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Inertial algorithm for solving split inclusion problem in Banach spaces

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ABSTRACT

The purpose of this paper is to propose an algorithm for finding a common element of the set of fixed points of relatively nonexpansive mapping and the set of solutions of split inclusion problem with a way of selecting the stepsize without prior knowledge of the operator norm in the framework of Banach spaces. Then, the main result is used to the common fixed point problems of a family of relatively nonexpansive mappings and split equilibrium problem. Finally, a numerical example is provided to illustrate the main result.

RESUMEN

El propósito de este artículo es proponer un algoritmo para encontrar un elemento común del conjunto de puntos fijos de aplicaciones relativamente no-expansivas y el conjunto de soluciones de problemas de inclusión escindidos con una manera de seleccionar el tamaño del paso sin conocimiento previo de la norma del operador en el contexto de espacios de Banach. Luego, el resultado principal se usa para los problemas de punto fijo común de una familia de aplicaciones relativamente no expansivas y el problemas del equilibrio escindido. Finalmente, se entrega un ejemplo numérico para ilustrar el resultado principal.

Keywords and Phrases: Strong convergence, split feasibility problem, uniformly convex, uniformly smooth, fixed point problem.

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1 Introduction

Let H_1 and H_2 be two Hilbert spaces. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. Consider the following split inclusion problem (SIP) introduced by Moudafi [25] in Hilbert space:

$$\text{To find} \quad x^* \in H_1 \quad \text{such that} \quad 0 \in B_1(x^*) \quad \text{and} \quad 0 \in B_2\,(Ax^*). \tag{1.1}$$

Let the solution set of (1.1) be denoted by Ω. In fact, we know that the SIP is a generalization of the inclusion problem and the split feasibility problem (SFP). Next, we have some special cases of SIP (1.1). Let $f : H_1 \to \mathbb{R} \cup \{\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{\infty\}$ be proper, lower semicontinuous and convex functions. If we take $B_1 = \partial f$ and $B_2 = \partial g$, where ∂f and ∂g are the sub-differential of f and g, then the SIP (1.1) becomes the following proximal split feasibility problem:

$$\text{To find} \quad x^* \in \arg\min f \quad \text{such that} \quad Ax^* \in \arg\min g, \tag{1.2}$$

where $\arg\min f = \{x \in H1 : f(x) \leq f(y), \forall y \in H_1\}$ and $\arg\min g = \{x \in H_2 : g(x) \leq g(y), \forall y \in H_2\}$. In particular, if we take $f(x) = \frac{1}{2}\|M(x) - b\|^2$ and $g(x) = \frac{1}{2}\|N(x) - c\|^2$, where M and N are matrices, and $b, c \in H_1$, then the (1.2) becomes the least square problem. This problem has been intensively studied, especially, in Hilbert spaces; see for instance [26].

Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. If $B_1 = N_C, B_2 = N_Q$, where N_C and N_Q are the normal cones of C and Q, respectively, then we have the SFP:

$$\text{To find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{1.3}$$

This problem was first introduced, in a finite dimensional Hilbert space, by Censor and Elfving [13] for modeling inverse problems in radiation therapy treatment, which arise from phase retrieval and in medical image reconstruction, especially intensity modulated therapy [12]. To solve the SIP (1.1) Byrne *et al.* [11] proved some weak convergence results in infinite dimensional Hilbert spaces and proposed the following algorithm for given $x_1 \in H_1$:

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad \forall n \geq 1, \tag{1.4}$$

where $\lambda > 0$, $\gamma \in \left(0, \dfrac{2}{\|A\|^2}\right)$ and $J_\lambda^{B_1}$, $J_\lambda^{B_2}$ are metric and resolvent operators of B_1 and B_2, respectively. In order to obtain strong convergence, Kazmi and Rizvi [19] proposed the following algorithm to solve SIP (1.1):

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n) \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, \forall\, n \geq 1, \end{cases}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. However, in order to achieve the solution, one has to obtain the operator norm $\|A\|$, which is not easy to calculate in general. To avoid this computation, López *et al.* [23] find a new way to select the stepsize as follows:

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1,$$

where P_Q is the metric projection of H_2 onto Q, $\rho_n \in (0,4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$. This method is a modification of the CQ method and is often called the self-adaptive method, which permits step-size being selected self adaptively, for more details see [30, 37].

To solve SIP (1.1) in p-uniformly convex and smooth Banach space, Bello Cruz *et al.* [9] proposed the following algorithm, for given $x_1 \in E_1$ and $\{\alpha_n\} \in (0,1)$:

$$\begin{cases} u_n = J_{E_1{}^*}^q \left[J_{E_1}^p(x_n) - t_n A^* J_{E_2}(I - J_\lambda^{B_2})Ax_n\right] \\ x_{n+1} = J_{E_1{}^*}^q \left[\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(J_\lambda^{B_1}(u_n))\right]. \end{cases} \tag{1.5}$$

Very recently, Cholamjiak *et al.* [14] proposed algorithm for finding common solution of fixed point problem of relatively nonexpansive mapping to solve SIP (1.1) in p-uniformly convex and smooth Banach space. An initial guess $u_1 \in E_1$, let $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by:

$$\begin{cases} x_n = J_{\lambda_1}^{B_1}\left(J_{E_1{}^*}^q(J_{E_1}^p(u_n) - \lambda_n A^* J_{E_2}^p(I - J_{\lambda_2}^{B_2})Au_n)\right) \\ u_{n+1} = J_{E_1{}^*}^q(\alpha_n J_{E_1}^p(\epsilon_n) + \beta_n J_{E_1}^p(x_n) + \gamma_n J_{E_1}^p(Tx_n)), \qquad n \geq 1, \end{cases} \tag{1.6}$$

where $J_{\lambda_1}^{B_1}$, $J_{\lambda_2}^{B_2}$ are metric and resolvent operators. The sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For more SIP related articles (see, [3, 6, 16, 28, 34, 36, 38, 42]).

In nonlinear analysis, to work with an algorithm that has a high rate of convergence is more useful, through adding inertial term in the algorithm. First it was proposed by Polyak [31]. The main purpose of this method is to make use of previous iterates to update the next iterate. Recently, many authors have shown interest to study inertial type algorithms, see [2, 4, 17, 37, 40, 41].

Intention of this paper is to propose an algorithm to solve SIP (1.1) and fixed point of relatively nonexpansive mapping in p-uniformly convex and uniformly smooth real Banach spaces, without prior knowledge of operator norm, so that it can be more efficiently implemented. As an application, we apply our result to the common fixed point problems of a family of relatively nonexpansive mappings and split equilibrium problem. A numerical example is given to illustrate the efficiency of our algorithm, also our results complement and extend many recent and important results in this direction.

2 Preliminaries

Let E be a real Banach space with dual E^* and let $A : E_1 \to E_2$ be a bounded linear operator and A^* is adjoint of A. The modulus of convexity $\delta_E : [0,2] \to [0,1]$ is defined as

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|, \|x-y\| \geq \varepsilon\right\}.$$

E is called uniformly convex if $\delta_E(\varepsilon) > 0$, for $\varepsilon \in (0,2]$ and p-uniformly convex if there exist a $C_p > 0$ such that $\delta_E(\varepsilon) \geq C_P \varepsilon^p$, for any $\varepsilon \in (0,2]$. The modulus of smoothness $\rho_E : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \sup\{\frac{\|x+\tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

E is called uniformly smooth if $\lim\limits_{\tau\to 0} \dfrac{\rho_E(\tau)}{\tau} = 0$, $q-$uniformly smooth if there exist $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$, for any $\tau > 0$. The duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\| = \|x\|^{p-1}\}.$$

The duality mapping J_E^p is one-to-one and single-valued (see [5, 15]).

The metric projection for a nonempty, closed and convex subset C of Banach space E is given by

$$P_C x = \arg\min_{y\in C} \|x - y\|, \quad x \in E.$$

For a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the Bregman distance with respect to f is defined as

$$\Delta_{f(x,y)} = f(y) - f(x) - \langle f'(x), y - x\rangle, \quad x, y \in E.$$

Since the duality mapping J_E^p is the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$. Then the Bregman distance with respect to f_p is,

$$\Delta_p(x,y) = \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p = \frac{1}{q}(\|y\|^p - \|x\|^p) - \langle J_E^p x - J_E^p y, x\rangle. \tag{2.1}$$

We define the Bregman projection as the unique minimizer of the Bregman distance,

$$\Pi_C x = \arg\min_{y\in C} \Delta_p(x,y), \quad x \in E.$$

It can also be characterized by a variational inequality,

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \le 0, \quad \forall z \in C, \tag{2.2}$$

also,

$$\Delta_p(\Pi_C x, z) \le \Delta_p(x,z) - \Delta_p(x, \Pi_C x), \quad \forall z \in C. \tag{2.3}$$

In real Hilbert space $\Pi_C = P_C$, for more detail, see [1, 18]. The function $V_p : E^* \times E \to [0, +\infty)$ with f_p is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p, \quad \forall x \in E,\, \bar{x} \in E^*.$$

Then $V_p \ge 0$ and also satisfy following property:

$$V_p(\bar{x}, x) = \Delta_p(J_E^q(\bar{x}), x), \quad \forall x \in E,\, \bar{x} \in E^*. \tag{2.4}$$

Moreover,

$$V_p(\bar{x}, x) + \langle \bar{y}, J_E^q(\bar{x}) - x\rangle \le V_p(\bar{x} + \bar{y}, x), \tag{2.5}$$

$\forall x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see [29]). Also, V_p is convex in the first variable. Thus, for all $z \in E$,

$$\Delta_p\left(J_E^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right), z\right) \le \sum_{i=1}^N t_i \Delta_p(x_i, z), \tag{2.6}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum\limits_{i=1}^N t_i = 1$, see [33].

Lemma 2.1 ([27])**.** *Let E be a p-uniformly convex and uniformly smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be bounded sequences in E, then $\lim\limits_{n\to\infty} \Delta_p(x_n, y_n) = 0$ if and only if $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.2 ([43])**.** *Let $x, y \in E$. If E is q-uniformly smooth, then there is a $C_q > 0$ so that*

$$\|x - y\|^q \le \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

A point $x^* \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to x^* and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $\hat{F}(T)$ is the set of asymptotic fixed points. Similarly a point $x^* \in C$ is a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to x^* and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. Set of strong asymptotic fixed points of

T is denoted by $\tilde{F}(T)$.

Definition 2.3 ([24])**.** *A mapping T from C to C is said to be,*

1. *Bregman relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and*

$$\Delta_p(x^*, Ty) \leq \Delta_p(x^*, y), \quad \forall y \in C,\, x^* \in F(T).$$

2. *Bregman weakly relatively nonexpansive if $\tilde{F}(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$ and*

$$\Delta_p(x^*, Ty) \leq \Delta_p(x^*, y), \quad \forall y \in C,\, x^* \in F(T).$$

For more details, see [32].

Definition 2.4 ([8])**.** *Let E be a p-uniformly convex and uniformly smooth Banach space and C a nonempty subset of E. A mapping $S : C \to E$ is said to be firmly nonexpansive-like if*

$$\langle J_E^p(x - Sx) - J_E^p(y - Sy), Sx - Sy\rangle \geq 0, \quad \forall x, y \in C. \tag{2.7}$$

If E is a Hilbert space, then S is firmly nonexpansive-like mapping if and only if it is firmly nonexpansive, *i.e.* $\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle$, $\forall x, y \in C$. We recall the following results:

Remark 2.5. *Let E be a p-uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Then the metric projection P_C is a firmly nonexpansive-like mapping.*

Lemma 2.6 ([8])**.** *Let E be a smooth Banach space, C be a closed and convex nonempty subset of E and $S : C \to E$ a firmly nonexpansive-like mapping then $F(S)$ is closed and convex and $\hat{F}(S) = F(S)$.*

Let $B : E \to 2^{E^*}$ be a mapping, the effective domain of B is denoted by $D(B)$, such that $D(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be monotone if

$$\langle u - v, x - y\rangle \geq 0, \quad \forall x, y \in D(B), \quad u \in Bx \quad \text{and} \quad v \in By.$$

A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E.

For $\lambda_2 > 0$ and $x \in E_2$, consider the metric resolvent $M_{\lambda_2}^{B_2} : E_2 \to D(B_2)$ of B_2 defined by

$$M_{\lambda_2}^{B_2}(x) = (I + \lambda_2 (J_{E_2}^p)^{-1} B_2)^{-1}(x), \quad \forall x \in E_2.$$

Set of null points of B_2 is defined by ${B_2}^{-1}(0) = \{z \in E_2 : 0 \in Bz\}$. Since ${B_2}^{-1}(0)$ is closed and convex, then we have

$$0 \in J^P_{E_2}(M^{B_2}_{\lambda_2}(x) - x) + \lambda_2 B_2 M^{B_2}_{\lambda_2}(x).$$

Next, $F(M^{B_2}_{\lambda_2}) = B_2^{-1}(0)$, for $\lambda_2 > 0$, from [22] we also have,

$$\langle M^{B_2}_{\lambda_2}(x) - M^{B_2}_{\lambda_2}(y), J^p_{E_2}(x - M^{B_2}_{\lambda_2}(x)) - J^p_{E_2}(y - M^{B_2}_{\lambda_2}(y))\rangle \geq 0,$$

for all $x, y \in E_2$ and if $B_2^{-1}(0) \neq 0$, then

$$\langle J^p_{E_2}(x - M^{B_2}_{\lambda_2}(x)) - (M^{B_2}_{\lambda_2}(x) - z)\rangle \geq 0,$$

for all $x \in E_2$ and $z \in {B_2}^{-1}(0)$.

The monotonicity of B_2 implies that $M^{B_2}_{\lambda_2}$ is a firmly nonexpansive-like mapping.

Now, we can define a mapping $N^{B_1}_{\lambda_1} : E_1 \to D(B_1)$ called the relative resolvent of B_1 [20], for $\lambda_1 > 0$ as

$$N^{B_1}_{\lambda_1} = (J^p_{E_1} + \lambda_1 B_1)^{-1} J^p_{E_1}(x), \quad \forall x \in E_1.$$

Since $N^{B_1}_{\lambda_1}$ is relatively nonexpansive mapping and $F(N^{B_1}_{\lambda_1}) = {B_1}^{-1}(0)$ for $\lambda_1 > 0$.

Lemma 2.7 ([20])**.** *Let $B : E \to 2^{E^*}$ be a maximal monotone operator with $B^{-1} \neq \emptyset$ and let N^B_λ be a resolvent operator of B for $\lambda > 0$. Then*

$$\Delta_p(N^B_\lambda(x), z) + \Delta_p(N^B_\lambda(x), x) \leq \Delta_p(x, z) \quad \textit{for all} \quad x \in E \quad \textit{and} \quad z \in B^{-1}(0).$$

Lemma 2.8 ([35])**.** *Let E_1, E_2 be two p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. Let $N^{B_1}_{\lambda_1}$ be the resolvent operator associated with maximal monotone operator B_1 for $\lambda_1 > 0$ and $M^{B_2}_{\lambda_2}$ be a metric resolvent operator of maximal monotone operator B_2 for $\lambda_2 > 0$. Assume $\Omega \neq \emptyset$, $\lambda > 0$ and $x^* \in E_1$. Then x^* is a solution of problem* (1.1) *if and only if*

$$x^* = N^{B_1}_{\lambda_1}(J^q_{E_1^*}(J^p_{E_1}(x^*) - \lambda A^* J^p_{E_2}(I - M^{B_2}_{\lambda_2})Ax^*)).$$

3 Main results

We assume the following assumptions for the rest of the paper, let E_1, E_2 be two p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. Let $C = C_1$ be nonempty closed and convex subset of E_1. Let $B_1 : E_1 \to 2^{{E_1}^*}$ and $B_2 : E_2 \to 2^{{E_2}^*}$ be maximal monotone operators such that $B_1^{-1}(0) \neq 0$, $B_2^{-1}(0) \neq 0$. Let $N^{B_1}_{\lambda_1}$ be the resolvent operator of B_1 for $\lambda_1 > 0$ and $M^{B_2}_{\lambda_2}$ is the metric resolvent operator of B_2 for $\lambda_2 > 0$. Let $T : E_1 \to E_1$ be a Bregman relatively nonexpansive mapping. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$ and $\{\alpha_n\} \in (0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\theta_n \in (-\infty, +\infty)$ and assuming $\Omega \cap F(T) \neq \emptyset$.

Algorithm 3.1. *Select $x_0, x_1 \in E_1$ and assuming that the sequence x_n is generated via the formula*

$$\begin{cases} v_n = J^q_{E_1^*}[J^p_{E_1} x_n + \theta_n (J^p_{E_1} x_n - J^p_{E_1} x_{n-1})] \\ z_n = N^{B_1}_{\lambda_1}[J^q_{E_1^*}(J^p_{E_1}(v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n))] \\ y_n = J^q_{E_1^*}[\alpha_n J^p_{E_1}(z_n) + (1-\alpha_n) J^p_{E_1} T(z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geq 1, \end{cases} \tag{3.1}$$

where $f(v_n) = \frac{1}{p}\|(I - M^{B_2}_{\lambda_2})Av_n\|^p$, $g(v_n) = A^* J^p_{E_2}(I - M^{B_2}_{\lambda_2})Av_n$ and $\{\rho_n\} \in (0,\infty)$ satisfy $\liminf_{n\to\infty} \rho_n(pq - C_q \rho_n^{q-1}) > 0$. Suppose that the set $\Psi = \{n \in \mathbb{N} : (I - M^{B_2}_{\lambda_2})Av_n \neq 0\}$, otherwise $z_n = v_n$.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* = \Pi_{\Omega\cap F(T)} x_0$.*

Proof. We divide the proof into four steps:

Step 1: To show $\Omega \cap F(T) \subseteq C_n$, for all $n \geq 1$ and Algorithm 3.1 is well defined. Let C_k is closed and convex for $k \geq 1$. Then

$$\begin{aligned} C_{k+1} &= \{u \in C_k : \Delta_p(y_n, u) \leq \Delta_p v_n, u)\} \\ &= \left\{ u \in C_k : \frac{\|u\|^p}{p} + \frac{\|y_k\|}{q} - \langle J^p_{E_1} y_k, u\rangle \leq \frac{\|u\|^p}{p} + \frac{\|v_k\|}{q} - \langle J^p_{E_1} v_k, u\rangle \right\} \\ &= \left\{ u \in C_k : \|y_k\|^p - \|v_k\|^p \leq q\langle J^p_{E_1} y_k - J^p_{E_1} v_k, u\rangle \right\}, \end{aligned}$$

which implies C_{k+1} is closed. Let $u_1, u_2 \in C_{k+1}$ and $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 + \lambda_2 = 1$. Then

$$\|y_k\|^p - \|v_k\|^p \leq q\langle J^p_{E_1} y_k - J^p_{E_1} v_k, u_1\rangle \quad \text{and} \quad \|y_k\|^p - \|v_k\|^p \leq q\langle J^p_{E_1} y_k - J^p_{E_1} v_k, u_2\rangle.$$

Combining these two, we get

$$\|y_k\|^p - \|v_k\|^p \leq \langle J^p_{E_1} y_k - J^p_{E_1} v_k, \lambda_1 u_1 + \lambda_2 u_2\rangle.$$

By convexity $\lambda_1 u_1 + \lambda_2 u_2 \in C_k$. Therefore, $\lambda_1 u_1 + \lambda_2 u_2 \in C_{k+1}$ and C_{k+1} is convex. Thus C_n is convex, $\forall n \geq 1$. Let $x^* \in \Omega \cap F(T)$, then

$$
\begin{aligned}
\Delta_p(y_n,x^*) &= \Delta_p((1-\alpha_n)J^p_{E_1}z_n+\alpha_n J^p_{E_1}T(z_n),x^*)\\
&\le (1-\alpha_n)\Delta_p(z_n,x^*)+\alpha_n\Delta_p(T(z_n),x^*)\le \Delta_p(z_n,x^*). \qquad (3.2)
\end{aligned}
$$

Set $w_n := J^q_{E_1^*}(J^p_{E_1}(v_n)-\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}g(v_n))$, for all $n\ge 1$. From Lemma 2.2 and (2.1), we have

$$
\begin{aligned}
\Delta_p(z_n,x^*) &\le \Delta_p(w_n,x^*)\\
&= \Delta_p\left(J^q_{E_1^*}\left[J^p_{E_1}(v_n)-\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}g(v_n)\right],x^*\right)\\
&= \frac{1}{p}\|x^*\|^p+\frac{1}{q}\left\|J^p_{E_1}(v_n)-\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}g(v_n)\right\|^q-\langle J^p_{E_1}(v_n),x^*\rangle\\
&+\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^*,g(v_n)\rangle\\
&\le \frac{1}{p}\|x^*\|^p+\frac{1}{q}\|v_n\|^p-\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle v_n,g(v_n)\rangle+\frac{C_q}{q}\rho_n^q\frac{f^p(v_n)}{\|g(v_n)\|^p}\\
&-\langle x^*,J^p_{E_1}v_n\rangle+\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^*,g(v_n)\rangle\\
&= \frac{1}{p}\|x^*\|^p+\frac{1}{q}\|v_n\|^p-\langle x^*,J^p_{E_1}v_n\rangle+\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^*-v_n,g(v_n)\rangle\\
&+\frac{C_q}{q}\rho_n^q\frac{f^p(v_n)}{\|g(v_n)\|^p}\\
&= \Delta_p(v_n,x^*)+\rho_n\frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^*-v_n,g(v_n)\rangle+\frac{C_q}{q}\rho_n^q\frac{f^p(v_n)}{\|g(v_n)\|^p}. \qquad (3.3)
\end{aligned}
$$

Since $g(v_n)=A^*J^p_{E_2}(I-M^{B_2}_{\lambda_2})Av_n$ and $\langle J^p_{E_2}(I-M^{B_2}_{\lambda_2})Av_n, M^{B_2}_{\lambda_2}Av_n-Ax^*\rangle\ge 0$, then

$$
\begin{aligned}
\langle g(v_n),x^*-v_n\rangle &= \langle A^*J^p_{E_2}(I-M^{B_2}_{\lambda_2})Av_n,x^*-v_n\rangle=\langle J^p_{E_2}(I-M^{B_2}_{\lambda_2})Av_n,Ax^*-Av_n\rangle\\
&= \langle J^p_{E_2}(I-M^{B_2}_{\lambda_2})Av_n,M^{B_2}_{\lambda_2}Av_n-Av_n\rangle\\
&+\langle J^p_{E_2}(I-M^{B_2}_{\lambda_2})Av_n,Ax^*-M^{B_2}_{\lambda_2}Av_n\rangle\\
&\le -\|Av_n-M^{B_2}_{\lambda_2}Av_n\|^p=-pf(v_n). \qquad (3.4)
\end{aligned}
$$

Using (3.3) and (3.4),

$$
\begin{aligned}
\Delta_p(z_n,x^*) &\le \Delta_p(v_n,x^*)-\rho_n p\frac{f^p(v_n)}{\|g(v_n)\|^p}+\frac{C_q}{q}\rho_n^q\frac{f^p(v_n)}{\|g(v_n)\|^p}\\
&= \Delta_p(v_n,x^*)-\left(\rho_n p-\frac{C_q}{q}\rho_n^q\right)\frac{f^p(v_n)}{\|g(v_n)\|^p}. \qquad (3.5)
\end{aligned}
$$

Since $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$,

$$\Delta_p(z_n, x^*) \leq \Delta_p(v_n, x^*), \quad n \geq 1. \tag{3.6}$$

Step 2: We prove that $\{x_n\}$ is a Cauchy sequence.

Since, $\{\Delta_p(x_n, x_0)\}$ is nondecreasing and bounded. So, the limit $\lim_{n\to\infty} \Delta_p(x_n, x_0)$ exists and from (2.3) we have,

$$\begin{aligned}\Delta_p(x_{n+1}, x_n) = \Delta_p(x_{n+1}, \Pi_{C_n}x_0) &\leq \Delta_p(x_{n+1}, x_0) - \Delta_p(\Pi_{C_n}x_0, x_0)\\ &= \Delta_p(x_{n+1}, x_0) - \Delta_p(x_n, x_0),\end{aligned} \tag{3.7}$$

which implies that,

$$\lim_{n\to\infty} \Delta_p(x_{n+1}, x_n) = 0. \tag{3.8}$$

So, it follows from Lemma 2.1 that,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since $x_n = \Pi_{C_n}x_0 \subseteq C_m$ and from Lemma 2.1, for some positive integers m, n with $m \leq n$, we have

$$\begin{aligned}\Delta_p(x_m, x_n) = \Delta_p(x_m, \Pi_{C_n}x_0) &\leq \Delta_p(x_m, x_0) - \Delta_p(\Pi_{C_n}x_0, x_0)\\ &\leq \Delta_p(x_m, x_0) - \Delta_p(x_n, x_0).\end{aligned} \tag{3.10}$$

Since $\lim_{n\to\infty} \Delta_p(x_n, x_0)$ exists, it follows from (3.10) that $\lim_{n\to\infty} \|x_n - x_m\| = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence.

Step 3: We prove that $\lim_{n\to\infty} \|Tz_n - z_n\| = 0$, $\lim_{n\to\infty} \|(I - M_{\lambda_2}^{B_2})Ax_n\| = 0$ and $\lim_{n\to\infty} \|N_{\lambda_1}^{B_1}v_n - v_n\| = 0$.
Since $v_n = J_{E_1^*}^q[J_{E_1}^p x_n + \theta_n(J_{E_1}^p x_n - J_{E_1}^p x_{n-1})]$. Then it follows that,

$$J_{E_1}^p v_n - J_{E_1}^p x_n = \theta_n(J_{E_1}^p x_n - J_{E_1}^p x_{n-1}).$$

By the uniform continuity of $J_{E_1}^p$ and from (3.9), we have

$$\|J_{E_1}^p v_n - J_{E_1}^p x_n\| = \|\theta_n(J_{E_1}^p x_n - J_{E_1}^p x_{n-1})\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, from the definition of C_{n+1}, we have

$$\Delta_p(x_{n+1}, z_n) \leq \Delta_p(x_{n+1}, v_n), \tag{3.12}$$

and

$$\Delta_p(x_{n+1}, y_n) \leq \Delta_p(x_{n+1}, v_n). \tag{3.13}$$

Hence, it follows from (3.12) and (3.13) that $\lim_{n\to\infty} \Delta_p(x_{n+1}, z_n) = 0$ and $\lim_{n\to\infty} \Delta_p(x_{n+1}, y_n) = 0$. By Lemma (2.1), we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \tag{3.14}$$

and so,

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.15}$$

From (3.5), we obtain

$$\begin{aligned}
\left(\rho_n p - \frac{C_q}{q}\rho_n^q\right)\frac{f^p(v_n)}{\|g(v_n)\|^p} &\leq \Delta_p(v_n, x^*) - \Delta_p(z_n, x^*) \\
&= \langle J_{E_1}^p z_n - J_{E_1}^p v_n, x^* - v_n\rangle - \Delta_p(z_n, v_n) \\
&\leq \langle J_{E_1}^p z_n - J_{E_1}^p v_n, x^* - v_n\rangle \\
&\leq \|x^* - v_n\| \|J_{E_1}^p z_n - J_{E_1}^p v_n\|.
\end{aligned} \tag{3.16}$$

Since E_1 is a p-uniformly convex and p-uniformly smooth real Banach space, thus $J_{E_1}^p$ is uniformly norm-to-norm continuous. By $\lim_{n\to\infty} \|v_n - z_n\| = 0$, we obtain $\|J_{E_1}^p z_n - J_{E_1}^p v_n\| \to 0$. From (3.16) and the fact that $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$, we have

$$\frac{f^p(v_n)}{\|g(v_n)\|^p} \to 0 \quad \text{as} \quad n \to \infty,$$

implies,

$$\lim_{n\to\infty} \|(I - M_{\lambda_2}^{B_2})Av_n\| = 0. \tag{3.17}$$

Also

$$\|A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Av_n\| \leq \|A\| \|(I - M_{\lambda_2}^{B_2})Av_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Thus

$$\|A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Av_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Again from (3.1), we get

$$\|J_{E_1}^p Tz_n - J_{E_1}^p z_n\| = \frac{1}{1-\alpha_n}\|J_{E_1}^p y_n - J_{E_1}^p z_n\|. \tag{3.18}$$

It follows from (3.15) that

$$\lim_{n\to\infty} \|J_{E_1}^p Tz_n - J_{E_1}^p z_n\| = 0, \tag{3.19}$$

which also implies that

$$\lim_{n\to\infty} \|Tz_n - z_n\| = 0. \tag{3.20}$$

By Lemma (2.7) and (3.16), we have

$$\begin{aligned}\Delta_p(z_n, w_n) = \Delta_p(N_{\lambda_1}^{B_1} w_n, w_n) &\leq \Delta_p(w_n, x^*) - \Delta_p(z_n, x^*)\\ &\leq \Delta_p(v_n, x^*) - \Delta_p(z_n, x^*) \ \to\ 0 \quad \text{as} \quad n \to\ \infty.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \|N_{\lambda_1}^{B_1} w_n - w_n\| = \lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.21}$$

Step 4: We show that $\{x_n\}$ converges strongly to an element $x^* = \Pi_{\Omega\cap F(T)} x_0$. Since $\{x_n\}$ is a Cauchy sequence, there exists $x^* \in E_1$ such that $\{x_n\}$ converges strongly to x^*. Since $z_n \to x^* \in E_1$, we also have $v_n \to x^* \in E_1$. From (3.21), we get $x^* \in F(N_{\lambda_1}^{B_1}) \in B_1^{-1}(0)$. From (3.20), $\lim_{n\to\infty} \|Tz_n - z_n\| = 0$ and the closeness of T that $x^* = Tx^*$ that is, $x^* \in F(T)$. Since A is a bounded linear operator, we have that $\lim_{n\to\infty} \|Ax_n - Ax^*\| = 0$. By (3.17) we get $\lim_{n\to\infty} \|(I - M_{\lambda_2}^{B_2})Ax_n\| = 0$, this implies that $Ax^* \in \hat{F}(M_{\lambda_2}^{B_2})$ and by Lemma 2.6 we have $Ax^* \in F(M_{\lambda_2}^{B_2})$. This means that $x^* \in \Omega \cap F(T)$.

Let $p \in \Omega \cap F(T) \subseteq C_n$ such that $p = \Pi_{\Omega\cap F(T)} x_0$ and by definition $x_n = \Pi_{C_n} x_0$, we have

$$\Delta_p(x_n, x_0) = \Delta_p(p, x_0). \tag{3.22}$$

This implies that

$$\Delta_p(x^*, x_0) \leq \lim_{n\to\infty} \Delta_p(x_n, x_0) \leq \Delta_p(p, x_0), \tag{3.23}$$

hence $x^* = p$. Therefore, $\{x_n\}$ converges strongly to $x^* \in \Omega \cap F(T)$, where $x^* = \Pi_{\Omega\cap F(T)} x_0$. This completes the proof. $\square$

We next present some consequences of our main results. Firstly, if $\theta_n = 0$, we obtain the following non-inertial shrinking projection result.

Corollary 3.2. *Let $\Omega \cap F(T) \neq \emptyset$. Select $x_0, x_1 \in E_1$ and the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = x_n \\ z_n = N_{\lambda_1}^{B_1}\left(J_{E_1^*}^q\left(J_{E_1}^p(v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n)\right)\right) \\ y_n = J_{E_1^*}^q[\alpha_n J_{E_1}^p(z_n) + (1-\alpha_n) J_{E_1}^p T(z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1. \end{cases} \tag{3.24}$$

where $f(v_n) = \frac{1}{p}\|(I - M_{\lambda_2}^{B_2})Av_n\|^p$, $g(v_n) = A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Av_n$ *and* $\{\rho_n\} \in (0,\infty)$ *satisfy* $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$. *Suppose that the set* $\Psi = \{n \in \mathbb{N} : (I - M_{\lambda_2}^{B_2})Av_n \neq 0\}$, *otherwise* $z_n = v_n$. *Then the sequence* $\{x_n\}$ *converges strongly to* $x^* = \Pi_{\Omega\cap F(T)}x_0$.

Also, by letting $M_{\lambda_2}^{B_2}$ be the metric projection mapping onto a closed convex subset Q of E_2 in Algorithm (3.1), *i.e.* $M_{\lambda_2}^{B_2} = P_Q$ and $N_{\lambda_1}^{B_1} = I$, we obtain the following result as a solution to split feasibility and fixed point problems.

Corollary 3.3. *With reference to the data in Algorithm* (3.1), *let* Q *be a nonempty closed convex subset of* E_2 *and* $M_{\lambda_2}^{B_2} = P_Q$. *Assuming* $\Gamma := \{x \in C : x \in F(T), Ax \in Q\} \neq \emptyset$. *Then the sequence* x_n *generated by Algorithm* (3.1) *converges strongly to* $u \in \Gamma$, *where* $u = \Pi_\Gamma x_0$.

4 A countable family of relatively nonexpansive mappings

In this section, we apply our result to the common fixed point problems of a family of relatively nonexpansive mappings and equilibrium problem.

Definition 4.1 ([7]). *Let* C *be a subset of a real* p*-uniformly convex and uniformly smooth Banach space* E. *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of mappings of* C *in to* E *such that* $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. *Then* $\{T_n\}_{n=1}^\infty$ *is said to satisfy the* $AKTT$*-condition if, for any bounded subset* B *of* C,

$$\sum_{n=1}^\infty \sup_{z\in B}\{\|J_p^E(T_{n+1}z) - J_p^E(T_nz)\|\} < \infty.$$

As in [36], we prove the following Proposition:

Proposition 4.2. *Let* C *be a nonempty, closed and convex subset of a real* p*-uniformly convex and uniformly smooth Banach space* E. *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of mappings of* C *such that* $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ *and* $\{T_n\}_{n=1}^\infty$ *satisfies the* $AKTT$*-condition. Then for any bounded subset* B *of* C *there exists a mapping* $T : B \to E$ *such that*

$$Tx = \lim_{n\to\infty} T_nx, \quad \forall x \in B, \tag{4.1}$$

and

$$\lim_{n\to\infty} \sup_{z\in B} \|J_p^E(Tz) - J_p^E(T_nz)\| = 0.$$

Proof. To complete the proof we show that $\{T_nx\}$ is Cauchy sequence for each $x \in C$. Let $\epsilon > 0$ be given and by the $AKKT$-condition $\exists l_0 \in \mathbb{N}$ such that,

$$\sum_{l_0}^\infty \sup\{\|T_{n+1}y - T_ny\| : y \in C\} < \epsilon.$$

Let $k > l \geq l_0$, then

$$\begin{aligned}\|T_k x - T_l x\| &\leq \sup\{\|T_k y - T_l y\| : y \in C\} \\ &\leq \sup\{\|T_k y - T_{k-1} y\| : y \in C\} + \sup\{\|T_{k-1} y - T_l y\| : y \in C\} \\ &\quad \vdots \\ &\leq \sum_{l}^{k-1} \sup\{\|T_{n+1} y - T_n y\| : y \in C\} \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1} y - T_n y\| : y \in C\} < \epsilon.\end{aligned}$$

Therefore we have that $\{T_n x\}$ is Cauchy sequence. Moreover (3.4) implies that,

$$\|Tx - T_l x\| = \lim_{k \to \infty} \|T_k x - T_l x\| \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1} y - T_n y\| : y \in C\},$$

for all $x \in C$. So,

$$\sup \|Tx - T_l x\| \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1} y - T_n y\| : y \in C\},$$

therefore, we conclude that $\lim_{l_0 \to \infty} \sup \|Tx - T_{l_0} x\| = 0$. □

In the sequel, we say that $(\{T_n\}, T)$ satisfies the $AKTT$-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition and T is defined by (4.1) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 4.3. *Let $\{T_n\}$ be a countable family of Bregman relatively nonexpansive mapping on E_1 such that $F(T_n) = \hat{F}(T_n)$ and assuming $\Omega_1 = \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \emptyset$. Select $x_0, x_1 \in E_1$ and the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = J_{E_1^*}^q [J_{E_1}^p x_n + \theta_n (J_{E_1}^p x_n - J_{E_1}^p x_{n-1})] \\ z_n = N_{\lambda_1}^{B_1} \left[J_{E_1^*}^q \left(J_{E_1}^p (v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n) \right) \right] \\ y_n = J_{E_1^*}^q [\alpha_n J_{E_1}^p (z_n) + (1 - \alpha_n) J_{E_1}^p T_n (z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p (y_n, u) \leq \Delta_p (v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geq 1, \end{cases} \tag{4.2}$$

where $f(v_n) = \frac{1}{p} \|(I - M_{\lambda_2}^{B_2}) A v_n\|^p$, $g(v_n) = A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2}) A v_n$ and suppose that the set $\Psi = \{n \in \mathbb{N} : (I - M_{\lambda_2}^{B_2}) A v_n \neq 0\}$, otherwise $z_n = v_n$. Suppose that in addition $(\{T_n\}_{n=1}^{\infty}, T)$ satisfy AKTT-Condition and $F(T) = \hat{F}(T)$, then the sequence generated by $\{x_n\}$ converges strongly to $x^* \in \Omega_1$, where $x^* = \Pi_{\Omega_1} x_0$.

Proof. To this end, it suffices to show that $\lim_{n \to \infty} \|x_n - T x_n\| = 0$. By following the method of proof in Theorem 3.1, we can show that $\{x_n\}$ is bounded and $\lim_{n \to \infty} \|x_n - T_n x_n\| = 0$. Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0.$$

By Proposition 4.2, we see that

$$\begin{aligned}\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \|J_p^{E_1}(T_n x_n) - J_p^{E_1}(Tx_n)\| \\ &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_n x) - J_p^{E_1}(Tx)\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

Since $J_p^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

4.1 Equilibrium problem

Let E be a real Banach space and let E^* be the dual space of E. Let C be a closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find:

$$x^* \in C \text{ such that } f(x^*, y) \ge 0, \ \forall y \in C. \tag{4.3}$$

The set of solutions of (4.3) is denoted by $EP(f)$. For a given mapping $T : C \to E^*$, define $f(x, y) = \langle Tx, y - x\rangle$, for all $x, y \in C$. Then, $x^* \in EP(f)$ if and only if $\langle Tx^*, y - x^*\rangle \ge 0$, for all $y \in C$ *i.e.* is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (4.3).

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A_1) $f(x, x) = 0$ for all $x \in C$,

(A_2) f is monotone, *i.e.* $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$,

(A_3) for all $x, y, z \in C$,

$$\limsup_{t\to 0} f(tz + (1-t)x, y) \le f(x, y),$$

(A_4) for all $x \in C, f(x, .)$ is convex and lower semicontinuous.

Lemma 4.4 ([10])**.** *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, J_E^P z - J_E^P x\rangle \geq 0 \quad \text{for all} \quad y \in C.$$

Lemma 4.5 ([39])**.** *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$, and let $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows*

$$T_r^f(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, J_E^P z - J_E^P x\rangle \geq 0 \quad \text{for all} \quad y \in C \right\},$$

for all $x \in E$. Then, the following hold:

1. *T_r^f is single-valued,*
2. *T_r^f is a firmly nonexpansive-type mapping [21], i.e., for all $x, y \in E$,*

$$\langle T_r^f x - T_r^f y, J_E^P T_r^f x - J_E^P T_r^f y\rangle \leq \langle T_r^f x - T_r^f y, J_E^P x - J_E^P y\rangle$$

3. *$F(T_r^f) = EP(f)$,*
4. *$EP(f)$ is closed and convex.*

We consider the following split equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{4.4}$$

and $y = Ax^* \in Q$ solves

$$f_2(y^*, y) \geq 0, \quad \forall y \in Q, \tag{4.5}$$

with the solution set $\Omega_2 = \{x^* \in EP(f_1) : Ax^* \in EP(f_2)\}$.

Theorem 4.6. *Let f_1, f_2 be bifunctions satisfying $(A_1) - (A_4)$ and assuming $\Omega_2 \cap F(T) \neq \emptyset$. Select $x_0, x_1 \in E_1$ and the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = J_{E_1^*}^q [J_{E_1}^p x_n + \theta_n (J_{E_1}^p x_n - J_{E_1}^p x_{n-1})] \\ z_n = T_r^{f_1} \left[J_{E_1^*}^q \left(J_{E_1}^p (v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n) \right) \right] \\ y_n = J_{E_1^*}^q [\alpha_n J_{E_1}^p (z_n) + (1 - \alpha_n) J_{E_1}^p T(z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{4.6}$$

where $f(v_n) = \frac{1}{p}\|(I - T_r^{f_2})Av_n\|^p$, $g(v_n) = A^* J_{E_2}^p (I - T_r^{f_2})Av_n$ *and* $\{\rho_n\} \in (0, \infty)$ *satisfy* $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$ *and suppose that the set* $\Psi = \{n \in \mathbb{N} : (I - T_r^{f_2})Av_n \neq 0\}$*, otherwise* $z_n = v_n$*. Then the sequence generated by* $\{x_n\}$ *converges strongly to* $x^* = \Pi_{\Omega_2 \cap F(T)} x_0$*.*

5 Numerical example

In this section, we present an example to show the behaviour of the Algorithm 3.1 presented in this paper and compare its performance with algorithm (1.6) of Cholamjiak *et al.* [14] and (1.5) of Bello Cruz *et al.* [9] by using MATLAB R2016(a). In numerical experiment, we will show that the sequence generated by Algorithm 3.1 via the self-adaptive technique converges faster than algorithms defined in (1.5) and (1.6) for different choices of the $\{\rho_n\}$ and initial values to see the convergence behaviour of Algorithm 3.1.

Example 1. *Let* $E_1 = E_2 = l_2(\mathbb{R})$*, where* $l_2(\mathbb{R}) := \left\{ r = (r_1, r_2, \dots, r_i, \dots), r_i \in \mathbb{R} : \sum_{i=1}^{\infty} | r_i |^2 < \infty \right\}$, $\|r_i\|_2 = \left(\sum_{i=1}^{\infty} | r_i |^2 \right)^{\frac{1}{2}}$, $\forall r \in E_1$*. Let* $C = C_1 := \{x \in E_1 : \|x\|_2 \leq 1\}$*. Let* $T : E_1 \to E_1$ *be defined by* $Tx = \dfrac{x}{2}$, $\forall x \in E_1$*. Let* $A : E_1 \to E_2$ *be a mapping defined by* $Ax = \dfrac{3x}{4}, \forall x \in E_1$*. Let* $\alpha_n = \dfrac{1}{2n}$ *and* $\theta_n = \dfrac{1+n}{5n}$ *and*

$$N_{\lambda_1}^{B_1} x = (1 + \lambda_1 B_1)^{-1} x = \frac{x}{1 + 3\lambda_1}, \quad \forall x \in E_1$$

and

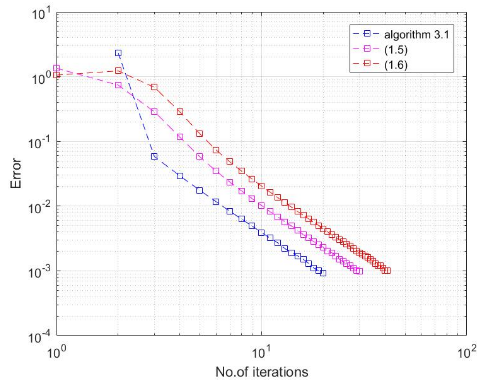
$$M_{\lambda_2}^{B_2} y = (1 + \lambda_2 B_2)^{-1} y = \frac{y}{1 + 5\lambda_2}, \quad \forall y \in E_2,$$

furthermore, it can be verified that for $\lambda_1, \lambda_2 \geq 0$*.*

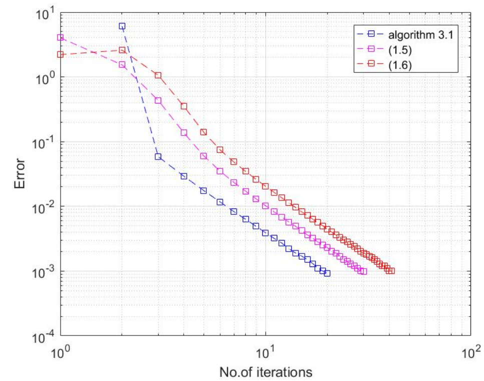
By choosing different ρ_n *and initial values with* $\lambda_1 = \lambda_2 = 1$ *for plotting the graphs of error* $= |x_{n+1} - x_n|$ *against number of iterations with stopping criteria* $|x_{n+1} - x_n| < 10^{-3}$ *for the following cases.*

1. $x_1 = x_0 = \left(2, 1, \frac{2}{3}, \dots\right)$, $\rho_n = \dfrac{n}{n+1}$.
2. $x_1 = x_0 = \left(5, \frac{5}{2}, \frac{5}{3}, \dots\right)$, $\rho_n = \dfrac{n}{n+1}$.
3. $x_1 = x_0 = \left(2, 1, \frac{2}{3}, \dots\right)$, $\rho_n = \dfrac{3n}{n+1}$.
4. $x_1 = x_0 = \left(5, \frac{5}{2}, \frac{5}{3}, \dots\right)$, $\rho_n = \dfrac{3n}{n+1}$.

Thus we see that sequences generated by our algorithm 3.1 converges to the solution set $\Omega \cap F(T)$*. The computational result can be found in Table 1 and Figure.1,2.*

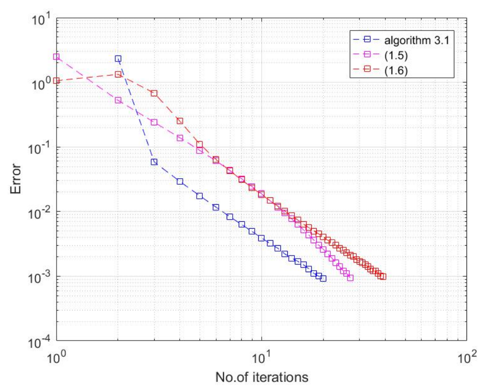


(a) Choice 1 in Example 1

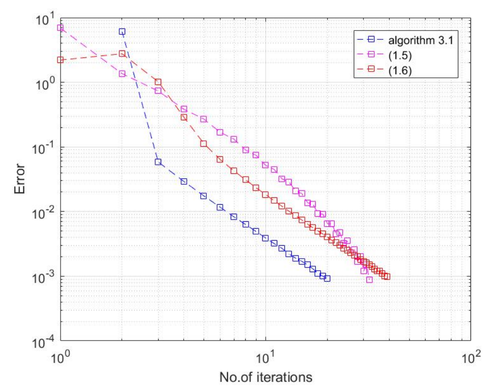


(b) Choice 2 in Example 1

Figure 1



(a) Choice 3 in Example 1



(b) Choice 4 in Example 1

Figure 2

Choice		Algorithm 3.1	(1.5)	(1.6)
1.	No. of Iteration	19	30	41
	CPU Time(s)	0.0313	0.0469	0.0564
2.	No. of Iteration	19	30	41
	CPU Time(s)	0.0524	0.0625	0.125
3.	No. of Iteration	18	27	39
	CPU Time(s)	0.1125	0.313	0.1250
4.	No. of Iteration	19	32	39
	CPU Time(s)	0.5938	0.0625	0.469

Table 1

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References

[1] T. O. Alakoya, L. O. Jolaoso and O. T. Mewomo, "Modified inertia subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems", Optimization, vol. 70, no. 3, pp. 545–574, 2021.

[2] T. O. Alakoya and O. T. Mewomo, "Viscosity S-iteration method with inertial technique and self-adaptive step size for split variational inclusion, equilibrium and fixed point problems", Comput. Appl. Math., vol. 41, no. 1, Paper No. 39, 31 pages, 2022.

[3] T. O. Alakoya, V. A. Uzor, O. T. Mewomo and J. C. Yao, "On a system of monotone variational inclusion problems with fixed-point constraint", J. Inequal. Appl., vol. 2022, Paper No. 47, 30 pages, 2022.

[4] T. O. Alakoya, V. A. Uzor and O. T. Mewomo, "A new projection and contraction method for solving split monotone variational inclusion, pseudomonotone variational inequality, and common fixed point problems", Comput. Appl. Math., vol. 42, no. 1, Paper No. 3, 33 pages, 2023.

[5] Y. I. Alber, "Metric and generalized projection operator in Banach spaces: Properties and applications", in *Theory and applications of nonlinear operators of accretive and monotone type*, Lecture Notes in Pure and Appl. Math. 178, A. G. Kartsatos, New York: Marcel Dekker, 1996, pp. 15–50.

[6] A. S. Alofi, S. M. Alsulami and W. Takahashi, "Strongly convergent iterative method for the split common null point problem in Banach spaces", J. Nonlinear Convex Anal., vol. 17, no. 2, pp. 311–324, 2016.

[7] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space", Nonlinear Anal., vol. 67, no. 8, pp. 2350–2360, 2007.

[8] K. Aoyama, F. Kohsaka and W. Takahashi, "Strong convergence theorems for a family of mappings of type (P) and applications", in *Asian Conference on Nonlinear Analysis and Optimization*, 2009, pp. 1–17.

[9] J. Y. Bello Cruz and Y. Shehu, "An iterative method for split inclusion problems without prior knowledge of operator norms", J. Fixed Point Theory Appl., vol. 19, no. 3, pp. 2017–2036, 2017.

[10] E. Blum, "From optimization and variational inequalities to equilibrium problems", Math. Student, vol. 63, no. 1–4, pp. 123–145, 1994.

[11] C. Byrne, Y. Censor, A. Gibali and S. Reich, "The split common null point problem", J. Nonlinear Convex Anal., vol. 13, no. 4, pp. 759–775, 2011.

[12] Y. Censor, T. Bortfeld, B. Martin and A. A. Trofimov, "A unified approach for inversion problems in intensity-modulated radiation therapy", Phys. Med. Biol., vol. 51, no. 10, pp. 2353–2365, 2006.

[13] Y. Censor and T. A. Elfving, "Multiprojection algorithm using Bregman projections in a product space", Numer. Algorithms, vol. 8, no. 2–4, pp. 221–239, 1994.

[14] P. Cholamjiak, S. Suantai and P. Sunthrayuth, "An iterative method with residual vectors for solving the fixed point and the split inclusion problems in Banach spaces", Comp. Appl. Math., vol. 38, no. 1, Paper No. 12, 25 pages, 2019.

[15] Q. L. Dong and H. B. Yuan, "Accelerated Mann and C_Q algorithms for finding a fixed point of nonexpansive mapping", Fixed Point Theory Appl., vol. 2015, Paper No. 125, 12 pages, 2015.

[16] E. C. Godwin, T. O. Alakoya, O. T. Mewomo and J. C. Yao "Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems", Appl. Anal., In Press, 2022.

[17] E. C. Godwin, C. Izuchukwu and O. T. Mewomo, "Image restoration using a modified relaxed inertial method for generalized split feasibility problems", Math. Methods Appl. Sci., vol. 45, no. 5, pp. 5521–5544, 2022.

[18] C. Izuchukwu, A. A. Mebawondu and O. T. Mewomo, "A new method for solving split variational inequality problems without co-coerciveness", J. Fixed Point Theory Appl., vol. 22, no. 4, Paper No. 98, 23 pages, 2020.

[19] K. R. Kazmi and H. Rizvi, "An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping", Optim. Lett., vol. 8, no. 3, pp. 1113–1124, 2014.

[20] F. Kohsaka and W. Takahashi, "Proximal point algorithm with Bregman function in Banach spaces", J. Nonlinear Convex Anal., vol. 6, no. 3, pp. 505–523, 2005.

[21] F. Kohsaka and W. Takahashi, "Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces", SIAM J. Optim., vol. 19, no. 2, pp. 824–835, 2008.

[22] L. W. Kuo and D. R. Sahu, "Bregman distance and strong convergence of proximal-type algorithms", Abstr. Appl. Anal., Art. ID 590519, 12 pages, 2013.

[23] G. López, V. Martín-Márquez, F. Wang and H. K. Xu, "Solving the split feasibility problem without prior knowledge of matrix norms", Inverse Probl., vol. 28, no. 8, 18 pages, 2012.

[24] S. Y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in Banach spaces", J. Approx. Theory, vol. 134, no. 2, pp. 257–266, 2005.

[25] A. Moudafi, "Split monotone variational inclusions", J. Optim. Theory Appl., vol. 150, no. 2, pp. 275–283, 2011.

[26] A. Moudafi and B. S. Thakur, "Solving proximal split feasibility problems without prior knowledge of operator norms", Optim. Lett., vol. 8 no. 7, pp. 2099-2110, 2014.

[27] E. Naraghirad and J. C. Yao, "Bregman weak relatively non expansive mappings in Banach space", Fixed Point Theory Appl., vol. 2013, Paper No. 141, 2013.

[28] F. U. Ogbuisi and O. T. Mewomo, "Iterative solution of split variational inclusion problem in a real Banach spaces", Afr. Mat., vol. 28, no. 1–2, pp. 295–309, 2017.

[29] R. P. Phelps, *Convex functions, monotone operators, and differentiability*, Lecture Notes in Mathematics 1364, Berlin: Springer Verlag, 1993.

[30] N. Pholasa, K. Kankam and P. Cholamjiak, "Solving the split feasibility problem and the fixed point problem of left Bregman firmly nonexpansive mappings via the dynamical step sizes in Banach spaces", Vietnam J. Math., vol. 49, no. 1, pp. 1011–1026, 2021.

[31] B. T. Polyak, "Some methods of speeding up the convergence of iteration methods", U.S.S.R. Comput. Math. Math. Phys, vol. 4, no. 5, pp. 1–17, 1964.

[32] S. Reich and S. Sabach, "Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces", D. H, Bailey *et al.* in *Fixed-point algorithms for inverse problems in science and engineering*, Springer Optimization and Its Applications, New York: Springer, 2011, pp. 301–316.

[33] Y. Shehu, F. U. Ogbuisi and O. S. Iyiola, "Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces", Optimization, vol. 65, no. 2, pp. 299–323, 2016.

[34] K. Sitthithakerngkiet, J. Deepho, J. Martínez-Moreno and P. Kumam, "Convergence analysis of a general iterative algorithm for finding a common solution of split variational inclusion and optimization problems", Numer. Algorithms, vol. 79, no. 3, pp. 801–824, 2018.

[35] S. Suantai, Y. Shehu and P. Cholamjiak, "Nonlinear iterative methods for solving the split common null point problem in Banach spaces", Optim. Methods Softw., vol. 34, no. 4, pp. 853–874, 2019.

[36] S. Suantai, Y. Shehu, P. Cholamjiak and O. S. Iyiola, "Strong convergence of a self-adaptive method for the split feasibility problem in Banach spaces", J. Fixed Point Theory Appl., vol. 20, no. 2, Paper No. 68, 21 pages, 2018.

[37] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, "Inertial-type algorithm for solving split common fixed point problems in Banach spaces", J. Sci. Comput., vol. 86, no. 1, Paper No. 12, 30 pages, 2021.

[38] S. Takahashi and W. Takahashi, "The split common null point problem and the shrinking projection method in Banach spaces", Optimization, vol. 65, no. 2, pp. 281–287, 2016.

[39] W. Takahashi and K. Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces", Nonlinear Anal., vol. 70, no. 1, pp. 45–57, 2009.

[40] D. V. Thong and D. Van Hieu, "An inertial method for solving split common fixed point problems", J. Fixed Point Theory Appl., vol. 19, no. 4, pp. 3029–3051, 2017.

[41] V. A. Uzor, T. O. Alakoya and O. T. Mewomo, "Strong convergence of a self-adaptive inertial Tseng's extragradient method for pseudomonotone variational inequalities and fixed point problems", Open Math., vol. 20, no. 1, pp. 234–257, 2022.

[42] V. A. Uzor, O. T. Alakoya and O. T. Mewomo, "On split monotone variational inclusion problem with multiple output sets with fixed point constraints", Comput. Methods Appl. Math., In Press, 2023.

[43] H. K. Xu, "Inequalities in Banach spaces with applications", Nonlinear Anal., vol. 16, no. 12, pp. 1127–1138, 1991.