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Double asymptotic inequalities for the generalized Wallis ratio

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ABSTRACT

Asymptotic estimates for the generalized Wallis ratio $W^*(x) := \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}$ are presented for $x \in \mathbb{R}^+$ on the basis of Stirling's approximation formula for the Γ function. For example, for an integer $p \geq 2$ and a real $x > -\frac{1}{2}$ we have the following double asymptotic inequality

$$A(p,x) < W^*(x) < B(p,x),$$

where

$$A(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{379(x+p)^3}\right),$$
$$B(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{191(x+p)^3}\right),$$
$$W_p(x) := \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}},$$

with $y^{(p)} \equiv y(y+1)\cdots(y+p-1)$, the Pochhammer rising (upper) factorial of order p.

RESUMEN

Se presentan estimaciones asintóticas para la razón generalizada de Wallis $W^*(x) := \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}$ para $x \in \mathbb{R}^+$ sobre la base de la fórmula de aproximación de Stirling para la función Γ. Por ejemplo, para un entero $p \geq 2$ y un real $x > -\frac{1}{2}$, tenemos la siguiente desigualdad doble asintótica

$$A(p,x) < W^*(x) < B(p,x),$$

donde

$$A(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{379(x+p)^3}\right),$$
$$B(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{191(x+p)^3}\right),$$
$$W_p(x) := \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}},$$

con $y^{(p)} \equiv y(y+1)\cdots(y+p-1)$, el factorial ascendiente de Pochhammer (superior) de orden p.

Keywords and Phrases: Approximation, asymptotic, estimate, generalized Wallis' ratio, double inequality.

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1 Introduction

In pure and applied mathematics, *e.g* in number theory, probability, combinatorics, statistics, and also in several exact sciences as, for example in statistical physics and quantum mechanics, we often encounter the Wallis ratios w_n,

$$\begin{aligned} w_n &:= \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} = 4^{-n}\frac{(2n)!}{(n!)^2} = 4^{-n}\binom{2n}{n} \\ &= \frac{2^n\prod_{k=1}^{n}(k-\frac{1}{2})}{2^n\cdot n!} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n+1)} \qquad (n\in\mathbb{N}). \end{aligned} \tag{1.1}$$

The sequence $n \mapsto W_n := \frac{1}{2n+1}\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)^2$, called the Wallis sequence, is closely connected to the sequence of the Wallis ratios w_n by the identity $W_n = w_n^{-2}/(2n+1)$. The Wallis sequence was intensively studied by several mathematicians, see *e.g.* [9–11, 14, 19].

According to (1.1), the continuous version $W^*(x)$ of the Wallis ratio is defined as

$$W^*(x) := \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} \qquad \left(x > -\tfrac{1}{2}\right). \tag{1.2}$$

Thus, we have $W^*(0) = 1$ and, referring to [11, 19], we have also

$$W^*(x) = \frac{2}{\pi}\cdot H(2x), \tag{1.3}$$

where $H(x)$ is the "Wallis-cos-sin" function, defined as

$$H(x) := \int_0^{\pi/2}\left(\cos t\right)^x \mathrm{d}t = \int_0^{\pi/2}\left(\sin t\right)^x \mathrm{d}t \qquad (x \geq -1). \tag{1.4}$$

Here, for $x > -1$, we have the derivatives

$$H'(x) = \int_0^{\pi/2}\left(\ln\cos t\right)\left(\cos t\right)^x \mathrm{d}t < 0\,, \quad H''(x) = \int_0^{\pi/2}\left(\ln\cos\, t\right)^2\left(\cos t\right)^x \mathrm{d}t > 0.$$

Consequently, using (1.3), we conclude that $W^*(x)$ is strictly decreasing and convex on the open interval $(-\frac{1}{2}, \infty)$.

Referring to (1.2), we have

$$W^*(x) = \frac{1}{\sqrt{\pi}}\cdot Q_\Gamma(x, \tfrac{1}{2}, 1) \qquad \left(x > -\tfrac{1}{2}\right), \tag{1.5}$$

where the ratio $Q_\Gamma(x, a, b)$ is defined as

$$Q_\Gamma(x, a, b) := \frac{\Gamma(x+a)}{\Gamma(x+b)}\,, \qquad \text{for } x > -\max\{a, b\}. \tag{1.6}$$

The ratio[1] $Q_\Gamma(x,a,b)$ was studied by many researchers, see *e.g.* the papers [2, 3, 5–7, 12, 13, 15–18, 20–27, 29, 30, 32]. Just recently several accurate estimates of $Q_\Gamma(x,a,b)$ were presented in [16], as for example in the following proposition.

Proposition 1 ([16, Theorem 1]). *For $a,b\in[0,1]$, $r\in\mathbb{N}\cup\{0\}$ and $x\in\mathbb{R}^+$ we have*[2]

$$Q_\Gamma(x,a,b)=\left(1+\frac{a}{x}\right)^x\left(1+\frac{b}{x}\right)^{-x}\frac{(x+a)^{a-1/2}}{(x+b)^{b-1/2}}\exp(b-a)\\ \cdot\exp\left(\sum_{i=1}^{r}\frac{B_{2i}}{2i(2i-1)}\left((x+a)^{1-2i}-(x+b)^{1-2i}\right)+\delta_r(x,a,b)\right),\tag{1.7}$$

where

$$\left|\delta_r(x,a,b)\right|<\Delta_r(x,a,b):=\frac{\left|B_{2r+2}\right|}{(2r+1)(2r+2)\left(x+\min\{a,b\}\right)^{2r+1}}\tag{1.8}$$

and the symbol B_k denotes the k-th Bernoulli coefficient [1, 23.1.2].

Thus, for $a=\frac{1}{2}$ and $b=1$, the Proposition produces the formula

$$W^*(x)=\frac{1}{\sqrt{\pi(x+1)}}\cdot\left(1+\frac{1}{2x}\right)^x\left(1+\frac{1}{x}\right)^{-x}\sqrt{e}\\ \cdot\exp\left(\sum_{i=1}^{r}\frac{B_{2i}}{2i(2i-1)}\left((x+1/2)^{1-2i}-(x+1)^{1-2i}\right)\right)\cdot\exp\left(\delta_r(x,\tfrac{1}{2},1)\right),\tag{1.9}$$

where

$$\left|\delta_r(x,\tfrac{1}{2},1)\right|<\frac{\left|B_{2r+2}\right|}{(2r+1)(2r+2)\left(x+\frac{1}{2}\right)^{2r+1}},\tag{1.10}$$

for integers $r\geq 0$ and $x>0$ with r being a parameter that affects the magnitude of the error term $\delta_r(x,1/2,1)$.

In this paper we will introduce a formula that is more compact than that given by (1.9)–(1.10). Our results are close to some formulas given in [4] and [31], where the main role is played by complete monotonicity of suitable functions. Unfortunately, using these articles, our results cannot be achieved easily/quickly. In this paper, we offer a simple and fast derivation using the Stirling approximation formula for the gamma function.

Remark 1.1. *In 2011, the Wallis quotient function $W(x,s,t):=\frac{\Gamma(x+t)}{\Gamma(x+s)}$ was introduced*[3] *in [2]. In this paper and also in the subsequent articles [3, 7], the authors investigate the qualitative profile of $W(x,s,t)$ using asymptotic expansions. Quantitative estimates were mostly not given there. However, for us, the quantitative estimates are essential.*

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[1] Instead of the symbol Q_Γ there was used in [2, 3, 7] the letter W: $Q_\Gamma(x,a,b)=W(x,a,b)$.
[2] Consider that $\sum_{i=1}^{0}x_i=0$, by definition.
[3] Clearly, $W^*(x)=W(x,1,\frac{1}{2})$.

2 Background

Using the definition (1.2) and the equality $\Gamma(y+1) = y\Gamma(y)$, valid for $y \in \mathbb{R}^+$, by induction we note the identity

$$W^*(x) = \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}}\, W^*(x+p), \tag{2.1}$$

valid for an integer $p \geq 0$ and real $x > -\frac{1}{2}$, where $y^{(p)}$ denotes the Pochhammer rising (upper) factorial, defined as

$$y^{(0)} := 1, \quad y^{(p)} := \prod_{i=0}^{p-1}(y+i) = y(y+1)\cdots(y+p-1) \qquad (\text{for } p \geq 1).$$

Using the duplication formula [1, 6.1.18], we have, for $x > 0$,

$$2x\Gamma(2x) = 2x \cdot (2\pi)^{-1/2}\, 2^{2x-1/2}\, \Gamma(x)\Gamma(x+\tfrac{1}{2}) = \pi^{-1/2} 2^{2x}\Gamma(x+1)\Gamma(x+\tfrac{1}{2}).$$

Hence, using (1.2), we obtain, for $x > 0$,

$$W^*(x) = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} = 2^{-2x}\frac{2x\,\Gamma(2x)}{\big(\Gamma(x+1)\big)^2} = 2^{-2x}\frac{2x\,\Gamma(2x)}{\big(x\,\Gamma(x)\big)^2}\,. \tag{2.2}$$

The continuous version of Stirling's factorial formula of order $r \geq 0$, for $x \in \mathbb{R}^+$, can be given in the following way [8, Sect. 9.5]

$$x\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \cdot \exp\big(s_r(x) + d_r(x)\big)\,, \tag{2.3}$$

where

$$s_0(x) \equiv 0 \quad \text{and} \quad s_r(x) = \sum_{i=1}^{r} \frac{c_i}{x^{2i-1}} \qquad \text{for } r \geq 1, \tag{2.4}$$

$$c_i = \frac{B_{2i}}{2i(2i-1)} \qquad \text{for } i \geq 1, \tag{2.5}$$

and, for some $\vartheta_r(x) \in (0,1)$,

$$d_r(x) = \vartheta_r(x) \cdot \frac{c_{r+1}}{x^{2r+1}}\,. \tag{2.6}$$

Here $B_2, B_4, B_6, \ldots$ are the Bernoulli coefficients, alternating in sign as

$$B_{2i} = (-1)^{i+1}\big|B_{2i}\big| \qquad \text{for } i \geq 1, \tag{2.7}$$

thanks to [1, 23.1.15, p. 805]. For example, using Mathematica [28],

$$B_2 = \frac{1}{6}, \quad B_4 = B_8 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2\,730}, \quad B_{14} = \frac{7}{6},$$

with the estimates $|B_{12}| < \frac{1}{3}$, $|B_{16}| < 7$, $B_{18} < 55$, $|B_{20}| < 530$, $B_{22} < 6200$.

3 Result

According to (2.2) and (2.3), we calculate, for $x > 0$,

$$\begin{aligned} W^*(x) &= 2^{-2x} \frac{2x\,\Gamma(2x)}{x\,\big(\Gamma(x)\big)^2} \\ &= 2^{-2x} \left(\frac{2x}{e}\right)^{2x} \sqrt{2\pi \cdot 2x} \cdot \exp\big(s_r(2x) + d_r(2x)\big) \cdot \left[\left(\frac{e}{x}\right)^x \frac{1}{\sqrt{2\pi x}} \cdot \exp\big(-s_r(x) - d_r(x)\big)\right]^2 \\ &= \frac{1}{\sqrt{\pi\, x}} \exp\Big(\underbrace{s_r(2x) - 2s_r(x)} + \underbrace{d_r(2x) - 2d_r(x)}\Big). \end{aligned} \tag{3.1}$$

Referring to (3.1) and (2.3)–(2.6), we derive the following lemma.

Lemma 3.1. *For any $r \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}^+$ we have*[4]

$$W^*(x) = \frac{1}{\sqrt{\pi\, x}} \cdot \exp\left(-\sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)x^{2i-1}}\right) \cdot \exp\big(\delta_r(x)\big), \tag{3.2}$$

where

$$\big|\delta_r(x)\big| < \frac{\big|B_{2r+2}\big|}{(r+1)(2r+1)x^{2r+1}}\,. \tag{3.3}$$

Proof. According to (2.4)–(2.5), we have

$$s_r(2x) - 2s_r(x) = \sum_{i=1}^{r} \frac{c_i}{(2x)^{2i-1}} - 2\sum_{i=1}^{r} \frac{c_i}{x^{2i-1}} = -\sum_{i=1}^{r} \frac{c_i}{x^{2i-1}}\big(2 - 2\cdot 4^{-i}\big) = -\sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)x^{2i-1}}.$$

Similarly, referring to (2.5)–(2.6), we have the error

$$\delta_r(x) := d_r(2x) - 2d_r(x) = \vartheta_r^*(x) \cdot \frac{c_{r+1}}{(2x)^{2r+1}} - 2\vartheta_r(x) \cdot \frac{c_{r+1}}{x^{2r+1}} = \frac{c_{r+1}}{x^{2r+1}} \left(\frac{\vartheta_r^*(x)}{2^{2r+1}} - 2\vartheta_r(x)\right),$$

for some $\vartheta_r(x), \vartheta_r^*(x) \in (0,1)$. Thus, using (2.5), we get, for $x > 0$,

$$\big|\delta_r(x)\big| < \frac{\big|B_{2r+2}\big|}{(2r+2)(2r+1)x^{2r+1}} \cdot 2\,. \qquad \square$$

Remark 3.2. *The formula for $W^*(x)$, given in (3.2)–(3.3), is more compact, but slightly less accurate, than the formula, given in (1.9)–(1.10), where $x = 0$ is a regular point as opposed to (3.2)–(3.3), where this point is seemingly singular.*

[4] Consider that $\sum_{i=1}^{0} x_i = 0$, by definition.

Thanks to (3.3), the absolute value of $\delta_r(x)$ is small for large x and any $r \geq 0$. But, for small $x > 0$, the formula in Lemma 3.1 becomes useless. This problem can be avoided by replacing x in Lemma 3.1 by $x+p$, for p large, $p \in \mathbb{N}$. In fact, using (2.1) and replacing x by $x' = x+p$ in Lemma 3.1, immediately follows the next theorem, with $\delta^*_{p,r}(x) = \delta_r(x+p, a, b)$.

Theorem 3.3. *For integers $p, r \geq 1$ and for $x > -\frac{1}{2}$, the ratio $W^*(x)$ can be expressed in the form*

$$W^*(x) = W^*_{p,r}(x) \cdot \exp\big(\delta^*_{p,r}(x)\big), \tag{3.4}$$

where

$$W^*_{p,r}(x) := \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}} \exp\left(-\sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)(x+p)^{2i-1}}\right) \tag{3.5}$$

and

$$\left|\delta^*_{p,r}(x)\right| < \frac{\left|B_{2r+2}\right|}{(r+1)(2r+1)(x+p)^{2r+1}}\,. \tag{3.6}$$

Here, p and r are parameters that affect the magnitude of the error term $\delta^*_{p,r}(x)$.

Example 3.4. *Setting $p = 3$ and $r = 5$ in Theorem 3.3, we obtain*

$$\begin{aligned} W^*(x) := &\frac{(x+1)(x+2)}{(x+\frac{1}{2})(x+\frac{3}{2})(x+\frac{5}{2})} \cdot \sqrt{\frac{x+3}{\pi}} \cdot \exp\left(-\frac{1}{8(x+3)} + \frac{1}{192(x+3)^3}\right. \\ &\left. - \frac{1}{640(x+3)^5} + \frac{17}{14\,336(x+3)^7} - \frac{31}{18\,432(x+3)^9}\right) \cdot \exp\big(\delta^*_{3,5}(x)\big), \end{aligned}$$

where $\left|\delta^*_{3,5}(x)\right| < \frac{1}{260(x+3)^{11}}$, for all $x > -\frac{1}{2}$. Consequently, $\left|\delta^*_{3,5}(x)\right| < 2 \cdot 10^{-7}$ for $x \in (-\frac{1}{2}, 0]$, $\left|\delta^*_{3,5}(x)\right| < 3 \cdot 10^{-8}$, for $x \in [0,1]$, and $\left|\delta^*_{3,5}(x)\right| < 10^{-9}$, for $x \geq 1$.

A direct, immediate consequence of Theorem 3.3 is the sequence of asymptotic expansions given in the following corollary.

Corollary 3.5. *For any integer $p \geq 1$ we have the asymptotic expansion*

$$\ln\big(W^*(x)\big) \sim \ln \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}} - \frac{1}{2}\ln\big(\pi(x+p)\big) - \sum_{i=1}^{\infty} \frac{(1-4^{-i})B_{2i}}{i(2i-1)(x+p)^{2i-1}}\,,$$

as $x \to \infty$.

Theorem 3.6. *For an integer $p \geq 2$ and real $x > -\frac{1}{2}$ there holds the following double asymptotic inequality*

$$A(p,x) \;<\; W^*(x) \;<\; B(p,x), \tag{3.7}$$

where

$$A(p,x) := W_p^*(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{379(x+p)^3}\right), \tag{3.8}$$

$$B(p,x) := W_p^*(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{191(x+p)^3}\right), \tag{3.9}$$

$$W_p^*(x) := W_{p,0}^*(x) = \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}}\,. \tag{3.10}$$

Proof. We use Theorem 3.3 with $r=2$, when $\left|\delta_{p,2}^*(x)\right| < \frac{1}{630(x+p)^5}$ and thus we estimate

$$y_-(p,x) < -\sum_{i=1}^{2} \frac{(1-4^{-i})B_{2i}}{i(2i-1)(x+p)^{2i-1}} + \delta_{p,2}^*(x) < y_+(p,x) < 0, \tag{3.11}$$

for $p \in \mathbb{N}$ and $x \in \mathbb{R}^+$, where

$$y_-(p,x) := -\frac{1}{8(x+p)} + \frac{1}{192(x+p)^3} - \frac{1}{630(x+p)^5}\,, \tag{3.12}$$

$$y_+(p,x) := -\frac{1}{8(x+p)} + \frac{1}{192(x+p)^3} + \frac{1}{630(x+p)^5}\,. \tag{3.13}$$

Furthermore, by Taylor's formula of orders 3 and 2 we have, for $y < 0$,

$$1 + y + \frac{y^2}{2} + \frac{y^3}{6} < e^y < 1 + y + \frac{y^2}{2}\,.$$

Thus, referring to (3.11)–(3.13), we have, for $p \in \mathbb{N}$,

$$\exp\big(y_-(p,x)\big) > 1 + y_-(p,x) + \frac{1}{2}y_-^2(p,x) + \frac{1}{6}y_-(p,x))^3, \tag{3.14}$$

$$\exp\big(y_+(p,x)\big) < 1 + y_+(p,x) + \frac{1}{2}y_+^2(p,x). \tag{3.15}$$

Now, due to (3.12), we estimate, for $x > -\frac{1}{2}$ and $x + p > -\frac{1}{2} + 2 > 1$, as follows:

$$\begin{aligned}
&1 + y_-(p,x) + \frac{1}{2}y_-^2(p,x) + \frac{1}{6}y_-(p,x))^3 \\
&= 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{5}{1\,024(x+p)^3} - \tfrac{1}{1\,536(x+p)^4} + \cancel{\tfrac{1}{24\,576(x+p)^5}} \\
&+ \cancel{\tfrac{1}{73\,728(x+p)^6}} - \tfrac{1}{589\,824(x+p)^7} + \cancel{\tfrac{1}{42\,467\,328(x+p)^9}} - \tfrac{1}{630(x+p)^5} \\
&> 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{5}{1\,024(x+p)^3} - \tfrac{1}{1\,536(x+p)^3} - \tfrac{1}{589\,824(x+p)^3} \\
&- \tfrac{1}{630(x+p)^3} > 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{1}{379(x+p)^3}\,,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&1 + y_+(p,x) + \frac{1}{2}y_+^2(p,x) \\
&= 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{1}{192(x+p)^3} - \cancel{\tfrac{1}{1\,536(x+p)^4}} + \tfrac{1}{73\,728(x+p)^6} \\
&< 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{1}{191(x+p)^3}\,. \qquad (3.17)
\end{aligned}$$

Using Theorem 3.3, (3.11), (3.14)–(3.15) and (3.16)–(3.17) we note the double inequality (3.7). $\square$

Example 3.7. *We have* $A(2, -\frac{49}{100}) = 32.25\ldots < W(-\frac{49}{100}) = 32.27\ldots < B(2, -\frac{49}{100}) = 32.28\ldots$ *However,* $A(1, -\frac{49}{100}) = 32.42\ldots > W(-\frac{49}{100}) = 32.27\ldots$

Example 3.8. *We have* $B(2, \frac{49}{100}) - A(2, \frac{49}{100}) < 3\cdot 10^{-2}$, $B(2,0) - A(2,0) < 4\cdot 10^{-4}$ *and* $B(2,\pi) - A(2,\pi) < 6\cdot 10^{-6}$.

Example 3.9. *We have exactly* $W(3) = w_3 = \frac{5}{16} = 0.3125$ *and, thanks to Theorem 3.6, we estimate* $0.312\,499\,6 < A(9,3) < \underline{W(3)} < B(9,3) < 0.312\,500\,1$.

Figure 1 illustrates the estimate (3.7) *by plotting*[5] *the graphs of the functions* $x \mapsto A(2,x)$, $x \mapsto W(x)$ *and* $x \mapsto B(2,x)$, *where all graphs practically coincide.*

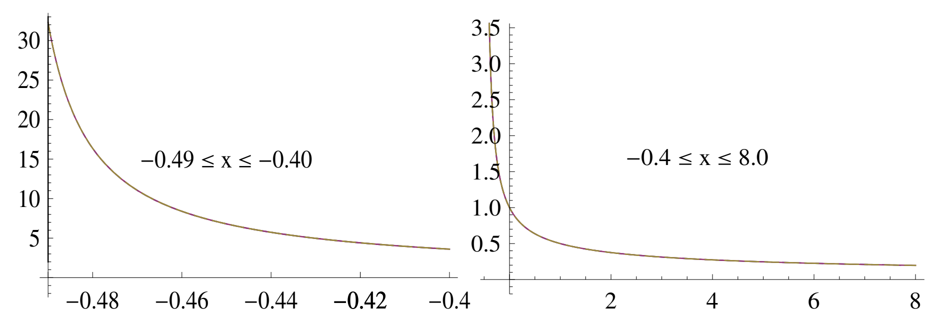

Figure 1: The graphs of the functions $x \mapsto A(2,x)$, $x \mapsto W(x)$ and $x \mapsto B(2,x)$.

Corollary 3.10. *For an integer* $p \geq 2$ *and* $x > -\frac{1}{2}$ *the approximation* $W^*(x) \approx A(p,x)$ *has the relative error*

$$\rho(p,x) := \frac{W^*(x) - A(p,x)}{W^*(x)}$$

estimated as

$$0 < \rho(p,x) < \frac{B(p,x) - A(p,x)}{A(p,x)} < \frac{1}{330(x+p)^3}\,.$$

Proof. Thanks to Theorem 3.6 we have

$$0 < \rho(p,x) < \frac{B(p,x) - A(p,x)}{A(p,x)} = \frac{B(p,x)}{A(p,x)} - 1 = \frac{S+\Delta_2}{S+\Delta_1} - 1,$$

[5]All figures and more demanding computations made in this paper were produced using Mathematica [28].

where

$$S = 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} \tag{3.18}$$

and

$$\Delta_1 = \frac{1}{379}(x+p)^{-3}, \quad \Delta_2 = \frac{1}{191}(x+p)^{-3}. \tag{3.19}$$

Thus,

$$0 < \rho(p,x) < \left(1 + \frac{\Delta_2 - \Delta_1}{S + \Delta_1}\right) - 1 < \frac{\Delta_2 - \Delta_1}{S},$$

where the assumptions $x > -\frac{1}{2}$ and $p \geq 2$ imply the estimate $x + p > 1$, which, due to (3.18), implies the inequalities

$$S \geq 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)} > 1 - \frac{1}{7(x+p)} \geq \frac{6}{7}\,.$$

Consequently, thanks to (3.19),

$$\frac{\Delta_2 - \Delta_1}{S} < \frac{7}{6}\left(\frac{1}{191} - \frac{1}{379}\right)\frac{1}{(x+p)^3} < \frac{1}{330(x+p)^3}\,. \qquad \square$$

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