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On a class of fractional $\Gamma(.)$-Kirchhoff-Schrödinger system type

Hamza El-Houari[1,✉]
Lalla Saádia Chadli[1]
Hicham Moussa[1]

[1] *Faculty of Sciences and Techniques, Research Laboratory "Applied Mathematics and Scientific Computing", Beni Mellal, Morocco.*
h.elhouari94@gmail.com
sa.chadli@yahoo.fr
hichammoussa23@gmail.com

ABSTRACT

This paper focuses on the investigation of a Kirchhoff-Schrödinger type elliptic system involving a fractional $\gamma(.)$-Laplacian operator. The primary objective is to establish the existence of weak solutions for this system within the framework of fractional Orlicz-Sobolev Spaces. To achieve this, we employ the critical point approach and direct variational principle, which allow us to demonstrate the existence of such solutions. The utilization of fractional Orlicz-Sobolev spaces is essential for handling the nonlinearity of the problem, making it a powerful tool for the analysis. The results presented herein contribute to a deeper understanding of the behavior of this type of elliptic system and provide a foundation for further research in related areas.

RESUMEN

Este artículo se enfoca en la investigación de sistemas elípticos de tipo Kirchhoff-Schrödinger que involucran un operador fraccionario $\gamma(.)$-Laplaciano. El objetivo principal es establecer la existencia de soluciones débiles para este sistema en el marco de espacios de Orlicz-Sobolev fraccionarios. Para lograrlo, empleamos el enfoque de punto crítico y el principio variacional directo, que nos permiten demostrar la existencia de dichas soluciones. El uso de espacios de Orlicz-Sobolev fraccionarios es esencial para lidiar con la nolinealidad del problema, convirtiéndolo en una herramienta poderosa para el análisis. Los resultados presentados contribuyen a una comprensión más profunda del comportamiento de este tipo de sistemas elípticos y entregan una base para investigación futura en áreas relacionadas.

Keywords and Phrases: Fractional Orlicz-Sobolev spaces, Kirchhoff-Schrödinger system, Critical point theorem.

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1 Introduction

The objective of this paper is to establish the existence of weak solutions for a non-local elliptic systems, as described below:

$$\begin{cases} K_1\big[\mathcal{F}_1(u)+\Upsilon_1(u)\big]\Big((-\Delta)^s_{\gamma_1}u+a_1(x)\gamma_1(u)u\Big)=F_u(x,u,v) & \text{in } \Omega,\\ K_2\big[\mathcal{F}_2(v)+\Upsilon_2(v)\big]\Big((-\Delta)^s_{\gamma_2}v+a_2(x)\gamma_2(v)v\Big)=F_v(x,u,v) & \text{in } \Omega,\\ u=v=0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $N\geq 2$, $s\in(0,1)$, $\mathcal{F}_{i=1,2},\Upsilon_{i=1,2}:E_i\to\mathbb{R}$ are two functionals, respectively defined by

$$\mathcal{F}_i(w)=\int_{\Omega^2}\Gamma_i\Big(\frac{|w(x)-w(y)|}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^N},\quad \Upsilon_i(w)=\int_\Omega a_i(x)\Gamma_i(|w|)dx,$$

and $K_{i=1,2}$ are two bounded continuous Kirchhoff functions, F belongs to $C^1(\Omega\times\mathbb{R}^2)$ and satisfies certain suitable growth assumptions, and F_u (respectively, F_v) is the partial derivative of F with respect to u (respectively, v). Additionally, a_i with $i=1,2$, are two continuous functions that satisfy the following conditions:

(A_1): $a_i\in C(\Omega,\mathbb{R})$ and $\inf\limits_{x\in\Omega}a_i(x)\geq a_0>0$.

(A_2): $meas(x\in\Omega:a_i(x)\leq H)<\infty$, for all $H>0$, where $meas(.)$ denotes the Lebesgue measure in Ω.

The stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2\Big)\frac{\partial^2u}{\partial x^2}=0, \tag{1.2}$$

presented by Kirchhoff [19] in 1883. Later (1.2) was developed to form

$$u_{tt}-K\Big(\int_\Omega|\nabla u|^2dx\Big)\Delta u=f(x,u),\quad x\in\Omega. \tag{1.3}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-K\Big(\int_\Omega|\nabla u|^2dx\Big)\Delta u=f(x,u),\quad x\in\Omega. \tag{1.4}$$

In recent years, considerable research attention has been dedicated to investigating the existence of solutions for elliptic problems within the fractional Sobolev space. This growing interest is evident in the works of various researchers, such as those referenced in [8,9,18,24]. In a similar vein, Azroul *et al.* explored the existence of a solution for the following fractional (p,q)-Schrödinger-Kirchhoff system type, as documented in [2].

$$\begin{cases} K_1\big[I_{M_p}\big]((-\Delta)_p^s u + a(x)|u|^{p-2}u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \mathbb{R}^N, \\ K_2\big[I_{M_q}\big]((-\Delta)_q^s v + a(x)|v|^{q-2}v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \mathbb{R}^N, \\ (u,v) \in W^p \times W^q, \end{cases} \tag{1.5}$$

where

$$I_{M_r}(w) = \int_{\mathbb{R}^N\times\mathbb{R}^N} |w(x)-w(y)|^r M_r(x-y)dx\,dy + \int_{\mathbb{R}^N} a(x)|w|^r dx,$$

when we take $M_r(x) = |x|^{-N-sr}$. In this case, problem (1.5) become

$$\begin{cases} K_1\big[I_r^s(u)\big]((-\Delta)_p^s u + |u|^{p-2}u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \mathbb{R}^N, \\ K_2\big[I_r^s(v)\big]((-\Delta)_q^s v + |v|^{q-2}v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u = v = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.6}$$

where

$$I_r^s(w) = \int_{\Omega^2} \frac{|w(x)-w(y)|^r}{|x-y|^{sr+N}}dx\,dy + \int_{\mathbb{R}^N} a(x)|w|^r dx.$$

In 2017, Bonder *et al.* in [17] made a significant advancement by introducing an extension of the fractional Sobolev space, known as the fractional Orlicz-Sobolev space. This extension involved the generalization of the conventional fractional Laplacian operator to the fractional $\gamma(.)$-Laplacian operator, which is defined as follows:

$$(-\Delta)^s_{\gamma(.)}u(x) = p.v. \int_{\mathbb{R}^N} \gamma\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^{s+N}}dy, \quad \text{for all} \quad x \in \mathbb{R}^N, \tag{1.7}$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and right continuous function, with

$$\gamma(0) = 0, \quad \gamma(t) > 0 \quad \text{for } t > 0 \quad \text{and} \quad \lim_{t\to\infty} \gamma(t) = \infty. \tag{1.8}$$

The replacement of the $\gamma(.)$-Laplace operator with a fractional $\gamma(.)$-Laplacian operator raises the question of what results can be achieved. Currently, there are only a few results available regarding the fractional Orlicz-Sobolev spaces. For instance, in [9], we studied a nonlocal Kirchhoff type problem within this space.

$$\begin{cases} K_1\big[\mathcal{F}_1(u)\big](-\Delta)^s_{\gamma_1} u = F_u(x,u,v) & \text{in } \Omega, \\ K_2\big[\mathcal{F}_2(v)\big](-\Delta)^s_{\gamma_2} v = F_u(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where K_i is the Kirchhoff function. In our problem (1.1), the function F is presumed to be a member of $C^1(\Omega \times \mathbb{R}^2)$ and complies with appropriate growth conditions, but notably does not satisfy the well-known Ambrosetti-Rabinowitz condition. For further problems related to the fractional Orlicz-Sobolev spaces, we refer to [6, 7, 10–16]. By setting $\Gamma_i(t) = \frac{|t|^r}{r}$, our problem (1.1) can be reduced to the fractional (p,q)-Schrödinger-Kirchhoff elliptic system given in (1.6). In this paper, preceding works of the Kirchhoff-Schrödinger system are extended in fractional Orlicz-Sobolev spaces.

This article is divided into four sections. In the second section, we offer a brief review of the fractional Orlicz-Sobolev spaces, outlining their essential properties and results. Following that, the third section presents the specific assumptions made on the data. In the fourth section, we present our primary result concerning the existence of a weak solution and its proof, which relies on a contradiction argument.

2 Some preliminary results and hypotheses

In this section, we will briefly introduce the definitions and fundamental properties of FOSS. For detailed information and proofs, interested readers can refer to [1, 17, 20].

We take notice of $\mathbf{N}$ the set of all N-functions. The function $\Gamma \in \mathbf{N}$ is defined for $z \in \mathbb{R}$ by setting $\Gamma(z) = \displaystyle\int_0^{|z|} t\gamma(t)dt$.

We point out that $\Gamma \in \Delta_2$ if for a certain constant $k > 0$,

$$\Gamma(2z) \leq k\,\Gamma(z), \quad \text{for every } z > 0. \tag{2.1}$$

We observe that Γ and $\overline{\Gamma}$ satisfy the following Young's inequality:

$$rz \leq \Gamma(r) + \overline{\Gamma}(z) \quad \text{for all } z,\, r \geq 0 \text{ and } x \in \Omega. \tag{2.2}$$

In the Orlicz space $L_\Gamma(\Omega)$ is well-know, the Hölder inequality

$$\int_\Omega |u(z)v(z)|\,\mathrm{d}z \leq ||u||_\Gamma ||v||_{(\overline{\Gamma})} \quad \text{for all } u \in L_\Gamma(\Omega) \text{ and } v \in L_{\overline{\Gamma}}(\Omega), \tag{2.3}$$

where $L_\Gamma(\Omega)$ is defined as the set of equivalence classes of measurable functions $u : \Omega \to \mathbb{R}$ such that:

$$\int_\Omega \Gamma\Big(\frac{u(z)}{\tau}\Big)\,\mathrm{d}z < +\infty \quad \text{for certain } \tau > 0.$$

where $||.||_{(\Gamma)}$ is the Orlicz norm defined by

$$||u||_{(\Gamma)} := \sup_{||v||_{\overline{\Gamma}\leq 1}} \int_\Omega u(z)v(z)\,\mathrm{d}z.$$

$L_\Gamma(\Omega)$ is a Banach space under the following norm,

$$\|u\|_\Gamma = \inf\Big\{\lambda > 0 \,/\, \int_\Omega \Gamma\Big(\frac{|u(x)|}{\lambda}\Big)\,\mathrm{d}x \leq 1\Big\}.$$

We assume that:

$$(A_0)\ \int_0^1 \frac{\Gamma^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t < \infty \quad \text{and} \quad (A_\infty)\int_1^{+\infty} \frac{\Gamma^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t = +\infty \quad \text{for } s \in (0,1).$$

Under the hypotheses (A_0) and (A_∞), we can insert an N-function Γ^*, given by the following expression of its inverse in $\mathbb{R}^+$:

$$(\Gamma^*)^{-1}(t) = \int_0^t \frac{\Gamma^{-1}(r)}{r^{\frac{N+s}{N}}}dr \quad \text{for } t \geq 0. \tag{2.4}$$

The fact that $\Gamma \in \Delta_2$-condition globally implies that:

$$u_k \to u \quad \text{in} \quad L_\Gamma(\Omega) \iff \int_\Omega \Gamma(|u_k - u|)dx \to 0. \tag{2.5}$$

Now we set an useful lemma which we need in the proof.

Lemma 2.1 ([4])**.** *Let $\overline{\Gamma}$ be the complementary of the N-functions Γ. Then we have*

$$\overline{\Gamma}(\gamma(t)) \leq (n-1)\Gamma(t), \quad \textit{for all } t > 0, \tag{2.6}$$

where $n = \sup_{t>0} \frac{t^2\gamma(t)}{\Gamma(t)}$.

We define the fractional Orlicz-Sobolev spaces as follows

$$W^{s,\Gamma}(\Omega) = \Big\{u \in L_\Gamma(\Omega) : \int_\Omega\int_\Omega \Gamma\Big(\frac{\lambda|u(x)-u(y)|}{|x-y|^s}\Big)|x-y|^{-N}dx\,dy < \infty \quad \text{for some } \lambda > 0\Big\}.$$

This space is equipped with the norm,

$$||u||_{s,\Gamma} = ||u||_\Gamma + [u]_{s,\Gamma}, \tag{2.7}$$

where $[.]_{s,\Gamma}$ defined by

$$[u]_{s,\Gamma} = \inf\Big\{\lambda > 0 : \int_\Omega\int_\Omega \Gamma\Big(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\Big)|x-y|^{-N}dx\,dy \leq 1\Big\}.$$

To deal with this problem, we choose

$$W_0^{s,\Gamma}(\Omega) = \big\{u \in W^{s,\Gamma}(\mathbb{R}^N) : u = 0 \quad \text{a.e.} \quad \mathbb{R}^N \setminus \Omega\big\},$$

which can be equivalently renormed by setting $||.|| = [.]_{s,\Gamma}$ and

$$E_i = \Big\{u \in W^{s,\Gamma_i}(\mathbb{R}^N) : \int_\Omega a_i(x)\Gamma_i(|u|)dx < \infty; \quad u = 0 \quad \text{a.e.} \quad \mathbb{R}^N \setminus \Omega\Big\},$$

equipped with the following norm $||.||_{E_i,\Gamma_i} = [.]_{s,\Gamma_i} + ||.||_{a_i,\Gamma_i}$, where

$$||u||_{a_i,\Gamma_i} = \inf\Big\{\lambda > 0, \int_\Omega a_i(x)\Gamma_i\Big(\frac{|u(x)|}{\lambda}\Big)dx \leq 1\Big\}.$$

Throughout this paper Ω is a bounded open subset of $\mathbb{R}^N$ and $s \in (0,1)$.

In $W_0^{s,\Gamma}(\Omega)$ we have the following Poincaré inequality

$$||u||_\Gamma \leq \tau[u]_{s,\Gamma}, \quad \forall u \in W_0^{s,\Gamma}(\Omega). \tag{2.8}$$

where τ is a positive constant.

Remark 2.2. $[.]_{s,\Gamma}$ *is a norm of* $W_0^{s,\Gamma}(\Omega)$ *equivalent to* $||.||_{s,\Gamma}$.

Lemma 2.3 ([7])**.** *The representation given by*

$$\Gamma_{i=1;2}(t) := \int_0^{|t|} r\gamma_i(r)dr \quad \textit{for all } t \in \mathbb{R}, \tag{2.9}$$

exists and it is an N*-function where* $\gamma_{i=1;2}$ *verified* (1.8).

3 Hypotheses

We use through our paper that $\Gamma_i \in \mathbf{N}$ defined in (2.9) and we suppose that $\Gamma_i \in \Delta_2$. Then by lemma 2.1 in [23] we have for all $t > 0$ that

$$1 < l_i := \inf_{t>0} \frac{t^2\gamma_i(t)}{\Gamma_i(t)} \leq \sup_{t>0} \frac{t^2\gamma_i(t)}{\Gamma_i(t)} := n_i < N. \tag{3.1}$$

Related to functions Γ_i, K_i and F our hypotheses are the following:

(ϕ_1)**:** The function $t \to \Gamma_i(\sqrt{t})$ where $t \in [0, +\infty)$ is convex.

(ϕ_2)**:** There exists $1 < \eta_i < l_i$, such that

$$\lim_{t\to+\infty} \frac{|t|^{\eta_i}}{\Gamma_i(t)} = 0,$$

and the Kirchhoff function $K_i : [0,\infty) \to (0,\infty)$ is a nondecreasing continuous function such that:

(A_3): There exist $\alpha_1, \alpha_2 > 0$ such that:

$$\alpha_2 \geq K_i(t) \geq \alpha_1 \quad \text{for all } t \in [0, \infty).$$

And F satisfies:

(F_1): $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $F(x,0,0) = 0$ for all $x \in \Omega$

$$\begin{cases} |F_u(x,u,v)| \leq c_1 |u|^{r_1 - 1} + c_2 |v|^{\frac{r_2(r_1-1)}{r_1}}, \\ |F_v(x,u,v)| \leq c_1 |u|^{\frac{r_1(r_2-1)}{r_2}} + c_2 |v|^{r_2 - 1}, \end{cases} \tag{3.2}$$

where $r_i \in (1, l_i)$.

(F_2): There exist an open set $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$, and positive constants $\alpha_0 \in [1, l_1)$, $\beta_0 \in [1, l_2)$, $c > 0$ and $\rho, \sigma \in \mathbb{R}$ with $\rho + \sigma \neq 0$ such that

$$F(x, \rho t, \sigma t) \geq c(|\rho t|^{\alpha_0} + |\sigma t|^{\beta_0}), \quad \text{for all } (x,t) \in \Omega \times [0,1).$$

Remark 3.1 ([17, Proposition 2.11])**.** $W_0^{s,\Gamma}(\Omega)$ *is a separable and reflexive Banach space.*

Lemma 3.2 ([5, Lemma 4.3])**.** *The following properties hold true:*

1) $\mathcal{F}_i\left(\frac{u}{[u]_{s,\Gamma_i}}\right) \leq 1$, *for all* $u \in E_i \setminus \{0\}$.

2) $\zeta_0([u]_{s,\Gamma_i}) \leq \mathcal{F}_i(u) \leq \zeta_1([u]_{s,\Gamma_i})$, *for all* $u \in E_i$.

3) $\zeta_0(||u||_{a_i,\Gamma_i}) \leq \Upsilon_i(u) \leq \zeta_1(||u||_{a_i,\Gamma_i})$, *for all* $u \in E_i$.

Lemma 3.3 ([5, Lemma 4.7])**.** $\mathcal{F}_i$ *and* Υ_i *are two weak lower semi-continuous functions.*

Lemma 3.4 ([7, Lemma 3.3])**.** *Under assumption* (ϕ_1) *we have that* $(E, ||.||_{E_i})$ *is a real uniformly convex Banach space.*

Now we state the embedding compactness result.

Theorem 3.5 ([5, Theorem 1.2.])**.** *Let* Γ *be an N-function.*

i) If (A_0), (A_∞) *and* (3.1) *hold, then the embedding* $W^{s,\Gamma_i}(\Omega) \hookrightarrow L_{\Gamma_i^*}(\Omega)$ *is continuous, and the embedding* $W^{s,\Gamma_i}(\Omega) \hookrightarrow L_\Phi(\Omega)$ *is compact for any* N*-function* $\Phi \ll \Gamma_i$.

ii) If (A_1), (A_2) *and* (3.1) *hold, then the embedding* $E_i \hookrightarrow L_{\Gamma_i}(\Omega)$ *is continuous, and the embedding* $E \hookrightarrow L_\Phi(\Omega)$ *is compact for any N-function* $\Phi \ll \Gamma_i$.

Remark 3.6. *The assumption* (ϕ_2) *implies that* $|t|^{\eta_i} \ll \Gamma_i$*, then by Theorem 3.5 the following embeddings* $E_i \to L^{\eta_i}(\Omega)$ *are compact, i.e., there exist constants* $C_{\eta_i} > 0$ *such that*

$$||u||_{\eta_i} \leq C_{\eta_i} ||u||_{E_i,\Gamma_i} \quad \textit{for all } u \in E_i. \tag{3.3}$$

4 Main results

In this section, we present the existence result.

Theorem 4.1. *Assume that* (A_1)-(A_3), (F_1)-(F_3), (3.1) *and* (ϕ_2) *hold true. Then system* (1.1) *possesses a nontrivial weak solution.*

In order to prove Theorem 4.1, we will use the following Lemma:

Lemma 4.2 ([22])**.** *Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$ satisfies (PS)-condition. If J is bounded from below, then $c = \inf_X J$ is a critical value of J.*

In fact, since

$$F(x,u,v) = \int_0^u F_p(x,p,v)dp + \int_0^v F_t(x,0,t)dt + F(x,0,0), \quad \forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}.$$

By (3.2) and the fact that $F(x,0,0) = 0$, we show that:

$$\begin{aligned} |F(x,u,v)| \leq & \int_0^{|u|} |F_p(x,p,v)|dp + \int_0^{|v|} |F_t(x,0,t)|dt \\ \leq & \; c_1 \int_0^{|u|} |p|^{r_1-1} dp + c_2 \int_0^{|u|} |v|^{\frac{r_2(r_1-1)}{r_1}} dp + c_2 \int_0^{|v|} |t|^{r_2-1} dt \\ = & \; \frac{c_1}{r_1}|u|^{r_1} + c_2|u||v|^{\frac{r_2(r_1-1)}{r_1}} + \frac{c_2}{r_2}|v|^{r_2} \\ \leq & \; \frac{c_1}{r_1}|u|^{r_1} + \frac{c_2}{r_1}|u|^{r_1} + c_2\frac{r_1-1}{r_1}|v|^{r_2} + \frac{c_2}{r_2}|v|^{r_2} \\ = & \; c_3|u|^{r_1} + c_4|v|^{r_2}, \end{aligned} \tag{4.1}$$

where $c_3 = \dfrac{c_1+c_2}{r_1}$ and $c_4 = \dfrac{c_2 r_2(r_1-1)+r_1}{r_1 r_2}$.

Now we have all tools to study our problem (1.1). For that we shall define our working space $W := E_1 \times E_2$ with the norm

$$||(u,v)|| := ||u||_{E_1,\Gamma_1} + ||v||_{E_2,\Gamma_2}.$$

We can show that W is a separable and reflexive Banach space. We observe that the energy functional I on W corresponding to system (1.1) is

$$I(u,v) := \tilde{K}_1\Big[\mathcal{F}_1(u) + \Upsilon_1(u)\Big] + \tilde{K}_2\Big[\mathcal{F}_2(v) + \Upsilon_2(v)\Big] - \int_\Omega F(x,u,v)dx, \quad \forall (u,v) \in W.$$

Where

$$\tilde{K}(t) := \int_0^t K(\tau)d\tau.$$

Denote by $I_i : W \to \mathbb{R}$, $i = 1, 2$, the functionals $I_1(u, v) = (\tilde{K}o\mathcal{H})_1(u) + (\tilde{K}o\mathcal{H})_2(v)$ where

$$(\tilde{K}o\mathcal{H})_i(w) := \tilde{K}_i\Big[\int_{\Omega\times\Omega}\Gamma_i\Big(\frac{w(x)-w(y)}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^N} + \int_\Omega a_i(x)\Gamma_i(w)dx\Big]$$

and

$$I_2(u, v) = \int_\Omega F(x, u, v)dx.$$

Then

$$I(u, v) = I_1(u, v) - I_2(u, v).$$

Lemma 4.3. *The function I is well define and it is $C^1(E_i, \mathbb{R})$ and we have*

$$\begin{aligned}\langle I'(u,v),(\overline{u},\overline{v})\rangle &= K_1\Big[\mathcal{F}_1(u)+\Upsilon_1(u)\Big]\Big(\int_{\Omega\times\Omega}\gamma_1(h_u)h_u h_{\overline{u}}\,d\mu + \int_\Omega a_1(x)\gamma_1(u)u\overline{u}\,dx\Big)\\ &+ K_2\Big[\mathcal{F}_2(v)+\Upsilon_2(v)\Big]\Big(\int_{\Omega\times\Omega}\gamma_2(h_v)h_v h_{\overline{v}}\,d\mu + \int_\Omega a_2(x)\gamma_2(v)v\overline{v}\,dx\Big)\\ &- \int_\Omega (F_u(x,u,v)\overline{u} + F_v(x,u,v)\overline{v})dx,\end{aligned}$$

for all $\overline{u}, \overline{v} \in E_i$, where $h_u = \dfrac{u(x)-u(y)}{|x-y|^s}$ and $d\mu = \dfrac{dx\,dy}{|x-y|^N}$ (i.e, regular Borel measure on the set $\Omega\times\Omega$).

Lemma 4.4. *The function I is well define and it is $C^1(E_i, \mathbb{R})$ and we have*

$$\begin{aligned}\langle I'(u,v),(\overline{u},\overline{v})\rangle &= K_1\Big[\mathcal{F}_1(u)+\Upsilon_1(u)\Big]\Big(\int_{\Omega\times\Omega}\gamma_1(h_u)h_u h_{\overline{u}}\,d\mu + \int_\Omega a_1(x)\gamma_1(u)u\overline{u}\,dx\Big)\\ &+ K_2\Big[\mathcal{F}_2(v)+\Upsilon_2(v)\Big]\Big(\int_{\Omega\times\Omega}\gamma_2(h_v)h_v h_{\overline{v}}\,d\mu + \int_\Omega a_2(x)\gamma_2(v)v\overline{v}\,dx\Big)\\ &- \int_\Omega (F_u(x,u,v)\overline{u} + F_v(x,u,v)\overline{v})dx,\end{aligned}$$

for all $\overline{u}, \overline{v} \in E_i$, where $h_u = \dfrac{u(x)-u(y)}{|x-y|^s}$ and $d\mu = \dfrac{dx\,dy}{|x-y|^N}$ (i.e, regular Borel measure on the set $\Omega\times\Omega$).

Proof. First, we can see that

$$\begin{aligned}\langle(\tilde{K}o\mathcal{H})'_i(u), v\rangle &= K_i\Big[\int_{\Omega\times\Omega}\Gamma_i(h_u)d\mu + \int_\Omega a_i(x)\Gamma_i(u)dx\Big]\\ &\times\Big(\int_{\Omega\times\Omega}\gamma_i(h_u)h_u h_v\,d\mu + \int_\Omega a_i(x)\gamma_i(u)uv\,dx\Big),\end{aligned} \tag{4.2}$$

for all $\overline{u}, \overline{v} \in E_i$. It follows from (4.2) that for each $u \in E_i$, $(\tilde{K}o\mathcal{H})'_i(u) \in (E_i)^*$.

Next we prove that $(\tilde{K}o\mathcal{H})_i \in C^1(E_i, \mathbb{R})$. Let $\{u_k\} \subset E_i$ with $u_k \to u$ strongly in E_i, for $v \in E_i$

we have $h_v \in L_{\Gamma_i}(\Omega\times\Omega, d\mu)$ and by Hölder inequality

$$\begin{aligned}&\left|\int_{\Omega\times\Omega}(\gamma_i(h_{u_k})h_{u_k}-\gamma_i(h_u)h_u)h_v\,d\mu+\int_{\Omega}(a_i(x)\gamma_i(u_k)u_k-a_i(x)\gamma_i(u)u)v\right|\\&\le 2\|\gamma_i(h_{u_k})h_{u_k}-\gamma_i(h_u)h_u\|_{L_{\overline{\Gamma}_i}}||h_v||_{L_{\Gamma_i}}+2||a||_\infty\|\gamma_i(u_k)u_k-\gamma_i(u)u\|_{L_{\overline{\Gamma}_i}}||v||_{L_{\Gamma_i}}.\end{aligned} \tag{4.3}$$

On the other hand, $u_k \to u$ in E_i, then $h_{u_k}\to h_u$ in $L_{\Gamma_i}(\Omega\times\Omega)$, so by dominated convergence theorem, there exists a subsequence $\{h_{u_{n_k}}\}$ and a function h in $L_{\Gamma_i}(\Omega\times\Omega)$ such that

$$|\gamma_i(h_{u_{n_k}})h_{u_{n_k}}|\le|\gamma_i(h)h|\in L_{\overline{\Gamma}_i}(\Omega\times\Omega)\quad\text{a.e. in }\Omega\times\Omega.$$

And

$$\gamma_i(h_{u_{n_k}})h_{u_{n_k}}\to\gamma_i(h_u)h_u\quad\text{a.e. in }\Omega\times\Omega. \tag{4.4}$$

Then by dominated convergence theorem we obtain that

$$\sup_{||v||_{s,\Gamma_i}\le 1}\left|\int_{\Omega\times\Omega}(\gamma_i(h_{u_k})h_{u_k}-\gamma_i(h_u)h_u)h_v\,d\mu\right|\to 0. \tag{4.5}$$

By same techniques we obtain that

$$\sup_{||v||_{\Gamma_i}\le 1}\left|\int_{\Omega}(\gamma_i(u_k)u_k-\gamma_i(u)u)v\,dx\right|\to 0. \tag{4.6}$$

By (A_1), (A_2), (2.5), ii) in Theorem 3.5 and boundedness of sequence $\{u_k, v_k\}$ and using similar argument in the proof of Lemma 3.3 in [7] we show that

$$\lim_{n\to\infty}\int_\Omega a_i(x)M_i\big(u_n\big)dx=\int_\Omega a_i(x)M_i\big(u\big)dx. \tag{4.7}$$

According to the last equation and the continuity of K_i, we have

$$K_i\bigg(\int_{\Omega\times\Omega}\Gamma_i(h_{u_k})d\mu+\int_\Omega a_i(x)\Gamma_i(u_k)dx\bigg)\to K_i\bigg(\int_{\Omega\times\Omega}\Gamma_i(h_u)d\mu+\int_\Omega a_i(x)\Gamma_i(u)dx\bigg). \tag{4.8}$$

Combining (4.5), (4.6) in (4.3) and with the fact (4.8) we get $(\tilde{K}o\mathcal{H})_i'$ is continuous. Now we turn to prove that

$$\langle I_2'(u,v),(\overline{u},\overline{v})\rangle=\int_\Omega(F_u(x,u,v)\overline{u}+F_v(x,u,v)\overline{v})dx\quad\text{for all }(u,v),\ (\overline{u},\overline{v})\in W. \tag{4.9}$$

By (4.1) and 3.3 we have

$$\begin{aligned}\int_\Omega F(x,u,v)dx \leq & \int_\Omega |F(x,u,v|dx \leq c_3\int_\Omega |u|^{r_1}dx + c_4\int_\Omega |v|^{r_2}dx \\ = & \; c_3||u||_{r_1}^{r_1} + c_4||v||_{r_2}^{r_2} \\ \leq & \; c_3C_{r_1}||u||_{E_1,\Gamma_1}^{r_1} + c_4C_{r_2}||v||_{E_2,\Gamma_2}^{r_2}.\end{aligned} \tag{4.10}$$

Then I_2 is well defined in W. Now by (3.2), (3.3) and the similar argument in [23, Lemma 3.1] we see that (4.9) holds. □

Lemma 4.5. *Suppose that* (ϕ_1) *is fulfilled. Moreover, we assume that the sequence* (u_k) *converges weakly to* u *in* E_1 *and*

$$\lim_{k\to\infty}\sup\langle(\tilde{K}o\mathcal{H})_1'(u_k), u_k-u\rangle \leq 0. \tag{4.11}$$

Then (u_k) *converge strongly to* $u\in E_1$.

Proof. Since (u_k) converges weakly to u in E_1, then $([u_k]_{s,\Gamma_1})$ and $(||u_k||_{a_1,\Gamma_1})$ are bounded sequences of real numbers. That fact and relations 2) and 3) from Lemma 3.2, imply that the sequences $(\mathcal{F}_i(u_k))$ and $(\Upsilon_i(u_k))$ are bounded. This means that the sequence $(\tilde{K}o\mathcal{H})_1(u_k)$ is bounded. Then, up to a subsequence, $(\tilde{K}o\mathcal{H})_1(u_k)\to c$. Furthermore, Lemma 3.3 implies

$$(\tilde{K}o\mathcal{H})_1(u) \leq \lim_{k\to\infty}\inf(\tilde{K}o\mathcal{H})_1(u_k) = c. \tag{4.12}$$

Since $(\tilde{K}o\mathcal{H})_1$ is convex, we have

$$(\tilde{K}o\mathcal{H})_1(u) \geq (\tilde{K}o\mathcal{H})_1(u_k) + \langle(\tilde{K}o\mathcal{H})_1'(u_k), u_k-u\rangle. \tag{4.13}$$

Therefore, combining (4.11), (4.12) and (4.13), we conclude that $(\tilde{K}o\mathcal{H})_1(u) = c$.

Taking into account that $\frac{u_k+u}{2}$ converges weakly to u in E_1 and using again the weak lower semicontinuity of $(\tilde{K}o\mathcal{H})_1$, we get

$$c = (\tilde{K}o\mathcal{H})_1(u) \leq \lim_{k\to\infty}\inf(\tilde{K}o\mathcal{H})_1\Big(\frac{u_k+u}{2}\Big). \tag{4.14}$$

We argue by contradiction, and suppose that (u_k) does not converge to u in E_1. Then, there exists $\beta>0$ and a subsequence (u_{k_r}) of (u_k) such that

$$\Big\|\frac{u_{k_r}-u}{2}\Big\|_{a_1,\Gamma_1} \geq \beta.$$

By 2) and 3) in Lemma 3.2 we infer that

$$\begin{aligned}(\tilde{K}o\mathcal{H})_1\Big(\frac{u_k+u}{2}\Big) &\geq \zeta_0\Big(\Big\|\frac{u_{k_r}-u}{2}\Big\|_{a_1,\Gamma_1}\Big)+\zeta_0\Big(\Big[\frac{u_{k_r}-u}{2}\Big]_{s,\Gamma_1}\Big)\geq \zeta_0\Big(\Big\|\frac{u_{k_r}-u}{2}\Big\|_{a_1,\Gamma_1}\Big)\\ &\geq \zeta_0(\beta).\end{aligned}$$

On the other hand, the Δ_2-condition and relation (ϕ_1) enable us to apply Theorem 1.2 in [21] to obtain

$$\frac{1}{2}(\tilde{K}o\mathcal{H})_1(u)+\frac{1}{2}(\tilde{K}o\mathcal{H})_1(u_{k_r})-(\tilde{K}o\mathcal{H})_1\Big(\frac{u_{k_r}+u}{2}\Big)\geq(\tilde{K}o\mathcal{H})_1\Big(\frac{u_{k_r}-u}{2}\Big)\geq\zeta_0(\beta), \tag{4.15}$$

for all $r\in\mathbb{N}$.

Letting $r\to\infty$ in the above inequality, we get

$$c-\zeta_0(\beta)\geq\lim_{r\to\infty}\sup(\tilde{K}o\mathcal{H})_1\Big(\frac{u_{k_r}+u}{2}\Big)\geq c. \tag{4.16}$$

That is a contradiction. It follows that (u_k) converges strongly to u in E_1.
Similary we can obtain that, $v_k\to v$ in E_2. Therefore $\{(u_k,v_k)\}\to(u,v)$ in W. □

Lemma 4.6. *If a sequence (u_k,v_k) converges to (u_0,v_0) in W weakly, then*

$$\int_\Omega\left(\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx\to 0 \tag{4.17}$$

and

$$\int_\Omega\left(\frac{1}{\alpha_1}F_v(x,u_k,v_k)-\frac{1}{\alpha_2}F_v(x,u_0,v_0)\right)(v_k-v_0)dx\to 0. \tag{4.18}$$

Proof. By (3.2), remark 3.6 and Hölder inequality we have:

$$\begin{aligned}&\int_\Omega\left(\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx\\
&\leq\int_\Omega\left|\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right||u_k-u_0|dx\\
&\leq\frac{1}{\alpha_1}\int_\Omega|F_u(x,u_k,v_k)||u_k|dx+\frac{1}{\alpha_1}\int_\Omega|F_u(x,u_k,v_k)||u_0|dx\\
&+\frac{1}{\alpha_2}\int_\Omega|F_u(x,u_0,v_0)||u_k|dx+\frac{1}{\alpha_2}\int_\Omega|F_u(x,u_0,v_0)||u_0|dx\\
&\leq\frac{c_1}{\alpha_1}\int_\Omega|u_k|^{r_1}dx+\frac{c_2}{\alpha_1}\int_\Omega|v_k|^{\frac{r_2(r_1-1)}{r_1}}|u_k|dx+\frac{c_1}{\alpha_1}\int_\Omega|u_k|^{r_1-1}|u_0|dx\\
&+\frac{c_2}{\alpha_1}\int_\Omega|v_k|^{\frac{r_2(r_1-1)}{r_1}}|u_0|dx+\frac{c_1}{\alpha_2}\int_\Omega|u_0|^{r_1-1}|u_k|dx+\frac{c_2}{\alpha_2}\int_\Omega|v_0|^{\frac{r_2(r_1-1)}{r_1}}|u_k|dx\\
&+\frac{c_1}{\alpha_2}\int_\Omega|u_0|^{r_1}dx+\frac{c_2}{\alpha_2}\int_\Omega|v_0|^{\frac{r_2(r_1-1)}{r_1}}|u_0|dx\end{aligned}$$

$$
\begin{aligned}
&\leq c_1'\Big(\int_\Omega |u_k|^{r_1}dx\Big) + c_2'\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{1}{r_1}}\Big(\int_\Omega |v_k|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}}\\
&+ c_1'\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{1}{r_1}} + c_2'\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{1}{r_1}}\\
&\times\Big(\int_\Omega |v_k|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}} + c_1'\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{1}{r_1}}\\
&+ c_2'\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{1}{r_1}}\Big(\int_\Omega |v_0|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}} + c_1'\Big(\int_\Omega |u_0|^{r_1}dx\Big)\\
&+ c_2'\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{1}{r_1}}\Big(\int_\Omega |v_0|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}}.
\end{aligned}
\tag{4.19}
$$

Since $(u_k, v_k) \rightharpoonup (u_0, v_0)$ in W, it is clear that $u_k \rightharpoonup u_0$ in E_1 and $v_k \rightharpoonup v_0$ in E_2, then $\{||u_k||_{E_1}\}$ and $\{||v_k||_{E_2}\}$ are bounded. By remark 3.6, there exists $G > 0$ such that

$$
||u_k||_{E_1}, ||v_k||_{E_2}, ||u_k||_{r_1}, ||v_k||_{r_2} \leq G, \quad \text{for } n \in \mathbb{N}. \tag{4.20}
$$

Combining (4.19) and (4.20), then we have

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{\alpha_1}F_u(x,u_k,v_k) - \frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx \leq c_1'||u_k||_{r_1}^{r_1} + c_2'||u_k||_{r_1}||v_k||_{r_2}^{\frac{r_2(r_1-1)}{r_1}}\\
&+ c_1'||u_k||_{r_1}^{\frac{r_2(r_1-1)}{r_1}}||u_0||_{r_1} + c_2'||u_0||_{r_1}||v_k||_{r_2}^{\frac{r_2(r_1-1)}{r_1}} + c_1'||u_0||_{r_1}^{\frac{r_2(r_1-1)}{r_1}}||u_k||_{r_1}\\
&+ c_2'||u_k||_{r_1}||v_0||_{r_2}^{\frac{r_2(r_1-1)}{r_1}} + c_1'||u_0||_{r_1}^{r_1} + c_2'||u_0||_{r_1}||v_0||_{r_2}^{\frac{r_2(r_1-1)}{r_1}}\\
&\leq c_1'G^{r_1} + c_2'G^{\frac{r_2(r_1-1)}{r_1}+1} + c_1'G^{\frac{r_2(r_1-1)}{r_1}+1} + c_2'G^{\frac{r_2(r_1-1)}{r_1}+1} + c_1'c_2'G^{\frac{r_2(r_1-1)}{r_1}+1}\\
&+ c_2'c_1'G^{\frac{r_2(r_1-1)}{r_1}+1} + c_1'G^{r_1} + c_2'c_1'G^{\frac{r_2(r_1-1)}{r_1}+1}.
\end{aligned}
\tag{4.21}
$$

Using again Remark 3.6, then for any positive ϵ, we can choose $n_0 \in \mathbb{N}$ such that

$$
\Big(\int_\Omega |u_k-u_0|^{r_1}dx\Big)^{\frac{1}{r_1}} < \epsilon \quad \text{for all } n > n_0.
$$

Furthermore,

$$
\begin{aligned}
&\int_\Omega \left|\frac{1}{\alpha_1}F_u(x,u_k,v_k) - \frac{1}{\alpha_2}F_u(x,u_0,v_0)\right| |u_k-u_0|dx\\
&\leq \Big(\int_\Omega \left|\frac{1}{\alpha_1}F_u(x,u_k,v_k) - \frac{1}{\alpha_2}F_u(x,u_0,v_0)\right|^{\frac{r_1}{r_1-1}}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_k-u_0|^{r_1}dx\Big)^{\frac{1}{r_1}}\\
&\leq C\Big(\int_\Omega |F_u(x,u_k,v_k)|^{\frac{r_1}{r_1-1}} + |F_u(x,u_0,v_0)|^{\frac{r_1}{r_1-1}}dx\Big)^{\frac{r_1-1}{r_1}}\epsilon\\
&\leq \epsilon C\Big(\int_\Omega \Big((c_1|u_k|^{r_1-1} + c_2|v_k|^{\frac{r_2(r_1-1)}{r_1}})^{\frac{r_1}{r_1-1}} + (c_1|u_0|^{r_1-1} + c_2|v_0|^{\frac{r_2(r_1-1)}{r_1}})^{\frac{r_1}{r_1-1}}\Big)dx\Big)^{\frac{r_1-1}{r_1}}
\end{aligned}
$$

$$\leq \epsilon C\Big(\int_\Omega \Big(c_1^{\frac{r_1}{r_1-1}}|u_k|^{r_1} + c_2^{\frac{r_1}{r_1-1}}|v_k|^{r_2} + c_1^{\frac{r_1}{r_1-1}}|u_0|^{r_1} + c_2^{\frac{r_1}{r_1-1}}|v_0|^{r_2}\Big)dx\Big)^{\frac{r_1-1}{r_1}}$$
$$\leq \epsilon C\Big(c_1^{\frac{r_1}{r_1-1}}G^{r_1} + c_2^{\frac{r_1}{r_1-1}}G^{r_2} + c_1^{\frac{r_1}{r_1-1}}G^{r_1} + c_2^{\frac{r_1}{r_1-1}}G^{r_2}\Big)^{\frac{r_1-1}{r_1}}. \tag{4.22}$$

for all $n > n_0$. As ϵ is arbitrary, combining (4.21) with (4.22), we conclude that (4.17) holds. With a similar discussion as above, we can prove that (4.18) holds. □

Lemma 4.7. *$I(u,v)$ is coercive on W, that is, $I(u,v) \to +\infty$ as $||(u,v)|| \to +\infty$.*

Proof. Using (A_3), (4.1), Remark 3.6, and Lemma 3.2 we obtain

$$\begin{aligned}
I(u,v) &= (\tilde{K}o\mathcal{H})_1(u) + (\tilde{K}o\mathcal{H})_2(v) - \int_\Omega F(x,u,v)dx\\
&\geq \alpha_1(\mathcal{F}_1 + \Upsilon_1)(u) + \alpha_1(\mathcal{F}_2 + \Upsilon_2)(v) - \int_\Omega F(x,u,v)dx\\
&\geq \alpha_1 \min\{[u]^{l_1}_{s,\Gamma_1}, [u]^{n_1}_{s,\Gamma_1}\} + \alpha_1 \min\{\|u\|^{l_1}_{a,\Gamma_1}, \|u\|^{n_1}_{a,\Gamma_1}\}\\
&+ \alpha_1 \min\{[v]^{l_2}_{s,\Gamma_2}, [v]^{n_2}_{s,\Gamma_2}\} + \alpha_1 \min\{\|v\|^{l_2}_{a,\Gamma_2}, \|v\|^{n_2}_{a,\Gamma_2}\} - c_1\int_\Omega |u|^{r_1}dx - c_2\int_\Omega |v|^{r_2}dx\\
&= \alpha_1\big([u]^{l_1}_{s,\Gamma_1} + \|u\|^{l_1}_{a,\Gamma_1}\big) + \alpha_1\big([v]^{l_2}_{s,\Gamma_2} + \|v\|^{l_2}_{a,\Gamma_2}\big) - c_3||u||^{r_1}_{E_1,\Gamma_1} - c_4||u||^{r_2}_{E_2,\Gamma_2}\\
&\geq \frac{\alpha_1}{2^{l_1-1}}\|u\|^{l_1}_{E_1,\Gamma_1} + \frac{\alpha_1}{2^{l_2-1}}\|v\|^{l_2}_{E_2,\Gamma_2} - c_3||u||^{r_1}_{E_1,\Gamma_1} - c_4||v||^{r_2}_{E_2,\Gamma_2}.
\end{aligned}$$

Since $r_i \in (1, l_i)$, then the last inequality implies that $I(u,v) \to +\infty$ as $||(u,v)|| = ||u||_{E_1,\Gamma_1} + ||v||_{E_2,\Gamma_2} \to +\infty$. □

Lemma 4.8. *Assume that $l_i = n_i$, (3.1), (4.11) (ϕ_1)-(ϕ_2) and (F_1) hold. Then the energy functional I satisfies (PS)-condition.*

Proof. Let $\{(u_k, v_k)\}$ be any (PS)-sequence in W for I. It follows from Lemma 4.7 that sequence $\{(u_k, v_k)\}$ is bounded in W. Therefore, going if necessary to a subsequence, we can assume that $(u_k, v_k) \rightharpoonup (u_0, v_0)$ in W. Then $u_k \rightharpoonup u_0$ in E_1 and $v_k \rightharpoonup v_0$ in E_2 respectively. Since (u_k, v_k) is a (PS)-sequence, there is $c \in \mathbb{R}$ such that

$$I(u_k, v_k) \to c \quad \text{as} \quad j \to \infty \quad \text{and} \quad \langle I'(u_k, v_k), (\phi, \psi)\rangle = 0 \quad \text{for all } \phi, \psi \in C_c^\infty(\mathbb{R}^n), \tag{4.23}$$

which follows that,

$$\begin{aligned}
o_k(1) &= \Big\langle \frac{1}{\alpha_1}I'(u_k, v_k) - \frac{1}{\alpha_2}I'(u_0, v_0), (u_k - u_0, v_k - v_0)\Big\rangle\\
&= \frac{1}{\alpha_1}K_1\big(\mathcal{F}_1(u_k) + \Upsilon_1(u_k)\big)\Big(\int_{\Omega\times\Omega} \gamma_1(h_{u_k})h_{u_k}h_{u_k-u_0}d\mu + \int_\Omega a_1(x)\gamma_1(u_k)u_k(u_k - u_0)dx\Big)\\
&+ \frac{1}{\alpha_1}K_2\big(\mathcal{F}_2(v_k) + \Upsilon_2(v_k)\big)\Big(\int_{\Omega\times\Omega} \gamma_2(h_{v_k})h_{v_k}h_{v_k-v_0}d\mu + \int_\Omega a_2(x)\gamma_2(v_k)v_k(v_k - v_0)dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\alpha_1}\int_\Omega \left(F_{u_k}(x,u_k,v_k)(u_k-u_0)+F_{v_k}(x,u_k,v_k)(v_k-v_0)\right)dx\\
&- \frac{1}{\alpha_2}K_1\big(\mathcal{F}_1(u_0)+\Upsilon_1(u_0)\big)\Big(\int_{\Omega\times\Omega}\gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu+\int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big)\\
&- \frac{1}{\alpha_2}K_1\big(\mathcal{F}_1(u_0)+\Upsilon_1(u_0)\big)\Big(\int_{\Omega\times\Omega}\gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu+\int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big)\\
&+ \frac{1}{\alpha_2}\int_\Omega \left(F_{u_0}(x,u_0,v_0)(u_k-u_0)+F_{v_0}(x,u_0,v_0)(v_k-v_0)\right)dx
\end{aligned}
$$

Using (A_3) we infer that

$$
\begin{aligned}
o_k(1) \geq{}& \Big(\int_{\Omega\times\Omega}\gamma_1(h_{u_k})h_{u_k}h_{u_k-u_0}d\mu+\int_\Omega a_1(x)\gamma_1(u_k)u_k(u_k-u_0)dx\Big)\\
&+\Big(\int_{\Omega\times\Omega}\gamma_2(h_{v_k})h_{v_k}h_{v_k-v_0}d\mu+\int_\Omega a_2(x)\gamma_2(v_k)v_k(v_k-v_0)dx\Big)\\
&- \frac{1}{\alpha_1}\int_\Omega \left(F_{u_k}(x,u_k,v_k)(u_k-u_0)+F_{v_k}(x,u_k,v_k)(v_k-v_0)\right)dx\\
&-\Big(\int_{\Omega\times\Omega}(\gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu+\int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big)\\
&-\Big(\int_{\Omega\times\Omega}\gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu+\int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big)\\
&+ \frac{1}{\alpha_2}\int_\Omega \left(F_{u_0}(x,u_0,v_0)(u_k-u_0)+F_{v_0}(x,u_0,v_0)(v_k-v_0)\right)dx\\
={}&\Big(\int_{\Omega\times\Omega}(\gamma_1(h_{u_k})h_{u_k}-\gamma_1(h_{u_0})h_{u_0})h_{u_k-u_0}d\mu+\int_\Omega a_1(x)(\gamma_1(u_k)u_k-\gamma_1(u_0)u_0)(u_k-u_0)dx\Big)\\
&+\Big(\int_{\Omega\times\Omega}(\gamma_2(h_{v_k})h_{v_k}-\gamma_2(h_{v_0})h_{v_0})h_{v_k-v_0}d\mu+\int_\Omega a_2(x)(\gamma_2(v_k)v_k-\gamma_2(v_0)v_0)(v_k-v_0)dx\Big)\\
&-\int_\Omega\left(\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx\\
&-\int_\Omega\left(\frac{1}{\alpha_1}F_v(x,u_k,v_k)-\frac{1}{\alpha_2}F_v(x,u_0,v_0)\right)(v_k-v_0)dx.
\end{aligned}
$$

Furthermore, last inequality, Lemma 4.6 and strictly monotone of the operator I_1' implies that

$$\int_{\Omega\times\Omega}(\gamma_1(h_{u_k})h_{u_k}-\gamma_1(h_{u_0})h_{u_0})h_{u_k-u_0}d\mu=\langle\mathcal{F}_1'(u_k)-\mathcal{F}_1'(u_0),u_k-u_0\rangle\to 0 \tag{4.24}$$

$$\int_{\Omega\times\Omega}(\gamma_2(h_{v_k})h_{v_k}-\gamma_2(h_{v_0})h_{v_0})h_{v_k-v_0}d\mu=\langle\mathcal{F}_2'(v_k)-\mathcal{F}_2'(v_0),v_k-v_0\rangle\to 0 \tag{4.25}$$

$$\int_\Omega a_1(x)(\gamma_1(u_k)u_k-\gamma_1(u_0)u_0)(u_k-u_0)dx=\langle\Upsilon_1'(u_k)-\Upsilon_1'(u_0),u_k-u_0\rangle\to 0 \tag{4.26}$$

and

$$\int_\Omega a_2(x)(\gamma_2(v_k)v_k-\gamma_2(v_0)v_0)(v_k-v_0)dx=\langle\Upsilon_2'(v_k)-\Upsilon_2'(v_0),v_k-v_0\rangle\to 0. \tag{4.27}$$

By Remark 3.6, (4.24) and (4.25) we have

$$u_k(x) \to u_0(x) \quad \text{a.e. } x \in \Omega,$$

and

$$v_k(x) \to v_0(x) \quad \text{a.e. } x \in \Omega.$$

Using Fatou's Lemma we have

$$\int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu \leq \lim_{k\to\infty} \inf \int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu, \tag{4.28}$$

$$\int_{\Omega\times\Omega} \Gamma_1(h_u)d\mu \leq \lim_{k\to\infty} \inf \int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu, \tag{4.29}$$

$$\int_{\Omega} a_1(x)\Gamma_1(u_k(x))dx \leq \lim_{k\to\infty} \inf \int_{\Omega} a_1(x)\Gamma_1(u_0(x))dx, \tag{4.30}$$

and

$$\int_{\Omega} a_2(x)\Gamma_2(v_k(x))dx \leq \lim_{k\to\infty} \inf \int_{\Omega} a_2(x)\Gamma_2(v_0(x))dx. \tag{4.31}$$

Moreover,

$$\lim_{k\to\infty} \langle \mathcal{F}_1'(u_k), u_k - u_0\rangle = \lim_{k\to\infty} \langle \mathcal{F}_1'(u_k) - \mathcal{F}_1'(u_0), u_k - u_0\rangle = 0, \tag{4.32}$$

$$\lim_{k\to\infty} \langle \mathcal{F}_2'(v_k), v_k - v_0\rangle = \lim_{k\to\infty} \langle \mathcal{F}_2'(v_k) - \mathcal{F}_2'(v_0), v_k - v_0\rangle = 0. \tag{4.33}$$

By using Hölder inequality and (2.6) we have

$$\begin{aligned}
\langle \mathcal{F}_1'(u_k), u_k - u_0\rangle &= \int_{\Omega\times\Omega} \gamma_1(h_{u_k})h_{u_k}h_{u_k-u_0}d\mu \\
&\geq l_1 \int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - \int_{\Omega\times\Omega} \overline{\Gamma}_1(\gamma_1(h_{u_k})h_{u_k})d\mu - \int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu \\
&\geq l_1 \int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - (n_1-1)\int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - \int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu \\
&= \int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - \int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu.
\end{aligned} \tag{4.34}$$

Again using Hölder inequality and (2.6) we have

$$\begin{aligned}
\langle \mathcal{F}_2'(v_k), v_k - v_0\rangle &= \int_{\Omega\times\Omega} \gamma_v(h_{v_k})h_{v_k}h_{v_k-v_0}d\mu \\
&\geq l_2 \int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - \int_{\Omega\times\Omega} \overline{\Gamma}_2(\gamma_2(h_{v_k})h_{v_k})d\mu - \int_{\Omega\times\Omega} \Gamma_2(h_{v_0})d\mu \\
&\geq l_2 \int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - (n_2-1)\int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - \int_{\Omega\times\Omega} \Gamma_2(h_{2_0})d\mu \\
&= \int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - \int_{\Omega\times\Omega} \Gamma_2(h_{v_0})d\mu.
\end{aligned} \tag{4.35}$$

According to (4.28), (4.32) and (4.34) we infer that

$$\lim_{k\to\infty}\int_{\Omega\times\Omega}\Gamma_1(h_{u_k})d\mu=\int_{\Omega\times\Omega}\Gamma_1(h_{u_0})d\mu. \tag{4.36}$$

And by (4.29), (4.33) and (4.35) we have that

$$\lim_{k\to\infty}\int_{\Omega\times\Omega}\Gamma_2(h_{v_k})d\mu=\int_{\Omega\times\Omega}\Gamma_2(h_{v_0})d\mu. \tag{4.37}$$

Also, by (4.7) we have

$$\lim_{k\to\infty}\int_{\Omega}a_1(x)\Gamma_1(u_k)dx=\int_{\Omega}a_1(x)\Gamma_1(u_0)dx, \tag{4.38}$$

$$\lim_{k\to\infty}\int_{\Omega}a_2(x)\Gamma_2(v_k)dx=\int_{\Omega}a_2(x)\Gamma_2(v_0)dx. \tag{4.39}$$

In conclusion, estimates (4.36), (4.37), (4.38) and (4.39) get the result. □

Lemma 4.9. *There is a point $(u,v)\in W$ such that $I(u,v)<0$.*

Proof. Let $u_0=\rho w_0$ and $v_0=\sigma w_0$ where $w_0\in C_0^\infty(\Omega)\backslash\{0\}$ with $w_0(x)\geq 0$, $supp(w_0)\subset\Omega$ and $||w_0||\leq 1$. Without loss of generality, we can assume that $\rho\neq 0$. It is clear that $(u_0,v_0)\in W$ (see [3, Theorem 3.7]). When t is small enough by (A_3), (F_2) and Lemma 3.2 we have

$$\begin{aligned}
I(tu_0,tv_0)&=(\tilde{K}o\mathcal{H})_1(tu_0)+(\tilde{K}o\mathcal{H})_2(tv_0)-\int_\Omega F(x,tu_0,tv_0)dx\\
&\leq \alpha_2\Big(\int_{\Omega\times\Omega}\Gamma_1(h_{tu_0})d\mu+\int_\Omega a_1(x)\Gamma_1(tu_0)dx\Big)\\
&+\alpha_2\Big(\int_{\Omega\times\Omega}\Gamma_2(h_{tv_0})d\mu+\int_\Omega a_2(x)\Gamma_2(tv_0)dx\Big)-\int_\Omega F(x,t\rho w_0,t\sigma w_0)dx\\
&\leq \alpha_2\big(\max\{[tu_0]^{l_1}_{s,\Gamma_1};[tu_0]^{n_1}_{s,\Gamma_1}\}+\max\{\,||tu_0||^{l_1}_{a_1,\Gamma_1};||tu_0||^{n_1}_{a_1,\Gamma_1}\}\big)\\
&+\alpha_2\big(\max\{[tv_0]^{l_2}_{s,\Gamma_2};[tu_0]^{n_2}_{s,\Gamma_2}\}+\max\{\,||tv_0||^{l_2}_{a_2,\Gamma_2};||tv_0||^{n_2}_{a_2,\Gamma_2}\}\big)\\
&-\int_\Omega c(|t\rho w_0|^{\alpha_0}+|t\sigma w_0|^{\beta_0})dx\\
&\leq \alpha_2[tu_0]^{l_1}_{s,\Gamma_1}+\alpha_2[tu_0]^{n_1}_{s,\Gamma_1}+\alpha_2||tu_0||^{l_1}_{a_1,\Gamma_1}+\alpha_2||tu_0||^{n_1}_{a_1,\Gamma_1}\\
&+\alpha_2[tv_0]^{l_2}_{s,\Gamma_2}+\alpha_2[tv_0]^{n_2}_{s,\Gamma_2}+\alpha_2||tv_0||^{l_2}_{a_2,\Gamma_2}+\alpha_2||tv_0||^{n_2}_{a_2,\Gamma_2}\\
&-ct^{\alpha_0}\int_\Omega|\rho w_0|^{\alpha_0}dx-ct^{\beta_0}\int_\Omega|\sigma w_0|^{\beta_0}dx\\
&\leq \alpha_2t^{l_1}||u_0||^{l_1}_{E_1,\Gamma_1}+\alpha_2t^{n_1}||u_0||^{n_1}_{E_1,\Gamma_1}+\alpha_2t^{l_2}||v_0||^{l_2}_{E_2,\Gamma_2}+\alpha_2t^{n_2}||v_0||^{n_2}_{E_2,\Gamma_2}\\
&-ct^{\alpha_0}\int_\Omega|u_0|^{\alpha_0}dx-ct^{\beta_0}\int_\Omega|v_0|^{\beta_0}dx.
\end{aligned}$$

Hence $\alpha_0\in[1,l_1)$ and $\beta_0\in[1,l_2)$, we can choose $t_0>0$ small enough such that $I(t_0u_0,t_0v_0)<0$. □

Proof of Theorem 4.1. Let $X = W$ and $J = I$. By Lemma 4.4, Lemma 4.7, Lemma 4.8 all conditions of Lemma 4.2 hold. Then system (1.1) possesses a critical point $(u, v) \in W$ which is a weak solution of system (1.1) satisfying $I(u, v) = \inf_W I$. Lemma 4.9 implies that $(u, v) \neq 0$. Thus system (1.1) possesses at least one nontrivial weak solution. $\square$

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