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Families of skew linear harmonic Euler sums involving some parameters

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ABSTRACT

In this study we investigate a family of skew linear harmonic Euler sums involving some free parameters. Our analysis involves using the properties of the polylogarithm function, commonly referred to as the Bose-Einstein integral. A reciprocity property is utilized to highlight an explicit representation for a particular skew harmonic linear Euler sum. A number of examples are also given which highlight the theorems. This work generalizes some results in the published literature and introduces some new results.

RESUMEN

En este estudio, investigamos una familia de sumas de Euler lineales alternantes armónicas involucrando algunos parámetros libres. Nuestro análisis involucra el uso de propiedades de la función polilogaritmo, comúnmente referida como la integral de Bose-Einstein. Se utiliza una propiedad de reciprocidad para destacar una representación explícita para una suma de Euler lineal alternante armónica. Se entregan ejemplos que dan luces de los teoremas. Este trabajo generaliza algunos resultados publicados en la literatura e introduce algunos nuevos resultados.

Keywords and Phrases: Skew linear harmonic Euler sum, Polygamma function, harmonic number, polylogarithm function, Bernoulli number.

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1 Introduction and preliminaries

In this paper we will investigate the closed form representation of skew linear harmonic Euler sums of the form:

$$\sum_{n\geqslant 1}(-1)^{n+1}\,A_n^{(t)}\left\{\frac{(-1)^p}{(2n+1+a)^{p+1}}+\frac{(-1)^{t+1}}{(2n+1-a)^{p+1}}\right\} \tag{1.1}$$

for the parameter $-1\leq a<1$ and integers p and t, and by reciprocity, also give a closed form expression for

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{A_n^{(p+1)}\left(\frac{1}{2}\right)}{n^t}. \tag{1.2}$$

The alternating, or skew, harmonic numbers $A_n^{(t)}$ of order t, in (1.1), are defined by

$$A_n^{(t)}:=\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j^t}\qquad (t\in\mathbb{C},\,n\in\mathbb{N}) \tag{1.3}$$

and $A_n:=A_n^{(1)}$. The polylogarithm

$$\mathrm{Li}_s(z)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}}{z^{-1}\exp(t)-1}\,\mathrm{d}t \tag{1.4}$$

in this context, is sometimes referred to as a Bose-Einstein integral [15]. The skew harmonic Euler sum (1.1) under investigation in this paper can be thought of as belonging to an extended family which has its origin in the early investigations of Goldbach and Euler in which they initiated the study of sums of the type, see Flajolet and Salvy [6]

$$\mathbb{S}_{p,t}=\sum_{n\geq 1}\frac{H_n^{(p)}}{n^t}$$

known as linear harmonic Euler sums of weight $p+t$. Nielsen [9] and many others, see [1,2,6,11,12], expanded this work and it is now known that $\mathbb{S}_{p,t}$ can be explicitly evaluated, in terms of special functions such as the Riemann zeta function, in the cases when $p=t\in\mathbb{N}$, $p+t$ of odd weight, $p+t$ of even weight in only the pair $\{(4,2),(2,4)\}$ with $p\neq t$. A reciprocity (or shuffle) relation

$$\mathbb{S}_{p,t}+\mathbb{S}_{t,p}=\zeta(p)\,\zeta(t)+\zeta(p+t)$$

exists to evaluate $\mathbb{S}_{t,p}$ in the case $\mathbb{S}_{p,t}$ is known (or vice-versa). The subsequent notion of four distinct classes of linear harmonic Euler sums, of the kind

$$
\begin{aligned}
\mathbb{S}^{++}_{p,t}(a,b;q) &:= \sum_{n=1}^{\infty} \frac{H^{(p)}_{qn}(a)}{(n+b)^t}, & \mathbb{S}^{+-}_{p,t}(a,b;q) &:= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H^{(p)}_{qn}(a)}{(n+b)^t}, \\
\mathbb{S}^{-+}_{p,t}(a,b;q) &:= \sum_{n=1}^{\infty} \frac{A^{(p)}_{qn}(a)}{(n+b)^t}, & \mathbb{S}^{--}_{p,t}(a,b;q) &:= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^{(p)}_{qn}(a)}{(n+b)^t},
\end{aligned} \tag{1.5}
$$

here $q \in \mathbb{Z}_{\geq 1}$, $a, b \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$ and $p, t \in \mathbb{C} \setminus \mathbb{Z}^-$ was identified by Flajolet and Salvy [6], in the case $a = 0$, $b = 0$ and $q = 1$. The case $a \in \mathbb{Z}_{\geq 1}$, $b \in \mathbb{Z}_{\geq 1}$ and $q = 1$ was examined by Alzer and Choi [1], and finally the case $a \in \mathbb{Z}_{\geq 1}$, $b \in \mathbb{Z}_{\geq 1}$ and $q \in \mathbb{Z}_{\geq 1}$ was examined by Sofo and Choi [13]. The skew linear harmonic Euler sums (1.1) for certain values of the parameters a, p and t belong to the family

$$
\mathbb{S}^{--}_{t,p+1}\left(0, \frac{1-a}{2}\right) := \mathbb{S}^{--}_{t,p+1}\left(0, \frac{1-a}{2}; 1\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^{(t)}_n}{(n + \frac{1-a}{2})^{p+1}} \tag{1.6}
$$

and will be explicitly represented in terms of special functions as described in (1.1). Using a reciprocity theorem due to Alzer and Choi [1], we also represent

$$
\mathbb{S}^{--}_{p+1,t}\left(\frac{1-a}{2}, 0\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^{(p+1)}_n\left(\frac{1-a}{2}\right)}{n^t}
$$

explicitly in terms of special functions. Interest in Euler sums and multiple zeta values has recently been intense (see, for example, [3–5,7,8,10,18]). Likewise various Euler sums with parameters have been researched (see, for example, [14,16,17]).

2 A parameterized integral

In the next Theorem we evaluate a particular integral which forms the basis in evaluating a family of skew linear harmonic Euler sums.

Theorem 2.1. *Let $t \in \mathbb{N}$ and let $-1 \leq a < 1$. The following integral formulas hold true:*

$$
X(a,t,\infty) = \int_0^{\infty} \frac{x^a \operatorname{Li}_t(x^2)}{1+x^2}\, \mathrm{d}x \tag{2.1}
$$

$$
= -\frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \left(\eta(t) + \frac{i \exp\left(-\frac{ia\pi}{2}\right)}{2^t} \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right)\right)\right), \tag{2.2}
$$

where $\operatorname{Li}_t(x^2)$ is the polylogarithm function, $\eta(t)$ is the Dirichlet eta function and $\zeta(t,\mu)$ is the generalized zeta function described in [5].

Proof. From (2.1), we apply the change of variable $x^2 = y$, (then rename $y = x$), so that

$$X(a,t,\infty) = \frac{1}{2}\int_0^\infty \frac{x^{\frac{a-1}{2}}\,\mathrm{Li}_t(x)}{1+x}\,\mathrm{d}x.$$

we now utilize the Bose-Einstein integral (1.4), described in [15] so that

$$X(a,t,\infty) = \frac{1}{2}\int_0^\infty \frac{x^{\frac{a-1}{2}}\,\mathrm{Li}_t(x)}{1+x}\,\mathrm{d}x = \frac{1}{2\Gamma(t)}\int_0^\infty y^{t-1}\int_0^\infty \frac{x^{\frac{a-1}{2}}}{(1+x)\left(x^{-1}\exp(y)-1\right)}\,\mathrm{d}x\,\mathrm{d}y$$

which yields

$$\begin{aligned}
&= \frac{\pi\sec\left(\frac{a\pi}{2}\right)}{2\Gamma(t)}\int_0^\infty \frac{y^{t-1}}{\exp(y)+1}\left(-1+(\sinh(y)-\cosh(y))^{-\frac{a+1}{2}}\right)\mathrm{d}y\\
&= \frac{\pi\sec\left(\frac{a\pi}{2}\right)}{2\Gamma(t)}\int_0^\infty \frac{y^{t-1}}{\exp(y)+1}\left(-1+(-\exp(-y))^{-\frac{a+1}{2}}\right)\mathrm{d}y\\
&= \frac{\pi\sec\left(\frac{a\pi}{2}\right)}{2\Gamma(t)}\int_0^\infty \frac{y^{t-1}}{\exp(y)+1}\left(-1+\exp\left(\frac{1}{2}(y-i\pi)(a+1)\right)\right)\mathrm{d}y\\
&= \pi\sec\left(\frac{a\pi}{2}\right)\left(-\left(1-2^{1-t}\right)\zeta(t)-\frac{i}{2^t}\exp\left(-\frac{ia\pi}{2}\right)\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right)\\
&= -\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right)\left(\eta(t)+\frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right),
\end{aligned}$$

this completes the proof and (2.2) is achieved. □

In the next Lemma we will demonstrate a series representation for the integral (2.3) on the unit interval $x \in (0,1)$.

Lemma 2.2. *Let $t \in \mathbb{N}$ and let $-1 \le a < 1$. The following integral formula holds true:*

$$X(a,t,1) = \int_0^1 \frac{x^a\,\mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\ge1}\frac{(-1)^{n+1}A_n^{(t)}}{2n+1+a}, \tag{2.3}$$

where $A_n^{(t)}$ are the skew harmonic numbers of order t.

Proof. A Taylor series expansion in the domain, $x \in (0,1)$ gives,

$$\mathrm{Li}_t(x^2) = \sum_{n\ge1}\frac{x^{2n}}{n^t}, \quad \frac{1}{1+x^2} = \sum_{n\ge0}(-1)^n x^{2n}.$$

By the Cauchy product of two convergent series, then it follows that

$$\frac{x^a\mathrm{Li}_t(x^2)}{1+x^2} = \sum_{n\ge1}(-1)^{n+1}A_n^{(t)}x^{2n+a}$$

and therefore

$$\int_0^1 \frac{x^a \operatorname{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\geq 1} (-1)^{n+1} A_n^{(t)} \int_0^1 x^{2n+a}\,\mathrm{d}x = \sum_{n\geq 1} \frac{(-1)^{n+1} A_n^{(t)}}{2n+1+a}.$$

In a similar fashion, it follows that

$$X(-a,t,1) = \int_0^1 \frac{x^{-a}\operatorname{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\geq 1} \frac{(-1)^{n+1} A_n^{(t)}}{2n+1-a}.$$

In subsequent evaluations we also require the following result, which may be evaluated as a standard integral.

$$\int_0^1 \frac{x^{-a}\log^j(x)}{1+x^2}\,\mathrm{d}x = \frac{(-1)^j\, j!}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right) - \zeta\left(j+1,\frac{3-a}{4}\right)\right). \qquad \square$$

In the next Theorem we establish an identity for a linear skew harmonic Euler sum of weight $(t+1)$, $t\in\mathbb{N}$, for the parameter $a\neq 0$.

Theorem 2.3. *Let $t\in\mathbb{N}$, $-1\leq a<1$ with $a\neq 0$. The following formulas hold true:*

$$S(a,t) = \sum_{n\geqslant 1} (-1)^{n+1} A_n^{(t)} \left\{\frac{1}{2n+1+a} + \frac{(-1)^{t+1}}{2n+1-a}\right\} \tag{2.4}$$

$$= \frac{1}{4}\sum_{n\geqslant 1} \frac{1}{n^t}\left\{H_{\frac{n}{2}-\frac{1-a}{4}} - H_{\frac{n}{2}-\frac{3-a}{4}} + (-1)^{t+1}\left(H_{\frac{n}{2}-\frac{1+a}{4}} - H_{\frac{n}{2}-\frac{3+a}{4}}\right)\right\} \tag{2.5}$$

$$\begin{aligned} &= -\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right)\left(\eta(t) + \frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right) - \zeta\left(t,\frac{3-a}{4}\right)\right)\right)\\ &\quad + \frac{(-1)^{t+1}\,2^{t-2}\,(\pi i)^t\, B_t}{t!}\left(\psi\left(\frac{1-a}{4}\right) - \psi\left(\frac{3-a}{4}\right)\right)\\ &\quad + \frac{(-1)^{t+1}\,2^{t-2}}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{(-1)^j\, j! B_{t-j}\,(\pi i)^{t-j}}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right) - \zeta\left(j+1,\frac{3-a}{4}\right)\right). \end{aligned} \tag{2.6}$$

Proof. Consider the following integral on the real half line $x\geq 0$

$$X(a,t,\infty) = \int_0^\infty \frac{x^a \operatorname{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x.$$

Putting

$$\Delta(a,t;x) := \frac{x^a \operatorname{Li}_t(x^2)}{1+x^2}$$

it may be seen that $\lim_{x\downarrow 0}\Delta(a,t;x)$, $\lim_{x\uparrow\infty}\Delta(a,t;x)$ and $\lim_{x\to 1}\Delta(a,t;x)$ exist, in fact $\lim_{x\to 1}\Delta(a,t;x) = \frac{1}{2}\operatorname{Li}_t(1) = \frac{1}{2}\zeta(t)$ and it may be expressed as

$$X(a,t,\infty) = \int_0^1 \Delta(a,t;x)\,\mathrm{d}x + \int_1^\infty \Delta(a,t;x)\,\mathrm{d}x. \tag{2.7}$$

Using the transformation $xy = 1$ in the last integral in (2.7) and recovering the variable x instead of y in the resultant integral, we obtain

$$X(a,t,\infty) = \int_0^1 \Delta(a,t;x)\, \mathrm{d}x + \int_0^1 \frac{x^{-a}\ \mathrm{Li}_t(\frac{1}{x^2})}{1+x^2}\, \mathrm{d}x.$$

From the properties of $\mathrm{Li}_t(\frac{1}{x^2})$, see [5], the last integral is expressed as

$$\begin{aligned} X(a,t,\infty) &= \int_0^1 \Delta(a,t;x)\, \mathrm{d}x + (-1)^{t+1} \int_0^1 \frac{x^{-a}}{1+x^2} \left(\mathrm{Li}_t(x^2) + \frac{(2\pi i)^t}{t!} B_t\left(\frac{\log x}{\pi i}\right)\right) \mathrm{d}x \\ &= \int_0^1 \Delta(a,t;x)\, \mathrm{d}x + (-1)^{t+1} \int_0^1 \Delta(-a,t;x)\, \mathrm{d}x + (-1)^{t+1} \frac{(2\pi i)^t}{t!} \int_0^1 \frac{x^{-a}}{1+x^2} B_t\left(\frac{\log x}{\pi i}\right) \mathrm{d}x \end{aligned}$$

where $B_t(\frac{\log x}{\pi i})$ are the Bernoulli polynomials. From the recurrence relation of the Bernoulli polynomials,

$$B_j(t) = \sum_{k=0}^{j} \binom{j}{k} B_k\ t^{j-k} = \sum_{k=0}^{j} \binom{j}{k} B_{j-k} t^k,$$

we can express the last integral in the form

$$X(a,t,\infty) = \int_0^1 \Delta(a,t;x)\, \mathrm{d}x + (-1)^{t+1} \int_0^1 \Delta(-a,t;x)\, \mathrm{d}x + (-1)^{t+1} \frac{(2\pi i)^t}{t!} \sum_{j=0}^{t} \binom{t}{j} \frac{B_{t-j}}{(\pi i)^j} \int_0^1 \frac{x^{-a}\ \log^j(x)}{1+x^2} \mathrm{d}x.$$

We now rearrange the above relation and use the results of Theorem 2.1 and Lemma 2.2 to obtain

$$\begin{aligned} S(a,t) &= \sum_{n\geqslant 1} (-1)^{n+1} A_n^{(t)} \left\{ \frac{1}{2n+1+a} + \frac{(-1)^{t+1}}{2n+1-a} \right\} \\ &= X(a,t,\infty) + \frac{(-1)^{t+1}\, 2^{t-2}}{t!} \sum_{j=0}^{t} \binom{t}{j} \frac{(-1)^j\, j! B_{t-j} (\pi i)^{t-j}}{4^j} \left(\zeta\left(j+1, \frac{1-a}{4}\right) - \zeta\left(j+1, \frac{3-a}{4}\right)\right). \end{aligned}$$

If we isolate the $j = 0$ term and put (2.2) for $X(a,t,\infty)$, we obtain

$$\begin{aligned} S(a,t) &= \sum_{n\geqslant 1} (-1)^{n+1} A_n^{(t)} \left\{ \frac{1}{2n+1+a} + \frac{(-1)^{t+1}}{2n+1-a} \right\} \\ &= X(a,t,\infty) + \frac{(-1)^{t+1}\, 2^{t-2} (\pi i)^t B_t}{t!} \left(\psi\left(\frac{1-a}{4}\right) - \psi\left(\frac{3-a}{4}\right)\right) \\ &+ \frac{(-1)^{t+1}\, 2^{t-2}}{t!} \sum_{j=0}^{t} \binom{t}{j} \frac{(-1)^j\, j! B_{t-j} (\pi i)^{t-j}}{4^j} \left(\zeta\left(j+1, \frac{1-a}{4}\right) - \zeta\left(j+1, \frac{3-a}{4}\right)\right), \end{aligned}$$

and (2.6) follows. The representation (2.5) is achieved in the following way.

$$\begin{aligned} X(a,t,1) &= \int_0^1 \frac{x^a \ \mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\geqslant 1}\frac{1}{n^t}\sum_{r\geqslant 0}(-1)^r\int_0^1 x^{2n+2r+a}\,\mathrm{d}x \\ &= \sum_{n\geqslant 1}\frac{1}{n^t}\sum_{r\geqslant 0}\frac{(-1)^r}{2n+2r+a+1} = \sum_{n\geqslant 1}\frac{1}{2n^t}\zeta\left(-1,1,\frac{1}{2}(2n+a+1)\right) \\ &= \sum_{n\geqslant 1}\frac{1}{4n^t}\left(\psi\left(\frac{2n+a+1}{4}\right)-\psi\left(\frac{2n+a+3}{4}\right)\right) \qquad (2.8) \\ &= \frac{1}{4}\sum_{n\geqslant 1}\frac{1}{n^t}\left(H_{\frac{n}{2}-\frac{1-a}{4}}-H_{\frac{n}{2}-\frac{3-a}{4}}\right). \end{aligned}$$

Following the same pattern we have,

$$X(-a,t,1) = \int_0^1 \frac{x^{-a}\ \mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \frac{1}{4}\sum_{n\geqslant 1}\frac{1}{n^t}\left(H_{\frac{n}{2}-\frac{1+a}{4}}-H_{\frac{n}{2}-\frac{3+a}{4}}\right)$$

and therefore $X(a,t,1)+(-1)^{t+1}X(-a,t,1)$ implies the result (2.5). □

In the next section we extend the results of the previous section by considering a more general version of the integral (2.1) thereby allowing an extension of the result (2.4).

3 The logarithmic case

This section establishes a number of general skew linear harmonic Euler sum identities.

Theorem 3.1. *Let $p,t\in\mathbb{N}$, $-1\leq a<1$ with $a\neq 0$. The following formulas hold true:*

$$\begin{aligned} S(a,t,p) :&= p!\sum_{n\geqslant 1}(-1)^{n+1}A_n^{(t)}\left\{\frac{(-1)^p}{(2n+1+a)^{p+1}}+\frac{(-1)^{t+1}}{(2n+1-a)^{p+1}}\right\} \qquad (3.1) \\ &= \sum_{n\geqslant 1}\frac{(-1)^p\,p!}{4^{p+1}n^t}\left(H^{(p+1)}_{\frac{n}{2}-\frac{1-a}{4}}-H^{(p+1)}_{\frac{n}{2}-\frac{3-a}{4}}+(-1)^{t+1}\left(H^{(p+1)}_{\frac{n}{2}-\frac{1+a}{4}}-H^{(p+1)}_{\frac{n}{2}-\frac{3+a}{4}}\right)\right) \qquad (3.2) \\ &= \frac{\partial^p}{\partial a^p}(X(a,t,\infty))+\frac{(-1)^{t+1}2^{t-2p-2}}{t!}\sum_{j=0}^{t}\frac{1}{4^j}\binom{t}{j}\binom{p+j}{j}(-1)^j j!B_{t-j}(\pi i)^{t-j} \\ &\times\left(\zeta\left(p+j+1,\frac{1-a}{4}\right)-\zeta\left(p+j+1,\frac{3-a}{4}\right)\right), \qquad (3.3) \end{aligned}$$

where $X(a,t,\infty)$ is given by (2.2), and $H^{(m)}_\alpha$ are harmonic numbers.

Proof. From Theorem 2.1, we note that

$$\frac{\partial^p}{\partial a^p}X(a,t,\infty) = \int_0^\infty \frac{x^a\ \ln^p(x)\,\mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x. \qquad (3.4)$$

From Theorem 2.3 consider (2.4) and differentiate p times with respect to the parameter a so that

$$\begin{aligned}
\frac{\partial^p}{\partial a^p}\left(S\left(a,t\right)\right) = S\left(a,p,t\right) &= p!\sum_{n\geqslant 1}\left(-1\right)^{n+1} A_n^{(t)}\left\{\frac{\left(-1\right)^p}{\left(2n+1+a\right)^{p+1}}+\frac{\left(-1\right)^{t+1}}{\left(2n+1-a\right)^{p+1}}\right\}\\
&= -\frac{\pi}{2}\frac{\partial^p}{\partial a^p}\left(\sec\left(\frac{a\pi}{2}\right)\left(\eta\left(t\right)+\frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right)\right)\\
&+\frac{\partial^p}{\partial a^p}\left(\frac{\left(-1\right)^{t+1}2^{t-2}\left(\pi i\right)^t B_t}{t!}\left(\psi\left(\frac{1-a}{4}\right)-\psi\left(\frac{3-a}{4}\right)\right)\right)\\
&+\frac{\partial^p}{\partial a^p}\left(\frac{\left(-1\right)^{t+1}2^{t-2}}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{\left(-1\right)^j j!B_{t-j}\left(\pi i\right)^{t-j}}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right)-\zeta\left(j+1,\frac{3-a}{4}\right)\right)\right).
\end{aligned}$$

After some simplification and rearrangement we obtain the identity (3.3). The representation (3.2) can be attained from the representation (2.8),

$$\begin{aligned}
\frac{\partial^p}{\partial a^p}\left(X\left(a,t,1\right)+\left(-1\right)^{t+1}X\left(-a,t,1\right)\right) &= \frac{\partial^p}{\partial a^p}\left(\sum_{n\geqslant 1}\frac{1}{4n^t}\left(\psi\left(\frac{2n+a+1}{4}\right)-\psi\left(\frac{2n+a+3}{4}\right)\right)\right)\\
&+\left(-1\right)^{t+1}\frac{\partial^p}{\partial a^p}\left(\sum_{n\geqslant 1}\frac{1}{4n^t}\left(\psi\left(\frac{2n-a+1}{4}\right)-\psi\left(\frac{2n-a+3}{4}\right)\right)\right)\\
&+\sum_{n\geqslant 1}\frac{1}{4^{p+1}n^t}\left(\psi^{(p)}\left(\frac{2n+a+1}{4}\right)-\psi^{(p)}\left(\frac{2n+a+3}{4}\right)\right)\\
&+\left(-1\right)^{p+t+1}\sum_{n\geqslant 1}\frac{1}{4^{p+1}n^t}\left(\psi^{(p)}\left(\frac{2n-a+1}{4}\right)-\psi^{(p)}\left(\frac{2n-a+3}{4}\right)\right),
\end{aligned}$$

where $H_\alpha^{(m)}$ are harmonic numbers of order $m\in\mathbb{N}$ with index $\alpha\in\mathbb{C}\backslash\mathbb{Z}_{\leqslant -1}$, and upon simplification of the above expression we obtain (3.2). □

There are some cases of the value of the parameter a of Theorem 3.1 which are worthy of investigation and these are given in the next Corollaries. In particular we examine the three cases of (1). $a=0$, (2). $t=1$ and p is an even integer, and (3). $p+1=t$, for $t\in\mathbb{N}_0$.

Corollary 3.2. *Let $p,t\in\mathbb{N}$, with $p+t$ of odd weight, and put $a=0$. The following formula holds true:*

$$S\left(0,t,p\right):=2\left(-1\right)^p p!\sum_{n\geqslant 1}\left(-1\right)^{n+1}\frac{A_n^{(t)}}{\left(2n+1\right)^{p+1}} \tag{3.5}$$

$$\begin{aligned}
&= -\left(\frac{\pi}{2}\right)^{p+1}\left|E_p\right|\eta\left(t\right)+2^t p!\binom{p+t}{t}\beta\left(p+t+1\right) \qquad (3.6)\\
&- 2^t\sum_{j=0}^{t-2}\frac{\left(-1\right)^{p+j}\ p!}{\left(t-j\right)!}\left(i\pi\right)^{t-j}\binom{p+j}{j}B_{t-j}\beta\left(p+j+1\right)\\
&- 2^t\sum_{r=0}^{p-1}\frac{\pi^{p+1-r}\ r!\left(2^{p+1-r}-1\right)}{p+1-r}\binom{p}{r}\binom{t+r-1}{r}\left|B_{p+1-r}\right|\beta\left(t+r\right).
\end{aligned}$$

where B_z are the Bernoulli numbers, E_z are the Euler numbers and $\beta(z)$ is the Dirichlet beta function.

Proof. For $a = 0$ and for odd weight $p + t$ Theorem 3.1 provides

$$\begin{aligned} S(0,t,p) :&= 2(-1)^p p! \sum_{n\geqslant 1} (-1)^{n+1} \frac{A_n^{(t)}}{(2n+1)^{p+1}} \\ &= -\frac{\pi}{2} \lim_{a\to 0} \frac{\partial^p}{\partial a^p} \left(\sec\left(\frac{a\pi}{2}\right) \left(\eta(t) + \frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t} \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right)\right)\right)\right) \\ &+ \lim_{a\to 0} \frac{\partial^p}{\partial a^p} \left(\frac{(-1)^{t+1} 2^{t-2}}{t!} \sum_{j=0}^{t} \binom{t}{j} \frac{(-1)^j j! B_{t-j} (\pi i)^{t-j}}{4^j} \left(\zeta\left(j+1, \frac{1-a}{4}\right) - \zeta\left(j+1, \frac{3-a}{4}\right)\right)\right). \end{aligned} \tag{3.7}$$

Consider

$$-\frac{\pi}{2}\eta(t) \lim_{a\to 0} \frac{\partial^p}{\partial a^p} \sec\left(\frac{a\pi}{2}\right) - \frac{\pi}{2} \lim_{a\to 0} \frac{\partial^p}{\partial a^p} \left(\sec\left(\frac{a\pi}{2}\right) \frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t} \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right)\right)\right), \tag{3.8}$$

now simplify the second term so that (3.8) can be written as

$$\begin{aligned} &= -\left(\frac{\pi}{2}\right)^{p+1} |E_p|\, \eta(t) - \frac{i\pi}{2^{t+1}} \lim_{a\to 0} \frac{\partial^p}{\partial a^p} \left(\left(i + \tan\left(\frac{a\pi}{2}\right)\right) \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right)\right)\right) \\ &= -\left(\frac{\pi}{2}\right)^{p+1} |E_p|\, \eta(t) - \frac{i\pi}{2^{t+1}} \lim_{a\to 0} \sum_{r=0}^{p} \binom{p}{r} \left(i + \tan\left(\frac{a\pi}{2}\right)\right)^{(r)} \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right)\right)^{(p-r)} \end{aligned} \tag{3.9}$$

where $F^{(r)}$ indicates the r^{th} derivative of F with respect to the parameter a. The term

$$\lim_{a\to 0} \frac{\partial^r}{\partial a^r} \left(i + \tan\left(\frac{a\pi}{2}\right)\right) = \begin{cases} 0, & \text{for } r \text{ even} \\ i, & \text{for } r = 0 \\ \frac{2^{r+1}(2^{r+1}-1)}{r+1} \left(\frac{\pi}{2}\right)^r |B_{r+1}|, & \text{for } r \text{ odd} \end{cases}$$

and

$$\lim_{a\to 0} \frac{\partial^m}{\partial a^m} \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right)\right) = 4^t \ (t)_m \ \beta(m+t), \quad \text{for } m \in \mathbb{N},$$

where $(t)_m$ is Pochhammer's symbol, see [5]. Now, substituting into (3.9) and simplifying we have

$$\begin{aligned} S(0,t,p) :&= 2(-1)^p p! \sum_{n\geqslant 1} (-1)^{n+1} \frac{A_n^{(t)}}{(2n+1)^{p+1}} \\ &= -\left(\frac{\pi}{2}\right)^{p+1} |E_p|\, \eta(t) - i\pi p! 2^{t-1} \binom{p+t-1}{p} \beta(p+t) \\ &- 2^t \sum_{j=0}^{t} \frac{(-1)^{p+j}\ p!}{(t-j)!} (i\pi)^{t-j} \binom{p+j}{j} B_{t-j} \beta(p+j+1) \\ &- 2^t \sum_{r=0}^{p-1} \frac{\pi^{p+1-r}\ r! \left(2^{p+1-r}-1\right)}{p+1-r} \binom{p}{r} \binom{t+r-1}{r} |B_{p+1-r}|\, \beta(t+r). \end{aligned} \tag{3.10}$$

In the second sum we isolate the $j = t$ term, with the value $B_0 = 1$ and the $j = t - 1$ term, with the value $B_1 = -\frac{1}{2}$, so that in simplifying we produce the result (3.6). $\square$

Corollary 3.3. *For $t = 1$, and p an even integer, the following formula is valid:*

$$S(0,1,p) := 2p! \sum_{n\geqslant 1} (-1)^{n+1} \frac{A_n}{(2n+1)^{p+1}} = 2(p+1)!\beta(p+2) - \left(\frac{\pi}{2}\right)^{p+1} |E_p| \ln(2) - 2\sum_{r=0}^{p-1} \frac{\pi^{p+1-r} r! \left(2^{p+1-r} - 1\right)}{p+1-r} \binom{p}{r} |B_{p+1-r}| \beta(r+1). \tag{3.11}$$

where B_z are the Bernoulli numbers, E_z are the Euler numbers and $\beta(\cdot)$ is the Dirichlet beta function.

Proof. The proof follows directly from Corollary 3.2. We remark that this case has been examined in the paper [18], but in slightly different form than Corollary 3.3. An equivalent expression, in less compact form, for (3.11) has been given by Stewart [18]. $\square$

Another special case worthy of mention is for the situation when $p + 1 = t$, this is detailed in the next Corollary.

Corollary 3.4. *Let $p + 1 = t \in \mathbb{N}$. The following relation is valid:*

$$S(0,t,t-1) : = 2(-1)^{t-1}(t-1)! \sum_{n\geqslant 1} (-1)^{n+1} \frac{A_n^{(t)}}{(2n+1)^t} \tag{3.12}$$

$$= -\left(\frac{\pi}{2}\right)^t |E_{t-1}| \eta(t) + 2^t (t-1)! \binom{2t-1}{t} \beta(2t) \tag{3.13}$$

$$+ 2^t \sum_{j=0}^{t-2} \frac{(-1)^{t+j}(t-1)!}{(t-j)!} (i\pi)^{t-j} \binom{t+j-1}{j} B_{t-j} \beta(t+j)$$

$$- 2^t \sum_{r=0}^{t-2} \frac{\pi^{t-r}\ r! \left(2^{t-r} - 1\right)}{t-r} \binom{t-1}{r} \binom{t+r-1}{r} |B_{t-r}| \beta(t+r),$$

where B_z are the Bernoulli numbers, E_z are the Euler numbers and $\beta(z)$ is the Dirichlet beta function.

Proof. Follows directly from (3.2). $\square$

4 Reciprocity identity

The following Theorem is enunciated by Alzer and Choi [1] regarding a general shuffle relation, and we shall utilize this result in the upcoming Corollary.

Theorem 4.1. *The following formula is given by Alzer and Choi [1, p.14]. Let $p, q \in N$, $\alpha, b \in \mathbb{C} \backslash \mathbb{Z}^-$, with $\alpha \neq b$, then,*

$$\mathbb{S}_{p,q}^{--}(\alpha, b) + \mathbb{S}_{q,p}^{--}(b, \alpha) = \eta(p, \alpha+1)\,\eta(q, b+1) + \sum_{k\geq 1} \frac{1}{(k+\alpha)^p (k+b)^q}. \tag{4.1}$$

The infinite sum can be expressed as finite linear combination of polygamma functions. Here $\eta(p, \alpha+1)$ is the generalized eta function.

Let us recall, from (3.1) and using the notation of (1.6), that

$$S(a,t,p) = \frac{(-1)^p\, p!}{2^{p+1}} \mathbb{S}_{t,p+1}^{--}\left(0, \frac{1+a}{2}\right) + \frac{(-1)^{t+1}\, p!}{2^{p+1}} \mathbb{S}_{t,p+1}^{--}\left(0, \frac{1-a}{2}\right).$$

The case $a = 0$ is described as

$$S(0,t,p) = \frac{(-1)^p\, p!}{2^p} \mathbb{S}_{t,p+1}^{--}\left(0, \frac{1}{2}\right)$$

and its closed form representation given by (3.3). We can now apply Theorem 4.1 to obtain reciprocity relations for some identities of Section 3. Consider the following Corollary.

Corollary 4.2. *Let $p, t \in \mathbb{N}$, with $p + t$ of odd weight. The following identity holds true:*

$$\begin{aligned}
\mathbb{S}_{p+1,t}^{--}\left(\frac{1}{2}, 0\right) &= \sum_{n\geqslant 1} \frac{(-1)^{n+1} A_n^{(p+1)}\left(\frac{1}{2}\right)}{n^t} \\
&= (-1)^{p+1}\, 2^{p+t-1} \binom{p+t-1}{t-1} (\ln 2 - 1) - \frac{(-1)^p\, 2^p}{p!} S(0,t,p+1) \\
&\quad + 2^{p+1}\left(1 - \beta(p+1)\right)\eta(t) + (-1)^{p+1} \sum_{j=1}^{p} \frac{(-1)^{j+1}\, 2^{p+t-j}}{j} \binom{p+t-j-1}{t-1} \zeta(j+1) \\
&\quad + (-1)^{p+1} \sum_{j=1}^{p} j! 2^{p+t+1} \binom{p+t-j-1}{p} \left(1 - \lambda(j+1)\right),
\end{aligned} \tag{4.2}$$

where $S(0,t,p+1)$ is the expression (3.6), $\eta\left(p+1, \frac{3}{2}\right)$ is the generalized eta function and $\lambda(j+1)$ is the Dirichlet lambda function, see [5].

Proof. Applying Theorem 4.1, we are able to express

$$\mathbb{S}_{p+1,t}^{--}\left(\frac{1}{2},0\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(p+1)}\left(\frac{1}{2}\right)}{n^t}=-\frac{(-1)^p\,2^p}{p!}S\left(0,t,p+1\right)+\eta\left(t,1\right)\eta\left(p+1,\frac{3}{2}\right)+\sum_{j\geq 1}\frac{1}{j^{p+1}\left(j+\frac{1}{2}\right)^t}. \tag{4.3}$$

The sum

$$\begin{aligned}\sum_{j\geq 1}\frac{1}{j^{p+1}\left(j+\frac{1}{2}\right)^t}&=(-1)^{p+1}\,2^{p+t}\binom{p+t-1}{t-1}\left(\psi(1)-\psi\left(\frac{3}{2}\right)\right)\\&=+(-1)^{p+1}\sum_{j=1}^{p}\frac{2^{p+t-j}}{j!}\binom{p+t-j-1}{t-1}\psi^{(j)}(1)\\&+(-1)^t\sum_{j=1}^{t-1}\frac{(-1)^{p+t-j}\,2^{p+t-j}}{j!}\binom{p+t-j-1}{p}\psi^{(j)}\left(\frac{3}{2}\right).\end{aligned} \tag{4.4}$$

The following relations apply, in simplifying the above expression,

$$\begin{aligned}&\psi(1)=-\gamma,\quad \psi\left(\frac{3}{2}\right)=-2\ln 2-\gamma,\quad \psi^{(j)}(1)=(-1)^{j-1}\,(j-1)!\zeta\left(j+1\right),\\&\psi^{(j)}\left(\frac{3}{2}\right)=\psi^{(j)}\left(\frac{1}{2}\right)+(-1)^j\,j!2^{j+1}=(-1)^j\,j!2^{j+1}\left(1-\lambda\left(j+1\right)\right),\\&\eta\left(t,1\right)=\eta\left(t\right),\quad \eta\left(p+1,\frac{3}{2}\right)=2^{p+1}\left(1-\beta\left(p+1\right)\right),\end{aligned}$$

where γ is the familiar Euler-Mascheroni constant (see, *e.g.*, [17, Section 1.2]), and therefore, using (4.4)

$$\begin{aligned}\sum_{j\geq 1}\frac{1}{j^{p+1}\left(j+\frac{1}{2}\right)^t}&=(-1)^{p+1}\,2^{p+t}\binom{p+t-1}{t-1}(2\ln 2-2)\\&=+(-1)^{p+1}\sum_{j=1}^{p}\frac{(-1)^{j+1}\,2^{p+t-j}}{j}\binom{p+t-j-1}{t-1}\zeta(j+1)\\&+(-1)^{p+1}\sum_{j=1}^{t-1}j!2^{p+t+1}\binom{p+t-j-1}{p}\left(1-\lambda\left(j+1\right)\right).\end{aligned}$$

Substituting this expression in (4.3) we arrive at the the expression (4.2) and the proof is finalized. □

5 Some examples

Example 5.1. *Let $(a,t,p)=(0,2,3)$, the following identity holds,*

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(2)}}{(2n+1)^4}=\frac{1}{16}\mathbb{S}_{2,4}^{--}\left(0,\frac{1}{2}\right)=\frac{\pi^4 G}{24}+\frac{5\pi^2\beta(4)}{3}-20\beta(6).$$

Example 5.2. *Let $(a,t,p)=(0,4,3)$, the following identity holds,*

$$\frac{1}{16}\mathbb{S}_{4,4}^{--}\left(0,\frac{1}{2}\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(4)}}{(2n+1)^4}=\frac{8\pi^4\beta(4)}{45}+\frac{80\pi^2\beta(6)}{3}+280\beta(8).$$

Example 5.3. *Let $(a,t,p)=(0,5,0)$, the following identity holds,*

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(5)}}{2n+1}=\frac{1}{2}\mathbb{S}_{5,1}^{--}\left(0,\frac{1}{2}\right)=-2G\beta(4)-8\zeta(2)\beta(4)+16\beta(6)-\frac{\pi\eta(5)}{4}.$$

Example 5.4. *Let $(a,t,p)=(0,2t,1)$, the following identity holds,*

$$\sum_{n\geqslant 1}(-1)^{n+1}\frac{A_n^{(2t)}}{(2n+1)^2}=\frac{1}{4}\mathbb{S}_{2t,2}^{--}\left(0,\frac{1}{2}\right)=\pi^2 2^{2t-3}\beta(2t)-(2t+1)2^{2t-1}\beta(2t+2)$$
$$+2^{2t-1}\sum_{j=0}^{2t-2}\frac{(-1)^{t+j}(j+1)}{(2t-j)!}(i\pi)^{2t-j}B_{2t-j}\beta(j+2).$$

Example 5.5. *As a last example we utilize the results of Example 5.4, so that the following identity holds,*

$$\mathbb{S}_{2,2t}^{--}\left(\frac{1}{2},0\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(2)}\left(\frac{1}{2}\right)}{n^{2t}}=\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^{2t}}\sum_{j=1}^{n}\frac{(-1)^{j+1}}{\left(j+\frac{1}{2}\right)^2}$$
$$=\eta\left(2t,\frac{3}{2}\right)\eta(2,1)+\sum_{k\geq 1}\frac{1}{k^2\left(k+\frac{1}{2}\right)^{2t}}-\mathbb{S}_{2t,2}^{--}\left(0,\frac{1}{2}\right).$$

Simplifying the algebra we arrive at:

$$\mathbb{S}_{2,2t}^{--}\left(\frac{1}{2},0\right)=2^{2t+3}\ t\ \ln(2)+2^{2t}\zeta(2)-\mathbb{S}_{2t,2}^{--}\left(0,\frac{1}{2}\right)-\sum_{j=1}^{2t-1}(2t-j)\ \ j!\ 2^{2t+3}(1-\lambda(j+1)).$$

6 Concluding remarks

In this paper we have offered an explicit representation for integrals with log-polylog integrand both in the unit domain and on the positive real half line $x\geq 0$, see (2.1) and (3.4). These explicit evaluations enabled the representation, in closed form, of families of skew linear harmonic Euler sums of the form (2.4) and (3.1) which are new in the literature. An application of a reciprocity theorem allowed further explicit evaluations of families of skew linear harmonic Euler sums. A number of pertinent examples were also given to highlight the theorems and corollaries. It is expected that further work will follow examining variant linear Euler sums incorporating parameters.

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