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Surjective maps preserving the reduced minimum modulus of products

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ABSTRACT

Suppose $\mathfrak{B}(H)$ is the Banach algebra of all bounded linear operators on a Hilbert space H with $\dim(H) \geq 3$. Let $\gamma(.)$ denote the reduced minimum modulus of an operator. We charaterize surjective maps φ on $\mathfrak{B}(H)$ satisfying

$$\gamma(\varphi(T)\varphi(S)) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)).$$

Also, we give the general form of surjective maps on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of Jordan triple products of operators.

RESUMEN

Suponga que $\mathfrak{B}(H)$ es el álgebra de Banach de todos los operadores lineales acotados en un espacio de Hilbert H con $\dim(H) \geq 3$. Denote por $\gamma(.)$ el módulo mínimo reducido de un operador. Caracterizamos las aplicaciones sobreyectivas φ en $\mathfrak{B}(H)$ que satisfacen

$$\gamma(\varphi(T)\varphi(S)) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)).$$

También entregamos la forma general de las aplicaciones sobreyectivas en $\mathfrak{B}(H)$ que preservan el módulo mínimo reducido de productos triples de Jordan de operadores.

Keywords and Phrases: Reduced minimum modulus, operator product, Jordan triple product, nonlinear preservers.

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1 Introduction and Preliminaries

Throughout the paper all Banach spaces are assumed over the field of complex numbers $\mathbb{C}$. For a given Banach space X, $\mathfrak{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. For $T \in \mathfrak{B}(X)$, $R(T)$ and $\ker(T)$ denote the range and the null space of T, respectively. The unit circle in $\mathbb{C}$ will be denoted by $\mathbb{T}$.

Mappings between Banach algebras or operator algebras who preserve various spectral properties have been widely studied. Suppose H is a Hilbert space. Mbekhta [10] characterized surjective linear maps on $\mathfrak{B}(H)$ preserving the generalized spectrum, and then deduced the form of all surjective unital linear maps on $\mathfrak{B}(H)$ preserving the reduced minimum modulus. See also the paper by Bourhim [2], the Banach space case is settled. This result was generalized by Skhiri [13] who, for an arbitrary Banach space X, determined the structure of surjective linear maps φ on $\mathfrak{B}(X)$ preserving the reduced minimum modulus, provided that $\varphi(I)$ is invertible. Bourhim *et. al.* [3] showed that a surjective linear map between C^*-algebras which preserves the reduced minimum modulus is a Jordan $*$-isomorphism multiplied by a unitary element. Consequently, the invertiblity assumption of $\varphi(I)$ in [13] is superfluous.

Let X and Y be Banach spaces. Mashreghi and Stepanyan [9], described a bicontinuous bijective (with no linearity assumption) map $\varphi : \mathfrak{B}(X) \to \mathfrak{B}(Y)$ which leaves invariant the reduced minimum modulus of sum/difference of operators. Later, Costara [5] showed that a bijective map on $M_n(\mathbb{C})$ which preserves the reduced minimum modulus of difference of operators is automatically bicontinuous. Cui and Hou [6] characterized maps on standard operator algebras on a Hilbert space H preserving functional values of operator products, where by a functional value on a standard operator algebra $\mathcal{A}$ we mean a function $F : \mathcal{A} \to [0, +\infty]$ satisfying the following conditions:

(i) $F(T) < \infty$ for each rank one $T \in \mathcal{B}(H)$,

(ii) F is unitary (and conjugate unitary) similarity invariant,

(iii) $F(\lambda T) = |\lambda| F(T)$ for all $T \in \mathfrak{B}(H)$ and $\lambda \in \mathbb{C}$,

(iv) $F(T) = 0$ if and only if $T = 0$.

The *reduced minimum modulus* of an operator $T \in \mathfrak{B}(X)$ is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, \ker(T)) \geq 1\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0. \end{cases} \tag{1.1}$$

(see *e.g.* [3, 8, 12]). This quantity measures the closeness of the range of an operator, that is for $T \in \mathfrak{B}(X)$, $\gamma(T) > 0$ if and only if $R(T)$, the range of T, is closed (see [12, Part 10, Chapter II]). It is proved that if T is invertible then $\gamma(T) = \|T^{-1}\|^{-1}$, see [3, 12]. Suppose H is a Hilbert space.

For $T \in \mathfrak{B}(H)$, let $\sigma(T)$ denote the spectrum of T, then

$$\gamma(T)^2 = \inf\{\lambda : \lambda \in \sigma(T^*T) \setminus \{0\}\}, \tag{1.2}$$

see [8, Theorem 4]. Consequently, $\gamma(T) = \gamma(T^*T)^{\frac{1}{2}} = \gamma(TT^*)^{\frac{1}{2}} = \gamma(T^*)$. So, $\gamma(T)^2 = \gamma(T^2)$ whenever $T = T^*$. Moreover, if $U, V \in \mathfrak{B}(H)$ are unitary operators, then $\gamma(UTV) = \gamma(T)$ for all $T \in \mathfrak{B}(H)$.

We denote by $\mathfrak{R}_1(H)$ the set of all bounded rank one operators on H. We recall that every rank one operator T in $\mathfrak{B}(H)$ is of the form $T = x \otimes y$ for some nonzero vectors $x, y \in H$, and $(x \otimes y)^* = y \otimes x$. So, $(x \otimes y)^*(x \otimes y) = (y \otimes x)(x \otimes y) = \|x\|^2 y \otimes y$. Thus, $\sigma((x \otimes y)^*(x \otimes y)) = \{0, \|x\|^2\|y\|^2\}$, and $\gamma(x \otimes y) = \|x\|\|y\|$.

In this paper, we study surjective maps preserving the reduced minimum modulus of products and Jordan triple products. Obviously, such maps preserve zero product/Jordan triple product in both directions. So, preserving zero product/Jordan triple product plays an important role in our arguments.

Recall that, another definition of $\gamma(\cdot)$ was given by C. Apostol in [1] which is different at $T = 0$. The advantage of Definition (1.1) is that it separates the zero operator from the others. So we would be able to use the results for zero product (resp. zero Jordan triple product) preservers. Therefore, in this article, we shall work with the definition of $\gamma(\cdot)$ given by (1.1).

In Section 2, we assume that H is a complex Hilbert space of dimension greater than or equal 3 and study surjective maps (no linearity and continuity are assumed) on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products. Note that the reduced minimum modulus is not a functional value in the sense of [6], as it does not satisfy Condition (iv) in the definition of a functional value. However, Condition (iv) in [6] is used to show zero product preserving property for the maps under consideration. So, the characterization given in [6] works here. We use this characterization to find a finer characterization for surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products. We show that a surjective map ϕ on $\mathfrak{B}(H)$ preserves the reduced minimum modulus of products if and only if ϕ is a linear or conjugate linear $*$-automorphism multiplied by partial isometries. More precisely, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$ for all $T \in \mathfrak{B}(H)$, where ψ is a linear or conjugate linear $*$-automorphism and U_T, V_T are partial isometries on $\overline{R(\psi(T))}$ and $\overline{R(\psi(T)^*)}$, respectively. We recall that by the general characterization of $*$-automorphisms (resp. $*$-anti-automorphisms) on $\mathfrak{B}(H)$ (see [11, Theorem A.8]), $\psi(T) = UTU^*$ (resp. $\psi(T) = UT^*U^*$), where U is a unitary (resp. anti-unitary) operator on H. Finally in Section 3, we consider surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of Jordan triple products of operators. If H is infinite dimensional, we prove that a surjective map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ preserves the reduced minimum modulus of Jordan triple products if and only if there is a unitary operator U on H and a function $\mu : \mathfrak{B}(H) \to \mathbb{T}$ such that either $\phi(T) = \mu(T)UTU^*$ or

$\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$. In finite dimensional case, we will show that such a map on $M_n(\mathbb{C})$ $(n \geq 3)$, has one of the forms $\phi(A) = \mu(A)Uf(A)U^*$ or $\phi(A) = \mu(A)Uf(A)^{tr}U^*$ for all $A \in M_n(\mathbb{C})$, where μ is a function from $M_n(\mathbb{C})$ to $\mathbb{T}$ and for a matrix $A = [a_{ij}]$, $f(A) = [f_0(a_{ij})]$, where $f_0 : \mathbb{C} \to \mathbb{C}$ is the identity or the conjugation map.

2 Preserving reduced minimum modulus of operator products

Let H be a complex Hilbert space of dimension ≥ 3 and let $\mathcal{U}(H)$ denote the set of unitaries on H. In this section we describe a surjective (with no linearity and continuity assumption) map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfying

$$\gamma\big(\phi(T)\phi(S)\big) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)). \tag{2.1}$$

Then obviously, for $T, S \in \mathfrak{B}(H)$, $TS = 0 \Rightarrow \phi(T)\phi(S) = 0$. So, ϕ preserves zero product. We recall that $\gamma(.)$ does not satisfy Condition (iv) in the definition of a functional value. However, in arguments leading to [6, Theorem 2.3 and Theorem 3.2], the only use of this condition is zero product preserving property. In addition, $\gamma(p) = \inf\{\lambda : \lambda \in \sigma(p^*p) \setminus \{0\}\}\big)^{\frac{1}{2}} = 1$ for all projections $p \in \mathfrak{B}(H)$. Particularly, $\gamma(.)$ is constant on the set of all rank one projections. So, we have the same characterization as in [6, Theorem 2.3] on $\mathfrak{R}_1(H)$. Hence by a similar discussion leading to [6, Theorem 3.2], we see that a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (2.1) if and only if there exist a unitary or an anti-unitary U_0 in $\mathfrak{B}(H)$ and functions $h_1, h_2 : \mathfrak{B}(H) \to \mathcal{U}(H)$ satisfying $h_1(T)T = Th_2(T)$ for all $T \in \mathfrak{B}(H)$, such that

$$\phi(T) = U_0 h_1(T) T {U_0}^* = U_0 T h_2(T) {U_0}^*, \tag{2.2}$$

for all $T \in \mathfrak{B}(H)$.

Here by using properties of γ, we are going to find further necessary and sufficient conditions for ϕ to satisfy (2.1).

To prove our main results, we need the following lemma.

Lemma 2.1. *Let $A, B \in \mathfrak{B}(H)$. Then the following statements are equivalent.*

(i) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{B}(H)$.

(ii) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii) $|A| = |B|$.

Similarly, the following statements are also equivalent.

(i)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{B}(H)$.

(ii)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii)′ $|A^*| = |B^*|$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Assume that $\gamma(AT) = \gamma(BT)$ for all $T \in \mathfrak{R}_1(H)$. Let $x, y \in H$ and $y \neq 0$, then

$$\|Ax\|\|y\| = \gamma(A(x \otimes y)) = \gamma(B(x \otimes y)) = \|Bx\|\|y\|.$$

Thus, $\|Ax\| = \|Bx\|$ for all $x \in H$. So, $\langle A^*Ax, x\rangle = \langle B^*Bx, x\rangle$ for all $x \in H$. Consequently, $|A| = |B|$ that is (ii) implies (iii). If $|A| = |B|$, then $A^*A = B^*B$ and

$$\gamma(AT)^2 = \gamma(T^*A^*AT) = \gamma(T^*B^*BT) = \gamma(BT)^2, \tag{2.3}$$

for all $T \in \mathfrak{B}(H)$. Thus, $\gamma(AT) = \gamma(BT)$ for all $T \in \mathfrak{B}(H)$.

Since $\gamma(T) = \gamma(T^*)$ for all $T \in \mathfrak{B}(H)$, the equivalence of the last three statements is an immediate consequence of the one of the previous statements. □

Proposition 2.2. *Let H be a complex Hilbert space with* $\dim H \geq 3$, *and* $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ *a surjective map. Then ϕ satisfies* (2.1) *if and only if there exists a linear or conjugate linear $*$-automorphism $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ such that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$.*

Proof. Assume that ϕ satisfies (2.1). Using (2.2), it is easy to see that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$, where $\psi(T) = U_0TU_0{}^*$ and U_0 is a unitary or anti-unitary operator on H.

Conversely, suppose that there exists a linear or conjugate linear $*$-automorphism ψ on $\mathfrak{B}(H)$ such that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$. Let $T \in \mathfrak{B}(H)$ be an arbitrary but fixed element, then by the implication (iii) $\Rightarrow$ (i) in Lemma 2.1, we have

$$\gamma(\phi(T)\phi(S)) = \gamma(\psi(T)\phi(S)) \quad (S \in \mathfrak{B}(H)). \tag{2.4}$$

On the other hand, since $|\phi(S)^*| = |\psi(S)^*|$ for all $S \in \mathfrak{B}(H)$, by the implication (iii)′ $\Rightarrow$ (i)′ in Lemma 2.1, we get

$$\gamma(\psi(T)\phi(S)) = \gamma(\psi(T)\psi(S)) = \gamma(\psi(TS)) = \gamma(TS) \quad (S \in \mathfrak{B}(H)). \tag{2.5}$$

Comparing (2.4) and (2.5) implies that $\gamma(\phi(T)\phi(S)) = \gamma(TS)$ for all $T, S \in \mathfrak{B}(H)$, and we are done. $\square$

Lemma 2.3. *Let $T \in \mathfrak{B}(H)$ and U be a partial isometry on $\overline{R(T)}$. Then, $\gamma(UTS) = \gamma(TS)$ for all $S \in \mathfrak{B}(H)$.*

Proof. Since U is a partial isometry on $\overline{R(T)}$,

$$\|UTx\| = \|Tx\| \tag{2.6}$$

for all $x \in H$ and $\ker UT = \ker T$. Let $S \in \mathfrak{B}(H)$, then

$$x \in \ker(UTS) \Longleftrightarrow UTSx = 0 \Longleftrightarrow Sx \in \ker UT \Longleftrightarrow Sx \in \ker T \Longleftrightarrow x \in \ker TS.$$

By (2.6), we get $\|UTSx\| = \|TSx\|$ for all $x \in H$. Now, using the above argument we have

$$\gamma(UTS) = \inf\{\|UTSx\| : \operatorname{dist}(x, \ker UTS) \geq 1\} = \inf\{\|TSx\| : \operatorname{dist}(x, \ker TS) \geq 1\} = \gamma(TS).$$

$\square$

Now we are ready to give a slightly finer characterization for surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products.

Theorem 2.4. *Let H be a complex Hilbert space with $\dim H \geq 3$, and $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ a surjective map. Then ϕ satisfies* (2.1) *if and only if*

$$\phi(T) = U_T\psi(T) = \psi(T)V_T^* \quad (T \in \mathfrak{B}(H)),$$

where ψ is a linear or conjugate linear $*$-automorphism on $\mathfrak{B}(H)$ and for each $T \in \mathfrak{B}(H)$, U_T , V_T are partial isometries on $\overline{R(\psi(T))}$, $\overline{R(\psi(T)^*)}$, respectively. As a consequence, there is a unitary or anti-unitary operator U on H such that

$$\phi(T) = U_T UTU^* = UTU^*V_T^* \quad (T \in \mathfrak{B}(H)).$$

Proof. First, we assume that ϕ satisfies (2.1). By Proposition 2.2, there exists a linear or conjugate linear $*$-automorphism $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ such that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$. Choose an arbitrary but fixed $T \in \mathfrak{B}(H)$. Note that $\|\phi(T)x\| = \|\psi(T)x\|$ and $\|\phi(T)^*x\| = \|\psi(T)^*x\|$ for all $x \in H$. Define

$$\begin{aligned} U_1 : \psi(T)(H) &\to \phi(T)(H) \\ \psi(T)x &\mapsto \phi(T)x, \end{aligned}$$

for all $x \in H$. Then, U_1 is well-defined. Indeed, if $y_1, y_2 \in \psi(T)(H)$, then there exist $x_1, x_2 \in H$ such that $\psi(T)x_1 = y_1$ and $\psi(T)x_2 = y_2$. Also,

$$\begin{aligned}\|U_1y_1 - U_1y_2\| &= \|U_1(\psi(T)x_1) - U_1(\psi(T)x_2)\| = \|\phi(T)x_1 - \phi(T)x_2\| \\ &= \|\phi(T)(x_1 - x_2)\| = \|\psi(T)(x_1 - x_2)\| = \|\psi(T)x_1 - \psi(T)x_2\| \\ &= \|y_1 - y_2\|.\end{aligned}$$

So $U_1y_1 = U_1y_2$, whenever $y_1 = y_2$ which means that U_1 is well-defined. It is easy to see that U_1 is a linear isometry. Hence, it has a linear isometric extension $\overline{U_1}$ to $\overline{R(\psi(T))}$. Define $U_T : H \to H$ by $U_T(x) = \overline{U_1}(x)$ whenever $x \in \overline{R(\psi(T))}$, and $U_T(x) = 0$ for $x \in \overline{R(\psi(T))}^{\perp}$. Therefore, U_T is a partial isometry with $\ker U_T = \overline{R(\psi(T))}^{\perp}$ and we have $\phi(T) = U_T\psi(T)$. By a similar argument, we find a partial isometry V_T such that $\ker V_T = \overline{R(\psi(T)^*)}^{\perp}$ and $\phi(T)^* = V_T\psi(T)^*$. So, $\phi(T) = \psi(T)V_T^*$. Consequently, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$ for all $T \in \mathfrak{B}(H)$.

Conversely, suppose that for $T \in \mathfrak{B}(H)$, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$, where $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ is a linear or conjugate linear $*$-automorphism and U_T , V_T are partial isometries on $\overline{R(\psi(T))}$, $\overline{R(\psi(T)^*)}$, respectively. Then by Lemma 2.3, for $T, S \in \mathfrak{B}(H)$,

$$\begin{aligned}\gamma\big(\phi(T)\phi(S)\big) &= \gamma(U_T\psi(T)\psi(S)V_S^*\big) = \gamma\big(\psi(T)\psi(S)V_S^*\big) = \gamma\big(V_S\psi(S)^*\psi(T)^*\big) = \gamma\big(\psi(S)^*\psi(T)^*\big) \\ &= \gamma\big(\psi(T)\psi(S)\big) = \gamma(TS).\end{aligned}$$

The last assertion follows by [11, Theorem A.8]. $\square$

3 Preserving reduced minimum modulus of Jordan triple product

In [7], authors studied preservers of zero Jordan triple products and found a characterization through some certain subsets of $\mathfrak{B}(X)$. We recall that the Jordan triple product of operators T, S is TST. In the sequel, we consider surjective maps ϕ on $\mathfrak{B}(H)$ satisfying

$$\gamma(\phi(T)\phi(S)\phi(T)) = \gamma(TST) \quad (T, S \in \mathfrak{B}(H)). \tag{3.1}$$

It is easily seen that such a map preserves zero Jordan triple product in both directions, that is

$$TST = 0 \Longleftrightarrow \phi(T)\phi(S)\phi(T) = 0. \tag{3.2}$$

We apply the characterization of maps satisfying (3.2), in [7], to find a finer characterization for maps satisfying (3.1).

Remark 3.1. *(1) Applying [7, Theorem 2.2], we conclude that if H is infinite dimensional and a surjection ϕ on $\mathfrak{B}(H)$ satisfies* (3.2)*, then there is a function $\mu : \mathfrak{B}(H) \to \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator $A : H \to H$ such that either*

$$(a)\ \phi(T) = \mu(T)ATA^{-1} \quad (T \in \mathfrak{B}(H)) \quad or \quad (b)\ \phi(T) = \mu(T)AT^{\star}A^{-1} \quad (T \in \mathfrak{B}(H)).$$

Here $T^{\star}$ denotes the Banach space adjoint of $T \in \mathfrak{B}(H)$. If J is the conjugate linear isomorphism from H onto its dual H^*, then it is easily seen that $T^{\star} = JT^*J^{-1}$, for all $T \in \mathfrak{B}(H)$. Therefore,

$$\phi(T) = \mu(T)AJT^*J^{-1}A^{-1} \quad (T \in \mathfrak{B}(H)).$$

Clearly, AJ is linear or conjugate linear depending on A is conjugate linear or linear, respectively. Renaming AJ by A, we arrive at

(b)′ $\phi(T) = \mu(T)AT^*A^{-1}$*, for all* $T \in \mathfrak{B}(H)$,

where A is a linear or conjugate linear invertible operator.

(2) Suppose that $H = \mathbb{C}^n$, $n \geq 3$, and that $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a surjective map satisfying (3.2)*. Applying [7, Theorem 2.1] shows that there exist an invertible matrix $S \in M_n(\mathbb{C})$, a field automorphism $f_0 : \mathbb{C} \to \mathbb{C}$, and a scalar function $\mu : M_n(\mathbb{C}) \to \mathbb{C} \setminus \{0\}$ such that one of the following holds:*

(c) $\phi(A) = \mu(A)Sf(A)S^{-1} \quad (A \in M_n(\mathbb{C}))$,
or

(d) $\phi(A) = \mu(A)Sf(A)^{tr}S^{-1} \quad (A \in M_n(\mathbb{C}))$*, where* $f([a_{ij}]) = [f_0(a_{ij})]$.

In the two following theorems, we show that if a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (3.1), then the invertible operators A and S in Remark 3.1 (1)-(2) can be replaced by unitaries and moreover, $|\mu| = 1$. As a consequence, ϕ is norm preserving.

Let H be a complex Hilbert space and let $\{e_i\}_i$ be a fixed orthonormal basis for H. If $x = \sum_i \langle x, e_i\rangle e_i$ is an arbitrary element in H, we define $Cx = \sum_i \overline{\langle x, e_i\rangle} e_i$ which is called the *conjugation* operator on H. It is evident that C is an anti-unitary operator with $C^* = C$. Hence, $C^{-1} = C$ and $C^2 = I$. Since $\sigma(CTC) = \sigma(T)$, we have $\sigma((CTC)^*(CTC)) = \sigma(CT^*TC) = \sigma(T^*T)$. Thus, $\gamma(CTC) = \gamma(T)$ for all $T \in \mathfrak{B}(H)$.

Theorem 3.2. *Let H be an infinite dimensional complex Hilbert space. A surjective map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfies* (3.1) *if and only if there exist a function $\mu : \mathfrak{B}(H) \to \mathbb{T}$ and a unitary or anti-unitary operator U on H such that either $\phi(T) = \mu(T)UTU^*$ or $\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$.*

Proof. The "if" part holds in an obvious way. Suppose that ϕ satisfies (3.1), then ϕ satisfies (3.2). Thus by Remark 3.1 (1)-(a), (b)$'$, there exists an invertible linear or conjugate linear operator $A \in \mathfrak{B}(H)$ such that either

$$\text{(i) } \phi(T) = \mu(T)ATA^{-1} \quad (T \in \mathfrak{B}(H)) \quad \textit{or} \quad \text{(ii) } \phi(T) = \mu(T)AT^*A^{-1} \quad (T \in \mathfrak{B}(H)).$$

It follows that for each $T \in \mathfrak{B}(H)$, $\gamma(\phi(T)) = \gamma(T)$. Indeed, $1 = \gamma(I) = \gamma(\phi(I)^3) = |\mu(I)^3|$ and so $|\mu(I)| = 1$. Therefore,

$$\gamma(T) = \gamma(\phi(I)\phi(T)\phi(I)) = |\mu(I)|^2\gamma(\phi(T)) = \gamma(\phi(T)) \quad (T \in \mathfrak{B}(H)).$$

Case 1. In either case, assume that A is linear and that $A = U|A|$ is the polar decomposition of A. Then U is unitary. Set $\phi_{_U}(T) = U^*\phi(T)U$ $(T \in \mathfrak{B}(H))$ and $R = |A|$, then

$$\phi_{_U}(T) = \mu(T)RTR^{-1} \quad (T \in \mathfrak{B}(H)) \quad \textit{or} \quad \phi_{_U}(T) = \mu(T)RT^*R^{-1} \quad (T \in \mathfrak{B}(H)).$$

For a unit vector $x \in H$, we have

$$1 = \gamma(x \otimes x) = \gamma\big(\phi(x \otimes x)\big) = \gamma\big(U^*\phi(x \otimes x)U\big) = \gamma\big(\phi_{_U}(x \otimes x)\big) = |\mu(x \otimes x)|\|Rx\|\|R^{-1}x\|.$$

On the other hand,

$$1 = \gamma\big((x \otimes x)I(x \otimes x)\big) = \gamma\big(\phi_{_U}(x \otimes x)^2\big) = |\mu(x \otimes x)|^2\|Rx\|\|R^{-1}x\|.$$

Therefore, $|\mu(x\otimes x)|^2 = |\mu(x\otimes x)|$. Since $\gamma(\phi_{_U}(x\otimes x)) = \gamma(x\otimes x) = 1$ is nonzero, $|\mu(x\otimes x)| = 1$. It follows that $|\mu| = 1$ on the set of rank one projections on H. Consequently, $\|Rx\|\|R^{-1}x\| = 1$ for all unit vectors $x \in H$. By [6, Lemma 2.4], there is $\alpha > 0$ such that $R = \alpha I$. So,

$$\phi_{_U}(T) = \mu(T)\alpha IT\alpha^{-1}I = \mu(T)T \quad (T \in \mathfrak{B}(H))$$

or

$$\phi_{_U}(T) = \mu(T)\alpha IT^*\alpha^{-1}I = \mu(T)T^* \quad (T \in \mathfrak{B}(H)).$$

In addition, for $T \in \mathfrak{B}(H)$

$$\gamma(T) = \gamma(\phi(T)) = \gamma(\phi_{_U}(T)) = |\mu(T)|\gamma(T).$$

Thus, $|\mu(T)| = 1$ for every $T \in \mathfrak{B}(H)$, and we infer that

$$\phi(T) = \mu(T)UTU^* \quad (T \in \mathfrak{B}(H)) \quad \textit{or} \quad \phi(T) = \mu(T)UT^*U^* \quad (T \in \mathfrak{B}(H)).$$

Case 2. Let A be conjugate linear (in (i) or (ii)), and let C be the conjugation operator on H. Define $\phi_C(T) = C\phi(T)C$ for all $T \in \mathfrak{B}(H)$. Then, ϕ_C satisfies (3.1). Since ϕ satisfies one of the conditions (i) or (ii) above, $\phi_C(T) = \mu(T)CATA^{-1}C$, or $\phi_C(T) = \mu(T)CAT^*A^{-1}C$, where CA is linear with inverse $A^{-1}C$. Now, by the first part of the proof, there is a unitary operator V on H such that either $\phi_C(T) = \mu(T)VTV^*$ or $\phi_C(T) = \mu(T)VT^*V^*$, for all $T \in \mathfrak{B}(H)$ and $|\mu(T)| = 1$ for all T. Putting $U = CV$, then U is an anti-unitary operator and either $\phi(T) = \mu(T)UTU^*$ or $\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$. □

The proof of the following theorem follows the same line as the proof of [7, Theorem 4.1]. We recall that A^{tr} denotes the transpose of a matrix A.

Theorem 3.3. *Suppose $n \geq 3$. Then $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies (3.1) if and only if there exists a unitary matrix U and a function $\mu : M_n(\mathbb{C}) \to \mathbb{T}$ such that either*

$$\text{(i) } \phi(A) = \mu(A)Uf(A)U^* \quad \textit{or} \quad \text{(ii) } \phi(A) = \mu(A)U(f(A))^{tr}U^*,$$

for all $A = [a_{ij}] \in M_n(\mathbb{C})$. We have $f([a_{ij}]) = [f_0(a_{ij})]$ where, $f_0 : \mathbb{C} \to \mathbb{C}$ is the identity or the complex conjugate on $\mathbb{C}$.

Remark 3.4. *(i) As we mentioned in Section 1, another definition of the reduced minimum modulus was given by C. Apostol in [1] which differs from (1.1) at $T = 0$. Let T be a bounded linear operator on a Banach space X. According to [1], the reduced minimum modulus of T which we denote by $\gamma_a(T)$, is defined by*

$$\gamma_a(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, \ker(T)) \geq 1\} & \textit{if } T \neq 0, \\ 0 & \textit{if } T = 0. \end{cases} \tag{3.3}$$

It is natural to ask whether our results remain valid when we replace (1.1) by (3.3). The advantage of Definition (1.1) is that it separates the zero operator from the others. So we would be able to use the properties of zero product (resp. zero Jordan triple product) preservers. Since positivity of $\gamma(T)$ (resp. $\gamma_a(T)$) is equivalent to the closeness of the range of T, and since in finite dimensional case every operator has closed range, so in this case $\gamma_a(T) = 0$ if and only if $T = 0$. Hence, our results hold true with convention (3.3). However, in the inifinite dimensional case, we still do not know whether the same characterizations remain valid with convention (3.3), and the problem remains open.

(ii) One of our main assumptions in this article is that $\dim H \geq 3$. In fact a principal key in our arguments is the characterization of zero product (resp. zero Jordan triple product) preservers on certain subalgebras of $\mathcal{B}(X)$ when X is a Banach space with $\dim X \geq 3$, given in [6, 7]. In general, this assumption on dimension is crucial for characterizing zero product preservers, see [4, Example 3.1]. It seems that characterizing the maps preserving the reduced

minimum modulus of products (resp. Jordan triple product) of complex 2×2 *matrices needs different arguments.*

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