CUBO, A Mathematical Journal
Vol. 25, no. 1, pp. 121–137, April 2023
DOI: 10.56754/0719-0646.2501.121

New upper estimate for positive solutions to a second order boundary value problem with a parameter

Liancheng Wang[1]
Bo Yang[1, ✉]

[1] *Department of Mathematics, Kennesaw State University, Kennesaw, GA 30144, USA.*
lwang5@kennesaw.edu
byang@kennesaw.edu ✉

ABSTRACT

We consider a second order boundary value problem with a parameter. A new upper bound for positive solutions and Green's function of the problem is obtained.

RESUMEN

Consideramos un problema de valor en la frontera de segundo orden con un parámetro. Se obtiene una nueva cota superior para soluciones positivas y la función de Green del problema.

Keywords and Phrases: Boundary value problem with a parameter, positive solution, upper and lower estimates.

2020 AMS Mathematics Subject Classification: 34B18, 34B27.

Accepted: 16 February, 2023
Received: 7 September, 2022

1 Introduction

Fourth order boundary value problems arise from the study of elasticity. They are models for the deflection or bending of elastic beams (see [15, 16]). Recently, fourth order boundary value problems for differential equations with parameters have received quite some attention in the literature. For example, in 2003, Li [5] considered the fourth order boundary value problem

$$\begin{aligned} &u^{(4)} + \beta u'' - \alpha u = f(t,u), \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned}$$

where α, β are parameters. For a partial list of some recent papers on boundary value problems with parameters, we refer the reader to the papers [1, 2, 3, 4, 6, 7, 8, 9, 11, 12].

In 2011, Webb and Zima [10] studied the existence of multiple positive solutions for a class of fourth order boundary value problems. They also studied in [10] a class of second order boundary value problems with a parameter, which are closely related to the fourth order ones. One of the problems that were considered in [10] consists of the equation

$$u''(t) + k^2 u(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

and the boundary conditions

$$u(0) = u(1) = 0, \tag{1.2}$$

where $k \in (0, \pi)$ is a positive constant. It is well-known that second order boundary value problems are important in their own right. Second order problems arise in a wide variety of mathematical models and have been studied extensively.

When $k \in (0, \pi)$, the Green function $G : [0,1] \times [0,1] \to [0, \infty)$ for the problem (1.1)-(1.2) is given by (see [10])

$$G(t,s) = \begin{cases} \dfrac{\sin(kt)\sin(k(1-s))}{k \sin k}, & t \le s, \\ \dfrac{\sin(ks)\sin(k(1-t))}{k \sin k}, & s \le t. \end{cases}$$

The problem (1.1)-(1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds, \quad 0 \le t \le 1. \tag{1.3}$$

It is easy to see that $G(t,s) \ge 0$ for $0 \le t, s \le 1$. Webb and Zima proved a number of results in [10]. In particular, in the case of $k \in (\pi/2, \pi)$, they obtained the following upper and lower estimates for the Green function $G(t,s)$.

Lemma 1.1 ([10, Lemma 2.1]). *If $k \in (\pi/2, \pi)$, then it holds that*

$$c_T(t)\Phi_T(s) \le G(t,s) \le \Phi_T(s), \quad 0 \le t, s \le 1, \tag{1.4}$$

where

$$\Phi_T(s) = \frac{1}{k \sin k} \begin{cases} \sin(ks), & s < 1 - \pi/(2k), \\ \sin(ks)\sin(k(1-s)), & 1 - \pi/(2k) \le s \le \pi/(2k), \\ \sin(k(1-s)), & s > \pi/(2k), \end{cases}$$

and

$$c_T(t) = \min\left\{\sin(kt), \sin(k(1-t))\right\}, \quad 0 \le t \le 1.$$

There are different approaches to solutions for boundary value problems. One important way of finding positive solutions for boundary value problems is to apply fixed point index theorems on a positive cone. To define a positive cone in a function space (for example, the space $C[0,1]$), we need some *a priori* upper and lower estimates for positive solutions of the boundary value problem. Through the years, we have learned that sharper estimates can help define a smaller cone, and, it is easier to search for the positive solution(s) in a smaller cone than in a larger cone. In other words, finer upper and lower estimates can help us establish sharper existence and nonexistence conditions. We refer the reader to the recent papers [14, 15] in which the author used a fixed point theorem on cones to solve fourth order boundary value problems. In both papers, upper and lower estimates for positive solutions play a crucial role in finding solutions for the boundary value problems.

The main purpose of this paper is to further improve the upper estimate in (1.4). Throughout this paper, we assume that

(H) $k \in (\pi/2, \pi)$ is a real number, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

This paper is organized as follows. In Section 2, we establish a new upper estimate for the Green function $G(t,s)$. In Section 3, we prove an interval estimate for points where a positive solution to the problem (1.1)-(1.2) can achieve its maximum. In Section 4, we establish a new upper estimate for positive solutions to the problem (1.1)-(1.2). Here, by a positive solution, we mean a solution $u(t)$ to the the problem (1.1)-(1.2) such that $u(t) > 0$ for $0 < t < 1$. In Section 5, we present an example to illustrate that our new upper estimates can help us solve fourth order boundary value problems.

We remark that some authors like to base their study on estimates for the Green function (like the authors of [10]), and some other authors choose to base their study on estimates for positive solutions (like we will do in Section 5 of this paper). Since both types of estimates have applications, we in this paper will present both types (one type in Section 2, and a second type in Section 4).

Though the two types are similar in form, usually they do not imply each other. This is a second reason we choose to present both types of upper estimates in this paper.

2 New upper estimate for $G(t, s)$

In this section, we will prove a new upper estimate for the Green function $G(t, s)$. Since the analysis is on $G(t, s)$ only, we will not mention any positive solution $u(t)$ to the problem (1.1)-(1.2) in this section.

We define the function $b : [0, 1] \to [0, 1]$ by

$$b(t) = \begin{cases} \sin(k(1-t)), & 0 \le t \le 1 - \frac{\pi}{2k}, \\ 1, & 1 - \frac{\pi}{2k} \le t \le \frac{\pi}{2k}, \\ \sin(kt), & \frac{\pi}{2k} \le t \le 1. \end{cases}$$

The function $b(t)$ will be used to give a new upper estimate for the Green function of the problem (1.1)-(1.2). Also, we define the function $\tau : [0, 1] \to [0, 1]$ by

$$\tau(t) = \min\left\{\frac{\pi}{2k}, \max\left\{t, 1 - \frac{\pi}{2k}\right\}\right\}.$$

In other words,

$$\tau(t) = \begin{cases} 1 - \frac{\pi}{2k}, & 0 \le t \le 1 - \frac{\pi}{2k}, \\ t, & 1 - \frac{\pi}{2k} \le t \le \frac{\pi}{2k}, \\ \frac{\pi}{2k}, & \frac{\pi}{2k} \le t \le 1. \end{cases}$$

With this notation, we can rewrite the function $\Phi_T(s)$ in Lemma 1.1 into a new form.

Lemma 2.1. *We have*

$$\Phi_T(s) = G(\tau(s), s), \quad 0 \le s \le 1.$$

Proof. If $0 \le s \le 1 - \dfrac{\pi}{2k}$, we have

$$\tau(s) = 1 - \frac{\pi}{2k}.$$

In this case, we have $\tau(s) \ge s$, therefore,

$$\begin{aligned} G(\tau(s), s) &= \frac{\sin(ks)\sin(k(1-\tau(s)))}{k\sin k} = \frac{\sin(ks)\sin(k\cdot(\pi/(2k)))}{k\sin k} = \frac{\sin(ks)\sin(\pi/2)}{k\sin k} \\ &= \frac{\sin(ks)}{k\sin k} = \Phi_T(s). \end{aligned}$$

The other two cases — the case where $1 - \dfrac{\pi}{2k} \le s \le \dfrac{\pi}{2k}$ and the case where $\dfrac{\pi}{2k} \le s \le 1$ — can be handled in a similar way. The proof of the lemma is now complete. □

As a consequence of Lemma 2.1, we can now rewrite the upper estimate for $G(t,s)$ in Lemma 1.1 as

$$G(t,s) \le G(\tau(s),s), \quad 0 \le t,s \le 1. \tag{2.1}$$

We will obtain a new upper estimate for $G(t,s)$, which is better than (2.1), in the next several lemmas.

Lemma 2.2. *If (H) holds and $0 \le t \le s \le 1$, then $G(t,s) \le b(t)G(\tau(s),s)$.*

Proof. We take six cases to prove the inequality.

Case 1: $0 \le t \le s \le 1 - \pi/(2k)$. In this case, we have

$$0 \le s - t \le 1 - \frac{\pi}{2k},$$

and, consequently,

$$0 \le k(s-t) \le k - \frac{\pi}{2} < \frac{\pi}{2}.$$

Hence, in this case, we then have

$$b(t)G(\tau(s),s) - G(t,s) = \frac{\sin(k-kt)\sin(ks)}{k\sin k} - \frac{\sin(kt)\sin(k(1-s))}{k\sin k} = \frac{1}{k}\sin(k(s-t)) \ge 0.$$

Case 2: $0 \le t \le 1 - \pi/(2k) \le s \le \pi/(2k)$. In this case, we have

$$0 \le kt \le k - \frac{\pi}{2} < \frac{\pi}{2}.$$

It follows that

$$\frac{\pi}{2} \le k - kt \le k < \pi,$$

and therefore,

$$\sin(k-kt) \ge 0, \quad \cos(k-kt) \le 0. \tag{2.2}$$

Also, since

$$k - \frac{\pi}{2} \le ks \le \frac{\pi}{2},$$

we have

$$\frac{\pi}{2} \le \pi - ks \le \frac{3\pi}{2} - k < \pi,$$

$$\sin(ks) = \sin(\pi - ks) \ge \sin\left(\frac{3\pi}{2} - k\right) = -\cos k. \tag{2.3}$$

By using (2.2) and (2.3), we have

$$\begin{aligned} b(t)G(\tau(s),s)-G(t,s) &= \frac{\sin(k-ks)}{k\sin k}(\sin(k-kt)\sin(ks)-\sin(kt)) \\ &\geq \frac{\sin(k-ks)}{k\sin k}(-\sin(k-kt)\cos k-\sin(kt)) \\ &= -\frac{\sin(k-ks)}{k\sin k}\cdot\cos(k-kt)\sin k \\ &= -\frac{\sin(k-ks)}{k}\cdot\cos(k-kt)\geq 0. \end{aligned}$$

Case 3: $0 \leq t \leq 1-\pi/(2k)$ and $\pi/(2k) \leq s \leq 1$. In this case, we have

$$0 \leq kt \leq k-\frac{\pi}{2} < \frac{\pi}{2},$$

from where it follows that

$$\frac{\pi}{2} \leq k-kt \leq k < \pi \quad \text{and} \quad \frac{\pi}{2} < \pi-kt \leq \pi.$$

In summary, we have

$$\frac{\pi}{2} \leq k-kt \leq \pi-kt \leq \pi,$$

which implies that

$$\sin(k-kt) \geq \sin(\pi-kt).$$

So, in this case, we have

$$\begin{aligned} b(t)G(\tau(s),s)-G(t,s) &= \frac{\sin(k-ks)}{k\sin k}\,(\sin(k-kt)-\sin(kt)) \\ &= \frac{\sin(k-ks)}{k\sin k}\,(\sin(k-kt)-\sin(\pi-kt)) \geq 0. \end{aligned}$$

Case 4: $1-\pi/(2k) \leq t \leq s \leq \pi/(2k)$. In this case, we have

$$0 \leq kt \leq ks \leq \frac{\pi}{2}$$

and

$$b(t)G(\tau(s),s)-G(t,s) = G(s,s)-G(t,s) = \frac{\sin(k-ks)}{k\sin k}(\sin(ks)-\sin(kt)) \geq 0.$$

Case 5: $1-\pi/(2k) \leq t \leq \pi/(2k)$ and $\pi/(2k) \leq s \leq 1$. In this case, we have

$$b(t)G(\tau(s),s)-G(t,s) = G(\pi/(2k),s)-G(t,s) = \frac{\sin(k-ks)}{k\sin k}(1-\sin(kt)) \geq 0.$$

Case 6: $\pi/(2k) \leq t \leq s \leq 1$. In this case,

$$b(t)G(\tau(s), s) - G(t, s) = 0.$$

The proof is now complete. □

Lemma 2.3. *If (H) holds and $0 \leq s \leq t \leq 1$, then $G(t, s) \leq b(t)G(\tau(s), s)$.*

Proof. First, we notice that, for all $t, s \in [0, 1]$,

$$G(t, s) = G(1 - t, 1 - s), \tag{2.4}$$

$$b(t) = b(1 - t), \tag{2.5}$$

$$\tau(t) = \tau(1 - t), \tag{2.6}$$

and

$$G(\tau(1 - s), 1 - s) = G(\tau(s), s). \tag{2.7}$$

Now, if $0 \leq s \leq t \leq 1$, then $0 \leq 1 - t \leq 1 - s \leq 1$, and, by Lemma 2.2,

$$G(1 - t, 1 - s) \leq b(1 - t)G(\tau(1 - s), 1 - s). \tag{2.8}$$

In this case, if we combine (2.8) together with the symmetry properties (2.4), (2.5), (2.6), and (2.7), we get

$$G(t, s) \leq b(t)G(\tau(s), s), \quad \text{for} \ \ 0 \leq s \leq t \leq 1.$$

The proof of the lemma is now complete. □

If we combine Lemmas 2.2 and 2.3, we get

Theorem 2.4. *If (H) holds, then, for all $t, s \in [0, 1]$, $G(t, s) \leq b(t)G(\tau(s), s)$.*

Since $b(t) \leq 1$ for $0 \leq t \leq 1$, it is clear that Theorem 2.4 improves the upper estimate (2.1) for $G(t, s)$ in Lemma 1.1.

3 Localization of the maximum

In this section, we shall prove some upper and lower estimates for the point where a solution to the problem (1.1)-(1.2) achieves its maximum on the interval $[0, 1]$. In other words, we shall find a subinterval of $[0, 1]$ which contains the point where the maximum is achieved by a solution.

Theorem 3.1. *Suppose that $k \in (\pi/2, \pi)$, and suppose that $u \in C^2[0,1]$. If*

$$u''(t) + k^2 u(t) \leq 0 \quad \textit{for} \quad 0 \leq t \leq 1, \tag{3.1}$$

$u(0) = u(1) = 0$, and $u(t) \not\equiv 0$ on $[0,1]$, then $u(t) > 0$ on $(0,1)$, and there exists a unique $t_0 \in (0,1)$ such that $u(t_0) = \|u\|$. Here,

$$\|u\| := \max_{t\in[0,1]} |u(t)|.$$

Proof. For convenience, we define the auxiliary function

$$h(t) := -u''(t) - k^2 u(t), \quad 0 \leq t \leq 1.$$

Then, by (3.1), we have

$$u(t) = \int_0^1 G(t,s)(-u''(s) - k^2 u(s))\, ds \geq 0, \quad 0 \leq t \leq 1.$$

Since $u(t) \not\equiv 0$, we have $\|u\| > 0$. Combining (3.1) and the fact that $u(t) \geq 0$, we have

$$u''(t) \leq -k^2 u(t) \leq 0, \quad 0 \leq t \leq 1.$$

Since $u''(t) \leq 0$, by Theorem 1.2 of [13], we have

$$u(t) \geq \min\{t, 1-t\}\|u\|, \quad 0 \leq t \leq 1.$$

This implies that

$$u(t) > 0 \ \text{ for } \ 0 < t < 1. \tag{3.2}$$

Again, by virtue of (3.1), we have

$$u''(t) \leq -k^2 u(t) < 0, \quad 0 < t < 1.$$

This implies there exists a unique $t_0 \in (0,1)$ such that $u(t_0) = \|u\| > 0$. The proof of the theorem is now complete. □

Theorem 3.2. *Suppose that $k \in (\pi/2, \pi)$, and suppose that $u \in C^2[0,1]$ satisfies (3.1), $u(0) = u(1) = 0$, and $u(t) > 0$ on $(0,1)$. If $t_0 \in (0,1)$ is such that $u(t_0) = \|u\|$, then*

$$1 - \frac{\pi}{2k} \leq t_0 \leq \frac{\pi}{2k}.$$

Proof. We define the auxiliary function $h(t)$ the same way as in the proof of Theorem 3.1, that is,

$h(t) = -u''(t) - k^2 u(t)$, $0 \le t \le 1$. It is clear that $h \in C[0,1]$ and $h(t) \ge 0$ on $[0,1]$, and

$$u(t) = \int_0^1 G(t,s)h(s)\,ds, \quad 0 \le t \le 1.$$

Since $u(t) \not\equiv 0$, we have $h(t) \not\equiv 0$ on $[0,1]$. Therefore, there exists a subinterval $[\alpha,\beta] \subset [0,1]$ such that

$$h(t) > 0, \quad \alpha \le t \le \beta. \tag{3.3}$$

It is clear that, for $0 \le t \le 1$,

$$u(t) = \int_0^1 G(t,s)h(s)\,ds = \int_0^t \frac{\sin(ks)\sin(k(1-t))}{k\sin k} h(s)\,ds + \int_t^1 \frac{\sin(kt)\sin(k(1-s))}{k\sin k} h(s)\,ds.$$

Taking the derivative, we get

$$u'(t) = -\int_0^t \frac{\sin(ks)\cos(k(1-t))}{\sin k} h(s)\,ds + \int_t^1 \frac{\cos(kt)\sin(k(1-s))}{\sin k} h(s)\,ds. \tag{3.4}$$

We note that

$$\sin(ks) > 0 \quad \text{for} \quad 0 < s < 1, \tag{3.5}$$

$$\sin(k(1-s)) > 0 \ \ \text{for} \ \ 0 < s < 1, \tag{3.6}$$

$$-\cos(k(1-t)) > 0 \ \ \text{for} \ \ 0 < t < 1 - \frac{\pi}{2k}, \tag{3.7}$$

$$\cos(kt) > 0 \quad \text{for} \quad 0 < t < 1 - \frac{\pi}{2k}. \tag{3.8}$$

If we apply (3.3), (3.5), (3.6), (3.7), and (3.8) in (3.4), we get

$$u'(t) > 0, \quad 0 \le t < 1 - \frac{\pi}{2k}. \tag{3.9}$$

So, if $t_0 \in (0,1)$ is such that $u(t_0) = \|u\|$, then $u'(t_0) = 0$ and therefore, in view of (3.9), it must hold that $t_0 \ge 1 - \frac{\pi}{2k}$. In a similar way, we can show that $t_0 \le \frac{\pi}{2k}$. The proof of the theorem is now complete. □

4 Upper estimate for positive solutions

In this section, we shall prove a new upper estimate for positive solutions to the problem (1.1)-(1.2). Note that this new upper estimate for positive solutions can not be derived directly from the upper estimate for the Green function $G(t,s)$ that was obtained in Section 2, though these upper estimates look similar.

Theorem 4.1. *Suppose that $k \in (\pi/2, \pi)$. If $u \in C^2[0,1]$ satisfies (3.1) and $u(0) = u(1) = 0$, then*

$$u(t) \le b(t)\|u\|, \quad 0 \le t \le 1. \tag{4.1}$$

Proof. Again, let $h(t) = -u''(t) - k^2 u(t)$. It is clear that $h(t) \ge 0$ for $0 \le t \le 1$.

If $u(t) \equiv 0$, then the theorem is trivially true. So, in the rest of the proof, we assume that $u(t) \not\equiv 0$ on $[0,1]$. In this case, by Theorem 3.1, we have $u(t) > 0$ on $(0,1)$, and there exists a unique $t_0 \in (0,1)$ such that $u(t_0) = \|u\| > 0$. By Theorem 3.2, the point t_0 satisfies

$$1 - \frac{\pi}{2k} \le t_0 \le \frac{\pi}{2k}.$$

Without loss of generality, we assume that $u(t_0) = \|u\| = 1$.

We will first show that

$$u(t) \le b(t)\|u\| = b(t), \quad 0 \le t \le 1 - \pi/(2k). \tag{4.2}$$

Assume, to the contrary, that there exists $\alpha \in (0, 1 - \pi/(2k))$ such that

$$u(\alpha) > b(\alpha) = \sin(k - k\alpha).$$

For easy reference, denote $\sigma = 1 - \pi/(2k)$. Then, we have $0 < \alpha < \sigma$. Define an auxiliary function

$$z(t) = \frac{u(t) - \sin(k - kt)}{\sin(kt + \frac{\pi - k}{2})}, \quad 0 \le t \le 1.$$

It is clear that

$$z(\alpha) > 0, \quad z(\sigma) \le 0, \quad z(1) = 0. \tag{4.3}$$

It follows that there exists $t_1 \in [\alpha, 1)$ such that $z'(t_1) = 0$, $z(t_1) \le 0$, and

$$z(t_1) \le z(t) \quad \text{for all} \quad \alpha \le t \le 1.$$

Direct calculations show that

$$z''(t) + p(t)z'(t) = q(t), \tag{4.4}$$

where

$$p(t) = \frac{2k\cos(kt + \frac{\pi - k}{2})}{\sin(kt + \frac{\pi - k}{2})}, \quad 0 \le t \le 1$$

and

$$q(t) = -\frac{h(t)}{\sin(kt + \frac{\pi - k}{2})}, \quad 0 \le t \le 1.$$

It is clear that $p(t)$ and $q(t)$ are continuous functions defined on $[0,1]$, and $q(t) \leq 0$ for $0 \leq t \leq 1$. Define

$$P(t) = \exp\left(\int_0^t p(s)ds\right), \quad 0 \leq t \leq 1.$$

Multiplying Equation (4.4) by $P(t)$, we get

$$(P(t)z'(t))' \leq 0, \quad 0 \leq t \leq 1.$$

Since $z'(t_1) = 0$, we have

$$P(t)z'(t) \geq 0, \quad 0 \leq t \leq t_1.$$

That is, $z(t)$ is non-decreasing on $[0, t_1]$. Since $z(t_1) \leq 0$ and $\alpha < t_1$, we have $z(\alpha) \leq 0$, which contradicts the first inequality in (4.3). Hence, (4.2) must be true.

In a similar way, we can show that

$$u(t) \leq b(t)\|u\|, \quad \pi/(2k) \leq t \leq 1.$$

And, it is obvious that

$$u(t) \leq \|u\| = b(t)\|u\|, \quad 1 - \pi/(2k) \leq t \leq \pi/(2k).$$

The proof of the theorem is now complete. $\square$

Corollary 4.2. *Suppose that (H) holds. If $u \in C^2[0,1]$ is a positive solution for the problem* (1.1)-(1.2)*, then $u(t)$ satisfies* (4.1)*.*

Proof. If $u \in C^2[0,1]$ is a positive solution for the problem (1.1)-(1.2), then $u(t)$ satisfies the boundary conditions (1.2), and, for $0 \leq t \leq 1$,

$$u''(t) + k^2 u(t) = -f(t, u(t)) \leq 0.$$

That is, $u(t)$ satisfies the inequality (3.1). By Theorem 4.1, $u(t)$ satisfies (4.1). This completes the proof of the corollary. $\square$

5 Example

We conclude this paper with a concrete example. Consider the fourth order boundary value problem

$$u''''(t) - \omega^4 u(t) = f(t, u(t)), \quad 0 \leq t \leq 1, \tag{5.1}$$

$$u(0) = u''(0) = u''(1) = u(1) = 0. \tag{5.2}$$

Here, the function $f : [0,1] \times [0,+\infty) \to [0,\infty)$ is defined as

$$f(t,u) = 15 \max\{(1+99u)/100, u^2\}, \quad u \geq 0. \tag{5.3}$$

It is clear that this function $f(t,u)$ is actually independent of t and continuous in u. Throughout the section, we fix $\omega = 3$.

We will adopt the same set of notations as in [10]. In particular, the symbols $m, \mu_1, f^0, f^\infty, f^{0,r}$ are all defined the same way as in [10] (see pages 233, 234 of [10]). Also, the Green's functions $G_0(t,s), G_T(t,s), G_H(t,s)$ are defined the same way as in [10] (see equations (2.18), (2.19), and (2.20) of [10]). Note that the function $G_T(t,s)$ of [10] is the same as the function $G(t,s)$ that was given in Section 1 of this paper. We know from [10] that all three functions $G_T(t,s), G_H(t,s)$, and $G_0(t,s)$ are non-negative functions.

For this special case (where $\omega = 3$), the following computational results are given in [10, page 235]:

$$m \approx 12.8961, \quad \mu_1 \approx 16.4091. \tag{5.4}$$

According to [10], these numerical values can be used together with the following existence result to solve the fourth order boundary value problem (5.1)-(5.2) for two positive solutions in the case where $\mu_1 < f^0, f^\infty \leq +\infty$.

Lemma 5.1 ([10, Theorem 2.4, Case (D_2)]). *If*

$$\mu_1 < f^0 \leq \infty, \quad f^{0,r} < m \quad \textit{for some} \quad r > 0 \quad \textit{and} \quad \mu_1 < f^\infty \leq \infty,$$

then the problem (5.1)-(5.2) *has at least two positive solutions.*

For the function $f(t,u)$ defined in (5.3), it is straightforward to verify that $f^0 = f^\infty = +\infty$ and, for each $r > 0$, $f^{0,r} \geq 15 > m$. Therefore, Lemma 5.1 does not apply to the problem (5.1)-(5.2).

On the other hand, by applying the new upper estimate that was obtained in this paper, we are able to show that the problem (5.1)-(5.2) has two positive solutions. For this purpose, we choose our function space $X = C[0,1]$, which is equipped with the supremum norm $\|\cdot\|$. Define a positive cone P of X by

$$P = \{u \in X \mid b(t)u(1/2)/c_T(1/2) \geq u(t) \geq c_T(t)\|u\| \text{ for } 0 \leq t \leq 1\}.$$

Define the operator $T : P \to X$ by

$$(Tu)(t) = \int_0^1 G_0(t,s) f(s,u(s))\,ds, \quad \forall\, t \in [0,1],\, \forall\, u \in P.$$

It is clear that T is completely continuous. It is also clear that, in order to show that the problem (5.1)-(5.2) has two positive solutions, we need only to show that the operator T has two distinct nonzero fixed points in P. Next, we shall prove that, for this particular cone P, it holds that T maps P into P. We will need the upper estimate given in Theorem 4.1 in the proof of this fact.

Lemma 5.2. *For each $u \in X$ such that $u(t) \geq 0$ for $0 \leq t \leq 1$, it holds that $Tu \in P$. In particular, $T(P) \subset P$.*

Proof. Let $z(t) = (Tu)(t)$ and let $h(t) = z''(t) + \omega^2 z(t)$ for $0 \leq t \leq 1$. Then, we have

$$z''''(t) - \omega^4 z(t) = f(t, u(t)), \quad 0 \leq t \leq 1,$$

$$z(0) = z''(0) = z''(1) = z(1) = 0.$$

It follows that $h(0) = h(1) = 0$, and

$$h''(t) - \omega^2 h(t) - f(t,u(t)) = 0, \quad 0 \leq t \leq 1.$$

Hence,

$$h(t) = \int_0^1 G_H(t,s)(-f(s,u(s)))\,ds \leq 0, \quad 0 \leq t \leq 1.$$

Since $z''(t) + \omega^2 z(t) - h(t) = 0$ and $z(0) = z(1) = 0$, we have

$$z''(t) + \omega^2 z(t) \leq 0, \quad 0 \leq t \leq 1,$$

$$z(t) = \int_0^1 G_T(t,s)(-h(s))\,ds \geq 0, \quad 0 \leq t \leq 1.$$

Note that $\omega = 3 \in (\pi/2, \pi)$. If we apply Theorem 4.1, we get

$$z(t) \leq b(t)\|z\|, \quad 0 \leq t \leq 1.$$

For all $t_1, t_2 \in [0,1]$, by Lemma 1.1, we have

$$\begin{aligned} z(t_1) &= \int_0^1 G_T(t_1,s)(-h(s))\,ds \geq \int_0^1 c_T(t_1)\Phi_T(s)(-h(s))\,ds = c_T(t_1)\int_0^1 \Phi_T(s)(-h(s))\,ds \\ &\geq c_T(t_1)\int_0^1 G_T(t_2,s)(-h(s))\,ds = c_T(t_1) z(t_2). \end{aligned}$$

Since $t_2 \in [0, 1]$ is arbitrary, we have

$$z(t_1) \geq c_T(t_1)\|z\|, \quad 0 \leq t_1 \leq 1.$$

In summary, we have, for all $0 \leq t \leq 1$,

$$z(t) \leq b(t)\|z\| \leq b(t)z(1/2)/c_T(1/2).$$

The proof of the lemma is now complete. □

Lemma 5.3. *For each $u \in P$ with $\|u\| = 1$, we have $\|Tu\| < \|u\|$.*

Proof. For each $u \in P$ with $\|u\| = 1$, we have $Tu \in P$, and

$$\begin{aligned}(G_T(1/2))\|Tu\| \leq (Tu)(1/2) &= \int_0^1 G_0(1/2, s)f(s, u(s))\,ds \\ &= \int_0^1 G_0(1/2, s)(15 \cdot (1 + 99u(s))/100)\,ds \\ &\leq \int_0^1 G_0(1/2, s)(15 \cdot (1 + 99b(s)/c_T(1/2))/100)\,ds.\end{aligned}$$

It follows that, for each $u \in P$ with $\|u\| = 1$,

$$\|Tu\| \leq (G_T(1/2))^{-1} \cdot (3/20) \cdot \int_0^1 G_0(1/2, s)(1 + 99b(s)/c_T(1/2))\,ds.$$

A direct calculation shows that the right hand side of the last inequality is approximately 0.978566. Thus, we have shown that, for each $u \in P$ with $\|u\| = 1$, it holds that

$$\|Tu\| < 0.979 < 1 = \|u\|.$$

The proof is complete. □

In a similar way, since $f^0 = f^\infty = +\infty$, we can show that

1. there exists a small positive number $\alpha \in (0, 1/2)$ such that, for each $u \in P$ with $\|u\| = \alpha$, it holds that $\|Tu\| \geq \|u\|$; and
2. there exists a positive number $\beta \in (2, +\infty)$ such that, for each $u \in P$ with $\|u\| = \beta$, it holds that $\|Tu\| \geq \|u\|$.

Now, by the norm type of the fixed point theorem of cone expansion and contraction (see Theorem 4 of [14]), the operator T has two fixed points u_1 and u_2 such that

$$0 < \alpha \leq \|u_1\| < 1 < \|u_2\| \leq \beta.$$

It follows that the problem (5.1)-(5.2) has two positive solutions. Note that we are able to achieve this because the new upper estimate (in terms of $b(t)$) from Section 4 can help us define a fine cone P, which makes the search for positive solution(s) easier.

6 Acknowledgments

The authors are grateful to the anonymous referees for their careful reading of the manuscript and valuable suggestions.

References

[1] Z. Bai, "Positive solutions of some nonlocal fourth-order boundary value problem", Appl. Math. Comput., vol. 215, no. 12, pp. 4191–4197, 2010.

[2] G. Chai, "Existence of positive solutions for second-order boundary value problem with one parameter", J. Math. Anal. Appl., vol. 330, no. 1, pp. 541–549, 2007.

[3] Y. Cui, "Multiple solutions to fourth-order boundary value problems", Comput. Math. Appl., vol. 57, no. 4, pp. 643–649, 2009.

[4] X. Dong and Z. Bai, "Positive solutions of fourth-order boundary value problem with variable parameters", J. Nonlinear Sci. Appl., vol. 1, no. 1, pp. 21–30, 2008.

[5] Y. Li, "Positive solutions of fourth-order boundary value problems with two parameters", J. Math. Anal. Appl., vol. 281, no. 2, pp. 477–484, 2003.

[6] Y. Li, "Positive solutions of fourth-order periodic boundary value problems", Nonlinear Anal., vol. 54, no. 6, pp. 1069–1078, 2003.

[7] X. Liu and W. Li, "Positive solutions of the nonlinear fourth-order beam equation with three parameters", J. Math. Anal. Appl., vol. 303, no. 1, pp. 150–163, 2005.

[8] R. Ma, "Nodal solutions for a fourth-order two-point boundary value problem", J. Math. Anal. Appl., vol. 314, no. 1, pp. 254–265, 2006.

[9] W. Shen and T. He, "Bifurcation from interval and positive solutions for a class of fourth-order two-point boundary value problem", Bound. Value Probl., vol. 2013, Paper No. 170, 12 pages, 2013.

[10] J. R. L. Webb and M. Zima, "Multiple positive solutions of resonant and non-resonant nonlocal fourth-order boundary value problems", Glasg. Math. J., vol. 54, no. 1, pp. 225–240, 2012.

[11] Z. L. Wei and C. C. Pang, "Positive solutions and multiplicity of fourth-order m-point boundary value problem with two parameters", Nonlinear Anal., vol. 67, no. 5, pp. 1586–1598, 2007.

[12] J. Xu and Z. Wei, "Positive solutions for multipoint boundary-value problem with parameters", Electron. J. Differential Equations, vol. 2008, Paper No. 106, 8 pages, 2008.

[13] B. Yang, "Upper and lower estimates for positive solutions of the higher order Lidstone boundary value problem", J. Math. Anal. Appl., vol. 382, no. 1, pp. 290–302, 2011.

[14] B. Yang, "Maximum principle for a fourth order boundary value problem", Differ. Equ. Appl., vol. 9, no. 4, pp. 495–504, 2017.

[15] B. Yang, “Positive solutions to a boundary value problem for the beam equation”, Z. Anal. Anwend., vol. 26, no. 2, pp. 221–230, 2007.

[16] B. Yang, “Estimates of positive solutions to a boundary value problem for the beam equation”, Commun. Math. Anal., vol. 2, no. 1, pp. 13–21, 2007.