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# On a class of evolution problems driven by maximal monotone operators with integral perturbation

Fatima Fennour[1]
Soumia Saïdi [1,✉]

[1] *LMPA Laboratory, Department of Mathematics, Faculty of Exact Sciences and Informatics, University of Jijel, 18000 Jijel, Algeria.*
fennourfatima38@gmail.com
soumiasaidi44@gmail.com[✉]

**ABSTRACT**

The present paper is dedicated to the study of a first-order differential inclusion driven by time and state-dependent maximal monotone operators with integral perturbation, in the context of Hilbert spaces. Based on a fixed point method, we derive a new existence theorem for this class of differential inclusions. Then, we investigate an optimal control problem subject to such a class, by considering control maps acting in the state of the operators and the integral perturbation.

**RESUMEN**

El presente artículo está dedicado al estudio de una inclusión diferencial de primer orden impulsada por operadores monótonos maximales dependiendo del tiempo y del estado con una perturbación integral, en el contexto de espacios de Hilbert. En base a un método de punto fijo, derivamos un nuevo teorema de existencia para esta clase de inclusiones diferenciales. A continuación investigamos un problema de control óptimo sujeto a dicha clase, considerando funciones de control actuando en el estado de los operadores y de la perturbación integral.

**Keywords and Phrases:** Integro-differential inclusion, maximal monotone operator, integral perturbation, optimal solution.

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# 1 Introduction

Sweeping processes with integral forcing term or integro-differential sweeping processes have been introduced in [8]. Later, the well-posedness result to the non-convex integro-differential sweeping process has been shown in [20]. Recent investigations on this topic have been developed in [5–7]. More recently, differential inclusions with integral perturbation involving $m$-accretive operators or subdifferentials or time-dependent maximal monotone operators have been studied in [4, 13, 14]. The aforementioned contributions find many areas of applications such as electrical circuits, nonlinear integro-differential complementarity systems, optimal control, fractional systems, etc.

We are concerned, in this paper, with the following Integro-Differential Problem with time and state-dependent maximal monotone operators $A(t,u)$

$$(IDP_{A(t,u)}) \qquad \begin{cases} -\dot{u}(t) \in A(t,u(t))u(t) + \int_{T_0}^{t} f(t,s,u(s))ds \quad \text{a.e. } t \in I := [T_0,T], \\ u(T_0) = u_0 \in D(A(T_0,u_0)), \end{cases}$$

where $H$ stands for a real Hilbert space, $A(t,x) : D(A(t,x)) \subset H \rightrightarrows H$ is a maximal monotone operator whose domain is denoted $D(A(t,x))$, for each $(t,x) \in I \times H$, and $f : I \times I \times H \to H$ is a single-valued map.

Our problem generalizes the Integro-Differential Problem with time-dependent maximal monotone operators $A(t)$

$$(IDP_{A(t)}) \qquad \begin{cases} -\dot{u}(t) \in A(t)u(t) + \int_{T_0}^{t} f(t,s,u(s))ds \quad \text{a.e. } t \in I, \\ u(T_0) = u_0 \in D(A(T_0)), \end{cases}$$

stated in [14]. So, we aim to study a more general case, that is, when the operator depends on both time and state variables.

Note that the evolution problem when a single-valued map $f(\cdot,\cdot)$ instead of the integral perturbation in $(IDP_{A(t,u)})$ has been discussed in [1, 28, 34]. Here, we use Schauder's fixed point theorem (see also [1]) to establish our main existence result. For this purpose, we make use of the uniqueness of the solution to $(IDP_{A(t)})$ and an estimate of the velocity. However, the papers [28, 34] have followed a discretization method.

In the next part of the paper, we deal with the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem

$$(\mathcal{OCP}) \qquad \min \phi[u,a,b] = \phi_1(u(T)) + \int_0^T \phi_2(t,u(t),a(t),b(t),\dot{u}(t),\dot{a}(t),\dot{b}(t))dt,$$

on the set of control maps $(a(\cdot),b(\cdot))$ and the associated solutions $u(\cdot)$ of the $\mathcal{C}$ontrolled $\mathcal{P}$roblem

$$(\mathcal{CP}_{a,b}) \qquad \begin{cases} -\dot{u}(t) \in A(t,a(t))u(t) + \displaystyle\int_0^t f(t,s,b(s),u(s))ds \quad \text{a.e. } t \in [0,T], \\ u(t) \in D(A(t,a(t))), \quad t \in [0,T], \\ (a(\cdot),b(\cdot)) \in W^{1,2}([0,T],\mathbb{R}^{n+m}), \\ a(0) = a_0, \quad u(0) = u_0 \in D(A(0,a_0)), \end{cases}$$

where the cost functional $\phi_1 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and the running cost $\phi_2 : [0,T] \times \mathbb{R}^{4n+2m} \to \overline{\mathbb{R}}$ satisfy convenient conditions.

This investigation is inspired by the related one on the controlled integro-sweeping process in [5], see also [10–12, 17–19, 21–23, 29–32, 37], among others, for further contributions on optimization problems subject to controlled sweeping processes or control problems governed by maximal monotone operators.

Let us give the two following motivating examples: the first-one consists of minimizing a Bolza-type functional subject to the controlled differential inclusion of the form

$$(\mathcal{CP}_{x,a,b}) \qquad -\dot{u}(t) \in N_{C(x(t))}(u(t)) + f_1(a(t),u(t)) + \int_0^t f_2(b(s),u(s))ds \quad \text{a.e. } t \in [0,T],$$

where $A(t,x(t)) = N_{C(x(t))}$ is the normal cone of a moving set $C(x(t))$, $(x(\cdot),a(\cdot),b(\cdot))$ are controls acting in the moving sets, additive perturbations, and the integral part of the sweeping dynamics (see [5]). The second example concerns an optimization problem subject to the controlled differential inclusion described by

$$(\mathcal{CP}_{x,a}) \qquad -\dot{u}(t) \in N_{C(t)}(u(t)) + f(a(t),u(t)) \quad \text{a.e. } t \in [0,T],$$

where $C(t) = C + x(t)$ and $(x(\cdot),a(\cdot))$ are control maps (see [12]).

The considered problem $(\mathcal{OCP})$ is new, since we minimize over the solution set to the controlled integro-differential inclusion $(\mathcal{CP}_{a,b})$, where the controls act in both the state of the (time and state-dependent) operator and the integral perturbation. To the best of our knowledge, this topic is new in the scientific literature.

The rest of the paper is organized as follows. After recalling some preliminaries in Section 2, we handle $(IDP_{A(t)})$. Then, we develop the case $(IDP_{A(t,u)})$. Section 4 applies the obtained results to show the well-posedness of $(\mathcal{CP}_{a,b})$ and establishes the existence of optimal solutions to $(\mathcal{OCP})$.

## 2 Notation and preliminaries

Let $I := [T_0, T]$ be an interval of $\mathbb{R}$ and let $H$ be a real separable Hilbert space whose inner product is denoted $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. Denote by $\overline{B}_H$ the closed unit ball of $H$ and $\overline{B}_H[x, r]$ its closed ball of center $x \in H$ and radius $r > 0$.

On the space $\mathcal{C}_H(I)$ of continuous maps $x : I \to H$, we consider the norm of uniform convergence on $I$, $\|x\|_\infty = \sup_{t \in I} \|x(t)\|$.

By $L^p_H(I)$, for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \leq p < +\infty$ (resp. endowed with the usual essential supremum norm $\| \cdot \|_{L^\infty_H(I)}$). Denote by $W^{1,2}(I, H)$, the space of absolutely continuous functions from $I$ to $H$ with derivatives in $L^2_H(I)$.

Recall the definition and some properties of maximal monotone operators, see [3, 9, 36].

Let $A : D(A) \subset H \rightrightarrows H$ be a set-valued operator whose domain, range and graph are defined by

$$
\begin{aligned}
D(A) &= \{x \in H : \; Ax \neq \emptyset\}, \\
R(A) &= \{y \in H : \; \exists\, x \in D(A), \; y \in Ax\} = \cup\{Ax : \; x \in D(A)\}, \\
Gr(A) &= \{(x, y) \in H \times H : \; x \in D(A), \; y \in Ax\}.
\end{aligned}
$$

The operator $A : D(A) \subset H \rightrightarrows H$ is monotone, if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ whenever $(x_i, y_i) \in Gr(A)$, $i = 1, 2$. It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\mu > 0$, $R(I_H + \mu A) = H$, where $I_H$ stands for the identity map of $H$. If $A$ is a maximal monotone operator then, for every $x \in D(A)$, $Ax$ is non-empty, closed and convex. Then, the projection of the origin onto $Ax$, $A^0(x)$, exists and is unique.

Associated with any maximal monotone operator $A$ is the so-called resolvent $J^A_\mu = (I_H + \mu A)^{-1}$, $\mu > 0$, which turns out to be a nice firmly non-expansive operator with full domain. Resolvents not only provide an alternative view on monotone operators because one can recover the underlying maximal monotone operator via $(J^A_\mu)^{-1} - I_H$ but they also are crucial for the formulation of algorithms for finding zeros of $A$ (*e.g.*, the celebrated proximal point algorithm).

Recall that the Yosida approximation of $A$ of index $\mu > 0$ is defined by $A_\mu = \frac{1}{\mu}\left(I_H - J^A_\mu\right)$. Yosida approximations are powerful tools to study monotone operators. They can be viewed as regularizations and approximations of $A$ because $A_\mu$ is a single-valued Lipschitz-continuous operator on $H$ and $A_\mu$ approximates $A$ in the sense that $A_\mu x \to A^0(x) \in Ax$ as $\mu \to 0^+$.

Let us summarize the following properties of these operators:

$$J_\mu^A x \in D(A) \text{ and } A_\mu(x) \in A(J_\mu^A x) \quad \text{for every } x \in H, \tag{2.1}$$
$$\|A_\mu(x)\| \leq \|A^0(x)\| \quad \text{for every } x \in D(A).$$

The normal cone to a non-empty closed convex set $S$ at $x \in H$ denoted $N_S(x)$ defined by

$$N_S(x) = \{y \in H : \langle y, z - x\rangle \leq 0 \quad \forall z \in S\}, \tag{2.2}$$

is a maximal monotone operator.

Let $A : D(A) \subset H \rightrightarrows H$ and $B : D(B) \subset H \rightrightarrows H$ be two maximal monotone operators, then, we denote by $\operatorname{dis}(A, B)$ (see [35]) the pseudo-distance between $A$ and $B$ defined by

$$\operatorname{dis}(A, B) = \sup\left\{\frac{\langle y - y', x' - x\rangle}{1 + \|y\| + \|y'\|} : (x, y) \in Gr(A), (x', y') \in Gr(B)\right\}.$$

Clearly, $\operatorname{dis}(A, B) \in [0, +\infty]$, $\operatorname{dis}(A, B) = \operatorname{dis}(B, A)$ and $\operatorname{dis}(A, B) = 0$ iff $A = B$.

Let us first recall some useful lemmas that will be used in what follows (see [27]).

The first one permits to prove some inclusions using a convergence in the sense of the pseudo-distance.

**Lemma 2.1.** *Let $A_n$ $(n \in \mathbb{N})$, $A$ be maximal monotone operators of $H$ such that* $\operatorname{dis}(A_n, A) \to 0$. *Suppose also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then, $x \in D(A)$ and $y \in A(x)$.*

The next lemma deals with some modes of convergence in the sense of the pseudo-distance and the element of minimal norm.

**Lemma 2.2.** *Let $A_n$ $(n \in \mathbb{N})$, $A$ be maximal monotone operators of $H$ such that* $\operatorname{dis}(A_n, A) \to 0$ *and $\|A_n^0(x)\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(A_n)$. Then, for every $\zeta \in D(A)$, there exists a sequence $(\zeta_n)$ such that*

$$\zeta_n \in D(A_n),\ \zeta_n \to \zeta \ \text{ and } \ A_n^0(\zeta_n) \to A^0(\zeta).$$

Another approach on how to prove some inclusions using an estimate involving the element of minimal norm is provided by the following lemma.

**Lemma 2.3.** *Let $A$ be a maximal monotone operator. If $x, y \in H$ are such that*

$$\langle A^0(z) - y, z - x\rangle \geq 0 \quad \forall z \in D(A),$$

*then,* $x \in D(A)$ *and* $y \in A(x)$.

In the last lemma, we provide an estimate by means of the pseudo-distance, the element of minimal norm, and the resolvent.

**Lemma 2.4.** *Let* $A$, $B$ *be maximal monotone operators of* $H$. *Then, for* $\mu > 0$ *and* $x \in D(A)$ *one has*

$$\|x - J_\mu^B(x)\| \leq \mu\|A^0(x)\| + \operatorname{dis}(A, B) + \sqrt{\mu\big(1 + \|A^0(x)\|\big)\operatorname{dis}(A, B)}.$$

Recall the classical definition of Komlós convergence (see [16, p. 128]).

**Definition 2.5.** *A sequence* $(u_n)$ *in* $L^1_H(I)$ *Komlós converges to a function* $u \in L^1_H(I)$ *if for any subsequence* $(v_n)$ *of* $(u_n)$, *one has*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} v_j(t) = u(t) \quad \text{a.e.}$$

We also need the following theorem about the relationship between Komlós convergence and bounded sequences in $L^1_H(I)$ (see [25, Theorem 3.1]).

**Proposition 2.6.** *Let* $(u_n)$ *be a bounded sequence in* $L^1_H(I)$. *Then, there exists a subsequence* $(v_n)$ *of* $(u_n)$ *and* $u \in L^1_H(I)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} w_j(t) = u(t) \quad \text{a.e.}$$

*for any subsequence* $(w_n)$ *of* $(v_n)$.

Let us recall the Schauder's fixed point theorem (see [24]).

**Theorem 2.7.** *Let* $C$ *be a non-empty closed bounded convex subset of a Banach space* $E$ *and let* $f : C \to C$ *be a continuous map. If* $f(C)$ *is relatively compact, then,* $f$ *has a fixed point.*

The discrete version of Gronwall's lemma (see [27]) is given as follows:

**Lemma 2.8.** *Let* $(\alpha_i), (\beta_i), (\gamma_i)$ *and* $(\eta_i)$ *be sequences of non-negative real numbers such that*

$$\eta_{i+1} \leq \alpha_i + \beta_i(\eta_0 + \eta_1 + \cdots + \eta_{i-1}) + (1 + \gamma_i)\eta_i \quad \textit{for } i \in \mathbb{N}.$$

*Then,*

$$\eta_k \leq \left(\eta_0 + \sum_{j=0}^{k-1}\alpha_j\right)\exp\left(\sum_{j=0}^{k-1}(j\beta_j + \gamma_j)\right) \quad \textit{for } k \in \mathbb{N}^*.$$

We end this section by recalling the Gronwall-like differential inequality proved in [6].

**Lemma 2.9.** *Let $y : I \to \mathbb{R}$ be a non-negative absolutely continuous function and let $h_1, h_2, g : I \to \mathbb{R}_+$ be non-negative integrable functions. Suppose for some $\varepsilon > 0$*

$$\dot{y}(t) \leq g(t) + \varepsilon + h_1(t)y(t) + h_2(t)(y(t))^{\frac{1}{2}} \int_0^t (y(s))^{\frac{1}{2}} ds \quad a.e.\ t \in I.$$

*Then, for all $t \in I$, one has*

$$\begin{aligned}(y(t))^{\frac{1}{2}} \leq{} & (y(0) + \varepsilon)^{\frac{1}{2}} \exp\left(\int_0^t (h(s)+1)ds\right) + \frac{\varepsilon^{\frac{1}{2}}}{2} \int_0^t \exp\left(\int_s^t (h(r)+1)dr\right)ds \\ & + 2\left[\left(\int_0^t g(s)ds + \varepsilon\right)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}} \exp\left(\int_0^t (h(r)+1)dr\right)\right] \\ & + 2\int_0^t \Big(h(s)+1\Big) \exp\left(\int_s^t (h(r)+1)dr\right)\left(\int_0^s g(r)dr + \varepsilon\right)^{\frac{1}{2}} ds,\end{aligned}$$

*where $h(t) = \max\left(\frac{h_1(t)}{2}, \frac{h_2(t)}{2}\right)$ a.e. $t \in I$.*

# 3 Main result

We start this section by giving some important details to [14, Proposition 4.4] which asserts the existence result to $(IDP_{A(t)})$. We succeed further to obtain the uniqueness of the solution and an estimate of its derivative.

**Theorem 3.1.** *Let $A(t) : D(A(t)) \subset H \rightrightarrows H$ be a maximal monotone operator for each $t \in I$, satisfying*

$(h_1)$ *there exists a function $\beta(\cdot) \in W^{1,2}(I, \mathbb{R})$ which is non-negative on $[T_0, T[$ and non-decreasing with $\beta(T_0) = 0$ and $\beta(T) < +\infty$ such that*

$$\operatorname{dis}(A(t), A(s)) \leq |\beta(t) - \beta(s)| \quad \textit{for all } t, s \in I;$$

$(h_2)$ *there exists a non-negative real constant $c$ such that*

$$\|A^0(t)x\| \leq c(1 + \|x\|) \quad \textit{for all } t \in I,\ x \in D(A(t));$$

$(h_3)$ *the set $D(A(t))$ is relatively ball-compact for any $t \in I$.*

*Let $f : I \times I \times H \longrightarrow H$ be a map such that*

$(i)$ *the map $f(\cdot, \cdot, x)$ is measurable on $I \times I$ for each $x \in H$;*

(ii) *the map $f(t,s,\cdot)$ is continuous on $H$ for each $(t,s) \in I \times I$, and for every $\eta > 0$, there exists a non-negative function $\xi_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that for all $t, s \in I$ and for any $x, y \in \overline{B}_H[0,\eta]$*

$$\|f(t,s,x) - f(t,s,y)\| \leq \xi_\eta(t)\|x - y\|;$$

(iii) *there exists a non-negative real constant $m$ such that for all $(t,s,x) \in I \times I \times H$, one has*

$$\|f(t,s,x)\| \leq m(1 + \|x\|).$$

*Then, for all $u_0 \in D(A(T_0))$, the Integro-Differential Problem $(IDP_{A(t)})$ has a unique absolutely continuous solution $u(\cdot)$ that satisfies*

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)) \quad a.e.\ t \in I, \tag{3.1}$$

*for the non-negative real constant $K = (2(T - T_0)m + \frac{3}{2}c)(K_1 + 1) + 2$ where*

$$K_1 = \left(\|u_0\| + \left(2(T - T_0)m + \frac{3}{2}c + 2\right)(T + \beta(T))\right) \exp\left(\left((T - T_0)m + \frac{3}{2}c\right)(T - T_0) + m(T + \beta(T))^2\right).$$

*Proof.* [14, Proposition 4.4] ensures the existence of a solution $u(\cdot)$. Our main concern is to find a suitable estimate of $\dot{u}(\cdot)$, then, to prove that $u(\cdot)$ is unique.

For any $n \geq 1$, define a subdivision of $I$ by $T_0 = t_0^n < t_1^n < \cdots < t_n^n = T$.

Set for any $n \geq 1$ and $i = 0, 1, \ldots, n-1$,

$$h_{i+1}^n = t_{i+1}^n - t_i^n, \quad \beta_{i+1}^n = \beta(t_{i+1}^n) - \beta(t_i^n).$$

Suppose that

$$h_i^n \leq h_{i+1}^n, \quad \beta_i^n \leq \beta_{i+1}^n.$$

Define the function $\gamma(t) = t + \beta(t)$, $t \in I$. Choose the subdivision such that for all $i = 0, \ldots, n-1$ and $n \geq 1$,

$$\gamma_{i+1}^n = \beta_{i+1}^n + h_{i+1}^n \leq \frac{\gamma(T)}{n} =: \eta_n. \tag{3.2}$$

Fix any integer $n \geq 1$. Let us start by setting $u_0^n := u_0$, for $i = 0, \ldots, n-1$ and $\tau \in ]t_i^n, t_{i+1}^n]$,

$$u_{i+1}^n = J_{i+1}^n\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds\right\} d\tau\right), \tag{3.3}$$

where

$$J_{i+1}^n := J_{h_{i+1}^n}^{A(t_{i+1}^n)} = (I_H + h_{i+1}^n A(t_{i+1}^n))^{-1}.$$

In view of (2.1) and (3.3), observe that

$$u_{i+1}^n \in D(A(t_{i+1}^n)), \tag{3.4}$$

and

$$u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds \right\} d\tau \in u_{i+1}^n + h_{i+1}^n A(t_{i+1}^n) u_{i+1}^n.$$

Then, one writes

$$-\frac{u_{i+1}^n - u_i^n}{h_{i+1}^n} \in A(t_{i+1}^n) u_{i+1}^n + \frac{1}{h_{i+1}^n} \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds \right\} d\tau. \tag{3.5}$$

Thanks to Lemma 2.4 and (3.3), one has

$$\begin{aligned}
&\|u_{i+1}^n - u_i^n\| \\
&= \left\| J_{i+1}^n \left( u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds \right\} d\tau \right) - u_i^n \right\| \\
&\le \left\| J_{i+1}^n \left( u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds \right\} d\tau \right) - J_{i+1}^n(u_i^n) \right\| \\
&+ \|J_{i+1}^n(u_i^n) - u_i^n\| \\
&\le \int_{t_i^n}^{t_{i+1}^n} \left\| \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds \right\| d\tau + h_{i+1}^n \|A^0(t_i^n) u_i^n\| \\
&+ \mathrm{dis}\,(A(t_i^n), A(t_{i+1}^n)) + \sqrt{h_{i+1}^n (1 + \|A^0(t_i^n) u_i^n\|) \mathrm{dis}\,(A(t_i^n), A(t_{i+1}^n))}.
\end{aligned}$$

Using the fact that $\sqrt{ab} \le \frac{1}{2}(a+b)$ for all $a, b \in \mathbb{R}_+$, one has

$$\begin{aligned}
\|u_{i+1}^n - u_i^n\| \le& \int_{t_i^n}^{t_{i+1}^n} \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \|f(\tau, s, u_j^n)\| ds\, d\tau + \int_{t_i^n}^{t_{i+1}^n} \int_{t_i^n}^{\tau} \|f(\tau, s, u_i^n)\| ds\, d\tau \\
&+ \frac{3}{2} h_{i+1}^n \|A^0(t_i^n) u_i^n\| + \frac{3}{2} \mathrm{dis}\,(A(t_{i+1}^n), A(t_i^n)) + \frac{1}{2} h_{i+1}^n.
\end{aligned}$$

Next, combining $(h_1)$, $(h_2)$ and $(iii)$, one obtains

$$\begin{aligned}
\|u_{i+1}^n - u_i^n\| \le& \frac{3}{2} h_{i+1}^n c(1 + \|u_i^n\|) + \frac{3}{2} \beta_{i+1}^n + \frac{1}{2} h_{i+1}^n + h_{i+1}^n m \sum_{j=0}^{i-1} h_{j+1}^n (1 + \|u_j^n\|) \\
&+ \int_{t_i^n}^{t_{i+1}^n} (\tau - t_i^n) m (1 + \|u_i^n\|) d\tau,
\end{aligned}$$

along with (3.2) and the fact that $\tau - t_i^n \leq T - T_0$, one simplifies

$$\|u_{i+1}^n - u_i^n\| \leq h_{i+1}^n \left((T-T_0)m + \frac{3}{2}c\right)\|u_i^n\| + \gamma_{i+1}^n \left((T-T_0)m + \frac{3}{2}c + 2\right) + h_{i+1}^n m \sum_{j=0}^{i-1} h_{j+1}^n (1 + \|u_j^n\|). \tag{3.6}$$

Remember that $h_{i+1}^n \leq \eta_n$ for $i = 0, \dots, n-1$, and $\sum_{j=0}^{i-1} h_{j+1}^n \leq T - T_0$, along with (3.2), one gets

$$\|u_{i+1}^n - u_i^n\| \leq h_{i+1}^n \left((T-T_0)m + \frac{3}{2}c\right)\|u_i^n\| + \gamma_{i+1}^n \left(2(T-T_0)m + \frac{3}{2}c + 2\right) + \eta_n m \sum_{j=0}^{i-1} h_{j+1}^n \|u_j^n\|.$$

This yields

$$\|u_{i+1}^n\| \leq \left(1 + h_{i+1}^n \left((T-T_0)m + \frac{3}{2}c\right)\right)\|u_i^n\| + \gamma_{i+1}^n \left(2(T-T_0)m + \frac{3}{2}c + 2\right) + \eta_n^2 m \sum_{j=0}^{i-1} \|u_j^n\|.$$

An application of Lemma 2.8, it follows that for all $n \geq 1$ and $i = 1, \dots, n$

$$\|u_i^n\| \leq K_1, \tag{3.7}$$

with

$$K_1 := \left(\|u_0\| + \left(2(T-T_0)m + \frac{3}{2}c + 2\right)\gamma(T)\right)\exp\left(\left((T-T_0)m + \frac{3}{2}c\right)(T-T_0) + m\gamma^2(T)\right).$$

Coming back to (3.6) with the help of (3.2), one gets

$$\|u_{i+1}^n - u_i^n\| \leq \gamma_{i+1}^n K, \tag{3.8}$$

with

$$K := \left(2(T-T_0)m + \frac{3}{2}c\right)(K_1 + 1) + 2.$$

For each $n \geq 1$, we define the map $u_n(\cdot) : I \to H$ by: for $t \in [t_i^n, t_{i+1}^n[, 0 \leq i \leq n-1$

$$u_n(t) = u_i^n + \frac{t - t_i^n}{h_{i+1}^n}\left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} \left\{\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds\right\} d\tau\right) - \int_{t_i^n}^{t} \left\{\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n) ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n) ds\right\} d\tau, \tag{3.9}$$

$$u_n(T) = u_n^n, \quad u_n(T_0) = u_0^n.$$

It is clear that the function $u_n(\cdot) : I \to H$ is absolutely continuous for each $n \geq 1$, with $u_n(t_i^n) = u_i^n$ and $u_n(t_{i+1}^n) = u_{i+1}^n$. Moreover, for all $t \in ]t_i^n, t_{i+1}^n[$

$$\begin{aligned}\dot{u}_n(t) = \frac{1}{h_{i+1}^n}\bigg(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} \bigg\{\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n)ds\bigg\}d\tau\bigg) \\ - \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(t, s, u_j^n)ds - \int_{t_i^n}^{t} f(t, s, u_i^n)ds. \qquad (3.10)\end{aligned}$$

Combining $(iii)$, (3.7), (3.8) and (3.9), it results

$$\|u_n(t) - u_i^n\| \leq \|u_{i+1}^n - u_i^n\| + 2(T - T_0)m(1 + K_1)h_{i+1}^n \leq \gamma_{i+1}^n(K + 2(T - T_0)m(1 + K_1)),$$

along with (3.2) yields

$$\|u_n(t) - u_i^n\| \leq L\eta_n \qquad (3.11)$$

where

$$L := K + 2(T - T_0)m(1 + K_1).$$

Fix $s \in [t_i^n, t_{i+1}^n[$ and $t \in [t_j^n, t_{j+1}^n[$ with $j > i$. Then, by (3.2), (3.8) and (3.11), one has

$$\begin{aligned}\|u_n(t) - u_n(s)\| &\leq \|u_n(t) - u_j^n\| + \|u_j^n - u_i^n\| + \|u_i^n - u_n(s)\| \\ &\leq \|u_j^n - u_i^n\| + 2L\eta_n \leq \sum_{p=0}^{j-i-1} \|u_{i+p+1}^n - u_{i+p}^n\| + 2L\eta_n \\ &\leq K \sum_{p=0}^{j-i-1} \gamma_{i+p+1}^n + 2L\eta_n = K\left(\gamma(t_j^n) - \gamma(t_i^n)\right) + 2L\eta_n \\ &\leq K\left(\gamma(t) - \gamma(t_i^n)\right) + 2L\eta_n = K\left(\gamma(t) - \gamma(s) + \gamma(s) - \gamma(t_i^n)\right) + 2L\eta_n \\ &\leq K\left(\gamma(t) - \gamma(s) + \gamma(t_{i+1}^n) - \gamma(t_i^n)\right) + 2L\eta_n \\ &= K\left(\gamma(t) - \gamma(s)\right) + K\gamma_{i+1}^n + 2L\eta_n \\ &\leq K\left(\gamma(t) - \gamma(s)\right) + (K + 2L)\eta_n.\end{aligned}$$

Then, for any $n \geq 1$ and $T_0 \leq s \leq t \leq T$, one gets

$$\|u_n(t) - u_n(s)\| \leq K\big(\gamma(t) - \gamma(s)\big) + (K + 2L)\eta_n = K\big(t - s + \beta(t) - \beta(s)\big) + (K + 2L)\eta_n. \qquad (3.12)$$

Combining (3.4)-(3.5) and (3.9)-(3.10), it results that

$$-\dot{u}_n(t) \in A(\delta_n(t))u_n(\delta_n(t)) + g_n(t) \quad \text{a.e. } t \in I, \quad u_n(\delta_n(t)) \in D(A(\delta_n(t))),$$

where $g_n(t) = \int_{T_0}^t f(t, s, u_n(\theta_n(s)))ds$ and the maps $\theta_n, \delta_n : I \to I$ are defined by $\theta_n(T_0) = T_0$,

$\theta_n(t) = t_i^n$ if $t \in ]t_i^n, t_{i+1}^n]$ and $\delta_n(T_0) = T_0$, $\delta_n(t) = t_{i+1}^n$ if $t \in ]t_i^n, t_{i+1}^n]$ for some $i \in \{0, \dots, n-1\}$.

By Arzelà-Ascoli theorem (with the help of $(h_3)$), it is easy to show that the constructed sequence $(u_n(\cdot))$ uniformly converges to some $u(\cdot) \in W^{1,2}(I, H)$. To verify that $u(\cdot)$ is a solution of the required integro-differential inclusion, we proceed as in Step 3 in the proof of [2, Theorem 3.2] with appropriate changes.

Finally, passing to the limit in (3.12) as $n \to \infty$ (noting that $\eta_n \to 0$) yields

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)) \quad \text{a.e. } t \in I.$$

**Uniqueness**. Let $u_1(\cdot)$ and $u_2(\cdot)$ be two solutions to $(IDP_{A(t)})$. Since $A(t)$ is monotone then, one has

$$\frac{1}{2}\frac{d}{dt}\|u_2(t) - u_1(t)\|^2 \leq \left\langle \int_{T_0}^{t} f(t, s, u_1(s))ds - \int_{T_0}^{t} f(t, s, u_2(s))ds, u_2(t) - u_1(t) \right\rangle. \tag{3.13}$$

By the estimate of the velocity above, there exists a non-negative real constant $\eta$ such that $\|u_1(t)\| \leq \eta$ and $\|u_2(t)\| \leq \eta$, for each $t \in I$, along with $(ii)$, there is $\xi_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that

$$\|f(t, s, u_1(s)) - f(t, s, u_2(s))\| \leq \xi_\eta(t)\|u_1(s) - u_2(s)\| \quad \text{for all } (t, s) \in I \times I,$$

so that coming back to (3.13), it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_2(t) - u_1(t)\|^2 \leq \xi_\eta(t)\|u_2(t) - u_1(t)\| \int_{T_0}^{t} \|u_2(s) - u_1(s)\|ds.$$

Hence, Lemma 2.9 with $\varepsilon > 0$ arbitrary yields $u_1 = u_2$ and guarantees the uniqueness of the solution to $(IDP_{A(t)})$. □

Now, we are able to prove our main result concerning $(IDP_{A(t,u)})$.

**Theorem 3.2.** *Let $A(t, x) : D(A(t, x)) \subset H \rightrightarrows H$ be a maximal monotone operator for each $(t, x) \in I \times H$ satisfying*

$(H_1)$ *there exist a non-negative and non-decreasing real function $\alpha(\cdot) \in W^{1,2}(I, \mathbb{R})$ and a non-negative real constant $\lambda < 1$ such that*

$$\operatorname{dis}(A(t, x), A(s, y)) \leq |\alpha(t) - \alpha(s)| + \lambda\|x - y\| \quad \forall t, s \in I \text{ and } \forall x, y \in H;$$

$(H_2)$ *there exists a non-negative real constant $c$ such that*

$$\|A^0(t, x)y\| \leq c(1 + \|x\| + \|y\|) \quad \text{for all } (t, x) \in I \times H \text{ and } y \in D(A(t, x));$$

$(H_3)$ *for any bounded subset $X$ of $H$, the set $D(A(I \times X))$ is relatively ball-compact.*

*Let $f : I \times I \times H \longrightarrow H$ be a map satisfying assumptions (i)-(ii)-(iii) of Theorem 3.1.*

*Put $d = c(2 + \|u_0\|)$, $S = \left(2(T - T_0)m + \frac{3}{2}d\right)(S_1 + 1) + 2$, where*

$$S_1 = \left(\|u_0\| + \left(2(T - T_0)m + \frac{3}{2}d + 2\right)(T + \alpha(T) + 1)\right)$$
$$\exp\left(\left((T - T_0)m + \frac{3}{2}d\right)(T - T_0) + m(T + \alpha(T) + 1)^2\right).$$

*If $\lambda S < 1$, then, the Integro-Differential Problem $(IDP_{A(t,u)})$ admits an absolutely continuous solution $u(\cdot)$ that satisfies*

$$\|\dot{u}(t)\| \le \dot{\varphi}(t) \quad a.e.\ t \in I, \tag{3.14}$$

*where $\varphi : I \to \mathbb{R}_+$ is the absolutely continuous solution to*

$$\dot{\varphi}(t) = \frac{L}{1 - \lambda L}(1 + \dot{\alpha}(t)), \quad \varphi(T_0) = 0,$$

*for the non-negative real constant $L = \left(2(T - T_0)m + \frac{3}{2}d\right)(L_1 + 1) + 2$, where*

$$L_1 = \left(\|u_0\| + \left(2(T - T_0)m + \frac{3}{2}d + 2\right)(T + \alpha(T) + \lambda)\right)$$
$$\exp\left(\left((T - T_0)m + \frac{3}{2}d\right)(T - T_0) + m(T + \alpha(T) + \lambda)^2\right).$$

*Proof.* Observe that $1 - \lambda L > 0$ (in the differential equation) noting that $\lambda S < 1$ by assumption and since $L < S$ then, $\lambda < \frac{1}{L}$.

Since $\varphi(\cdot)$ is absolutely continuous, then, there exists some non-negative real constant $\delta > 0$ such that

$$\int_{T_0}^{T} \dot{\varphi}(s)ds < \delta \quad \text{for all } t \in I.$$

Let us just take $\delta = 1$ (for simplicity) and suppose that

$$\int_{T_0}^{T} \dot{\varphi}(s)ds < 1 \quad \text{for all } t \in I. \tag{3.15}$$

Let us consider the convex bounded closed subset $Y$ of the Banach space $\mathcal{C}_H(I)$ defined by

$$Y := \left\{u \in \mathcal{C}_H(I) : u(t) = u_0 + \int_{T_0}^{t} \dot{u}(s)ds,\ \|\dot{u}(t)\| \le \dot{\varphi}(t),\ t \in I\right\}.$$

Let $h \in Y$, and define the time-dependent maximal monotone operator $B_h(t) = A(t, h(t))$, $t \in I$

(as in [15, Lemma 5]). For all $T_0 \leq \tau \leq t \leq T$, one has using $(H_1)$

$$\begin{aligned} \operatorname{dis}(B_h(t), B_h(\tau)) = \operatorname{dis}(A(t,h(t)), A(\tau,h(\tau))) &\leq \alpha(t) - \alpha(\tau) + \lambda\|h(t) - h(\tau)\| \\ &\leq \int_\tau^t \dot\alpha(s)ds + \lambda\int_\tau^t \|\dot h(s)\|ds \leq \int_\tau^t [\dot\alpha(s) + \lambda\dot\varphi(s)]\, ds = \beta(t) - \beta(\tau), \end{aligned}$$

where $\beta(\cdot) \in W^{1,2}(I,\mathbb{R})$ is given by

$$\beta(t) = \int_{T_0}^t [\dot\alpha(s) + \lambda\dot\varphi(s)]\, ds, \quad \forall t \in I.$$

Furthermore, one writes using $(H_2)$ and (3.15)

$$\begin{aligned} \|B_h^0(t)x\| = \|A^0(t,h(t))x\| &\leq c(1 + \|h(t)\| + \|x\|) \\ &\leq c\left(1 + \|u_0\| + \int_{T_0}^t \dot\varphi(s)ds + \|x\|\right) \\ &\leq c(2 + \|u_0\| + \|x\|) \leq d(1 + \|x\|), \end{aligned}$$

for all $t \in I$ and $x \in D(A(t,h(t)))$, where $d = c(2 + \|u_0\|)$.

In view of Theorem 3.1, there exists a unique absolutely continuous solution $u_h : I \to H$ to the integro-differential inclusion

$$(\mathcal{I}_h) \qquad \begin{cases} -\dot u_h(t) \in B_h(t)u_h(t) + \displaystyle\int_{T_0}^t f(t,s,u_h(s))ds \quad \text{a.e. } t \in I,\ h \in Y, \\ u_h(t) \in D(B_h(t)) = D(A(t,h(t))), \quad \forall t \in I \\ u_h(T_0) = u_0 \in D(B_h(T_0)) = D(A(T_0,u_0)), \end{cases}$$

with

$$\|\dot u_h(t)\| \leq \rho(1 + \dot\alpha(t) + \lambda\dot\varphi(t)) \quad \text{a.e. } t \in I, \tag{3.16}$$

for the non-negative real constant $\rho = (2(T-T_0)m + \frac{3}{2}d)(\rho_1 + 1) + 2$, where

$$\begin{aligned} \rho_1 = &\left(\|u_0\| + \left(2(T-T_0)m + \frac{3}{2}d + 2\right)(T + \beta(T))\right) \\ &\exp\left(\left((T-T_0)m + \frac{3}{2}d\right)(T-T_0) + m(T+\beta(T))^2\right). \end{aligned}$$

Now, for each $h \in Y$, let us consider the map $\Phi$ defined on $Y$ by

$$\Phi(h)(t) := u_h(t), \quad t \in I,$$

where $u_h(\cdot)$ is the unique absolutely continuous solution to the latter integro-differential inclusion, namely $(\mathcal{I}_h)$.

Observe that $\rho < L$. Indeed, note by $(H_1)$ that $\alpha(\cdot)$ is a non-decreasing and non-negative function, along with the definition of $\beta(\cdot)$, one writes

$$\beta(T) = \int_{T_0}^{T} [\dot{\alpha}(s) + \lambda\dot{\varphi}(s)]\, ds \leq \alpha(T) + \lambda \int_{T_0}^{T} \dot{\varphi}(s)ds \leq \alpha(T) + \lambda,$$

using the fact that $\int_{T_0}^{T} \dot{\varphi}(s)ds < 1$ by (3.15). Then, from the definition of $\rho_1$ and $L_1$, this just shows that $\rho_1 < L_1$. We return therefore to the expression of $\rho$ and $L$ to compare.

Thus, coming back to (3.16), one writes

$$\|\dot{u}_h(t)\| \leq L(1 + \dot{\alpha}(t) + \lambda\dot{\varphi}(t)) = \dot{\varphi}(t). \tag{3.17}$$

As a result, $\Phi(h) \in Y$.

Also, note that using (3.15) for any $h \in Y$, one gets

$$\|u_h(t)\| \leq \|u_0\| + \varphi(T) \quad \text{for all } t \in I. \tag{3.18}$$

Let us prove that $\Phi(Y)$ is relatively compact in $\mathcal{C}_H(I)$.

On the one hand, note by (3.18) that for any $h \in Y$

$$h(t) \in (\|u_0\| + \varphi(T))\overline{B}_H.$$

On the other hand, since $u_h(t) \in D(A(t, h(t)))$ for each $t \in I$ then,

$$u_h(t) \in D(A(I \times (\|u_0\| + \varphi(T))\overline{B}_H)) \cap (\|u_0\| + \varphi(T))\overline{B}_H.$$

Using the ball-compactness assumption in $(H_3)$, one deduces that for each $t \in I$, $\{\Phi(h)(t),\ h \in Y\}$ is relatively compact in $H$, for any $t \in I$. Moreover, $(\Phi(h))$ is equi-continuous. By Arzelà-Ascoli theorem, $\Phi(Y)$ is relatively compact in $\mathcal{C}_H(I)$.

Now, we check that $\Phi$ is continuous. It is sufficient to show that: if $(h_n)$ uniformly converges to $h$ in $Y$, then, the sequence of absolutely continuous solutions $u_{h_n}$ associated with $h_n$ to the integro-differential inclusion

$$\begin{cases} -\dot{u}_{h_n}(t) \in A(t, h_n(t))u_{h_n}(t) + \displaystyle\int_{T_0}^{t} f(t, s, u_{h_n}(s))ds \quad \text{a.e. } t \in I,\ h_n \in Y, \\ u_{h_n}(T_0) = u_0 \in D(A(T_0, u_0)), \end{cases}$$

uniformly converges to the absolutely continuous solution $u_h$ associated with $h$ to the integro-differential inclusion $(\mathcal{I}_h)$.

As $(u_{h_n}(t))$ is relatively compact in $H$, for any $t \in I$ (from above) and $(u_{h_n})$ is equi-absolutely continuous, along with the estimate (3.16), we may assume that there exists some map $z \in W^{1,2}(I, H)$ such that

$$(u_{h_n}) \text{ uniformly converges to } z(\cdot), \tag{3.19}$$

and

$$(\dot{u}_{h_n})\ \sigma(L^1_H(I), L^\infty_H(I))\text{-converges to } w \in L^1_H(I) \text{ with } w = \dot{z} \text{ a.e.} \tag{3.20}$$

Put $\eta := \|u_0\| + \varphi(T)$. Then, by $(ii)$, there exists a non-negative function $\xi_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that for all $t, s \in I$

$$\|f(t, s, u_{h_n}(s)) - f(t, s, z(s))\| \leq \xi_\eta(t)\|u_{h_n}(s) - z(s)\|.$$

This along with the pointwise convergence of $(u_{h_n})$ to $z$ gives

$$\lim_{n\to\infty} \|f(t, s, u_{h_n}(s)) - f(t, s, z(s))\| = 0. \tag{3.21}$$

Note by (3.18) and $(iii)$ that for any $n$ and any $t, s \in I$

$$\|f(t, s, u_{h_n}(s))\| \leq m(1 + \eta), \tag{3.22}$$

along with (3.21), it follows from the Lebesgue dominated convergence theorem that

$$\left\|\int_{T_0}^t f(t, s, u_{h_n}(s))ds - \int_{T_0}^t f(t, s, z(s))ds\right\| \leq \int_{T_0}^t \|f(t, s, u_{h_n}(s)) - f(t, s, z(s))\|ds \to 0,$$

as $n \to \infty$.

Moreover, thanks to (3.22), we note that for any $t, s \in I$

$$\left\|\int_{T_0}^t f(t, s, u_{h_n}(s))ds\right\| \leq m(T - T_0)(1 + \eta). \tag{3.23}$$

This along with the convergence above, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{T_0}^T \left\|\int_{T_0}^t f(t, s, u_{h_n}(s))ds - \int_{T_0}^t f(t, s, z(s))ds\right\| dt = 0. \tag{3.24}$$

Define for any $n \geq 1$, the functions $g_n, g$ on $I$ by

$$g_n(t) = \int_{T_0}^t f(t, s, u_{h_n}(s))ds, \quad g(t) = \int_{T_0}^t f(t, s, z(s))ds \quad \text{for any } t \in I.$$

As $u_{h_n}(t) \in D(A(t, h_n(t)))$ for all $t \in I$ and $u_{h_n}(t) \to z(t)$, $(A^0(t, h_n(t))u_{h_n}(t))$ is bounded by $(H_2)$

and the boundedness of the sequences $(u_{h_n})$ and $(h_n)$ in $\mathcal{C}_H(I)$, for every $t \in I$

$$\operatorname{dis}(A(t, h_n(t)), A(t, h(t))) \leq \lambda \|h_n(t) - h(t)\| \to 0, \quad \text{as} \quad n \to \infty, \tag{3.25}$$

by $(H_1)$ and the uniform convergence of $(h_n)$ to $h$ in $\mathcal{C}_H(I)$. Thus, from Lemma 2.1, one deduces that $z(t) \in D(A(t, h(t)))$, for each $t \in I$.

Now, let us verify that $z$ satisfies the integro-differential inclusion

$$-\dot{z}(t) \in A(t, h(t))z(t) + \int_{T_0}^{t} f(t, s, z(s))ds \quad \text{a.e. } t \in I.$$

From (3.20) and (3.24), one deduces that $(\dot{u}_{h_n}(\cdot) + g_n(\cdot))$ $\sigma(L^1_H(I), L^\infty_H(I))$-converges to $\dot{z}(\cdot) + g(\cdot)$. Hence, $(\dot{u}_{h_n}(\cdot) + g_n(\cdot))$ Komlós-converges to $\dot{z}(\cdot) + g(\cdot)$, and there is a negligible set $V$ such that for $t \in I \setminus V$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\dot{u}_{h_j}(t) + g_j(t)) = \dot{z}(t) + g(t), \tag{3.26}$$

and

$$-\dot{u}_{h_n}(t) \in A(t, h_n(t))u_{h_n}(t) + g_n(t). \tag{3.27}$$

Let $x \in D(A(t, h(t)))$. From $(H_2)$ and (3.25) along with Lemma 2.2, there is a sequence $(x_n)$ such that $x_n \in D(A(t, h_n(t)))$,

$$x_n \to x \quad \text{and} \quad A^0(t, h_n(t))x_n \to A^0(t, h(t))x. \tag{3.28}$$

In view of (3.27), by the monotonicity of the operators $A(t, h_n(t))$ for each $n$ and $t \in I$, one has

$$\langle \dot{u}_{h_n}(t) + g_n(t), u_{h_n}(t) - x_n \rangle \leq \left\langle A^0(t, h_n(t))x_n, x_n - u_{h_n}(t) \right\rangle. \tag{3.29}$$

Note that

$$\begin{aligned}\langle \dot{u}_{h_n}(t) + g_n(t), z(t) - x \rangle &= \langle \dot{u}_{h_n}(t) + g_n(t), u_{h_n}(t) - x_n \rangle \\ &\quad + \langle \dot{u}_{h_n}(t) + g_n(t), z(t) - u_{h_n}(t) \rangle + \langle \dot{u}_{h_n}(t) + g_n(t), x_n - x \rangle,\end{aligned}$$

then,

$$\begin{aligned}\frac{1}{n} \sum_{j=1}^{n} \langle \dot{u}_{h_j}(t) + g_j(t), z(t) - x \rangle &= \frac{1}{n} \sum_{j=1}^{n} \langle \dot{u}_{h_j}(t) + g_j(t), u_{h_j}(t) - x_j \rangle \\ &\quad + \frac{1}{n} \sum_{j=1}^{n} \langle \dot{u}_{h_j}(t) + g_j(t), z(t) - u_{h_j}(t) \rangle + \frac{1}{n} \sum_{j=1}^{n} \langle \dot{u}_{h_j}(t) + g_j(t), x_j - x \rangle.\end{aligned}$$

Hence, combining (3.17), (3.23) and (3.29), one deduces that

$$\frac{1}{n}\sum_{j=1}^{n}\langle \dot{u}_{h_j}(t)+g_j(t), z(t)-x\rangle \leq \frac{1}{n}\sum_{j=1}^{n}\left\langle A^0(t,h_j(t))x_j, x_j-u_{h_j}(t)\right\rangle$$
$$+\left(\dot{\varphi}(t)+(T-T_0)m(1+\eta)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\|z(t)-u_{h_j}(t)\|+\frac{1}{n}\sum_{j=1}^{n}\|x_j-x\|\right).$$

Passing to the limit when $n\to\infty$, using (3.19), (3.26), (3.28), this last inequality yields

$$\langle \dot{z}(t)+g(t), z(t)-x\rangle \leq \langle A^0(t,h(t))x, x-z(t)\rangle \quad \text{a.e. } \forall x\in D(A(t,h(t))).$$

It results from Lemma 2.3 that

$$-\dot{z}(t)\in A(t,h(t))z(t)+g(t) \quad \text{a.e. } t\in I,$$

with $z(T_0)=u_0\in D(A(T_0,u_0))$ and by uniqueness $z=u_h$.

Therefore, one just checks that $\Phi(h_n)-\Phi(h)\to 0$ in $\mathcal{C}_H(I)$ as $n\to\infty$. Consequently, $\Phi: Y\to Y$ is continuous from the bounded convex closed subset $Y$ of the Banach space $\mathcal{C}_H(I)$ with $\Phi(Y)$ is relatively compact. Applying Schauder's fixed point theorem (see Theorem 2.7) there exists $h\in Y$ such that $h=\Phi(h)$, that is, $h(t)=u_h(t)$. Furthermore, the estimation (3.14) holds true on $I$. The proof of the theorem is then complete. □

We derive from Theorem 3.2, the particular case of the sweeping process, that is, $A(t,x)=N_{C(t,x)}$, for $(t,x)\in I\times H$.

**Corollary 3.3.** *Let $C: I\times H\rightrightarrows H$ be a set-valued mapping satisfying:*

($H_1'$) *For each $(t,y)\in I\times H$, $C(t,y)$ is a non-empty closed convex subset of $H$.*

($H_2'$) *There exist a non-negative real constant $\lambda<1$, and a function $\alpha\in W^{1,2}(I,\mathbb{R})$ which is non-negative on $[T_0,T[$ and non-decreasing such that*

$$|d(x,C(t,u))-d(x,C(s,v))|\leq|\alpha(t)-\alpha(s)|+\lambda||v-u|| \quad \forall t,s\in I, \quad \forall x,v,u\in H.$$

($H_3'$) *For any bounded subset $X$ of $H$, the set $C(I\times X)$ is relatively ball-compact.*

*Let $f: I\times I\times H\longrightarrow H$ be a map satisfying assumptions of Theorem 3.2.*

*Choose any* $d > 0$ *and put* $S = \left(2(T-T_0)m + \frac{3}{2}d\right)(S_1+1)+2$, *where*

$$S_1 = \left(\|u_0\| + \left(2(T-T_0)m + \frac{3}{2}d + 2\right)(T+\alpha(T)+1)\right)$$
$$\exp\left(\left((T-T_0)m + \frac{3}{2}d\right)(T-T_0) + m(T+\alpha(T)+1)^2\right).$$

*If* $\lambda S < 1$, *then, the integro-differential sweeping process*

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}u(t) + \displaystyle\int_{T_0}^{t} f(t,s,u(s))ds \quad a.e.\ t \in I, \\ u(T_0) = u_0 \in C(T_0,u_0), \end{cases}$$

*has an absolutely continuous solution* $u(\cdot)$. *Moreover, an appropriate estimate of* $\dot{u}(\cdot)$ *holds true.*

*Proof.* We follow the arguments used in the proof of [33, Corollary 8].

Let $A(t,x) = N_{C(t,x)}$, for each $(t,x) \in I \times H$. Then, for any $(t,x) \in I \times H$, $A(t,x) : D(A(t,x)) \subset H \rightrightarrows H$ is a maximal monotone operator with $D(A(t,x)) = C(t,x)$ and since the projection of the origin onto $N_{C(t,x)}y$ equals 0 then $\|A^0(t,x)y\| = 0$ for any $(t,x) \in I \times H$ and any $y \in C(t,x)$ (keeping in mind (2.2) and $(H_1')$). So, $(H_2)$ holds true for any non-negative real constant $c$. Moreover, it is easily seen that $(H_3)$ is satisfied. Let us verify $(H_1)$.

On the one hand, from [26], one has

$$d_H(C(t,u), C(s,v)) = \sup_{x \in H} |d(x, C(t,u)) - d(x, C(s,v))|, \tag{3.30}$$

where $d_H(\cdot,\cdot)$ denotes the Hausdorff distance between two closed subsets of $H$.

On the other hand, it is known from [35] that since $C(t,u)$, $C(s,v)$ are convex closed sets, then

$$\operatorname{dis}(N_{C(t,u)}, N_{C(s,v)}) = d_H(C(t,u), C(s,v)). \tag{3.31}$$

Combining (3.30) and (3.31) with $(H_2')$, then, $(H_1)$ holds true.

Hence, all assumptions of Theorem 3.2 are satisfied. The latter ensures the existence of a solution to the integro-differential sweeping process under consideration.

Furthermore, in view of (3.14), an appropriate estimate of $\dot{u}$ is obtained. $\square$

# 4 An optimal control problem

In this section, we focus on the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem ($\mathcal{OCP}$).

First, let us prove the existence and uniqueness of the solution to problem ($\mathcal{CP}_{a,b}$).

**Proposition 4.1.** *Let $H = \mathbb{R}^n$ and $I := [0, T]$. Fix a couple $(a(\cdot), b(\cdot)) \in W^{1,2}(I, \mathbb{R}^{n+m})$. Assume that for any $(t, y) \in I \times \mathbb{R}^n$, $A(t, y) : D(A(t, y)) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator satisfying assumptions $(H_1)$-$(H_2)$. Let $f : I \times I \times \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a map such that $f(\cdot, \cdot, x, y)$ is measurable on $I \times I$ for each $(x, y) \in \mathbb{R}^{m+n}$, $f(t, s, \cdot, \cdot)$ is continuous on $\mathbb{R}^{m+n}$ for each $(t, s) \in I \times I$ and satisfying the following assumptions*

($\underline{i}$) *there exists a non-negative real constant $M$, for any $b(\cdot) \in W^{1,2}(I, \mathbb{R}^m)$ such that*

$$\|f(t, s, b(s), x)\| \leq \|b(s)\| + M\|x\|, \quad \forall t, s \in I, \quad \forall x \in \mathbb{R}^n;$$

($\underline{ii}$) *for a non-negative real constant $\eta$ and any $b(\cdot) \in W^{1,2}(I, \mathbb{R}^m)$, there exists a non-negative real constant $l$ such that*

$$\|f(t, s, b(s), x_1) - f(t, s, b(s), x_2)\| \leq l\|x_1 - x_2\|, \quad \forall t, s \in I, \quad \forall x_1, x_2 \in \overline{B}_{\mathbb{R}^n}[0, \eta].$$

*Then, this couple control generates a unique solution $u(\cdot) \in W^{1,2}(I, \mathbb{R}^n)$ to the $\mathcal{C}$ontrolled $\mathcal{P}$roblem ($\mathcal{CP}_{a,b}$). Moreover, one has for a.e. $t \in I$*

$$\left\|\dot{u}(t) + \int_0^t f(t, s, b(s), u(s))ds\right\| \leq K(1 + \dot{\beta}(t)) + (1 + L)\zeta, \tag{4.1}$$

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)), \tag{4.2}$$

*where $\zeta = \max\left(\|b\|_{L^1_{\mathbb{R}^m}(I)}, TM\right)$, $L = \|u_0\| + K\int_0^T (1 + \dot{\beta}(s))ds$, and the function $\beta$ is defined by*

$$\beta(t) = \int_0^t [\dot{\alpha}(s) + \lambda\|\dot{a}(s)\|]ds, \quad t \in I,$$

*and $K$ is a non-negative real constant which depends on $\|u_0\|$, $\|a_0\|$, $c$, $\zeta$, $T$, and $\beta$.*

*Proof.* For any $t \in I$ and any fixed $a(\cdot) \in W^{1,2}(I, \mathbb{R}^n)$, define the time-dependent maximal monotone operators $B_a(t) := A(t, a(t))$ and proceed as in the first part of the proof of Theorem 3.2.

Let $\tau, t \in I$ such that $0 \leq \tau \leq t \leq T$. Then, one has by $(H_1)$

$$\operatorname{dis}(B_a(t), B_a(\tau)) = \operatorname{dis}(A(t, a(t)), A(\tau, a(\tau))) \leq \beta(t) - \beta(\tau),$$

and clearly $\beta(\cdot) \in W^{1,2}(I, \mathbb{R})$ is defined by

$$\beta(t) = \int_0^t [\dot{\alpha}(s) + \lambda \|\dot{a}(s)\|] ds, \quad t \in I.$$

Now, in view of $(H_2)$, there exists a non-negative real number $c$ such that for $t \in I$, $z \in D(A(t, a(t)))$

$$\|B_a^0(t)z\| = \|A^0(t, a(t))z\| \leq c(1 + \|a(t)\| + \|z\|) \leq c\left(1 + \left\|a_0 + \int_0^t \dot{a}(s)ds\right\| + \|z\|\right) \leq c_1(1 + \|z\|),$$

where $c_1 = c(1 + \|a_0\| + \int_0^T \|\dot{a}(s)\|ds)$.

Hence, the operator $B_a(t)$ satisfies $(h_1)$-$(h_2)$ of Theorem 3.1.

Next, for $b(\cdot) \in W^{1,2}(I, \mathbb{R}^m)$ fixed, define the function $f_b$ on $I \times I \times \mathbb{R}^n$ by

$$f_b(t, s, u) = f(t, s, b(s), u) \quad \text{for all } (t, s, u) \in I \times I \times \mathbb{R}^n.$$

It is clear that the function $f_b(\cdot, \cdot, u)$ is measurable on $I \times I$ for any fixed $u \in \mathbb{R}^n$, by assumption and by continuity of $b(\cdot)$. Moreover, from $(\underline{i})$ one gets

$$\|f_b(t, s, u)\| \leq \|b(s)\| + M\|u\| \leq \zeta(1 + \|u\|), \tag{4.3}$$

for all $(t, s, u) \in I \times I \times \mathbb{R}^n$, where $\zeta = \max(\|b\|_\infty, M)$.

Now, by $(\underline{ii})$ for a non-negative real constant $\eta$, there exists a non-negative real constant $l$ such that

$$\|f_b(t, s, u_1) - f_b(t, s, u_2)\| \leq l\|u_1 - u_2\|, \quad \forall t \in I, \quad \forall u_1, u_2 \in \overline{B}_{\mathbb{R}^n}[0, \eta].$$

Thus, the map $f_b$ satisfies assumptions of Theorem 3.1. Consequently, it follows the existence and uniqueness of the solution to the considered integro-differential inclusion.

Furthermore, in view of (3.1) and (4.3) along with the absolute continuity of $u(\cdot)$, estimates (4.1)-(4.2) hold true. The velocity $\dot{u}(\cdot)$ is clearly in $L^2_{\mathbb{R}^n}(I)$, and $u(\cdot) \in W^{1,2}(I, \mathbb{R}^n)$. The proof of the proposition is therefore finished. □

We are going to impose convenient assumptions that guarantee the existence of (global) optimal solutions to the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem $(\mathcal{OCP})$ subject to the solution set of the $\mathcal{C}$ontrolled $\mathcal{P}$roblem $(\mathcal{CP}_{a,b})$.

**Theorem 4.2** (Existence of optimal solutions)**.** *Assume that for any $(t, y) \in I \times \mathbb{R}^n$, $A(t, y) : D(A(t, y)) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator satisfying assumptions $(H_1)$-$(H_2)$. Let $f : I \times I \times \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a continuous map satisfying assumptions of Proposition 4.1. Suppose that the terminal cost functional $\phi_1 : \mathbb{R}^n \to \overline{\mathbb{R}}$ is lower semi-continuous, while the running cost $\phi_2 : I \times \mathbb{R}^{4n+2m} \to \overline{\mathbb{R}}$ is lower semi-continuous with respect to $t$ and is majorized by a summable* *function on $I$ along reference curves. Moreover, assume that $\phi_2(t,\cdot)$ is bounded from below on bounded sets for a.e. $t \in I$. Let the running cost $\phi_2$ be convex with respect to velocity variables $\dot{u}$, $\dot{a}$, $\dot{b}$, and that there is a minimizing sequence $(u^k(\cdot), a^k(\cdot), b^k(\cdot))$ of $(\mathcal{OCP})$, which $(a^k(\cdot), b^k(\cdot))$ is bounded in $W^{1,2}(I, \mathbb{R}^{n+m})$. Then, the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem $(\mathcal{OCP})$ admits an optimal solution in the space $W^{1,2}(I, \mathbb{R}^{2n+m})$.*

*Proof.* From Proposition 4.1, one deduces that the set of feasible solutions to the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem $(\mathcal{OCP})$ is non-empty. Let us fix the minimizing sequence of feasible solutions $(u^k(\cdot), a^k(\cdot), b^k(\cdot))$ for $(\mathcal{OCP})$ (from the statement of the theorem), which is bounded in $W^{1,2}(I, \mathbb{R}^{2n+m})$. This implies in particular that there exists a couple $(a_0, b_0) \in \mathbb{R}^{n+m}$ such that $(a^k(0), b^k(0)) \to (a_0, b_0)$ in this space as $k \to \infty$, while the triple $(u_0, a_0, b_0) = (u(0), a(0), b(0))$ clearly satisfies the initial conditions. It is readily seen that the sequence $(\dot{a}^k(\cdot), \dot{b}^k(\cdot))$ is bounded in $L^2_{\mathbb{R}^{n+m}}(I)$. Then, up to a subsequence that we do not relabel, there exists a couple $(v^a(\cdot), v^b(\cdot)) \in L^2_{\mathbb{R}^{n+m}}(I)$ such that

$$(\dot{a}^k(\cdot), \dot{b}^k(\cdot)) \text{ weakly converges in } L^2_{\mathbb{R}^{n+m}}(I) \text{ to } (v^a(\cdot), v^b(\cdot)).$$

Define now the functions

$$(\hat{a}(t), \hat{b}(t)) = (a_0, b_0) + \int_0^t (v^a(s), v^b(s))ds, \text{ for all } t \in I,$$

and observe that $(\dot{\hat{a}}(t), \dot{\hat{b}}(t)) = (v^a(t), v^b(t))$ for a.e. $t \in I$, and that the couple $(\hat{a}(\cdot), \hat{b}(\cdot))$ belongs to the space $W^{1,2}(I, \mathbb{R}^{n+m})$. It follows from above and the estimates of Proposition 4.1 that the sequence of the corresponding solutions $(u^k(\cdot))$ is uniformly bounded and equi-continuous on $I$. By Arzelà-Ascoli theorem, up to a subsequence that we do not relabel, $(u^k(\cdot))$ uniformly converges on $I$ to some $\hat{u}(\cdot) \in \mathcal{C}_{\mathbb{R}^n}(I)$ which is absolutely continuous on this interval. It follows from (4.2) that $(\dot{u}^k(\cdot))$ is bounded in $L^2_{\mathbb{R}^n}(I)$ and hence it weakly converges in $L^2_{\mathbb{R}^n}(I)$ up to a subsequence, to some function $w(\cdot)$ with $\dot{\hat{u}}(t) = w(t)$ for a.e. $t \in I$, that is,

$$(\dot{u}^k(\cdot)) \text{ weakly converges in } L^2_{\mathbb{R}^n}(I) \text{ to } \dot{\hat{u}}(\cdot). \tag{4.4}$$

The next step is to check that the limiting triple $\hat{z}(\cdot) = (\hat{u}(\cdot), \hat{a}(\cdot), \hat{b}(\cdot))$ satisfies the differential inclusion $(\mathcal{CP}_{a,b})$.

Since $f$ is continuous by assumption along with the preceding modes of convergence above, then, one has

$$f(t, s, b^k(s), u^k(s)) \to f(t, s, \hat{b}(s), \hat{u}(s)) \quad \text{as } k \to \infty, \quad t, s \in I.$$

By $(\underline{i})$, one has

$$\|f(t, s, b^k(s), u^k(s))\| \leq \|b^k(s)\| + M\|u^k(s)\|, \quad t, s \in I.$$

which is uniformly bounded since $(b^k(\cdot))$ and $(u^k(\cdot))$ are bounded in $\mathcal{C}_H(I)$.

From the Lebesgue dominated convergence theorem, it results

$$\lim_{k\to\infty}\left\|\int_0^t f(t,s,b^k(s),u^k(s))ds - \int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds\right\| = 0.$$

Moreover, note that

$$\left\|\int_0^t f(t,s,b^k(s),u^k(s))ds\right\| < \|b^k\|_{L^1_{\mathbb{R}^m}(I)} + M\|u^k\|_{L^1_{\mathbb{R}^n}(I)},$$

is uniformly bounded, then, the Lebesgue dominated convergence theorem yields

$$\lim_{k\to\infty}\int_0^T \left\|\int_0^t f(t,s,b^k(s),u^k(s))ds - \int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds\right\|^2 dt = 0. \tag{4.5}$$

Observe that $u^k(t) \in D(A(t,a^k(t)))$, $a^k(t) \to \hat{a}(t)$, $u^k(t) \to \hat{u}(t)$, for all $t \in I$, the sequence $(A^0(t,a^k(t))u^k(t))$ is bounded by $(H_2)$ for all $t \in I$, and

$$\operatorname{dis}(A(t,a^k(t)),A(t,\hat{a}(t))) \le \lambda\|a^k(t)-\hat{a}(t)\| \to 0, \quad \text{when } k\to\infty, \tag{4.6}$$

using $(H_1)$. Then, from Lemma 2.1 one deduces that $\hat{u}(t) \in D(A(t,\hat{a}(t))), \forall t \in I$.

Now, we are going to verify that $\hat{u}(\cdot)$ satisfies the integro-differential inclusion

$$-\dot{\hat{u}}(t) \in A(t,\hat{a}(t))\hat{u}(t) + \int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds \quad \text{a.e. } t\in I.$$

Define the maps $g^k$ and $g$ on $I$ by

$$g^k(t) = \int_0^t f(t,s,b^k(s),u^k(s))ds, \quad g(t) = \int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds, \quad \text{for any } t\in I.$$

In view of (4.4) and (4.5),

$$(\dot{u}^k(\cdot)+g^k(\cdot)) \text{ weakly converges in } L^2_{\mathbb{R}^n}(I) \text{ to } \dot{\hat{u}}(\cdot)+g(\cdot).$$

Hence, $(\dot{u}^k(\cdot)+g^k(\cdot))$ Komlós-converges to $\dot{\hat{u}}(\cdot)+g(\cdot)$ (see Proposition 2.6). So, there is a negligible set $Y$ such that for $t\in I\setminus Y$: $\dot{u}^k(\cdot)+g^k(\cdot)\to\dot{\hat{u}}(\cdot)+g(\cdot)$ Komlós, that is,

$$\lim_{k\to\infty}\frac{1}{k}\sum_{p=1}^{k}\left(\dot{u}^p(t)+\int_0^t f(t,s,b^p(s),u^p(s))ds\right) = \dot{\hat{u}}(t)+\int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds, \tag{4.7}$$

and

$$-\dot{u}^k(t) \in A(t,a^k(t))u^k(t) + \int_0^t f(t,s,b^k(s),u^k(s))ds.$$

Let $y \in D(A(t,\hat{a}(t)))$. Applying Lemma 2.2 to the maximal monotone operators $A(t,a^k(t))$ and $A(t,\hat{a}(t))$ that satisfy (4.6), ensures the existence of a sequence $(y^k)$ such that $y^k \in D(A(t,a^k(t)))$

$$y^k \to y \text{ and } A^0(t,a^k(t))y^k \to A^0(t,\hat{a}(t))y. \tag{4.8}$$

Since

$$-\dot{u}^k(t) \in A(t,a^k(t))u^k(t) + \int_0^t f(t,s,b^k(s),u^k(s))ds \quad \text{a.e.},$$

and $A(t,a^k(t))$ is monotone, one has

$$\langle \dot{u}^k(t) + g^k(t), u^k(t) - y^k \rangle \leq \langle A^0(t,a^k(t))y^k, y^k - u^k(t) \rangle. \tag{4.9}$$

Note that

$$\langle \dot{u}^k(t) + g^k(t), \hat{u}(t) - y \rangle = \langle \dot{u}^k(t) + g^k(t), u^k(t) - y^k \rangle + \langle \dot{u}^k(t) + g^k(t), \hat{u}(t) - u^k(t) - (y - y^k) \rangle,$$

then,

$$\begin{aligned}\frac{1}{k}\sum_{p=1}^{k}\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - y \rangle &= \frac{1}{k}\sum_{p=1}^{k}\left\langle \dot{u}^p(t) + g^p(t), y^p - y \right\rangle + \frac{1}{k}\sum_{p=1}^{k}\left\langle \dot{u}^p(t) + g^p(t), u^p(t) - y^p \right\rangle \\ &+ \frac{1}{k}\sum_{p=1}^{k}\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - u^p(t) \rangle.\end{aligned}$$

Thus, one gets using (4.9)

$$\begin{aligned}\frac{1}{k}\sum_{p=1}^{k}\left\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - y \right\rangle &\leq \frac{1}{k}\sum_{p=1}^{k}\left\langle \dot{u}^p(t) + g^p(t), y^p - y \right\rangle + \frac{1}{k}\sum_{p=1}^{k}\left\langle A^0(t,a^p(t))y^p, y^p - u^p(t) \right\rangle \\ &+ \frac{1}{k}\sum_{p=1}^{k}\left\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - u^p(t) \right\rangle.\end{aligned}$$

A passage to the limit as $k \to \infty$ with the use of (4.7)-(4.8), the boundedness of $(\dot{u}^p(\cdot) + g^p(\cdot))$ in $\mathbb{R}^n$, and the preceding modes of convergence above, yields

$$\left\langle \dot{\hat{u}}(t) + \int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds, \hat{u}(t) - y \right\rangle \leq \left\langle A^0(t,\hat{a}(t))y, y - \hat{u}(t) \right\rangle \quad \text{a.e.}$$

Hence, Lemma 2.3 guarantees that

$$-\dot{\hat{u}}(t) \in A(t,\hat{a}(t))\hat{u}(t) + \int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds \quad \text{a.e. } t \in I,$$

with $\hat{u}(t) \in D(A(t,\hat{a}(t)))$ for all $t \in I$. By uniqueness, it follows that $\hat{u}$ is the unique solution to $(\mathcal{CP}_{\hat{a},\hat{b}})$ associated to the couple control maps $(\hat{a}(\cdot),\hat{b}(\cdot))$. To justify further the optimality of $(\hat{u}(\cdot),\hat{a}(\cdot),\hat{b}(\cdot))$ in $(\mathcal{OCP})$, it is sufficient to show that

$$\phi[\hat{u},\hat{a},\hat{b}] \le \liminf_{k\to\infty} \phi[u^k,a^k,b^k] \tag{4.10}$$

for the Bolza-type functional in $(\mathcal{OCP})$. The latter (4.10) readily follows from the assumptions on the cost functions $\phi_1$ and $\phi_2$ due to the Mazur weak closure theorem and the Lebesgue dominated convergence theorem. Indeed, Mazur's theorem ensures that the weak convergence of $\{\dot{u}^k,\dot{a}^k,\dot{b}^k\}$ to $\{\dot{\hat{u}},\dot{\hat{a}},\dot{\hat{b}}\}$ in $L^2_{\mathbb{R}^{2n+m}}(I)$ yields the $L^2_{\mathbb{R}^{2n+m}}(I)$ strong convergence of convex combinations of $(\dot{u}^k,\dot{a}^k,\dot{b}^k)$ to $(\dot{\hat{u}},\dot{\hat{a}},\dot{\hat{b}})$, and thus the a.e. convergence of a subsequence of these convex combinations on $I$ to the limiting triple.

Employing finally the assumed convexity of the running cost $\phi_2$ with respect to the velocity variables verifies (4.10) and hence completes the proof of the theorem. $\square$

We derive from Theorem 4.2, the particular case of the controlled sweeping process.

**Corollary 4.3.** *Let $C : I\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with non-empty closed convex values. Suppose that there exist a non-negative real constant $\lambda<1$, and a function $\beta\in W^{1,2}(I,\mathbb{R})$ which is non-negative on $[0,T[$ and non-decreasing with $\beta(T)<\infty$ and $\beta(0)=0$ such that*

$$|d(u,C(t,y))-d(u,C(s,z))| \le |\beta(t)-\beta(s)|+\lambda\|y-z\| \quad \forall t,s\in I, \quad \forall u,y,z\in\mathbb{R}^n.$$

*Let $f : I\times I\times\mathbb{R}^{m+n}\to\mathbb{R}^n$, $\phi_1:\mathbb{R}^n\to\overline{\mathbb{R}}$ and $\phi_2 : I\times\mathbb{R}^{4n+2m}\to\overline{\mathbb{R}}$ be defined as in Theorem 4.2. The optimal control problem is*

$$\min \phi[u,a,b] = \phi_1(u(T)) + \int_0^T \phi_2(t,u(t),a(t),b(t),\dot{u}(t),\dot{a}(t),\dot{b}(t))dt,$$

*on the set of controls $(a(\cdot),b(\cdot))$ and the associated solutions $u(\cdot)$ of the controlled integro-sweeping process*

$$\begin{cases} -\dot{u}(t)\in N_{C(t,a(t))}u(t)+\displaystyle\int_0^t f(t,s,b(s),u(s))ds \quad a.e.\ t\in I,\\ u(t)\in C(t,a(t)), \quad t\in I,\\ (a(\cdot),b(\cdot))\in W^{1,2}(I,\mathbb{R}^{n+m}),\\ a(0)=a_0, \quad u(0)=u_0\in C(0,a_0).\end{cases}$$

*Then, the minimizing problem above admits an optimal solution in the space $W^{1,2}(I,\mathbb{R}^{2n+m})$.*

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