end{pmatrix}.$$

One can easily compute that

$$\varphi^{-1}(t)\varphi(s)=e^{-\frac{1}{5(1+t)}}\begin{pmatrix} e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} & 0 \\ 0 & e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} \end{pmatrix},$$

and

$$\varphi^{-1}(t)\varphi'(s) = -\frac{e^{-\frac{1}{5(1+t)}}}{10(1+s)^2}\begin{pmatrix} e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} & 0 \\ 0 & e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} \end{pmatrix}.$$

Thus,

$$|\varphi^{-1}(t)\varphi(0)| \leq 2e^{\frac{t}{10(1+t)}} \leq 2e^{\frac{1}{10}} =: K, \quad \textit{for all} \quad t \geq 0.$$

Moreover,

$$\int_0^t |\varphi^{-1}(t)\varphi'(s)|\, ds \leq \frac{1}{5}e^{-\frac{1}{10}\frac{1}{1+t}} \int_0^t \frac{1}{(1+s)^2} e^{\frac{1}{10}\frac{1}{1+s}}\, ds = 2e^{\frac{1}{10}-\frac{1}{10}\frac{1}{1+t}} - 2 \leq 2e^{\frac{1}{10}} - 2,$$

for all $t \geq 0$. *Similarly,*

$$\begin{aligned}\int_0^t |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]\, ds &\leq \frac{2}{5}e^{-\frac{1}{10(1+t)}} \int_0^t \frac{1}{(1+s)^2} e^{\frac{1}{10}\frac{1}{1+s}}\, ds \\ &= 4e^{\frac{1}{10}-\frac{1}{10}\frac{1}{1+t}} - 4 \leq 4e^{\frac{1}{10}} - 4, \quad \textit{for all} \quad t \geq 0.\end{aligned}$$

Finally,

$$\int_0^\infty \big[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]\, ds \leq 6e^{\frac{1}{10}} - 6 \leq 0.31 =: \alpha.$$

Moreover, one can easily check that

$$|\varphi^{-1}(t)\varphi(0)| \leq e^{-\frac{1}{10}}.$$

Thus, all conditions of Theorems 3.2 and 3.1 are satisfied which implies that the unique solution of (3.7) *is bounded and its zero solution is uniformly stable.*

In Examples 2 and 3, the considered equations were totally nonlinear and therefore, the arguments of fundamental matrix solution, linearization or the use of Laplace transform would be impossible. Then, we are left with the construction of Liapunov function which is almost impossible. This shows the significance of our novel method.

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