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Fixed points of set-valued mappings satisfying a Banach orbital condition

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ABSTRACT

In this note, we prove a fixed point existence theorem for set-valued functions by extending the usual Banach orbital condition concept for single valued mappings. As we show, this result applies to various types of set-valued contractions existing in the literature.

RESUMEN

En esta nota, demostramos un teorema de existencia de un punto fijo para funciones a valores en conjuntos extendiendo el concepto de la condición orbital de Banach usual para funciones univaluadas. Como mostramos, este resultado aplica a diversos tipos de contracciones a valores en conjuntos existentes en la literatura.

Keywords and Phrases: Banach orbital condition; continuity of set-valued mappings; fixed point; Hausdorff upper semicontinuity; set-valued contraction.

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1 Introduction

Several authors, among others, Berinde [1], Berinde and Păcurar [3], Cho [4], Hicks and Rhoades [8], Kasahara [9] and Kirk and Shahzad [10] studied the existence of fixed points of single and set-valued operators, by stating conditions on the orbits of these operators. In the current work, we are interested in investigating the existence of fixed points, for set-valued mappings or correspondences, by a type of the so called Banach orbital condition. This condition is an adaptation of the usual one, which we introduce motivated by the work of Hicks and Rhoades in [8].

The main result of this note establishes the existence of fixed points for set-valued mappings satisfying the mentioned condition. Moreover, we show that this result and variants of it apply to various multi-valued mappings existing in the literature.

The presentation of this work is subdivided into three sections. Apart of this introduction, in Section 2, some notations and preliminary definitions are presented. The main result and its consequences are introduced in Section 3. Finally, Section 4 is devoted to some examples existing in the literature and satisfying the Banach orbital condition for set-valued mappings.

2 Preliminaries

In the sequel, (X,d) stands for a complete metric space and, for $a \in X$ and $r > 0$, we denote $B(a,r) = \{x \in X : d(x,a) < r\}$. A subset A is said to be bounded, whenever there exist $a \in X$ and $r > 0$ such that $A \subset B(a,r)$. We denote by $\mathcal{B}(X)$ the family of all bounded sets of X and by $\mathcal{C}(X)$ the family of all nonempty and closed subsets of X. In what follows, $\mathcal{CB}(X) = \mathcal{C}(X) \cap \mathcal{B}(X)$ and $B(A,r) = \bigcup_{a\in A} B(a,r)$, for each $A \in \mathcal{B}(X)$ and $r > 0$.

Let $T : X \to \mathcal{CB}(X)$ be a set-valued mapping, $x \in X$ and B be a subset of X. We denote $T(B) = \bigcup_{y\in B} Ty$ and for each $n \in \mathbb{N}$, $T^{n+1}x = T(T^n x)$, with $T^0 x = \{x\}$. The *orbit* of x under T is defined as

$$\mathcal{O}(x,T) = \bigcup_{n=0}^{\infty} T^n x.$$

Let $x_0 \in X$. A function $G : X \to \mathbb{R}$ is said to be (x_0,T)-*orbitally lower semicontinuous* at $x^* \in X$, if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{O}(x_0,T)$ converging to x^*, we have $G(x^*) \le \liminf G(x_n)$. In the sequel, $G_T : X \to \mathbb{R}$ stands for the function defined as $G_T(x) = d(x,Tx)$ and for $\xi : X \to X$, we denote $G_\xi = G_{\{\xi\}}$.

Given a set-valued mapping $T : X \to \mathcal{CB}(X)$, $x_0 \in X$, and $k \in [0,1)$, we say T satisfies the *multivalued Banach orbital* (MBO) condition at x_0 with constant k, whenever for all $x \in \mathcal{O}(x_0,T)$, $\inf_{y\in Tx} d(y,Ty) \le kd(x,Tx)$, and that, T satisfies the *strong multivalued Banach orbital* (SMBO) condition at x_0 with constant k, whenever for all $x \in \mathcal{O}(x_0,T)$, $\sup_{y\in Tx} d(y,Ty) \le kd(x,Tx)$.

3 Main results

Theorem 3.1. *Let $T : X \to \mathcal{CB}(X)$ be a set-valued mapping satisfying the MBO condition at $x_0 \in X$ with constant k. Then, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that, for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$, and the following two conditions hold:*

(i) $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) \leq k^n d(x_0, Tx_0)$ *and*

(ii) $d(x^*, Tx_n) \leq \{k^{n+1}/(1-k)\}d(x_0, Tx_0)$, *for all* $n \in \mathbb{N}$.

Moreover, the following conditions are equivalent:

(iii) $x^* \in Tx^*$

(iv) G_T *is* (x_0, T)*-orbitally lower semicontinuous at* x^*, *and*

(v) the function $h : X \to \mathbb{R}$, *defined by* $h(x) = d(x, Tx)$, *is lower semicontinuous at* x^*.

Proof. Let $\rho \in (k, 1)$. If $d(x_0, Tx_0) = 0$, we define $x_n = x_0$, for all $n \geq 1$. Otherwise, from assumption, there exists $x_1 \in Tx_0$ such that $d(x_1, Tx_1) < \rho d(x_0, Tx_0)$. If $d(x_1, Tx_1) = 0$, we define $x_n = x_1$, for all $n \geq 2$. Otherwise, there exists $x_2 \in Tx_1$ such that $d(x_2, Tx_2) < \rho d(x_1, Tx_1) < \rho^2 d(x_0, Tx_0)$. It follows by induction that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that, for all $n \in \mathbb{N}$, $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) \leq \rho^n d(x_0, Tx_0)$ and $x_{n+1} \in Tx_n$. Hence, condition (i) holds.

For all $n \in \mathbb{N}$ and $m \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\leq \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) \leq \sum_{k=0}^{m-1} \rho^{n+k} d(x_0, Tx_0) = \rho^n \sum_{k=0}^{m-1} \rho^k d(x_0, Tx_0) \\ &\leq \rho^n \sum_{k=0}^{m-1} \rho^k d(x_0, Tx_0). \end{aligned}$$

Hence, $d(x_n, x_{n+m}) \leq \{\rho^n/(1-\rho)\}d(x_0, Tx_0)$. In particular, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and consequently there exists $x^* \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. By taking limit, as $m \to \infty$, in the last inequality, we have

$$d(x^*, Tx_{n-1}) \leq d(x^*, x_n) \leq \{\rho^n/(1-\rho)\}d(x_0, Tx_0), \text{ for all } n \geq 1,$$

and consequently condition (ii) holds.

Suppose $x^* \in Tx^*$. Since $G_T(x^*) = 0$, it is clear that G_T is (x, T)-orbitally lower semicontinuous at x^*, for all $x \in X$. This proves that condition (iii) implies condition (iv). Next, conditions (iv) and (v) are equivalent, by the first axiom of countability. Finally, by assuming the lower semicontinuity of h, we have $d(x^*, Tx^*) = h(x^*) \leq \liminf h(x_n) = 0$, by condition (i). Since Tx^* is closed, this proves that condition (v) implies condition (iii) and the proof is complete. □

Remark 3.2. Any lower semicontinuity set-valued mapping, $T : X \to \mathcal{CB}(X)$, satisfying assumptions of Theorem 3.1, also satisfies the equivalent conditions $(iii)-(v)$. Indeed, let h be the function defined in condition (v) and $a > 0$. Hence, $\{x \in X : h(x) < a\} = \{x \in X : Tx \cap B(x, a) \neq \emptyset\}$. That is, h is upper semicontinuous.

Given $x_0 \in X$ and a single valued function, $f : X \to X$, we denote $\mathcal{O}(x_0, f) = \mathcal{O}(x_0, \{f\})$. As usual, $\{f^n\}_{n\in\mathbb{N}}$ denotes the sequence of functions defined recursively as f^0 the identity function and $f^{n+1} = f \circ f^n$, for all $n \in \mathbb{N}$. The following corollary is an equivalent version of the main result of Hicks and Rhoades in [8].

Corollary 3.3. *Let $\xi : X \to X$ be a function and $k \in [0, 1)$. Suppose there exists $x_0 \in X$ such that, for all $x \in \mathcal{O}(x_0, \xi)$, $d(\xi(x), \xi^2(x)) \leq kd(x, \xi(x))$. Then, there exists $x^* \in X$ such that the following two conditions hold:*

(i) $\lim_{n\to\infty} d(x^*, \xi^n(x_0)) = 0$ *and*

(ii) $d(x^*, \xi^n(x_0)) \leq \{k^n/(1-k)\}d(x_0, \xi(x_0))$, *for all* $n \in \mathbb{N}$.

Moreover, $x^* = \xi(x^*)$, if and only if, the function $x \in X \mapsto d(x, \xi(x)) \in \mathbb{R}$ is (x_0, ξ)-orbitally lower semicontinuous at x^*.

Proof. By Theorem 3.1, there exist $x_k^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x_k^* such that $x_{n+1} = \xi(x_n) = \xi^n(x_0)$. Since the sequence $\{x_n\}_{n\in\mathbb{N}}$ only depends on x_0 and not on k, neither does x_k^* depend on k. Therefore, conditions (i) and (ii) follow from Theorem 3.1 and the proof is complete. □

A set-valued mapping $T : X \to \mathcal{CB}(X)$ is said to be *Hausdorff upper semicontinuous*, if for each $x \in X$ and $\epsilon > 0$, there exists a neighborhood U of x such that $Ty \subset B(Tx, \epsilon)$, for all $y \in U$. This concept is weaker that the upper semicontinuity ot T. However, as we see below, it contributes to obtaining orbital lower semicontinuity for T.

Theorem 3.4. *Let $T : X \to \mathcal{CB}(X)$ be a Hausdorff upper semicontinuous set-valued mapping and suppose T satisfies the MBO condition at $x_0 \in X$ with constant k. Then, there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Proof. By Theorem 3.1, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{O}(x_0, T)$, converging to x^* such that, for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$. Let $\epsilon > 0$. From assumption, there exists a neighborhood U

of x^* such that $Tx \subset B(Tx^*, \epsilon)$, for all $x \in U$. Let $N \in \mathbb{N}$ such that $x_n \in U$, for all $n \geq N$. Hence $Tx_n \subset B(Tx^*, \epsilon)$, which implies that $\sup\limits_{y \in Tx_n} d(y, Tx^*) \leq \epsilon$, for all $n \geq N$. We have

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \leq d(x^*, x_{n+1}) + \epsilon, \text{ for all } n \geq N.$$

By taking inf-limit in n and considering that $\epsilon > 0$ is arbitrary, we obtain $d(x^*, Tx^*) = 0$. Since Tx^* is closed, we have $x^* \in Tx^*$, which completes the proof. □

We denote by $\mathcal{H}$ the Pompeiu-Hausdorff metric (see [3]) associate to d, *i.e.*, $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ is defined as

$$\mathcal{H}(U, V) = \inf \{\epsilon > 0 : U \subset B(V, \epsilon) \text{ and } V \subset B(U, \epsilon)\}.$$

Corollary 3.5. *Let $T : X \to \mathcal{CB}(X)$ be a continuous set-valued mapping with respect to the Pompeiu-Hausdorff metric, i.e. $\lim\limits_{n\to\infty} \mathcal{H}(Tx_n, Tx) = 0$, for all sequence, $\{x_n\}_{n\in\mathbb{N}}$, in X converging to $x \in X$. Suppose T satisfies the MBO condition at $x_0 \in X$ with constant k. Then, there exist $x^* \in X$ such that $x^* \in Tx^*$.*

Proof. It is a consequence of Theorem 3.4, and the Pompeiu-Hausdorff continuity of T implies its Hausdorff upper semicontinuity. □

Remark 3.6. Let $T : X \to \mathcal{CB}(X)$ be a set-valued mapping, $x_0 \in X$ and $k \in [0, 1)$. Notice that, a sufficient condition to T satisfies the MBO condition is $d(y, Ty) \leq kd(x, y)$, for all $x \in \mathcal{O}(x_0, T)$ and $y \in Tx$, and a sufficient condition to T satisfies the SMBO condition is $d(y, Ty) \leq kd(x, Tx)$, for all $y \in Tx$.

4 Some examples

In this section, we introduce some special types of set-valued mappings, which satisfy the MBO condition.

1. (Nadler contraction [6, 11]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ is a Nadler contraction, if for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq kd(x, y)$, for some $k \in [0, 1)$. Let $x \in X$ and $y \in Tx$. Hence,

$$d(y, Ty) \leq \sup_{z \in Tx} d(z, Ty) \leq \mathcal{H}(Tx, Ty) \leq kd(x, y),$$

   and consequently T satisfies the MBO condition. In this case, there exists $x^* \in X$ such that $x^* \in Tx^*$, by Corollary 3.5.

2. (Kannan contraction [12]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Kannan contraction, if and only if, there exists $k \in [0, 1/2)$ such that $\mathcal{H}(Tx, Ty) \leq k(d(x, Tx) +$

$d(y, Ty))$, for all $x, y \in X$. Let $k \in [0, 1/2)$ such that $\mathcal{H}(Tx, Ty) \le k(d(x, Tx) + d(y, Ty))$, for all $x, y \in X$. We have

$$d(y, Ty) \le \mathcal{H}(Tx, Ty) \le k(d(x, Tx) + d(y, Ty)),$$

and hence, $(1 - k)d(y, Ty) \le kd(x, Tx)$. Accordingly,

$$d(y, Ty) \le \{k/(1 - k)\}d(x, Tx), \text{ for all } x \in X \text{ and } y \in Tx.$$

Since $k/(1 - k) \in [0, 1)$, we have T satisfies the SMBO condition with constant $k/(1 - k)$.

3. (Kannan generalized contraction [7, 12]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the generalized Kannan contraction, if and only if, there exists $k \in [0, 1)$ such that $\mathcal{H}(Tx, Ty) \le k \max\{d(x, Tx), d(y, Ty)\}$, for all $x, y \in X$. In this case, if for some $y \in Tx$, $d(x, Tx) \le d(y, Ty)$, then $d(y, Ty) = 0$, otherwise $d(y, Ty) \le kd(x, Tx)$, for all $y \in Tx$. Consequently, T satisfies the SMBO condition with constant k.

4. (Chatterjea contraction [13]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Chatterjea contraction, if there exists $k \in [0, 1/2)$ such that for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \le k(d(x, Ty) + d(y, Tx))$. Let $x \in X$ and $y \in Tx$. Hence,

$$d(y, Ty) \le \mathcal{H}(Tx, Ty) \le k(d(x, Ty) + d(y, Tx)) = kd(x, Ty).$$

This fact along with the inequality $d(x, Ty) \le d(x, y) + d(y, Ty)$ implies that

$$d(y, Ty) \le \{k/(1 - k)\}d(x, y), \text{ for all } x \in X \text{ and } y \in Tx.$$

Consequently, T satisfies the multivalued Banach orbital condition with constant $k/(1-k) \in [0, 1)$.

5. (Chatterjea generalized contraction) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the generalized Chatterjea contraction, if there exists $k \in [0, 1/2)$ such that, for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \le k \max\{d(x, Ty), d(y, Tx)\}$. Let $x \in X$ and $y \in Tx$. Hence,

$$d(y, Ty) \le \mathcal{H}(Tx, Ty) \le kd(x, Ty),$$

and accordingly, T satisfies the SMBO condition with constant $k/(1 - k) \in [0, 1)$.

6. (Berinde contraction [2]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Berinde contraction if there exist $k \in [0, 1)$ and $L \ge 0$ such that, for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \le$

$kd(x,y) + Ld(y,Tx)$. Let $x \in X$ and $y \in Tx$. We have

$$\mathcal{H}(Tx,Ty) \leq kd(x,y) + L(y,Tx) = kd(x,y), \text{ for all } x \in X \text{ and } y \in Tx,$$

and since $y \in Tx$, we obtain $d(y,Ty) \leq kd(x,y)$ and hence T satisfy the MBO condition with constant k.

7. (Ciric-Reich-Rus contraction [2]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ is said to verify the Ciric-Reich-Rus contraction if and only if, there exists $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma \in [0,1)$ and, for all $x, y \in X$, $\mathcal{H}(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty)$. Let $x \in X$, $y \in Tx$. We will prove that any Ciric-Reich-Rus contraction is a Berinde contraction. Let $x, y \in X$. As we observed previously, we have the inequality

$$d(y,Ty) \leq d(y,z) + (z,Ty),$$

for all $z \in Tx$. Replacing this, and by the fact $d(x,Tx) \leq d(x,z)$, we have

$$\begin{aligned}
\mathcal{H}(Tx,Ty) &\leq \alpha d(x,y) + \beta d(x,z) + \gamma d(y,Ty)) \\
&\leq \alpha d(x,y) + \beta(d(x,y) + d(y,z)) + \gamma(d(y,z) + d(z,Ty)) \\
&= (\alpha+\beta)d(x,y) + (\beta+\gamma)d(y,z) + \gamma d(z,Ty) \\
&\leq (\alpha+\beta)d(x,y) + (\beta+\gamma)d(y,z) + \gamma\mathcal{H}(Tx,Ty).
\end{aligned}$$

Hence,

$$\mathcal{H}(Tx,Ty) \leq (\{\alpha+\beta)/(1-\gamma)\}d(x,y) + \{(\beta+\gamma)/(1-\gamma)\}d(y,Tx), \text{ for all } x \in X \text{ and } y \in Tx,$$

and since $\alpha+\beta+\gamma < 1$, it follows that $(\alpha+\beta)/(1-\gamma) < 1$ and $(\beta+\gamma)/(1-\gamma) \geq 0$. Therefore, T is a Berinde contraction, and accordingly T satisfies the MBO condition.

8. (Ciric contraction [5]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Ciric contraction, if there exist $\alpha \in [0,1/2)$ such that for all $x, y \in X$,

$$\mathcal{H}(Tx,Ty) \leq \alpha \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

We have T satisfies the multivalued Banach orbital condition. Indeed, let $x \in X$ and $y \in Tx$. Hence, for some $\alpha \in [0,1/2)$, we have

$$\mathcal{H}(Tx,Ty) \leq \alpha \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

but, since $y \in Tx$ and $d(x,Tx) \leq d(x,y)$, we obtain

$$\mathcal{H}(Tx,Ty) \leq \alpha \max\{d(x,y), d(y,Ty), d(x,Ty)\} \leq k(d(x,y) + d(y,Ty)).$$

Consequently,

$$d(y,Ty) \leq \{k/(1-k)\}d(x,y), \text{ for all } x \in X \text{ and } y \in Tx,$$

and therefore, T satisfies the MBO condition.

9. We introduce a new type of contraction, which satisfies the SMBO condition. Indeed, let $T : X \to \mathcal{CB}(X)$ be given as follows:

$$\mathcal{H}(Tx,Ty) \leq \alpha(d(x,Ty) + d(y,Ty)), \text{ for all } x, y \in X,$$

where $\alpha \in [0,1)$. Observe that, for all $y \in Tx$ and $x \in X$, we have $d(y,Ty) \leq \alpha d(x,Tx)$. Consequently, T satisfies the SMBO condition with constant α.

It is worth noting that the existence of a fixed point for contractions (1)-(6) was proved in [2].

Remark 4.1. Although the nine contraction set-valued mappings in this section satisfy the MBO condition, only the Nadler contraction has a fixed point without additional assumptions. The MBO condition for the other contractions is insufficient to have a fixed point.

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