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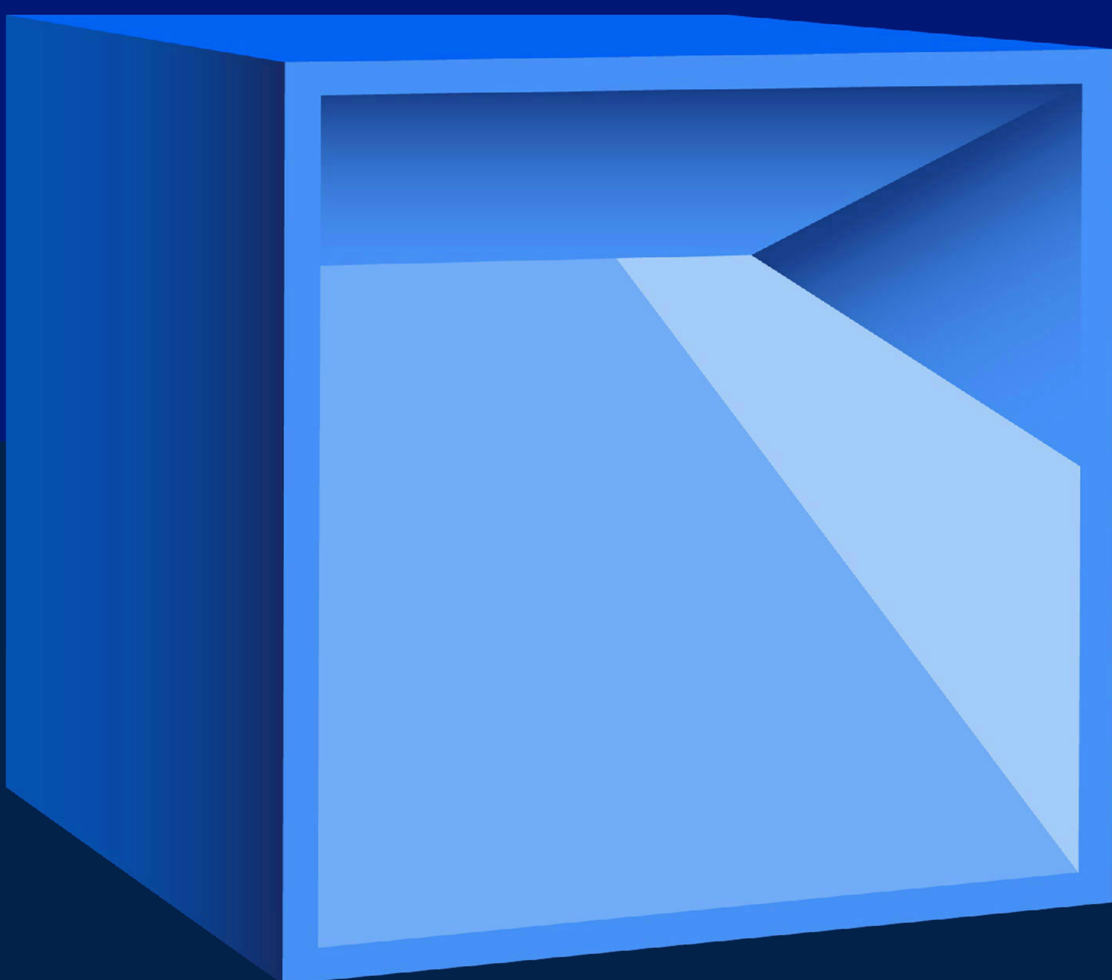

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SUMMARY

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Caputo fractional Iyengar type Inequalities

George A. Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

ABSTRACT

Here we present Caputo fractional Iyengar type inequalities with respect to L_p norms, with $1 \le p \le \infty$. The method is based on the right and left Caputo fractional Taylor's formulae.

RESUMEN

Aquí presentamos desigualdades de tipo Caputo fraccional Iyengar con respecto a las normas L_p, con $1 \le p \le \infty$. El método se basa en las fórmulas de Taylor fraccionales de Caputo derecha e izquierda.

Keywords and Phrases: Iyengar inequality, right and left Caputo fractional, Taylor formulae, Caputo fractional derivative.

2010 AMS Mathematics Subject Classification: 26A33, 26D10, 26D15.

1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [4].

Theorem 1. *Let* f *be a differentiable function on* $[a,b]$ *and* $|f'(x)| \leq M$*. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \tag{1}$$

We need

Definition 2. *([1], p. 394) Let* $\nu > 0$*,* $n = \lceil \nu \rceil$ *(*$\lceil \cdot \rceil$ *the ceiling of the number),* $f \in AC^n([a,b])$ *(i.e.* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$*). The left Caputo fractional derivative of order* ν *is defined as*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \tag{2}$$

$\forall\ x \in [a,b]$*, and it exists almost everywhere over* $[a,b]$.

We need

Definition 3. *([2], p. 336-337) Let* $\nu > 0$*,* $n = \lceil \nu \rceil$*,* $f \in AC^n([a,b])$*. The right Caputo fractional derivative of order* ν *is defined as*

$$D_{b-}^{\nu} f(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (z-x)^{n-\nu-1} f^{(n)}(z)\,dz, \tag{3}$$

$\forall\ x \in [a,b]$*, and exists almost everywhere over* $[a,b]$.

2 Main Results

We present the following Caputo fractional Iyengar type inequalities:

Theorem 4. *Let* $\nu > 0$*,* $n = \lceil \nu \rceil$ *(*$\lceil \cdot \rceil$ *is the ceiling of the number), and* $f \in AC^n([a,b])$ *(i.e.* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$*). We assume that* $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_{\infty}([a,b])$*. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_{\infty}([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \tag{4}$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$*, the right hand side of (4) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_\infty([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{5}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$*, for all* $k = 0, 1, \ldots, n-1$*, we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_\infty([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{6}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \ldots,\ N \in \mathbb{N}$*, it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_\infty([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{7}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$*,* $k = 1, \ldots, n-1$*, from (7) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_\infty([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{8}$$

$j = 0, 1, 2, \ldots, N$,

vi) when $N = 2$ *and* $j = 1$*, (8) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_\infty([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{9}$$

vii) when $0 < \nu \leq 1$*, inequality (9) is again valid without any boundary conditions.*

Proof. Let $\nu > 0$, $n = \lceil \nu \rceil$, and $f \in AC^n([a,b])$. Then by ([3], p. 54) left Caputo fractional Taylor's formula we have

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^{\nu} f(t)\, dt, \tag{10}$$

$\forall\, x \in [a,b]$.

Also by ([2], p. 341) right Caputo fractional Taylor's formula we get:

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k = \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} D_{b-}^{\nu} f(z)\, dz, \tag{11}$$

$\forall\, x \in [a,b]$.

By (10) we derive

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\|D_{*a}^{\nu} f\|_{L_\infty([a,b])}}{\Gamma(\nu+1)} (x-a)^{\nu}, \tag{12}$$

and by (11) we obtain

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\left\|D_{b-}^{\nu} f\right\|_{L_\infty([a,b])}}{\Gamma(\nu+1)} (b-x)^{\nu}, \tag{13}$$

$\forall\, x \in [a,b]$.

Call

$$\gamma_1 := \frac{\|D_{*a}^{\nu} f\|_{L_\infty([a,b])}}{\Gamma(\nu+1)}, \tag{14}$$

and

$$\gamma_2 := \frac{\left\|D_{b-}^{\nu} f\right\|_{L_\infty([a,b])}}{\Gamma(\nu+1)}. \tag{15}$$

Set

$$\gamma := \max(\gamma_1, \gamma_2). \tag{16}$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \gamma (x-a)^{\nu}, \tag{17}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \gamma (b-x)^{\nu}, \tag{18}$$

$\forall\, x \in [a,b]$.

Hence it holds

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k-\gamma(x-a)^{\nu}\leq f(x)\leq\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k+\gamma(x-a)^{\nu} \tag{19}$$

and

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k-\gamma(b-x)^{\nu}\leq f(x)\leq\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k+\gamma(b-x)^{\nu}, \tag{20}$$

$\forall\ x\in[a,b]$.

Let any $t\in[a,b]$, then by integration over $[a,t]$ and $[t,b]$, respectively, we obtain

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{(k+1)!}(t-a)^{k+1}-\frac{\gamma}{(\nu+1)}(t-a)^{\nu+1}\leq\int_a^t f(x)\,dx\leq$$

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{(k+1)!}(t-a)^{k+1}+\frac{\gamma}{(\nu+1)}(t-a)^{\nu+1}, \tag{21}$$

and

$$-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{(k+1)!}(t-b)^{k+1}-\frac{\gamma}{(\nu+1)}(b-t)^{\nu+1}\leq\int_t^b f(x)\,dx\leq$$

$$-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{(k+1)!}(t-b)^{k+1}+\frac{\gamma}{(\nu+1)}(b-t)^{\nu+1}. \tag{22}$$

Adding (21) and (22), we obtain

$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}-f^{(k)}(b)(t-b)^{k+1}\right]\right\}-$$

$$\frac{\gamma}{(\nu+1)}\left[(t-a)^{\nu+1}+(b-t)^{\nu+1}\right]\leq\int_a^b f(x)\,dx\leq$$

$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}-f^{(k)}(b)(t-b)^{k+1}\right]\right\}+$$

$$\frac{\gamma}{(\nu+1)}\left[(t-a)^{\nu+1}+(b-t)^{\nu+1}\right], \tag{23}$$

$\forall\ t\in[a,b]$.

Consequently we derive:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\gamma}{(\nu+1)}\left[(t-a)^{\nu+1}+(b-t)^{\nu+1}\right], \tag{24}$$

$\forall\ t \in [a, b]$.

Let us consider

$$g(t) := (t-a)^{\nu+1} + (b-t)^{\nu+1}, \ \forall\ t \in [a, b].$$

Hence

$$g'(t) = (\nu+1)\left[(t-a)^{\nu} - (b-t)^{\nu}\right] = 0,$$

giving $(t-a)^{\nu} = (b-t)^{\nu}$ and $t-a = b-t$, that is $t = \frac{a+b}{2}$ the only critical number here.

We have $g(a) = g(b) = (b-a)^{\nu+1}$, and $g\left(\frac{a+b}{2}\right) = \frac{(b-a)^{\nu+1}}{2^{\nu}}$, which the minimum of g over $[a, b]$.

Consequently the right hand side of (24) is minimized when $t = \frac{a+b}{2}$, with value $\frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^{\nu}}$.

Assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0, 1, ..., n-1$, then we obtain that

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{25}$$

which is a sharp inequality.

When $t = \frac{a+b}{2}$, then (24) becomes

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right| \leq$$

$$\frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{26}$$

Next let $N \in \mathbb{N}$, $j = 0, 1, 2, ..., N$ and $t_j = a + j\left(\frac{b-a}{N}\right)$, that is $t_0 = a$, $t_1 = a + \frac{b-a}{N}, ..., t_N = b$.

Hence it holds

$$t_j - a = j\left(\frac{b-a}{N}\right), \quad (b - t_j) = (N-j)\left(\frac{b-a}{N}\right), \quad j = 0, 1, 2, ..., N. \tag{27}$$

We notice that

$$(t_j - a)^{\nu+1} + (b - t_j)^{\nu+1} = \left(\frac{b-a}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1}\right], \tag{28}$$

$j = 0, 1, 2, ..., N$,

and (for $k = 0, 1, ..., n-1$)

$$\left[f^{(k)}(a)(t_j - a)^{k+1} + (-1)^k f^{(k)}(b)(b - t_j)^{k+1}\right] =$$

$$\left[f^{(k)}(a)\, j^{k+1} \left(\frac{b-a}{N}\right)^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1} \left(\frac{b-a}{N}\right)^{k+1}\right] =$$

$$\left(\frac{b-a}{N}\right)^{k+1}\left[f^{(k)}(a)\,j^{k+1}+(-1)^k f^{(k)}(b)\,(N-j)^{k+1}\right], \tag{29}$$

$j = 0, 1, 2, ..., N$.

By (24) we get

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[f^{(k)}(a)\,j^{k+1}+(-1)^k f^{(k)}(b)\,(N-j)^{k+1}\right]\right|$$

$$\leq\frac{\gamma}{(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{30}$$

$j = 0, 1, 2, ..., N$.

If $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, then (30) becomes

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)\,f(b)]\right|\leq$$

$$\frac{\gamma}{(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{31}$$

$j = 0, 1, 2, ..., N$.

When $N = 2$ and $j = 1$, then (31) becomes

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\gamma}{(\nu+1)}2\left(\frac{b-a}{2}\right)^{\nu+1}=\frac{\gamma}{(\nu+1)}\frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{32}$$

Let $0 < \nu \leq 1$, then $n = \lceil \nu \rceil = 1$. In that case, without any boundary conditions, we derive from (32) again that

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq\frac{\gamma}{(\nu+1)}\frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{33}$$

The theorem is proved in all cases. ☐

We give

Theorem 5. *Let* $\nu \geq 1$, $n = \lceil \nu \rceil$, *and* $f \in AC^n([a,b])$. *We assume that* $D_{*a}^{\nu}f, D_{b-}^{\nu}f \in L_1([a,b])$. *Then*

i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)\,(t-a)^{k+1}+(-1)^k f^{(k)}(b)\,(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left[(t-a)^{\nu}+(b-t)^{\nu}\right], \tag{34}$$

$\forall\ t\in[a,b]$,

ii) when $\nu=1$*, from (34), we have*

$$\left|\int_a^b f(x)\,dx-\left[f(a)(t-a)+f(b)(b-t)\right]\right|\leq$$

$$\left\|f'\right\|_{L_1([a,b])}(b-a),\quad \forall\ t\in[a,b], \tag{35}$$

iii) from (35), we obtain ($\nu=1$ *case)*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq\left\|f'\right\|_{L_1([a,b])}(b-a), \tag{36}$$

iv) at $t=\frac{a+b}{2}$, $\nu>1$*, the right hand side of (34) is minimized, and we get:*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{37}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$*, for all* $k=0,1,...,n-1$*;* $\nu>1$*, from (37), we obtain*

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{38}$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,\ N\in\mathbb{N}$*, it holds*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{39}$$

vii) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n-1$*, from (39) we get:*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{40}$$

$j = 0, 1, 2, ..., N$,

viii) when $N = 2$ *and* $j = 1$, *(40) turns to*

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{2}(f(a)+f(b))\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}. \tag{41}$$

Proof. Here $\nu \geq 1$ and $D_{*a}^{\nu}f, D_{b-}^{\nu}f \in L_1([a,b])$. By (10) we get

$$\left|f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \frac{1}{\Gamma(\nu)}(x-a)^{\nu-1}\int_a^x \left|D_{*a}^{\nu}f(t)\right|dt$$

$$\leq \frac{(x-a)^{\nu-1}}{\Gamma(\nu)}\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])}, \tag{42}$$

$\forall\, x \in [a,b]$.

That is

$$\left|f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \frac{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])}}{\Gamma(\nu)}(x-a)^{\nu-1}, \tag{43}$$

$\forall\, x \in [a,b]$.

By (11) we get

$$\left|f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{1}{\Gamma(\nu)}(b-x)^{\nu-1}\int_x^b \left|D_{b-}^{\nu}f(z)\right|dz$$

$$\leq \frac{(b-x)^{\nu-1}}{\Gamma(\nu)}\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}, \tag{44}$$

$\forall\, x \in [a,b]$.

That is

$$\left|f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}}{\Gamma(\nu)}(b-x)^{\nu-1}, \tag{45}$$

$\forall\, x \in [a,b]$.

Call

$$\delta_1 := \frac{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])}}{\Gamma(\nu)}, \tag{46}$$

and

$$\delta_2 := \frac{\left\| D_{b-}^{\nu} f \right\|_{L_1([a,b])}}{\Gamma(\nu)}. \tag{47}$$

Set

$$\delta := \max(\delta_1, \delta_2). \tag{48}$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \delta (x-a)^{\nu-1}, \tag{49}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \delta (b-x)^{\nu-1}, \tag{50}$$

$\forall\, x \in [a, b]$.

As in the proof of Theorem 4, we get:

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\delta}{\nu} \left[(t-a)^{\nu} + (b-t)^{\nu} \right], \tag{51}$$

$\forall\, t \in [a, b]$.

The rest of the proof is similar to the proof of Theorem 4. ☐

We continue with

Theorem 6. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$; $f \in AC^n([a,b])$, *with* $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_q([a,b])$. *Then*

i)

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}} \left[(t-a)^{\nu + \frac{1}{p}} + (b-t)^{\nu + \frac{1}{p}} \right], \tag{52}$$

$\forall\, t \in [a, b]$,

ii) at $t = \frac{a+b}{2}$, *the right hand side of (52) is minimized, and we get:*

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{53}$$

iii) if $f^{(k)}(a)=f^{(k)}(b)=0$*, for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{54}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,$ $N\in\mathbb{N}$*, it holds*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq \frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{55}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$*,* $k=1,...,n-1$*, from (55) we get:*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)f(b)]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{56}$$

for $j=0,1,2,...,N$*,*

vi) when $N=2$ *and* $j=1$*, (56) turns to*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{57}$$

vii) when $1/q<\nu\leq 1$*, inequality (57) is again valid but without any boundary conditions.*

Proof. Here $\nu>0$, $n=\lceil\nu\rceil$, $f\in AC^n([a,b])$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, with $D_{*a}^{\nu}f, D_{b-}^{\nu}f\in L_q([a,b])$. By (10) we have

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right|\leq\frac{1}{\Gamma(\nu)}\int_a^x(x-t)^{\nu-1}\left|D_{*a}^{\nu}f(t)\right|dt\leq$$

$$\frac{1}{\Gamma(\nu)}\left(\int_a^x (x-t)^{p(\nu-1)}\,dt\right)^{\frac{1}{p}}\left(\int_a^x |D_{*a}^{\nu}f(t)|^q\,dt\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\nu)}\frac{(x-a)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])}. \tag{58}$$

Here we assume that $\nu > \frac{1}{q} \Leftrightarrow p(\nu-1)+1>0$. So, we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \frac{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}(x-a)^{\nu-\frac{1}{q}}, \tag{59}$$

$\forall\, x \in [a,b]$.

By (11) we have

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{1}{\Gamma(\nu)}\int_x^b (z-x)^{\nu-1}\left|D_{b-}^{\nu}f(z)\right|dz \leq$$

$$\frac{1}{\Gamma(\nu)}\left(\int_x^b (z-x)^{p(\nu-1)}\,dz\right)^{\frac{1}{p}}\left(\int_x^b \left|D_{b-}^{\nu}f(z)\right|^q dz\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\nu)}\frac{(b-x)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}. \tag{60}$$

So, we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}(b-x)^{\nu-\frac{1}{q}}, \tag{61}$$

$\forall\, x \in [a,b]$.

Call

$$\rho_1 := \frac{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}, \tag{62}$$

and

$$\rho_2 := \frac{\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}. \tag{63}$$

Set

$$\rho := \max(\rho_1,\rho_2), \quad m := \nu-\frac{1}{q} > 0. \tag{64}$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \rho(x-a)^m, \tag{65}$$

and

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \rho(b-x)^m, \tag{66}$$

$\forall\ x \in [a, b]$.

As in the proof of Theorem 4, we obtain:

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right| \leq$$

$$\frac{\rho}{(m+1)}\left[(t-a)^{m+1} + (b-t)^{m+1}\right] =$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu} f\right\|_{L_q([a,b])}, \left\|D_{b-}^{\nu} f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}\left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}}\right], \tag{67}$$

$\forall\ t \in [a, b]$.

The rest of the proof is similar to the proof of Theorem 4. $\square$

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Z_k-Magic Labeling of Path Union of Graphs

P. Jeyanthi[1] K. Jeya Daisy[2] and Andrea Semaničová-Feňovčíková[3]

[1] *Research Centre, Department of Mathematics,*
Govindammal Aditanar College for Women,
Tiruchendur 628215, Tamilnadu, India
jeyajeyanthi@rediffmail.com

[2] *Department of Mathematics,*
Holy Cross College, Nagercoil, Tamilnadu, India
jeyadaisy@yahoo.com

[3] *Department of Applied Mathematics and Informatics,*
Technical University, Košice, Slovak Republic
andrea.fenovcikova@tuke.sk

ABSTRACT

For any non-trivial Abelian group A under addition a graph G is said to be A-*magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-*magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k and these graphs are referred as k-*magic* graphs. In this paper we prove that the graphs such as path union of cycle, generalized Petersen graph, shell, wheel, closed helm, double wheel, flower, cylinder, total graph of a path, lotus inside a circle and n-pan graph are Z_k-magic graphs.

RESUMEN

Para cualquier grupo Abeliano no-trivial A bajo adición, un grafo G se dice A-*mágico* si existe un etiquetado $f\colon E(G) \to A - \{0\}$ tal que el etiquetado de un vértice f^+ definido como $f^+(v) = \sum f(uv)$, tomado sobre todos los ejes uv incidentes en v, es constante. Un grafo A-*mágico* G se dice Z_k-mágico si el grupo A es Z_k, el grupo de enteros módulo k y estos se llaman grafos k-*mágicos*. En este paper demostramos que los grafos tales como la unión por caminos de ciclos, grafos de Petersen generalizados, concha, rueda, casco cerrado, rueda doble, flor, cilindro, el grafo total de un camino, lotos dentro de un círculo y n-sartenes son todos grafos Z_k-mágicos.

Keywords and Phrases: A-magic labeling, Z_k-magic labeling, Z_k-magic graph, generalized Petersen graph, shell, wheel, closed helm, double wheel, flower, cylinder, total graph of a path, lotus inside a circle, n-pan graph.

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1 Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian in [1]. If the labels of edges are distinct positive integers and for each vertex v the sum of the labels of all edges incident with v is the same for every vertex v in the given graph then the labeling is called a magic labeling. Sedláček [10] introduced the concept of A-magic graphs. A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices. Low and Lee [9] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu, Lam and Sun [12] proved that the product and composition of A-magic graphs were also A-magic.

For any non-trivial Abelian group A under addition a graph G is said to be A-*magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-*magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k. These Z_k-magic graphs are referred to as k-*magic graphs*. Shiu and Low [13] determined all positive integers k for which fans and wheels have a Z_k-magic labeling with a magic constant 0. Kavitha and Thirusangu [8] obtained a Z_k-magic labeling of two cycles with a common vertex. Motivated by the concept of A-magic graph in [10] and the results in [9, 12, 13] Jeyanthi and Jeya Daisy [2, 3, 4, 5, 6, 7] proved that some standard graphs admit Z_k-magic labeling. We use the following definitions in the subsequent section.

Definition 1.1. *Let $G_1, G_2, \ldots, G_n$, $n \geq 2$, be copies of a graph G. Let $v_i \in V(G_i)$, $i = 1, 2, \ldots, n$, be the vertex corresponding to the vertex $v \in V(G)$ in the i^{th} copy of G_i. We denoted by $P(n.G^v)$ the graph obtained by adding the edge v_iv_{i+1}, to G_i and G_{i+1}, $1 \leq i \leq n-1$, and we call $P(n.G^v)$ the path union of n copies of the graph G.*

Note, that up to isomorphism, we obtain $|V(G)|$ graphs $P(n.G^v)$. This operation was defined in [11].

Definition 1.2. *A generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$ is a 3-regular graph with the vertex set $\{u_i, v_i : i = 1, 2, \ldots, n\}$ and the edge set $\{u_iv_i, u_iu_{i+1}, v_iv_{i+m} : i = 1, 2, \ldots, n\}$, where the indices are taken over modulo n.*

Definition 1.3. *A shell graph S_n, $n \geq 4$, is obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called an apex.*

Definition 1.4. *A wheel graph W_n, $n \geq 3$, is obtained by joining the vertices of a cycle C_n to an extra vertex called the centre. The vertices of degree three are called rim vertices.*

Definition 1.5. *A helm graph H_n, $n \geq 3$, is obtained from a wheel W_n by adjoining a pendant edge at each vertex of the wheel except the center.*

Definition 1.6. *A closed helm graph CH_n, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to form a cycle.*

Definition 1.7. *A double wheel graph DW_n, $n \geq 3$, is obtained by joining the vertices of two cycles C_n to an extra vertex called the hub.*

Definition 1.8. *A flower graph Fl_n, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.*

Definition 1.9. *A Cartesian product of a cycle C_n, $n \geq 3$, and a path on two vertices is called a cylinder graph $C_n \square P_2$.*

Definition 1.10. *A total graph $T(G)$ is a graph with the vertex set $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G.*

Definition 1.11. *A lotus inside a circle LC_n, $n \geq 3$, is a graph obtained from a wheel W_n by subdividing every edge forming the outer cycle and joining these new vertices to form a cycle.*

Definition 1.12. *An n-pan graph, $n \geq 3$, is obtained by attaching a pendent edge to a vertex of a cycle C_n.*

2 Z_k-Magic Labeling of Path Union of Graphs

In this section we prove that the graphs such as path union of cycle, generalized Petersen graph, shell, wheel, closed helm, double wheel, flower, cylinder, total graph of a path, lotus inside a circle and n-pan graph are Z_k-magic graphs.

Let v be a vertex of a cycle C_r, $r \geq 3$. According to the symmetry all $P(n.C_r^v)$ are isomorphic. Thus we use the notation $P(n.C_r)$.

Theorem 2.1. *Let $r \geq 3$ and $n \geq 2$ be integers. The path union of a cycle $P(n.C_r)$ is Z_k-magic for $k \geq 3$ when r is odd.*

Proof. Let the vertex set and the edge set of $P(n.C_r)$ be $V(P(n.C_r)) = \{v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.C_r)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_1^j v_1^{j+1} : \ 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

For r is odd, we define an edge labeling $f : E(P(n.C_r)) \to Z_k - \{0\}$ as follows:

$$f(v_i^1 v_{i+1}^1) = f(v_i^n v_{i+1}^n) = \begin{cases} k-a, & \text{for } i = 1,3,\dots,r, \\ a, & \text{for } i = 2,4,\dots,r-1, \end{cases}$$
$$f(v_i^j v_{i+1}^j) = \begin{cases} k-2a, & \text{for } i = 1,3,\dots,r,\ j = 2,3,\dots,n-1, \\ 2a, & \text{for } i = 2,4,\dots,r-1,\ j = 2,3,\dots,n-1, \end{cases}$$
$$f(v_1^j v_1^{j+1}) = 2a, \quad \text{for } j = 1,2,\dots,n-1.$$

Then the induced vertex labeling $f^+ : V(P(n.C_r)) \to Z_k$ is $f^+(v) \equiv 0 \pmod k$ for every vertex v in $V(P(n.C_r))$. □

An example of a Z_{10}-magic labeling of $P(4.C_5)$ is shown in Figure 1.

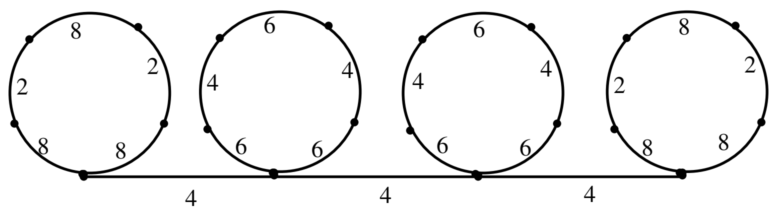

Figure 1: A Z_{10}-magic labeling of $P(4.C_5)$.

Up to isomorphism there are two graphs obtained by attaching n copies of a generalized Petersen graph $P(r,m)$, $r \geq 3$, $1 \leq m \leq \frac{r-1}{2}$ to a path P_n to get a graph $P(n.P(r,m)^v)$. We deal with the case when v is a vertex in the outer polygon of $P(r,m)$.

Theorem 2.2. *Let $r \geq 3$, $m \leq \frac{r-1}{2}$ and $n \geq 2$ be positive integers. The path union of a generalized Petersen graph $P(n.P(r,m)^v)$, where v is a vertex in the outer polygon of $P(r,m)$, is Z_k-magic for $k \geq 5$ when r is odd.*

Proof. Let the vertex set and the edge set of $P(n.P(r,m)^v)$ be $V(P(n.P(r,m)^v)) = \{u_i^j, v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.P(r,m)^v)) = \{u_i^j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\} \cup \{v_i^j v_{i+m}^j : 1 \leq i \leq r, 1 \leq j \leq n\}$, where the index i is taken over modulo r.

Let a, k be positive integers, $k > 4a$. Thus $k \geq 5$.

Define an edge labeling $f : E(P(n.P(r,m)^v)) \to Z_k - \{0\}$ as follows:

$$f(v_i^j v_{i+m}^j) = a, \quad \text{for } i = 1,2,\dots,r,\ j = 1,2,\dots,n-1,$$

$$f(u_i^j v_i^j) = k - 2a, \quad \text{for } i = 1,2,\dots,r,\ j = 1,2,\dots,n-1,$$

$$f(u_i^1 u_{i+1}^1) = \begin{cases} k-a, & \text{for } i = 1,3,\dots,r, \\ 3a, & \text{for } i = 2,4,\dots,r-1, \end{cases}$$

$$f(u_i^j u_{i+1}^j) = a, \quad \text{for } i = 1,2,\dots,r,\ j = 2,3,\dots,n-1,$$

$$f(v_i^n v_{i+m}^n) = \begin{cases} k-a, & \text{for } n \text{ is odd}, \\ a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(u_i^n v_i^n) = \begin{cases} 2a, & \text{for } n \text{ is odd}, \\ k-2a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(u_i^n u_{i+1}^n) = \begin{cases} a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is odd}, \\ k-3a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is even}, \\ 3a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_1^j u_1^{j+1}) = \begin{cases} 4a, & \text{for } j = 1,3,\dots \text{ and } j \le n-1, \\ k-4a, & \text{for } j = 2,4,\dots \text{ and } j \le n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.P(r,m)^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.P(r,m)^v))$. Thus $V(P(n.P(r,m)^v))$ is a Z_k-magic graph. □

An example of a Z_{15}-magic labeling of $P(5.P(5,2)^v)$ is shown in Figure 2.

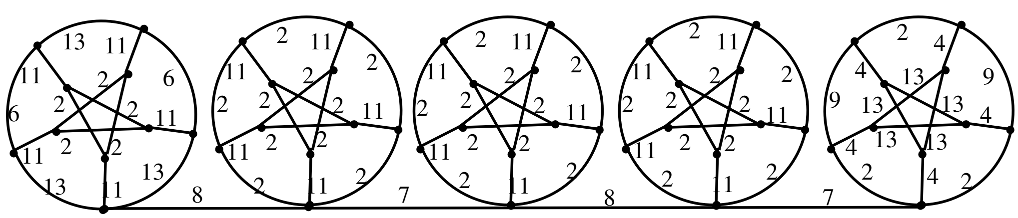

Figure 2: A Z_{15}-magic labeling of $P(5.P(5,2)^v)$.

Theorem 2.3. *Let $r \ge 4$ and $n \ge 2$ be positive integers. The path union of a shell graph $P(n.S_r^v)$, where $v \in V(S_r)$ is the vertex of degree $r-1$, is Z_k-magic for $k \ge 2r-3$ when r is odd and for $k \ge r-1$ when k is even.*

Proof. Let the vertex set and the edge set of $P(n.S_r^v)$ be $V(P(n.S_r^v)) = \{v_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(P(n.S_r^v)) = \{v_i^j v_{i+1}^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_1^j v_i^j : 3 \le i \le r, 1 \le j \le n\} \cup \{v_1^j v_1^{j+1} : 1 \le j \le n-1\}$ with the index i taken over modulo r.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > 2(r-2)a$. Thus $k \geq 2r-3$.

Define an edge labeling $f : E(P(n.S_r^v)) \to Z_k - \{0\}$ as follows:

$$f(v_1^1 v_i^1) = 2a, \quad \text{for } i = 3, 4, \dots, r-1,$$

$$f(v_1^1 v_2^1) = f(v_r^1 v_1^1) = a,$$

$$f(v_i^1 v_{i+1}^1) = k - a, \quad \text{for } i = 2, 3, \dots, r-1,$$

$$f(v_1^j v_1^{j+1}) = \begin{cases} k - 2a(r-2), & \text{for } j = 1, 3, \dots \text{ and } j \leq n-1, \\ 2a(r-2), & \text{for } j = 2, 4, \dots \text{ and } j \leq n-1, \end{cases}$$

$$f(v_1^j v_i^j) = a, \quad \text{for } i = 3, 4, \dots, r-1,\ j = 2, 3, \dots, n-1,$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{(r-3)a}{2}, & \text{for } i = 2, 4, \dots, r-1,\ j = 2, 3, \dots, n-1, \\ k - \frac{(r-1)a}{2}, & \text{for } i = 3, 5, \dots, r-2,\ j = 2, 3, \dots, n-1, \end{cases}$$

$$f(v_1^j v_2^j) = f(v_r^j v_1^j) = k - \frac{(r-3)a}{2}, \quad \text{for } j = 2, 3, \dots, n-1,$$

$$f(v_1^n v_i^n) = \begin{cases} k - 2a, & \text{for } i = 3, 4, \dots, r-1 \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 3, 4, \dots, r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_1^n v_2^n) = f(v_r^n v_1^n) = \begin{cases} k - a, & \text{for } n \text{ is odd}, \\ a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(v_i^n v_{i+1}^n) = \begin{cases} a, & \text{for } i = 2, 3, \dots, r-1 \text{ and } n \text{ is odd}, \\ k - a, & \text{for } i = 2, 3, \dots, r-1 \text{ and } n \text{ is even}. \end{cases}$$

Then the induced vertex labeling $f^+ : V(p(n.S_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in V(P(n.S_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > (r-2)a$. Thus $k \geq r-1$.

Define an edge labeling $f : E(P(n.S_r^v)) \to Z_k - \{0\}$ in the following way.

$$\begin{aligned}
f(v_1^1 v_i^1) &= a, \quad \text{for } i = 3, 4, \dots, r-1,\\
f(v_1^1 v_2^1) &= k - a,\\
f(v_r^1 v_1^1) &= 2a,\\
f(v_i^1 v_{i+1}^1) &= \begin{cases} a, & \text{for } i = 2, 4, \dots, r-2,\\ k - 2a, & \text{for } i = 3, 5, \dots, r-1, \end{cases}\\
f(v_1^j v_1^{j+1}) &= \begin{cases} k - a(r-2), & \text{for } j = 1, 3, \dots \text{ and } j \le n-1,\\ a(r-2), & \text{for } j = 2, 4, \dots \text{ and } j \le n-1, \end{cases}\\
f(v_1^j v_i^j) &= \tfrac{k}{2}, \quad \text{for } i = 3, 4, \dots, r-1,\ j = 2, 3, \dots, n-1,\\
f(v_i^j v_{i+1}^j) &= \begin{cases} \frac{3k}{4}, & \text{for } i = 2, 3, \dots, r-1,\ j = 2, 3, \dots, n-1 \text{ and } k \equiv 0 \pmod 4,\\ \frac{3k+2}{4}, & \text{for } i = 2, 4, \dots, r-2,\ j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4,\\ \frac{3k-2}{4}, & \text{for } i = 3, 5, \dots, r-1,\ j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}\\
f(v_1^j v_2^j) &= \begin{cases} \frac{k}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 0 \pmod 4,\\ \frac{k-2}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}\\
f(v_r^j v_1^j) &= \begin{cases} \frac{k}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 0 \pmod 4,\\ \frac{k+2}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}\\
f(v_1^n v_i^n) &= \begin{cases} k - a, & \text{for } i = 3, 4, \dots, r-1 \text{ and } n \text{ is odd},\\ a, & \text{for } i = 3, 4, \dots, r-1 \text{ and } n \text{ is even}, \end{cases}\\
f(v_1^n v_2^n) &= \begin{cases} a, & \text{for } n \text{ is odd},\\ k - a, & \text{for } n \text{ is even}, \end{cases}\\
f(v_r^n v_1^n) &= \begin{cases} k - 2a, & \text{for } n \text{ is odd},\\ 2a, & \text{for } n \text{ is even}, \end{cases}\\
f(v_i^n v_{i+1}^n) &= \begin{cases} k - a, & \text{for } i = 2, 4, \dots, r-2 \text{ and } n \text{ is odd}\\ 2a, & \text{for } i = 3, 5, \dots, r-1 \text{ and } n \text{ is odd},\\ a, & \text{for } i = 2, 4, \dots, r-2 \text{ and } n \text{ is even},\\ k - 2a, & \text{for } i = 3, 5, \dots, r-1 \text{ and } n \text{ is even}. \end{cases}
\end{aligned}$$

Then the induced vertex labeling $f^+ : V(P(n.S_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in V(P(n.S_r^v))$. Thus $P(n.S_r^v)$ is a Z_k-magic graph for r is even. □

An example of a Z_{11}-magic labeling of $P(3.S_7^v)$ is shown in Figure 3.

According to the symmetry of wheels there exist two non isomorphic graphs $P(n.W_r^v)$. We deal with the case when v is a rim vertex, that is a vertex of degree three in W_r.

Theorem 2.4. *Let $r \ge 4$ and $n \ge 2$ be integers. The path union of a wheel graph $P(n.W_r^v)$, where $v \in V(W_r)$ is a vertex of degree 3, is Z_k-magic for $k \ge r$ when r is odd and for $k \ge 2r-1$ when r*

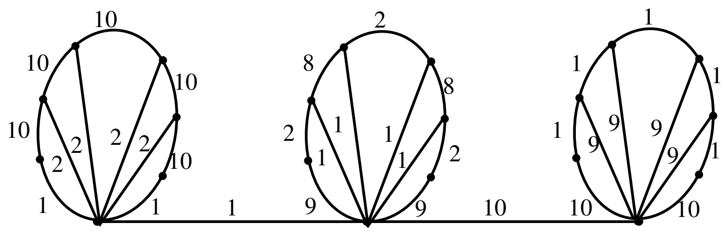

Figure 3: A Z_{11}-magic labeling of $P(3.S_7^v)$.

is even.

Proof. Let the vertex set and the edge set of $P(n.W_r^v)$ be $V(P(n.W_r^v)) = \{w_j, v_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(P(n.W_r^v)) = \{v_i^j v_{i+1}^j : 1 \le i \le r, 1 \le j \le n\} \cup \{w_j v_i^j : 1 \le i \le r, 1 \le j \le n\} \cup \{u_1^j u_1^{j+1} : 1 \le j \le n-1\}$, where the index i is taken over modulo r.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > (r-1)a$. This implies $k \ge r$.

Define an edge labeling $f : E(P(n.W_r^v)) \to Z_k - \{0\}$ as follows:

$$f(w_j v_i^j) = a, \quad \text{for } i = 2, 3, \ldots, r,\ j = 1, 2, \ldots, n-1,$$

$$f(w_j v_1^j) = k - (r-1)a, \quad \text{for } j = 1, 2, \ldots, n-1,$$

$$f(v_i^1 v_{i+1}^1) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r, \\ k - 2a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{(r-1)a}{2}, & \text{for } i = 1, 3, \ldots, r,\ j = 2, 3, \ldots, n-1, \\ k - \frac{(r+1)a}{2}, & \text{for } i = 2, 4, \ldots, r-1,\ j = 2, 3, \ldots, n-1, \end{cases}$$

$$f(w_n v_1^n) = \begin{cases} (r-1)a, & \text{for } n \text{ is odd}, \\ k - (r-1)a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(w_n v_i^n) = \begin{cases} k - a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_i^n v_{i+1}^n) = \begin{cases} k - a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ k - 2a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_1^j v_1^{j+1}) = \begin{cases} a(r-3), & \text{for } j = 1, 3, \ldots \text{ and } j \le n-1, \\ k - a(r-3), & \text{for } j = 2, 4, \ldots \text{ and } j \le n-1. \end{cases}$$

This means that for the induced vertex labeling $f^+ : V(P(n.W_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in V(P(n.W_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > 2(r-1)a$.

Define an edge labeling $f : E(P(n.W_r^v)) \to Z_k - \{0\}$ in the following way.

$$
\begin{aligned}
f(w_1 v_1^j) &= f(w_n v_1^n) = k - (r-1)a, \\
f(w_1 v_i^1) &= f(w_n v_i^n) = a, \quad \text{for } i = 2, 3, \dots, r, \\
f(v_i^1 v_{i+1}^1) &= f(v_i^n v_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 3, \dots, r-1, \\ k - 2a, & \text{for } i = 2, 4, \dots, r, \end{cases} \\
f(w_j v_1^j) &= k - 2(r-1)a, \quad \text{for } j = 2, 3, \dots, n-1, \\
f(w_j v_i^j) &= 2a, \quad \text{for } i = 2, 3, \dots, r,\ j = 2, 3, \dots, n-1, \\
f(v_i^j v_{i+1}^j) &= k - a, \quad \text{for } i = 1, 2, \dots, r,\ j = 2, 3, \dots, n-1, \\
f(v_1^j v_1^{j+1}) &= ra, \quad \text{for } j = 1, 2, \dots, n-1.
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(P(n.W_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.W_r^v))$. Hence f^+ is constant that means $P(n.W_r^v)$ admits a Z_k-magic labeling. □

An example of a Z_{12}-magic labeling of $P(3.W_6^v)$ is shown in Figure 4.

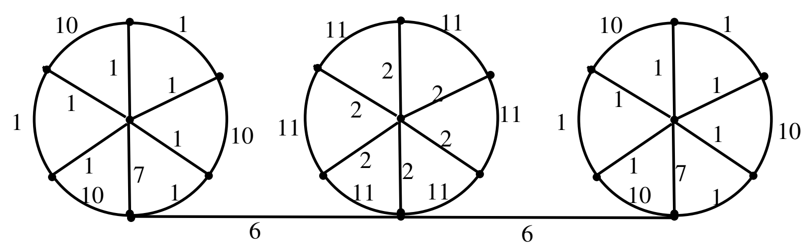

Figure 4: A Z_{12}-magic labeling of $P(3.W_6^v)$.

In the next theorem we deal with the path union of a closed helm graph $P(n.CH_r^v)$, where v is a vertex of degree three in CH_r.

Theorem 2.5. *Let $r \geq 4$ and $n \geq 2$ be integers. The path union of a closed helm graph $P(n.CH_r^v)$, where v is a vertex of degree 3 in CH_r, is Z_k-magic for $k \geq r$ when r is odd and for even $k \geq r$ when r is even.*

Proof. Let the vertex set and the edge set of $P(n.CH_r^v)$ be $V(P(n.CH_r^v)) = \{w_j, v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.CH_r^v)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{w_j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

Case (i): when r is odd.

Let a, k be positive integers, $k > (r-1)a$. Thus $k \geq r$.

Define an edge labeling $f : E(P(n.CH_r^v)) \to Z_k - \{0\}$ as follows:

$$f(w_j v_1^j) = k - (r-1)a, \quad \text{for } j = 1, 2, \ldots, n-1,$$

$$f(w_j v_i^j) = a, \quad \text{for } i = 2, 3, \ldots, r,\ j = 1, 2, \ldots, n-1,$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} (r-1)a, & \text{for } i = 1, 3, \ldots, r,\ j = 1, 2, \ldots, n-1, \\ k - (r-1)a, & \text{for } i = 2, 4, \ldots, r-1,\ j = 1, 2, \ldots, n-1, \end{cases}$$

$$f(u_i^1 u_{i+1}^1) = \begin{cases} (r-1)a, & \text{for } i = 1, 3, \ldots, r, \\ k - (r-2)a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases}$$

$$f(v_i^j u_i^j) = k - a, \quad \text{for } i = 2, 3, \ldots, r,\ j = 1, 2, \ldots, n-1,$$

$$f(v_1^j u_1^j) = k - (r-1)a, \quad \text{for } j = 1, 2, \ldots, n-1,$$

$$f(u_i^j u_{i+1}^j) = \begin{cases} \frac{(r-1)a}{2}, & \text{for } i = 1, 3, \ldots, r,\ j = 2, 3, \ldots, n-1, \\ k - \frac{(r-3)a}{2}, & \text{for } i = 2, 4, \ldots, r-1,\ j = 2, 3, \ldots, n-1, \end{cases}$$

$$f(w_n v_1^n) = \begin{cases} (r-1)a, & \text{for } n \text{ is odd}, \\ k - (r-1)a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(w_n v_i^n) =, \begin{cases} k - a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_1^n u_1^n) = \begin{cases} (r-1), & \text{for } n \text{ is odd}, \\ k - (r-1)a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(v_i^n u_i^n) = \begin{cases} a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is odd}, \\ k - a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_i^n v_{i+1}^n) = \begin{cases} k - (r-1)a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ k - (r-1)a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_i^n u_{i+1}^n) = \begin{cases} k - (r-1)a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ (r-2)a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ k - (r-2)a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_1^j u_1^{j+1}) = \begin{cases} k - (r-1)a, & \text{for } j = 1, 3, \ldots \text{ and } j \leq n-1, \\ (r-1)a, & \text{for } j = 2, 4, \ldots \text{ and } j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.CH_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.CH_r^v))$.

Case (ii): when r is even.

Let a be a positive integer and $k > (r-2)a$ be an even integer. Thus $k \geq r$.

Define an edge labeling $f : E(P(n.CH_r^v)) \to Z_k - \{0\}$ such that

$$f(w_1v_1^1) = k-(r-1)a,$$

$$f(w_1v_i^1) = a, \quad \text{for } i = 2,3,\dots,r,$$

$$f(v_i^1v_{i+1}^1) = \begin{cases} (r-1)a, & \text{for } i = 1,3,\dots,r-1, \\ k-(r-1)a, & \text{for } i = 2,4,\dots,r, \end{cases}$$

$$f(u_i^1u_{i+1}^1) = \begin{cases} (r-1)a, & \text{for } i = 1,3,\dots,r-1, \\ k-(r-2)a, & \text{for } i = 2,4,\dots,r-1, \end{cases}$$

$$f(v_1^1u_1^1) = (r-1)a,$$

$$f(v_i^1u_i^1) = k-a, \quad \text{for } i = 2,3,\dots,r,$$

$$f(w_jv_i^j) = f(v_i^ju_i^j) = f(v_i^jv_{i+1}^j) = \tfrac{k}{2}, \quad \text{for } i = 1,2,\dots,r,\ j = 2,3,\dots,n-1,$$

$$f(u_i^ju_{i+1}^j) = \begin{cases} \frac{k}{4}, & \text{for } i = 1,2,\dots,r,\ j = 2,3,\dots,n-1 \text{ and } k \equiv 0 \pmod 4, \\ \frac{k-2}{4}, & \text{for } i = 1,3,\dots,r-1,\ j = 2,3,\dots,n-1 \text{ and } k \equiv 2 \pmod 4, \\ \frac{k+2}{4}, & \text{for } i = 2,4,\dots,r,\ j = 2,3,\dots,n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}$$

$$f(w_nv_1^n) = \begin{cases} (r-1)a, & \text{for } n \text{ is odd}, \\ k-(r-1)a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(w_nv_i^n) = \begin{cases} k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_1^nu_1^n) = \begin{cases} k-(r-1)a, & \text{for } n \text{ is odd}, \\ (r-1)a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(v_i^nu_i^n) = \begin{cases} a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_i^nv_{i+1}^n) = \begin{cases} k-(r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is even}, \\ k-(r-1)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_i^nu_{i+1}^n) = \begin{cases} k-(r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ (r-2)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is even}, \\ k-(r-2)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_1^ju_1^{j+1}) = \begin{cases} k-ra, & \text{for } j = 1,3,\dots \text{ and } j \leq n-1, \\ ra, & \text{for } j = 2,4,\dots \text{ and } j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.CH_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in$

$V(P(n.CH_r^v))$. Hence f^+ is constant equal to 0 (mod k). Therefore $P(n.CH_r^v)$ is a Z_k-magic graph. $\square$

An example of a Z_6-magic labeling of $P(3.CH_6^v)$ is shown in Figure 5.

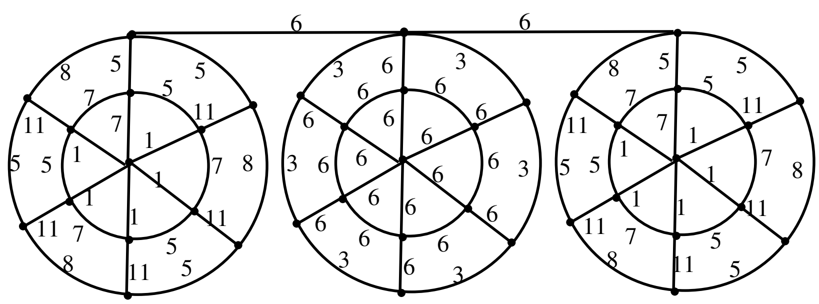

Figure 5: A Z_{12}-magic labeling of $P(3.CH_6^v)$.

Theorem 2.6. *Let $r \geq 3$ and $n \geq 2$ be integers. The path union of a double wheel graph $P(n.DW_r^v)$, where $v \in V(DW_r)$ is a vertex of degree 3, is Z_k-magic for $k \geq 5$ when r is odd.*

Proof. Let the vertex set and the edge set of $C(n.DW_r^v)$ be $V(P(n.DW_r^v)) = \{v_j, v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.DW_r^v)) = \{v_j v_i^j, v_j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$ with index i taken over modulo r.

Let a, k be positive integers, $k > 4a$. Thus $k \geq 5$.

For r is odd we define an edge labeling $f : E(P(n.DW_r^v)) \to Z_k - \{0\}$ as follows:

$$f(v_j v_i^j) = 2a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n-1,$$
$$f(v_j u_i^j) = k - 2a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n-1,$$
$$f(v_i^j v_{i+1}^j) = k - a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n-1,$$
$$f(u_i^1 u_{i+1}^1) = \begin{cases} k-a, & \text{for } i = 1, 3, \ldots, r, \\ 3a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases}$$
$$f(u_i^j u_{i+1}^j) = a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 2, 3, \ldots, n-1,$$
$$f(v_n v_i^n) = \begin{cases} k-2a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_n u_i^n) = \begin{cases} 2a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is odd}, \\ k-2a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_i^n v_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 2, \ldots, r-1 \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 1, 2, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_r^n v_1^n) = \begin{cases} a, & \text{for } n \text{ is odd}, \\ k-a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(u_i^n u_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ k-3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ 3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_1^j u_1^{j+1}) = \begin{cases} 4a, & \text{for } j = 1, 3, \ldots \text{ and } j \le n-1, \\ k-4a, & \text{for } j = 2, 4, \ldots \text{ and } j \le n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.DW_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.DW_r^v))$. □

An example of a Z_7-magic labeling of $P(3.DW_7^v)$ is shown in Figure 6.

Theorem 2.7. *Let $r \ge 3$ and $n \ge 2$ be positive integers. The path union of a flower graph $P(n.Fl_r^v)$, where $v \in V(Fl_r)$ is the vertex of degree 4, is Z_k-magic for $k \ge 5$ when r is odd and for $k \ge 3$ when k is even.*

Proof. Let the vertex set and the edge set of $P(n.Fl_r^v)$ be $V(P(n.Fl_r^v)) = \{w_j, v_i^j, u_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(P(n.Fl_r^v)) = \{w_j v_i^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_i^j u_i^j : 1 \le i \le r, 1 \le j \le n\} \cup \{w_j u_i^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_i^j v_{i+1}^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_1^j v_1^{j+1} : 1 \le j \le n-1\}$, with index i taken over modulo r.

Case (i): when r is odd.

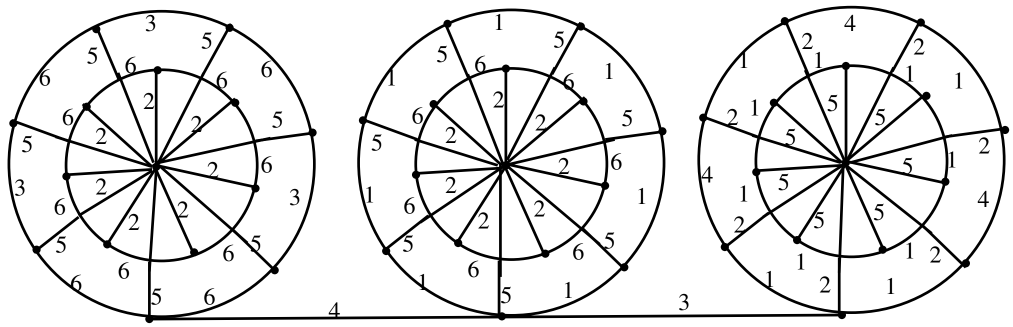

Figure 6: A Z_7-magic labeling of $P(3.DW_7^v)$.

Let a, k be positive integers, $k > 4a$. This means $k \geq 5$.

Define an edge labeling $f : E(P(n.Fl_r^v)) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(w_j v_i^j) &= f(v_i^j u_i^j) = a, \quad \text{for } i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, n-1, \\
f(u_i^j w_j) &= k - a, \quad \text{for } i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, n-1, \\
f(v_i^1 v_{i+1}^1) &= \begin{cases} a, & \text{for } i = 1, 3, \ldots, r, \\ k - 3a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases} \\
f(v_i^j v_{i+1}^j) &= k - a, \quad \text{for } i = 1, 2, \ldots, r, \ j = 2, 3, \ldots, n-1, \\
f(w_n v_i^n) &= f(v_i^n u_i^n) = \begin{cases} k - a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is even}, \end{cases} \\
f(u_i^n w_n) &= \begin{cases} a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is odd}, \\ k - a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is even}, \end{cases} \\
f(v_i^n v_{i+1}^n) &= \begin{cases} k - a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ 3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ k - 3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases} \\
f(v_1^j v_1^{j+1}) &= \begin{cases} k - 4a, & \text{for } j = 1, 3, \ldots \text{ and } j \leq n-1, \\ 4a, & \text{for } j = 2, 4, \ldots \text{ and } j \leq n-1. \end{cases}
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(P(n.Fl_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.Fl_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

Define an edge labeling $f : E(P(n.Fl_r^v)) \to Z_k - \{0\}$ as follows:

$$\begin{aligned}
f(w_1v_1^1) &= f(v_1^1u_1^1) = 2a,\\
f(u_1^1w_1) &= k-2a,\\
f(v_i^jv_{i+1}^j) &= k-a, \quad \text{for } i = 1,2,\dots,r,\ j = 2,3,\dots,n-1,\\
f(w_jv_i^j) &= f(v_i^ju_i^j) = a, \quad \text{for } i = 2,3,\dots,r,\ j = 1,2,\dots,n-1,\\
f(w_ju_i^j) &= k-a, \quad \text{for } i = 2,3,\dots,r,\ j = 1,2,\dots,n-1,\\
f(w_nv_1^n) &= f(v_1^nu_1^n) = \begin{cases} k-2a, & \text{for } n \text{ is odd},\\ 2a, & \text{for } n \text{ is even},\end{cases}\\
f(w_nu_1^n) &= \begin{cases} 2a, & \text{for } n \text{ is odd},\\ k-2a, & \text{for } n \text{ is even},\end{cases}\\
f(w_nv_i^n) &= f(v_i^nu_i^n) = \begin{cases} k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(w_nu_i^n) &= \begin{cases} a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd},\\ k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_i^nv_{i+1}^n) &= \begin{cases} a, & \text{for } i = 1,2,\dots,r \text{ and } n \text{ is odd},\\ k-a, & \text{for } i = 1,2,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_1^jv_1^{j+1}) &= \begin{cases} k-2a, & \text{for } j = 1,3,\dots \text{ and } j \le n-1,\\ 2a, & \text{for } j = 2,4,\dots \text{ and } j \le n-1.\end{cases}
\end{aligned}$$

The induced vertex labeling $f^+ : V(P(n.Fl_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in V(P(n.Fl_r^v))$. $\square$

An example of a Z_{10}-magic labeling of $P(4.Fl_3^v)$ is shown in Figure 7.

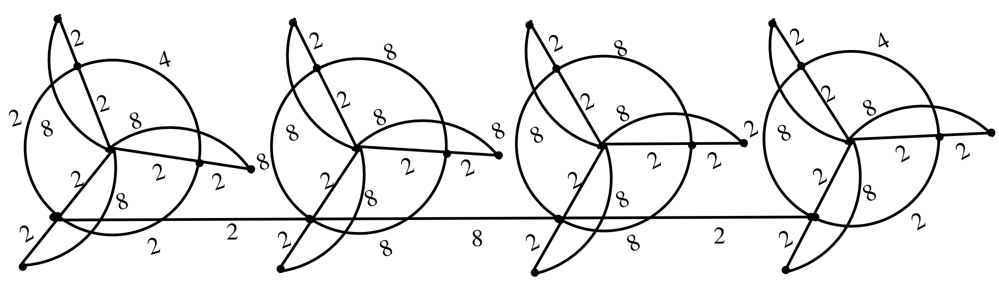

Figure 7: A Z_{10}-magic labeling of $P(4.Fl_3^v)$.

Let v be a vertex of a cylinder graph $C_r \square P_2$, $r \ge 3$. According to the symmetry all $P(n.(C_r \square P_2)^v)$ are isomorphic. Thus we use the notation $P(n.(C_r \square P_2))$.

Theorem 2.8. *Let $r \ge 3$, $n \ge 2$ be integers. The path union of a cylinder graph $P(n.(C_r \square P_2))$ is Z_k-magic for $k \ge 5$ when r is odd.*

Proof. Let the vertex set and the edge set of $P(n.(C_r\square P_2))$ be $V(P(n.(C_r\square P_2))) = \{v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.(C_r\square P_2))) = \{u_i^j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$, with index i taken over modulo r.

Let a, k be positive integers, $k > 4a$. Thus $k \geq 5$.

For r odd we define an edge labeling $f : E(P(n.(C_r\square P_2))) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(v_i^j u_i^j) &= k-2a, \quad \text{for } i = 1,2,\ldots,r,\ j = 1,2,\ldots,n-1,\\
f(v_i^j v_{i+1}^j) &= a, \quad \text{for } i = 1,2,\ldots,r,\ j = 1,2,\ldots,n-1,\\
f(u_i^1 u_{i+1}^1) &= \begin{cases} k-a, & \text{for } i = 1,3,\ldots,r,\\ 3a, & \text{for } i = 2,4,\ldots,r-1, \end{cases}\\
f(u_i^j u_{i+1}^j) &= a, \quad \text{for } i = 1,2,\ldots,r,\ j = 2,3,\ldots,n-1,\\
f(v_i^n v_{i+1}^n) &= \begin{cases} k-a, & \text{for } n \text{ is odd},\\ a, & \text{for } n \text{ is even}, \end{cases}\\
f(v_i^n u_i^n) &= \begin{cases} 2a, & \text{for } n \text{ is odd},\\ k-2a, & \text{for } n \text{ is even}, \end{cases}\\
f(u_i^n u_{i+1}^n) &= \begin{cases} a, & \text{for } i = 1,3,\ldots,r \text{ and } n \text{ is odd},\\ k-3a, & \text{for } i = 2,4,\ldots,r-1 \text{ and } n \text{ is odd},\\ k-a, & \text{for } i = 1,3,\ldots,r \text{ and } n \text{ is even},\\ 3a, & \text{for } i = 2,4,\ldots,r-1 \text{ and } n \text{ is even}, \end{cases}\\
f(u_1^j u_1^{j+1}) &= \begin{cases} 4a, & \text{for } j = 1,3,\ldots,\ j \leq n-1,\\ k-4a, & \text{for } j = 2,4,\ldots,\ j \leq n-1. \end{cases}
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(P(n.(C_r\square P_2))) \to Z_k$ is $f^+(v) \equiv 0 \equiv k$ for all $v \in V(P(n.(C_r\square P_2)))$. Hence f^+ is constant and is equal to $0 \equiv k$. $\square$

An example of a Z_9-magic labeling of $P(3.(C_7\square P_2))$ is shown in Figure 8.

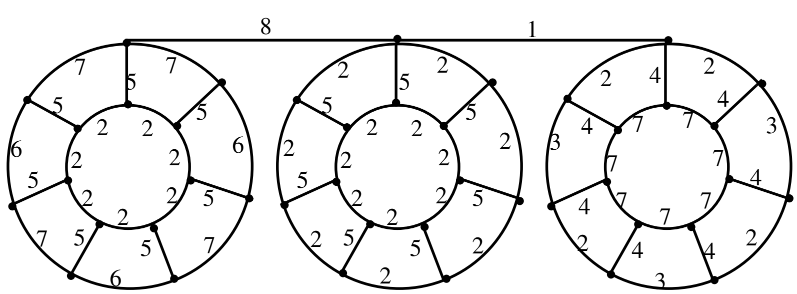

Figure 8: A Z_9-magic labeling of $P(3.(C_7\square P_2)^v)$.

Theorem 2.9. *Let $r \geq 5$ and $n \geq 2$ be positive integers. The path union of a total graph of a path $P(n.T(P_r)^v)$, where $v \in V(T(P_r))$ is a vertex of degree two, is Z_k-magic for $k \geq 3$.*

Proof. Let the vertex set and the edge set of $P(n.T(P_r)^v)$ be $V(P(n.T(P_r)^v)) = \{u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j : 1 \leq i \leq r-1, 1 \leq j \leq n\}$ and $E(P(n.T(P_r)^v)) = \{u_i^j u_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq r-2, 1 \leq j \leq n\} \cup \{u_{i+1}^j v_i^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \cup \{u_i^j v_i^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

Define an edge labeling $f : E(P(n.T(P_r)^v)) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(u_i^1 u_{i+1}^1) &= \begin{cases} a, & \text{for } i = 1, 3, \dots, r, \\ 2a, & \text{for } i = 2, 4, \dots, r-3, \end{cases} \\
f(u_{r-1}^1 u_r^1) &= f(v_1^1 v_2^1) = a, \\
f(v_i^1 v_{i+1}^1) &= \begin{cases} 2a, & \text{for } i = 3, 5, \dots, r, \\ a, & \text{for } i = 2, 4, \dots, r-1, \end{cases} \\
f(u_1^1 v_1^1) &= a, \\
f(u_2^1 v_2^1) &= k - a, \\
f(u_i^1 v_i^1) &= k - 2a, \quad \text{for } i = 3, 4, \dots, r-2, \\
f(u_{r-1}^1 v_{r-1}^1) &= k - a, \\
f(v_1^1 u_2^1) &= k - 2a, \\
f(v_i^1 u_{i+1}^1) &= k - a, \quad \text{for } i = 2, 3, \dots, r-1, \\
f(u_1^j v_1^j) &= f(u_2^j v_1^j) = a, \quad \text{for } j = 2, 3, \dots, n-1, \\
f(u_1^j u_2^j) &= f(u_{r-1}^j u_r^j) = k - a, \quad \text{for } j = 2, 3, \dots, n-1, \\
f(u_i^j u_{i+1}^j) &= k - 2a, \quad \text{for } i = 2, 3, \dots, r-2,\ j = 2, 3, \dots, n-1, \\
f(v_i^j v_{i+1}^j) &= k - 2a, \quad \text{for } i = 1, 2, \dots, r-2,\ j = 2, 3, \dots, n-1, \\
f(u_i^j v_i^j) &= f(u_{i+1}^j v_i^j) = 2a, \quad \text{for } i = 2, 3, \dots, r-2,\ j = 2, 3, \dots, n-1, \\
f(u_r^j v_{r-1}^j) &= f(u_{r-1}^j v_{r-1}^j) = a, \quad \text{for } j = 2, 3, \dots, n-1,
\end{aligned}
$$

$$f(u^n_{r-1}u^n_r) = \begin{cases} k-a, & \text{for } n \text{ is odd,} \\ a, & \text{for } n \text{ is even,} \end{cases}$$

$$f(u^n_i u^n_{i+1}) = \begin{cases} k-a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is odd,} \\ k-2a, & \text{for } i = 2,4,\dots,r-3 \text{ and } n \text{ is odd,} \\ a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is even,} \\ 2a, & \text{for } i = 2,4,\dots,r-3 \text{ and } n \text{ is odd,} \end{cases}$$

$$f(v^n_1 v^n_2) = \begin{cases} k-a, & \text{for } n \text{ is odd,} \\ a, & \text{for } n \text{ is even,} \end{cases}$$

$$f(v^n_i v^n_{i+1}) = \begin{cases} k-2a, & \text{for } i = 3,5,\dots,r \text{ and } n \text{ is odd,} \\ k-a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is odd,} \\ 2a, & \text{for } i = 3,5,\dots,r \text{ and } n \text{ is even,} \\ a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is even,} \end{cases}$$

$$f(u^n_1 v^n_1) = \begin{cases} k-a, & \text{for } n \text{ is odd,} \\ a, & \text{for } n \text{ is even,} \end{cases}$$

$$f(u^n_2 v^n_2) = \begin{cases} a, & \text{for } n \text{ is odd,} \\ k-a, & \text{for } n \text{ is even,} \end{cases}$$

$$f(u^n_i v^n_i) = \begin{cases} 2a, & \text{for } i = 3,4,\dots,r-2 \text{ and } n \text{ is odd,} \\ k-2a, & \text{for } i = 3,4,\dots,r-2 \text{ and } n \text{ is even,} \end{cases}$$

$$f(u^n_{r-1} v^n_{r-1}) = \begin{cases} a, & \text{for } n \text{ is odd,} \\ k-a, & \text{for } n \text{ is even,} \end{cases}$$

$$f(v^n_1 u^n_2) = \begin{cases} 2a, & \text{for } n \text{ is odd,} \\ k-2a, & \text{for } n \text{ is even,} \end{cases}$$

$$f(v^n_i u^n_{i+1}) = \begin{cases} a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is odd,} \\ k-a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is even,} \end{cases}$$

$$f(u^j_1 u^{j+1}_1) = \begin{cases} k-2a, & \text{for } j = 1,3,\dots \text{ and } j \le n-1, \\ 2a, & \text{for } j = 2,4,\dots \text{ and } j \le n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.T(P_r)^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.T(P_r)^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

Define an edge labeling $f : E(P(n.T(P_r)^v)) \to Z_k - \{0\}$ as follows:

$$f(u_i^1 u_{i+1}^1) = f(v_i^1 v_{i+1}^1) = \begin{cases} k-a, & \text{for } i = 1,3,\dots,r-1, \\ k-2a, & \text{for } i = 2,4,\dots,r, \end{cases}$$

$$f(v_1^1 u_1^1) = k-a,$$

$$f(v_i^1 u_i^1) = a, \quad \text{for } i = 2,3,\dots,r-1,$$

$$f(v_i^1 u_{i+1}^1) = 2a, \quad \text{for } i = 1,2,\dots,r-2,$$

$$f(v_{r-1}^1 u_r^1) = a,$$

$$f(u_1^j v_1^j) = f(u_2^j v_1^j) = a, \quad \text{for } j = 2,3,\dots,n-1,$$

$$f(u_1^j u_2^j) = f(u_{r-1}^j u_r^j) = k-a, \quad \text{for } j = 2,3,\dots,n-1,$$

$$f(u_i^j u_{i+1}^j) = k-2a, \quad \text{for } i = 2,3,\dots,r-2,\ j = 2,3,\dots,n-1,$$

$$f(v_i^j v_{i+1}^j) = k-2a, \quad \text{for } i = 1,2,\dots,r-2,\ j = 2,3,\dots,n-1,$$

$$f(u_i^j v_i^j) = f(u_{i+1}^j v_i^j) = 2a, \quad \text{for } i = 2,3,\dots,r-2,\ j = 2,3,\dots,n-1,$$

$$f(u_r^j v_{r-1}^j) = f(u_{r-1}^j v_{r-1}^j) = a, \quad \text{for } j = 2,3,\dots,n-1,$$

$$f(u_i^n u_{i+1}^n) = f(v_i^n v_{i+1}^n) = \begin{cases} a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is even}, \\ k-2a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_1^n v_1^n) = \begin{cases} a, & \text{for } n \text{ is odd}, \\ k-a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(u_i^n v_i^n) = \begin{cases} k-a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_i^n u_{i+1}^n) = \begin{cases} k-2a, & \text{for } i = 1,2,\dots,r-2 \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 1,2,\dots,r-2 \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_{r-1}^n u_r^n) = \begin{cases} k-a, & \text{for } n \text{ is odd}, \\ a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(u_1^j u_1^{j+1}) = \begin{cases} 2a, & \text{for } j = 1,3,\dots \text{ and } j \leq n-1, \\ k-2a, & \text{for } j = 2,4,\dots \text{ and } j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.T(P_r)^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.T(P_r)^v))$. Hence $P(n.T(P_r)^v)$ is a Z_k-magic graph. □

An example of a Z_5-magic labeling of $P(5.T(P_6)^v)$ is shown in Figure 9.

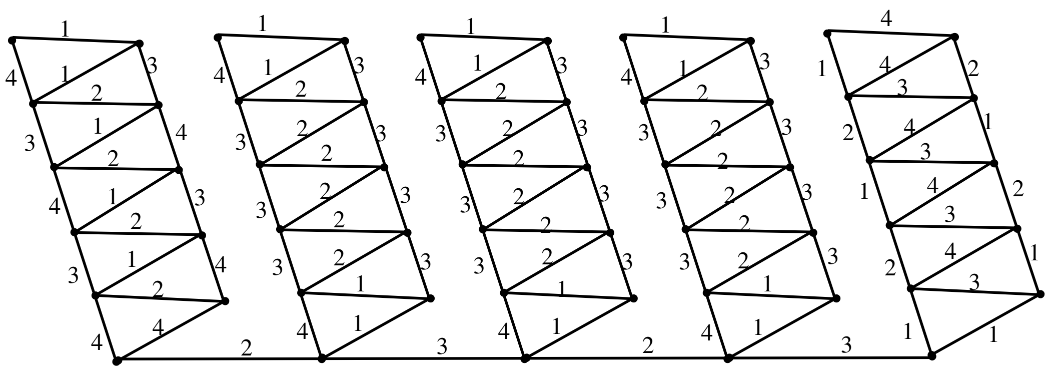

Figure 9: A Z_5-magic labeling of $P(5.T(P_6)^v)$.

Theorem 2.10. *Let $r \geq 3$ and $n \geq 2$ be integers. Let v is a vertex of degree 2 in LC_r. The path union of a lotus inside a circle graph $P(n.LC_r^v)$, is Z_k-magic for $k \geq r$.*

Proof. Let the vertex set and the edge set of $P(n.LC_r^v)$ be $V(P(n.LC_r^v)) = \{w_j, v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.LC_r^v)) = \{w_j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > (r-1)a$. Thus $k \geq r$.

Define an edge labeling $f : E(P(n.LC_r^v)) \to Z_k - \{0\}$ in the following way.

$$
\begin{aligned}
f(w_j v_1^j) &= k-(r-1)a, \quad \text{for } j=1,2,\dots,n-1,\\
f(w_j v_i^j) &= a, \quad \text{for } i=2,3,\dots,r,\ j=1,2,\dots,n-1,\\
f(v_1^j u_1^j) &= (r-2)a, \quad \text{for } j=1,2,\dots,n-1,\\
f(v_i^j u_i^j) &= k-2a, \quad \text{for } i=2,3,\dots,r,\ j=1,2,\dots,n-1,\\
f(v_i^j u_{i+1}^j) &= a, \quad \text{for } i=1,2,\dots,r,\ j=1,2,\dots,n-1,\\
f(u_i^1 u_{i+1}^1) &= \begin{cases} k-a, & \text{for } i=1,3,\dots,r,\\ 2a, & \text{for } i=2,4,\dots,r-1,\end{cases}\\
f(u_i^j u_{i+1}^j) &= \begin{cases} k-\frac{(r-1)a}{2}, & \text{for } i=1,3,\dots,r,\ j=2,3,\dots,n-1,\\ \frac{(r+1)a}{2}, & \text{for } i=2,4,\dots,r-1,\ j=2,3,\dots,n-1,\end{cases}\\
f(u_1^j u_1^{j+1}) &= \begin{cases} k-(r-3)a, & \text{for } j=1,3,\dots,\ j\le n-1,\\ (r-3)a, & \text{for } j=2,4,\dots,\ j\le n-1,\end{cases}\\
f(w_n v_1^n) &= \begin{cases} (r-1)a, & \text{for } n \text{ is odd},\\ k-(r-1)a, & \text{for } n \text{ is even},\end{cases}\\
f(w_n v_i^n) &= \begin{cases} k-a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_1^n u_1^n) &= \begin{cases} k-(r-2)a, & \text{for } n \text{ is odd},\\ (r-2)a, & \text{for } n \text{ is even},\end{cases}\\
f(v_i^n u_i^n) &= \begin{cases} 2a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_i^n u_{i+1}^n) &= \begin{cases} k-a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(u_i^n u_{i+1}^n) &= \begin{cases} a, & \text{for } i=1,3,\dots,r \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i=2,4,\dots,r-1 \text{ and } n \text{ is odd},\\ k-a, & \text{for } i=1,3,\dots,r \text{ and } n \text{ is even},\\ 2a, & \text{for } i=2,4,\dots,r-1 \text{ and } n \text{ is even}.\end{cases}
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(P(n.LC_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in V(P(n.LC_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > (r-1)a$. Thus $k \ge r$.

Define an edge labeling $f : E(P(n.LC_r)) \to Z_k - \{0\}$ as follows:

$$f(w_1 v_1^1) = k - (r-1)a,$$
$$f(w_1 v_i^1) = a, \quad \text{for } i = 2, 3, \ldots, r,$$
$$f(v_1^1 u_1^1) = (r-2)a,$$
$$f(v_i^1 u_i^1) = k - 2, \quad \text{for } i = 2, 3, \ldots, r,$$
$$f(v_i^1 u_{i+1}^1) = a, \quad \text{for } i = 1, 2, \ldots, r,$$
$$f(u_i^1 u_{i+1}^1) = \begin{cases} k - a, & \text{for } i = 1, 3, \ldots, r-1, \\ 2a, & \text{for } i = 2, 4, \ldots, r, \end{cases}$$
$$f(w_j v_i^j) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r-1,\ j = 2, 3, \ldots, n-1, \\ k - a, & \text{for } i = 2, 4, \ldots, r,\ j = 2, 3, \ldots, n-1, \end{cases}$$
$$f(v_i^j u_i^j) = \begin{cases} k - 2a, & \text{for } i = 1, 3, \ldots, r-1,\ j = 2, 3, \ldots, n-1, \\ k - a, & \text{for } i = 2, 4, \ldots, r,\ j = 2, 3, \ldots, n-1, \end{cases}$$
$$f(v_i^j u_{i+1}^j) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r-1,\ j = 1, 2, \ldots, n-1, \\ 2a, & \text{for } i = 2, 4, \ldots, r,\ j = 1, 2, \ldots, n-1, \end{cases}$$
$$f(u_i^j u_{i+1}^j) = \begin{cases} k - a, & \text{for } i = 1, 3, \ldots, r-1,\ j = 2, 3, \ldots, n-1, \\ a, & \text{for } i = 2, 4, \ldots, r,\ j = 2, 3, \ldots, n-1, \end{cases}$$
$$f(w_n v_1^n) = \begin{cases} (r-1)a, & \text{for } n \text{ is odd}, \\ k - (r-1)a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(w_n v_i^n) = \begin{cases} k - a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_1^n u_1^n) = \begin{cases} k - (r-2)a, & \text{for } n \text{ is odd}, \\ (r-2)a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(v_i^n u_i^n) = \begin{cases} 2a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is odd}, \\ k - 2a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_i^n u_{i+1}^n) = \begin{cases} k - a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_i^n u_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r-1 \text{ and } n \text{ is odd}, \\ k - 2a, & \text{for } i = 2, 4, \ldots, r \text{ and } n \text{ is odd}, \\ k - a, & \text{for } i = 1, 3, \ldots, r-1 \text{ and } n \text{ is even}, \\ 2a, & \text{for } i = 2, 4, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_1^j u_1^{j+1}) = \begin{cases} k - ra, & \text{for } j = 1, 3, \ldots, j \le n-1, \\ ra, & \text{for } j = 2, 4, \ldots, j \le n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.LC_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in$

$V(P(n.LC_r^v))$. Hence f^+ is constant and is equal to $\equiv 0 \pmod{k}$. $\square$

An example of a Z_{10}-magic labeling of $P(3.LC_6^v)$ is shown in Figure 10.

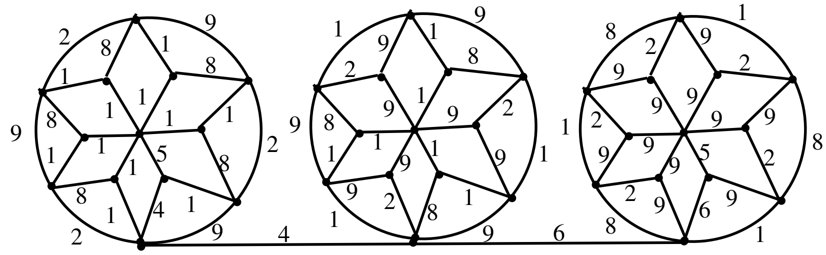

Figure 10: A Z_{10}-magic labeling of $P(3.LC_6^v)$.

In the last theorem we deal with the path union of an r-pan graph $P(n.(r\text{-pan})^v)$, where v is a vertex of degree two in an r-pan graph.

Theorem 2.11. *Let $r \geq 3$, $n \geq 2$ be integers. The path union of an r-pan graph $P(n.(r\text{-pan})^v)$, where v is a vertex of degree two in an r-pan graph, is Z_k-magic for $k \geq 5$ when r is odd.*

Proof. Let v be a vertex of degree two in an r-pan graph. Let the vertex set and the edge set of $P(n.(r\text{-pan})^v)$ be $V(P(n.(r\text{-pan})^v)) = \{w_j, v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.(r\text{-pan})^v)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_1^j w_j : 1 \leq j \leq n\} \cup \{w_1^j w_1^{j+1} : 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 5$.

For r odd we define an edge labeling $f : E(P(n.(r\text{-pan})^v)) \to Z_k - \{0\}$ as follows:

$$f(v_i^1 v_{i+1}^1) = f(v_i^n v_{i+1}^n) = \begin{cases} k-a, & \text{for } i = 1, 3, \dots, r, \\ a, & \text{for } i = 2, 4, \dots, r-1, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} k-2a, & \text{for } i = 1, 3, \dots, r,\ j = 2, 3, \dots, n-1, \\ 2a, & \text{for } i = 2, 4, \dots, r-1,\ j = 2, 3, \dots, n-1, \end{cases}$$

$$f(v_1^1 w_1) = f(v_1^n w_n) = 2a,$$

$$f(v_1^j w_j) = 4a, \quad \text{for } j = 2, 3, \dots, n-1,$$

$$f(w_1^j w_1^{j+1}) = k-2a, \quad \text{for } j = 1, 2, \dots, n-1.$$

Then the induced vertex labeling $f^+ : V(P(n.(r\text{-pan})^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.(r\text{-pan})^v))$. This means that $P(n.(r\text{-pan})^v)$ is a Z_k-magic graph. $\square$

An example of a Z_9-magic labeling of $P(4.(5\text{-pan})^v)$ is illustrated in Figure 11.

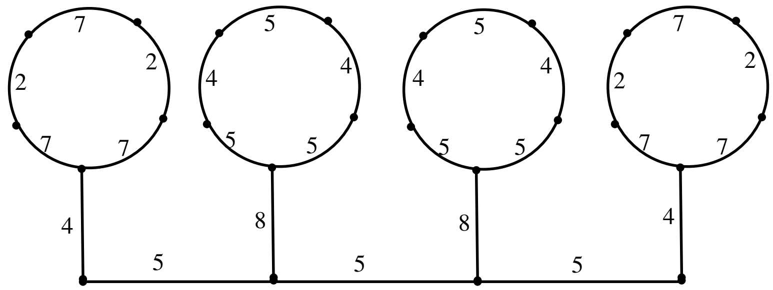

Figure 11: A Z_9-magic labeling of $P(4.(5\text{-pan})^v)$.

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Totally umbilical proper slant submanifolds of para-Kenmotsu manifold

M.S. Siddesha[1], C.S. Bagewadi[2] and D. Nirmala[3]

[1] *Department of mathematics,*
New Horizon College of Engineering,
Bangalore, India
[2,3] *Department of Mathematics,*
Kuvempu University,
Shankaraghatta, Shimoga, Karnataka, India
mssiddesha@gmail.com, prof_bagewadi@yahoo.co.in, nirmaladraj14@gmail.com

ABSTRACT

In this paper, we study slant submanifolds of a para-Kenmotsu manifold. We prove that totally umbilical slant submanifold of a para-Kenmotsu manifold is either invariant or anti-invariant or dimension of submanifold is 1 or the mean curvature vector H of the submanifold lies in the invariant normal subbundle.

RESUMEN

En este paper estudiamos subvariedades inclinadas en variedades para-Kenmotsu. Demostramos que una subvariedad inclinada en una variedad para-Kenmotsu totalmente umbilical es invariante, o anti-invariante, o una subvariedad de dimensión 1, o el vector de curvatura media H de la subvariedad vive en el fibrado normal invariante.

Keywords and Phrases: para-Kenmotsu manifold; totally umbilical; slant submanifold.
2010 AMS Mathematics Subject Classification: 53C25, 53C40, 53D15.

1 Introduction

The study of submanifolds of an almost contact manifold is one of the utmost interesting topics in differential geometry. According to the behaviour of the tangent bundle of a submanifold with respect to action of the almost contact structure ϕ of the ambient manifold, there are two well known classes of submanifolds, namely, invariant and anti-invariant submanifolds. Chen [4], introduced the notion of slant submanifolds of the almost Hermitian manifolds. The contact version of slant submanifolds were given by Lotta [12]. Since then many research articles have been appeared on the existence of different contact and lorentzian manifolds (See. [1, 3, 7, 14, 15]).

Motivated by the above studies, in the present paper we study slant submanifolds of almost para-Kenmotsu manifold and give a classification of results. Also we prove that totally umbilical slant submanifolds of para-Kenmotsu manifolds are totally geodesic.
The paper is organized as follows: In section 2, we review some basic concepts of para-Kenmotsu manifold and submanifold theory. Section 3 is the main section of this paper. In this section we give the classification result of totally umbilical slant submanifolds of para-Kenmotsu manifold. Further, we prove that totally umbilical slant submanifolds of a para-Kenmotsu manifold are totally geodesic.

2 Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional smooth manifold, ϕ a tensor field of type $(1,1)$, ξ a vector field and η a 1-form. We say that (ϕ,ξ,η) is an almost para contact structure on $\tilde{M}$ if [18]

$$\phi\xi = 0, \quad \eta\cdot\phi = 0, \quad rank(\phi) = 2m, \tag{2.1}$$

$$\phi^2 = I - \eta\otimes\xi, \quad \eta(\xi) = 1. \tag{2.2}$$

If an almost paracontact manifold admits a pseudo Riemannian metric g of signature $(m+1,m)$ satisfying

$$g(\phi\cdot,\phi\cdot) = -g + \eta\otimes\eta \tag{2.3}$$

called almost para contact metric manifold. Examples of almost para contact metric structure are given in [6] and [9].

Analogous to the definition of Kenmotsu manifold [10], Welyczko [17] introduced para-Kenmotsu structure for three dimensional normal almost para contact metric structures. The similar notion called p-Kenmotsu structure appears in the Sinha and Sai Prasad [16].

Definition 2.1. *An almost para contact metric manifold* $M(\phi,\xi,\eta,g)$ *is para-Kenmotsu manifold if the Levi-Civita connection* $\tilde{\nabla}$ *of* g *satisfies*

$$(\tilde{\nabla}_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for any $X, Y \in \chi(M)$, *(where* $\chi(M)$ *is the set of all differential vector fields on* M*).*

From (2.4), we have

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Assume M is a submanifold of a para-Kenmotsu manifold $\tilde{M}$. Let g and ∇ be the induced Riemannian metric and connections of M, respectively. Then the Gauss and Weingarten formulae are given respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.7}$$

for all X, Y on TM and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the normal connection and A is the shape operator of M with respect to the unit normal vector N. The second fundamental form σ and the shape operator A are related by:

$$g(\sigma(X, Y), N) = g(A_N X, Y). \tag{2.8}$$

Now for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we write

$$\phi X = PX + FX, \tag{2.9}$$

$$\phi V = pV + fV. \tag{2.10}$$

For $X, Y \in \Gamma(TM)$, it is easy to observe from (2.1) and (2.9) that

$$g(PX, Y) = -g(X, PY). \tag{2.11}$$

The covariant derivatives of the endomorphisms ϕ, P and F are defined respectively as

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y, \ \ \forall X, Y \in \Gamma(T\tilde{M}), \tag{2.12}$$

$$(\tilde{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \ \ \forall X, Y \in \Gamma(TM), \tag{2.13}$$

$$(\tilde{\nabla}_X F)Y = \nabla_X FY - F\nabla_X Y, \ \ \forall X, Y \in \Gamma(TM). \tag{2.14}$$

The structure vector field ξ has been considered to be tangential to M throughout this paper, else M is simply anti-invariant [12]. Since $\xi \in TM$, for any $X \in \Gamma(TM)$ by virtue of (2.5) and (2.6), we have

$$\nabla_X \xi = X - \eta(X)\xi \ \text{ and } \ \sigma(X, \xi) = 0. \tag{2.15}$$

Making use of (2.4), (2.6), (2.7), (2.9), (2.10) and (2.12)-(2.14), we obtain

$$(\tilde{\nabla}_X P)Y = p\sigma(X, Y) + A_{FY}X + g(PX, Y)\xi - \eta(Y)PX, \tag{2.16}$$

$$(\tilde{\nabla}_X F)Y = f\sigma(X, Y) - \sigma(X, PY) - \eta(Y)FX. \tag{2.17}$$

A submanifold M of an almost para contact metric manifold $\tilde{M}$ is said to be totally umbilical if

$$\sigma(X, Y) = g(X, Y)H, \tag{2.18}$$

where H is the mean curvature vector of M. Further M is totally geodesic if $\sigma(X, Y) = 0$ and minimal if $H = 0$.

3 Slant submanifolds of an almost contact metric manifold

For any $x \in M$ and $X \in T_xM$ such that X, ξ are linearly independent, the angle $\theta(x) \in [0, \frac{\pi}{2}]$ between ϕX and T_xM is a constant θ, that is θ does not depend on the choice of X and $x \in M$. θ is called the slant angle of M in $\tilde{M}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively [5]. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.
We mention the following results for later use.

Theorem 3.1. *[1] Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi). \tag{3.1}$$

Further more, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Corolary 1. *[1] Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, for any $X, Y \in TM$, we have*

$$g(PX, PY) = -\cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{3.2}$$

$$g(FX, FY) = -\sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{3.3}$$

Theorem 3.2. *Let M be a totally umbilical slant submanifold of a para-Kenmotsu manifold $\tilde{M}$. Then either one of the following statements is true:*
(i) M is invariant;
(ii) M is anti-invariant;
(iii) M is totally geodesic;
(iv) dim$M$= 1;
(v) If M is proper slant, then $H \in \Gamma(\mu)$;
where H is the mean curvature vector of M.

Proof. Suppose M is totally umbilical slant submanifold, then we have

$$\sigma(PX, PX) = g(PX, PX)H = \cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H.$$

By virtue of (2.6), one can get

$$\cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H = \tilde{\nabla}_{PX}PX - \nabla_{PX}PX.$$

From (2.9), we have

$$\cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H = \tilde{\nabla}_{PX}\phi X - \tilde{\nabla}_{PX}FX - \nabla_{PX}PX.$$

Applying (2.7) and (2.12), we get

$$\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H=(\tilde{\nabla}_{PX}\phi)X+\phi\tilde{\nabla}_{PX}X+A_{FX}PX-\nabla^{\perp}_{PX}FX-\nabla_{PX}PX.$$

Using (2.4) and (2.6), we obtain

$$\begin{aligned}\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H &= g(\phi PX,X)\xi-\eta(X)\phi PX+\phi(\nabla_{PX}X+\sigma(X,PX))\\ &\quad +A_{FX}PX-\nabla^{\perp}_{PX}FX-\nabla_{PX}PX.\end{aligned}$$

From (2.9), (2.11), (2.18) and the fact that X and PX are orthogonal vector fields on M, we arrive at

$$\begin{aligned}\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H &= -g(PX,PX)\xi-\eta(X)P^2X-\eta(X)FPX+P\nabla_{PX}X+F\nabla_{PX}X\\ &\quad +A_{FX}PX-\nabla^{\perp}_{PX}FX-\nabla_{PX}PX.\end{aligned}$$

Then applying (3.1) and (3.2), we obtain

$$\begin{aligned}\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H &= \cos^2\theta\{\|X\|^2-\eta^2(X)\}\xi+\cos^2\theta\eta(X)\{X-\eta(X)\}\xi-\eta(X)FPX\\ &\quad +P\nabla_{PX}X+F\nabla_{PX}X+A_{FX}PX-\nabla^{\perp}_{PX}FX-\nabla_{PX}PX. \qquad (3.4)\end{aligned}$$

Taking inner product with PX in (3.4), we get

$$0=g(P\nabla_{PX}X,PX)+g(A_{FX}PX,PX)-g(\nabla_{PX}PX,PX). \qquad (3.5)$$

By virtue of (3.2), the first term of (3.5) can be written as

$$g(P\nabla_{PX}X,PX)=-\cos^2\theta\{g(\nabla_{PX}X,X)-\eta(X)g(\nabla_{PX}X,\xi)\}. \qquad (3.6)$$

We simplify the third term of (3.5) as follows

$$\begin{aligned}g(\nabla_{PX}PX,PX) &= g(\tilde{\nabla}_{PX}PX,PX)=\frac{1}{2}PXg(PX,PX).\\ &= \frac{1}{2}PX[-\cos^2\theta\{(g(X,X)-\eta^2(X))\}]\\ &= -\frac{1}{2}\cos^2\theta[PXg(X,X)-P(X)(g(X,\xi)g(X,\xi))]\\ &= -\frac{1}{2}\cos^2\theta[PXg(X,X)-2\eta(X)P(X)g(X,\xi)]\\ &= -\frac{1}{2}\cos^2\theta[2g(\tilde{\nabla}_{PX}X,X)-2\eta(X)\{g(\tilde{\nabla}_{PX}X,\xi)+g(X,\tilde{\nabla}_{PX}\xi)\}].\end{aligned}$$

Using (2.5), (2.6), (3.6) and the fact that X and PX are orthogonal vector fields on M, we derive

$$\begin{aligned}g(\nabla_{PX}PX,PX) &= -\cos^2\theta[g(\nabla_{PX}X,X)-\eta(X)g(\nabla_{PX}X,\xi)\\ &\quad -\eta(X)g(X,PX-\eta(PX)\xi)]\\ &= -\cos^2\theta[g(\nabla_{PX}X,X)-\eta(X)g(\nabla_{PX}X,\xi)]\\ \rightarrow g(\nabla_{PX}PX,PX) &= g(P\nabla_{PX}X,PX).\end{aligned}$$

Using this fact in (3.5), we obtain

$$0 = g(A_{FX}PX, PX) = g(\sigma(PX, PX), FX).$$

As M is totally umbilical slant, then from (2.18) and (3.2), we obtain

$$0 = -\cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX). \tag{3.7}$$

Thus from (3.7), we conclude that either $\theta = \frac{\pi}{2}$ that is M is anti-invariant which part (ii) or the vector field X is parallel to the structure vector field ξ, i.e., M is 1-dimensional submanifold which is fourth part of the theorem or $H \perp FX$, for all $X \in \Gamma(TM)$, i.e., $H \in \Gamma(\mu)$ which is the last part of the theorem or $H = 0$, i.e., M is totally geodesic which is (iii) or $FX = 0$, i.e., M is invariant which is part (i). This completes the proof of the theorem. □

Theorem 3.3. *Every totally umbilical proper slant submanifold of a para-Kenmotsu manifold is totally geodesic.*

Proof. Let M be a totally umbilical proper slant submanifold of a para-Kenmotsu manifold $\tilde{M}$, then for any $X, Y \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X\phi Y - \phi\tilde{\nabla}_X Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Using equations (2.6) and (2.9), we get

$$\tilde{\nabla}_X PY + \tilde{\nabla}_X FY - \phi(\nabla_X Y + \sigma(X, Y)) = g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX.$$

Again from (2.6), (2.7) and (2.9), we obtain

$$\begin{aligned} g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX &= \nabla_X PY + \sigma(X, PY) - A_{FY}X \\ &\quad +\nabla^\perp_X FY - P\nabla_X Y - F\nabla_X Y - \phi\sigma(X, Y). \end{aligned}$$

As M is totally umbilical, then

$$\begin{aligned} g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX &= \nabla_X PY + g(X, PY)H - A_{FY}X + \nabla^\perp_X FY \\ &\quad -P\nabla_X Y - F\nabla_X Y - g(X, Y)\phi H. \end{aligned} \tag{3.8}$$

Taking the inner product with ϕH in (3.8) and from the fact that $H \in \Gamma(\mu)$, we obtain

$$g(\nabla^\perp_X FY, \phi H) = -g(X, Y)\|H\|^2.$$

Applying (2.7) and the property of Riemannian connection the above equation takes the form

$$g(FY, \nabla^\perp_X \phi H) = g(X, Y)\|H\|^2. \tag{3.9}$$

Now for any $X \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X \phi H = (\tilde{\nabla}_X \phi)H + \phi \tilde{\nabla}_X H.$$

Using the fact $H \in \Gamma(\mu)$ and by virtue of (2.4), (2.7) and (2.9), we obtain

$$-A_{\phi H}X + \nabla^{\perp}_X \phi H = -PA_H X - FA_H X + \phi \nabla^{\perp}_X H. \tag{3.10}$$

Also for any $X \in \Gamma(TM)$, we have

$$g(\nabla^{\perp}_X H, FX) = g(\tilde{\nabla}_X H, FX) = -g(H, \tilde{\nabla}_X FX).$$

Using (2.9), we obtain

$$g(\nabla^{\perp}_X H, FX) = -g(H, \tilde{\nabla}_X \phi X) + g(H, \tilde{\nabla}_X PX).$$

Further from (2.6) and (2.12), we derive

$$g(\nabla^{\perp}_X H, FX) = -g(H, (\tilde{\nabla}_X \phi)X) - g(H, \phi \tilde{\nabla}_X X) + g(H, \sigma(X, PX)).$$

Using (2.4) and (2.18), the first and last term of right hand side of the above equation are identically zero and hence by (2.3), the second term gives

$$g(\nabla^{\perp}_X H, FX) = g(\phi H, \tilde{\nabla}_X X).$$

Again by using (2.6) and (2.18), we obtain

$$g(\nabla^{\perp}_X H, FX) = g(\phi H, H)\|X\|^2 = 0.$$

This means that

$$\nabla^{\perp}_X H \in \Gamma(\mu). \tag{3.11}$$

Now, taking the inner product in (3.10) with FY for any $Y \in \Gamma(TM)$, we get

$$g(\nabla^{\perp}_X \phi H, FY) = -g(FA_H X, FY) + g(\phi \nabla^{\perp}_X H, FY).$$

Using (3.11) and from (3.3) and (3.9), we obtain

$$g(X, Y)\|H\|^2 = \sin^2\theta\{g(A_H X, Y) - \eta(Y)g(A_H X, \xi)\}. \tag{3.12}$$

Hence by (2.8) and (2.18), the above equation reduces to

$$g(X, Y)\|H\|^2 = \sin^2\theta\{g(X, Y)\|H\|^2 - \eta(Y)g(\sigma(X, Y), H)\}. \tag{3.13}$$

Since for a para-Kenmotsu manifold $\tilde{M}$, $\sigma(X, \xi) = 0$ for any X tangent to $\tilde{M}$, thus we obtain

$$g(X, Y)\|H\|^2 = \sin^2\theta\{g(X, Y)\|H\|^2.$$

Therefore, the above equation can be written as

$$\cos^2\theta g(X, Y)\|H\|^2 = 0. \tag{3.14}$$

Since M is proper slant submanifold, thus from (3.14) we conclude that $H = 0$, i.e., M is totally geodesic in $\tilde{M}$. This completes the proof. ∎

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The perimeter of a flattened ellipse can be estimated accurately even from Maclaurin's series

Vito Lampret
University of Ljubljana, 386 Slovenia
vito.lampret@guest.arnes.si

ABSTRACT

For the perimeter $P(a,b)$ of an ellipse with the semi-axes $a \geq b \geq 0$ a sequence $Q_n(a,b)$ is constructed such that the relative error of the approximation $P(a,b) \approx Q_n(a,b)$ satisfies the following inequalities

$$\begin{aligned} 0 \leq -\frac{P(a,b)-Q_n(a,b)}{P(a,b)} &\leq \frac{(1-q^2)^{n+1}}{(2n+1)^2} \\ &\leq \frac{1}{(2n+1)^2}\, e^{-q^2(n+1)}, \end{aligned}$$

true for $n \in \mathbb{N}$ and $q = \frac{b}{a} \in [0,1]$.

RESUMEN

Para el perímetro $P(a,b)$ de una elipse con semiejes $a \geq b \geq 0$, se construye una sucesión $Q_n(a,b)$ tal que el error relativo de la aproximación $P(a,b) \approx Q_n(a,b)$ satisface las siguientes desigualdades

$$\begin{aligned} 0 \leq -\frac{P(a,b)-Q_n(a,b)}{P(a,b)} &\leq \frac{(1-q^2)^{n+1}}{(2n+1)^2} \\ &\leq \frac{1}{(2n+1)^2}\, e^{-q^2(n+1)}, \end{aligned}$$

válidas para $n \in \mathbb{N}$ y $q = \frac{b}{a} \in [0,1]$.

Keywords and Phrases: approximation, elementary, ellipse, estimate, Maclaurin series, mathematical validity, perimeter, simple.

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1 Introduction

Injective parametric equations of the border of an ellipse having semi-axes of lengths a and $b \leq a$ are given as $x = x(t) = a\cos(t)$, $y = y(t) = b\sin(t)$, where $t \in [0, 2\pi)$. Its perimeter $P(a,b)$ is determined as

$$\begin{aligned} P(a,b) &= \int_0^{2\pi} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\,\mathrm{d}t = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\,\mathrm{d}t \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\cos^2(t)}\,\mathrm{d}t \underbrace{=}_{t=\pi/2-\tau} 4a\int_{\frac{\pi}{2}}^{0} \sqrt{1-\epsilon^2\sin^2(\tau)}(-\,\mathrm{d}\tau). \end{aligned}$$

Thus, the perimeter $P(a,b)$ of an ellipse having semi-axes of lengths a and $b \leq a$, is given as

$$P(a,b) = 4a\,E(\epsilon), \tag{1.1}$$

where

$$E(\epsilon) := \int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau \tag{1.2}$$

is complete elliptic integral of the second kind and

$$\epsilon := \sqrt{1-\left(\tfrac{b}{a}\right)^2} = \sqrt{\tfrac{a^2-b^2}{a^2}} \in [0,1), \tag{1.3}$$

is the eccentricity of an ellipse.

For $b \approx 0$ it is intuitively evident that $P(a,b) > 2\times 2a = 4a$. Moreover, since the functions $\epsilon \mapsto 1-\epsilon^2\sin^2(\tau)$ are decreasing on the interval $[0,1]$ for any $\tau \in [0,\pi/2]$, the function $E(\epsilon)$ is decreasing too. Therefore, we have

$$1 = \int_0^{\frac{\pi}{2}} \cos(\tau)\,\mathrm{d}\tau = E(1) \leq E(\epsilon) \leq E(0) = \frac{\pi}{2},$$

for $0 \leq \epsilon \leq 1$. Consequently, due to (1.1),

$$\inf_{0<b\leq a} P(a,b) = 4a < P(a,b) \leq P(a,a) = 2a\pi. \tag{1.4}$$

The first exact formula for an ellipse perimeter was presented 277 years ago by Collin Maclaurin [24], given as the sum of infinite series:

$$\begin{aligned} P(a,b) &= 2\pi a\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}^2 (1-2k)\,\epsilon^{2k} \\ &= 2\pi a\sum_{k=0}^{\infty}\left(\frac{(2k)!}{(2^k\,k!)^2}\right)^2 \frac{\left(-\,\epsilon^{2k}\right)}{2k-1} \\ &= 2\pi a\left\{1-\sum_{k=0}^{\infty}\left[\frac{1}{4^k}\binom{2k}{k}\right]^2 \frac{\epsilon^{2k}}{2k-1}\right\}, \end{aligned} \tag{1.5}$$

valid for $0 \le \epsilon \le 1$, where $\epsilon = \left(1 - b^2/a^2\right)^{1/2}$, called the eccentricity[1] of an ellipse. This series originates from the integral (1.2). Later, Ivory [13] discovered a faster converging series for the integral (1.2), which was later significantly improved by Gauss and Kummer. Additionally, Gauss developed very efficient, swiftly convergent method of arithmetic-geometric means for computation of the integral (1.2), see [1]. Subsequently, a lot of approximations of the ellipse perimeter have been found. For example, among them is very popular Ramanujan's "extraordinarily unusual and exotic" approximation [2]. Motivated by the Barnard–Pearce–Schovanec approximations [3] and Villarino's contribution on the accuracy of a Ramanujan's approximation [29] and his paper [28], we shall derive elementarily[2] an asymptotic estimate of the ellipse perimeter, based on the oldest Maclaurin series expansion. The result obtained surpasses most of the previous approximations.

2 Background

2.1 The binomial approximation

Using Taylor's formula (see for example [15, p. 111] with $x_0 = 0$, $h = x$ and $p = n$),

$$f(x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i)}(0)}{i!} x^i + \frac{x^{n+1}}{n!} \int_0^1 (1-t)^n f^{(n+1)}(t\,x)\ \mathrm{d}t\ ,$$

(true for $a, b \in \mathbb{R}$, $a < b$, $n \in \mathbb{N}$, $x \in [a, b]$ and $f \in C^{n+1}[a, b]$) for the function $f(x) \equiv (1+x)^{\frac{1}{2}}$, we obtain[3]

$$\begin{aligned}(1+x)^{\frac{1}{2}} =& 1 + \sum_{i=1}^{n} \binom{\frac{1}{2}}{i} x^i \\ &+ x^{n+1} \int_0^1 (1-t)^n \binom{\frac{1}{2}}{n+1} (n+1)(1+tx)^{\frac{1}{2}-n-1}\ \mathrm{d}t\,,\end{aligned} \tag{2.1}$$

valid for $x \in (-1, 1]$ and $n \in \mathbb{N}$.

Introducing w_i, called the i-th Wallis ratio, for[4] $i \ge 0$,

$$w_i := \prod_{j=1}^{i} \frac{2j-1}{2j} = \frac{(2i)!}{4^i (i!)^2} = \frac{1}{4^i} \binom{2i}{i}, \tag{2.2}$$

[1] We have $\epsilon = \sqrt{1-q^2}$, where $q := b/a$ is called the aspect ratio of an ellipse.

[2] not using complex analysis and absolute and uniform convergence of a series, as was used, for example, in [18]

[3] considering the identity $f^{(i)}(x) \equiv \binom{\frac{1}{2}}{i}(i!)(1+x)^{\frac{1}{2}-i}$

[4] $\prod_{j=m}^{n} x_j := 1$, for $m > n$; consequently $w_0 = 1$

we obtain

$$\binom{\frac{1}{2}}{i} = \frac{\prod_{j=0}^{i-1}(\frac{1}{2}-j)}{i!} = (-1)^{i-1}\frac{1}{2^i}\cdot\frac{\prod_{j=1}^{i-1}(2j-1)}{\prod_{j=1}^{i} j}$$

$$= (-1)^{i-1}\frac{1}{2i-1}\prod_{j=1}^{i}\frac{2j-1}{2j} = \underline{\underline{(-1)^{i-1}\frac{w_i}{2i-1}}}\,. \tag{2.3}$$

Thus, thanks to (2.1), replacing x by $-x$, we get

$$(1-x)^{\frac{1}{2}} = 1 - \sum_{i=1}^{n}\frac{w_i}{2i-1}x^i + r_n(x), \tag{2.4}$$

with the remainder

$$r_n(x) = -x^{n+1}\frac{w_{n+1}}{2n+1}(n+1)\int_0^1\left(\frac{1-t}{1-tx}\right)^n\frac{\mathrm{d}t}{(1-tx)^{\frac{1}{2}}}\,,$$

estimated, for $x \in (0,1)$, as

$$0 < -r_n(x) = \frac{x^{n+1}}{(1-x)^{\frac{1}{2}}}\cdot\frac{w_{n+1}}{2n+1}(n+1)\int_0^1\left(\frac{1-t}{1-tx}\right)^n\mathrm{d}t$$

$$< \frac{w_{n+1}}{(1-x)^{\frac{3}{2}}(2n+1)}x^{n+1}\,. \tag{2.5}$$

Indeed, using the substitution $\tau = \frac{1-t}{1-tx}$, i.e. $t = \frac{1-\tau}{1-\tau x}$ we have (considering $x \in (0,1)$)

$$\int_0^1\left(\frac{1-t}{1-tx}\right)^n\mathrm{d}t = \int_1^0\tau^n\left(-\frac{1-x}{(1-\tau x)^2}\right)\mathrm{d}\tau = \int_0^1\tau^n\cdot\frac{1-x}{(1-\tau x)^2}\,\mathrm{d}\tau$$

$$< \int_0^1\tau^n\cdot\frac{1-x}{(1-x)^2}\,\mathrm{d}\tau = \frac{1}{(1-x)(n+1)}\,.$$

2.2 Wallis ratios estimates

The integrals

$$I_n := \int_0^{\frac{\pi}{2}}\sin^n(x)\,\mathrm{d}x \qquad (n \ge 0), \tag{2.6}$$

satisfy the recurrence relation

$$I_n = \frac{n-1}{n}\,I_{n-2}, \qquad \text{for } n \ge 2,$$

where, obviously, we have $I_0 = \frac{\pi}{2}$ and $I_1 = 1$. Consequently, by induction we find

$$I_{2i} = \prod_{j=1}^{i}\frac{2j-1}{2j}\cdot\frac{\pi}{2} \;=\; w_i\cdot\frac{\pi}{2} \tag{2.7}$$

and

$$I_{2i+1} = \prod_{j=1}^{i} \frac{2j}{2j+1} = \frac{1}{(2i+1)w_i}\,. \tag{2.8}$$

Obviously, we estimate

$$0 < \sin^{2i+2}(x) < \sin^{2i+1}(x) < \sin^{2i}(x) < 1,$$

for $x \in \left(0, \frac{\pi}{2}\right)$ and $i \in \mathbb{N}$. Integrating, we obtain

$$0 \;<\; I_{2i+2} \;<\; I_{2i+1} \;<\; I_{2i} \;<\; 1,$$

for all $i \in \mathbb{N}$. Hence, thanks to (2.7)–(2.8), we get

$$\frac{2i+1}{2i+2} w_i \cdot \frac{\pi}{2} = w_{i+1} \cdot \frac{\pi}{2} \;<\; \frac{1}{(2i+1)w_i} \;<\; w_i \cdot \frac{\pi}{2}\,.$$

Consequently,

$$\frac{2}{\pi} \cdot \frac{1}{2i+1} < w_i^2 < \frac{2}{\pi} \cdot \frac{1}{2i-1} \qquad (i \in \mathbb{N}). \tag{2.9}$$

We remark that there exists a huge literature on useful, more accurate estimates for w_n, e.g. [4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 19, 20, 21, 22, 23, 25, 26, 27, 31]. However, for our needs, there suffice rather rough estimates (2.9).

2.3 Some logarithmic formula expansion

For $p \geq 1$ and $-1 < t < 1$ we have

$$\begin{aligned} 2\sum_{j=0}^{p-1} t^{2j} &= \sum_{k=0}^{2(p-1)} \left(t^k + (-t)^k\right) = \sum_{k=0}^{2(p-1)} t^k + \sum_{k=0}^{2(p-1)} (-t)^k \\ &= \frac{1-t^{2p-1}}{1-t} + \frac{1-(-t)^{2p-1}}{1+t}\,. \end{aligned}$$

Consequently, integrating from 0 to $x \in (-1,1)$, the first and the last part of these equalities, we obtain

$$\begin{aligned} 2\sum_{j=0}^{p-1} \frac{x^{2j+1}}{2j+1} &= \int_0^x \frac{1}{1-t}\,\mathrm{d}t - \int_0^x \frac{t^{2p-1}}{1-t}\,\mathrm{d}t + \int_0^x \frac{1}{1+t}\,\mathrm{d}t + \int_0^x \frac{t^{2p-1}}{1+t}\,\mathrm{d}t \\ &= -\ln(1-x) + \ln(1+x) - \underbrace{\int_0^x \left(\frac{1}{1-t} - \frac{1}{1+t}\right) t^{2p-1}\,\mathrm{d}t}_{=R_p^*(x)}. \end{aligned}$$

Thus,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{i=1}^{p} \frac{x^{2i-1}}{2i-1} + R_p^*(x), \tag{2.10}$$

with the remainder $R_p^*(x)$,

$$R_p^*(x) := \int_0^x \frac{2t^{2p}}{1-t^2}\,\mathrm{d}t \geq \int_0^x 2t^{2p}\,\mathrm{d}t. \qquad (0<x<1),$$

estimated as

$$\frac{2x^{2p+1}}{2p+1} < R_p^*(x) < \frac{2x^{2p+1}}{(1-x^2)(2p+1)} \qquad (p\in\mathbb{N},\ 0<x<1) \tag{2.11}$$

From (2.10)–(2.11) we end up with the expansion

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{i=1}^{\infty}\frac{x^{2i-1}}{2i-1}, \tag{2.12}$$

true for $x\in(0,1)$ and, consequently, also for $x\in(-1,0]$.

3 The Maclaurin series

3.1 Derivation

Referring to (2.4)–(2.5) and (2.6)–(2.7), we have, for any $n\in\mathbb{N}$,

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\sqrt{1-\underbrace{\epsilon^2\sin^2(\tau)}}\,\mathrm{d}\tau &= \frac{\pi}{2} - \sum_{i=1}^{n}\frac{w_i\,\epsilon^{2i}}{2i-1}\int_0^{\frac{\pi}{2}}\sin^{2i}(\tau)\,\mathrm{d}\tau + r_n^*(\epsilon)\\ &= \frac{\pi}{2} - \sum_{i=1}^{n}\frac{w_i\,\epsilon^{2i}}{2i-1}\left(w_i\frac{\pi}{2}\right) + r_n^*(\epsilon).\end{aligned}$$

Hence

$$\int_0^{\frac{\pi}{2}}\sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau = \frac{\pi}{2}\left(1-\sum_{i=1}^{n}\frac{w_i^2}{2i-1}\,\epsilon^{2i}\right) + r_n^*(\epsilon), \tag{3.1}$$

where w_i is the i-th Wallis' ratio and the error term $r_n^*(\epsilon) := \int_0^{\pi/2} r_n\big(\epsilon^2\sin^2(\tau)\big)\,\mathrm{d}\tau$ is estimated, due to (2.5) and considering (2.6)–(2.7), as

$$\begin{aligned}0 \leq -r_n^*(\epsilon) &\leq \frac{\epsilon^{2n+2}}{1-\epsilon^2}\cdot\frac{w_{n+1}}{2n+1}\int_0^{\frac{\pi}{2}}\sin^{2n+2}(\tau)\,\mathrm{d}\tau\\ &= \frac{\epsilon^{2n+2}\,w_{n+1}}{(1-\epsilon^2)(2n+1)}\cdot w_{n+1}\,\frac{\pi}{2}\,.\end{aligned}$$

Thus, according to (2.9),

$$0 \leq -r_n(\epsilon) \leq \frac{\pi}{2}\cdot\frac{1}{1-\epsilon^2}\cdot\frac{w_{n+1}^2}{2n+1}\,\epsilon^{2n+2} \leq \frac{1}{1-\epsilon^2}\cdot\frac{\epsilon^{2n+2}}{(2n+1)^2}\ . \tag{3.2}$$

This estimate is not usable for $\epsilon\approx 1$, i.e. for $b\approx 0$ (for a very flattened ellipse).

As $w_n^2 \le 1$, we have $\lim\limits_{n\to\infty} r_n(\epsilon) = 0$ for any $\epsilon < 1$, which is always true for ordinary ellipse, due to the equivalence $\epsilon = 1 \Leftrightarrow b = 0$. Hence, there holds the so-called Maclaurin series expansion[5]

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau = \frac{\pi}{2}\left(1-\sum_{i=1}^{\infty}\frac{w_i^2}{2i-1}\,\epsilon^{2i}\right), \tag{3.3}$$

valid for $0 \le \epsilon < 1$. In addition, the series on the right is convergent also for $\epsilon = 1$ due to (2.9). Indeed, we have $\frac{w_i^2}{2i-1} < \frac{1}{i^2}$, which implies the convergence of the series $\sum_{i=1}^{\infty}\frac{w_i^2}{2i-1}$.

Remark 3.1. *About fifty years after Maclaurin's book [24], including the series* (3.3)*, Ivory published article [13], where he presented the expansion*

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau = \frac{\pi(a+b)}{4a}\left(1+\sum_{i=1}^{\infty}\frac{w_i^2}{(2i-1)^2}\,\lambda^{2i}\right)\qquad\left(\lambda=\frac{a-b}{a+b}\right),$$

where the series on the right converges slightly faster than the series in (3.3)*.*

Applying (2.9) for the partial sums

$$\mu_n(\epsilon) := \sum_{i=1}^{n}\frac{w_i^2}{2i-1}\,\epsilon^{2i}\qquad (n\in\mathbb{N}\cup\{\infty\}), \tag{3.4}$$

we shall estimate the series $\mu_\infty(\epsilon)$ figuring in (3.3).

3.2 Approximating $\mu_\infty(\epsilon)$

Using (2.9) we estimate,

$$\frac{2}{\pi(2i-1)(2i+1)} < \frac{w_i^2}{2i-1} < \frac{2}{\pi(2i-1)^2}\qquad (i\in\mathbb{N}). \tag{3.5}$$

Therefore

$$\mu_\infty(\epsilon) \approx \sum_{i=1}^{\infty}\frac{2\epsilon^{2i}}{\pi(2i-1)(2i+1)}\quad (0\le\epsilon<1).$$

This idea produces the next theorem.

Theorem 3.2. *We have*

$$\mu_\infty(\epsilon) = M_n(\epsilon) + \delta_n(\epsilon), \tag{3.6}$$

where

$$M_n(\epsilon) = A(\epsilon) + B_n(\epsilon), \tag{3.7}$$

$$A(\epsilon) := \frac{1}{2\pi}\left[\left(\epsilon-\frac{1}{\epsilon}\right)\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)+2\right] \in \left(0,\tfrac{1}{\pi}\right), \tag{3.8}$$

[5]The coefficients of the original Maclaurin series [24] have a visually more complicated form.

$$B_n(\epsilon) := \sum_{i=1}^{n} \left(w_i^2 - \frac{2}{\pi(2i+1)} \right) \frac{\epsilon^{2i}}{2i-1}\,, \tag{3.9}$$

and

$$0 < \delta_n(\epsilon) < \delta_n^*(\epsilon) := \frac{2\epsilon^{2n+2}}{\pi(2n+1)^2}\,, \tag{3.10}$$

valid for any integer $n \geq 1$ *and every* $0 < \epsilon < 1$.

The basic function $A(\epsilon)$ *is strictly increasing having the range* $\left(0, \frac{1}{\pi}\right)$, *where* $\lim\limits_{\epsilon \downarrow 0} A(\epsilon) = 0$ *and* $\lim\limits_{\epsilon \uparrow 1} A(\epsilon) = \frac{1}{\pi}$. *Both sequences,* $n \mapsto B_n(\epsilon)$ *and* $n \mapsto \delta_n(\epsilon)$, *are strictly increasing, for every* $\epsilon \in (0,1)$.

The sequence $n \mapsto M_n(\epsilon)$ *converges strictly increasingly to* $\mu_\infty(\epsilon)$, *for any* $\epsilon \in (0,1)$. *Additionally, for every* $n \in \mathbb{N}$, *the functions* $\epsilon \mapsto M_n(\epsilon)$ *and* $\epsilon \mapsto \delta_n(\epsilon)$ *are strictly increasing on the interval* $(0,1)$.

Figure 1 shows, on the left, the graph[6] of the basic function $A(\epsilon)$, and, on the right, the graphs of the functions $M_1^*(\epsilon)$ and $\mu_\infty(\epsilon)$. As an example, we present $B_4^*(\epsilon)$ and $\delta_4^*(\epsilon)$ as follows:

$$\begin{aligned}
B_4^*(\epsilon) &= \left(\tfrac{1}{4} - \tfrac{2}{3\pi}\right)\epsilon^2 + \frac{1}{3}\left(\tfrac{9}{64} - \tfrac{2}{5\pi}\right)\epsilon^4 + \frac{1}{5}\left(\tfrac{25}{256} - \tfrac{2}{7\pi}\right)\epsilon^6 + \frac{1}{7}\left(\tfrac{1225}{16384} - \tfrac{2}{9\pi}\right)\epsilon^8 \\
&\approx 0.037\,793\,409\,\epsilon^2 + 0.004\,433\,682\,\epsilon^4 + 0.001\,342\,114\,\epsilon^6 + 0.000\,576\,077\,\epsilon^8, \\
\delta_4^*(\epsilon) &\leq \frac{2\epsilon^{10}}{81\pi} \leq 0.00786\,\epsilon^{10} \qquad \left(\epsilon = \sqrt{1 - \left(\tfrac{b}{a}\right)^2}\right).
\end{aligned}$$

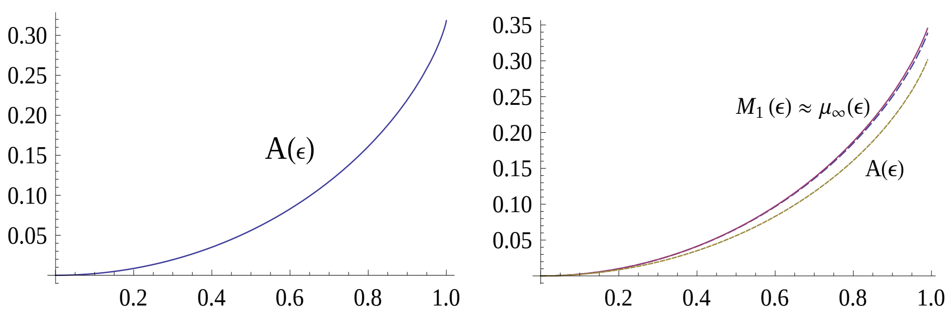

Figure 1: The graph of the basic function $A(\epsilon)$ (left) and the graphs of the functions $M_1(\epsilon)$, $\mu_\infty(\epsilon)$ and $A(\epsilon)$ (right).

Proof of Theorem 3.2. We have, for $0 < \epsilon < 1$,

$$\begin{aligned}
\sum_{i=1}^{\infty} w_i^2 \frac{\epsilon^{2i}}{2i-1} &= \sum_{i=1}^{\infty} \frac{2\,\epsilon^{2i}}{\pi(2i-1)(2i+1)} \\
&\quad + \sum_{i=1}^{n} \left(\frac{w_i^2}{2i-1} - \frac{2}{\pi(2i-1)(2i+1)} \right) \epsilon^{2i} + \delta_n(\epsilon),
\end{aligned} \tag{3.11}$$

[6] All the graphics and calculations in this paper are made using the Mathematica [30] computer system.

where

$$\delta_n(\epsilon) = \sum_{i=n+1}^{\infty} \left(w_i^2 - \frac{2}{\pi(2i+1)} \right) \frac{\epsilon^{2i}}{2i-1}. \tag{3.12}$$

Moreover, using (2.12), we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \frac{2}{\pi(2i-1)(2i+1)} \epsilon^{2i} \\
=&\frac{1}{\pi} \sum_{i=1}^{\infty} \left(\frac{1}{2i-1} - \frac{1}{2i+1} \right) \epsilon^{2i} \\
=&\frac{1}{\pi} \left(\frac{\epsilon}{2} \cdot 2 \sum_{i=1}^{\infty} \frac{\epsilon^{2i-1}}{2i-1} - \frac{1}{2\epsilon} \cdot 2 \sum_{i=1}^{\infty} \frac{\epsilon^{2i+1}}{2i+1} \right) \\
=&\frac{1}{\pi} \left[\frac{\epsilon}{2} \ln\left(\frac{1+\epsilon}{1-\epsilon} \right) - \frac{1}{2\epsilon} \left(\ln\left(\frac{1+\epsilon}{1-\epsilon} \right) - 2\epsilon \right) \right] \\
=&\frac{1}{2\pi} \left[\left(\epsilon - \frac{1}{\epsilon} \right) \ln\left(\frac{1+\epsilon}{1-\epsilon} \right) + 2 \right] = A(\epsilon).
\end{aligned}$$

Concerning $A(\epsilon) = \frac{1}{2\pi}\big(f(\epsilon)+2\big)$, the function $f(\epsilon) := \big(\epsilon - \frac{1}{\epsilon}\big) \ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$ $(0 < \epsilon < 1)$ has the derivative $f'(\epsilon) = g(\epsilon)/\epsilon^2$, where $g(\epsilon) = (1+\epsilon^2)\ln\left(\frac{1+\epsilon}{1-\epsilon}\right) - 2\epsilon$, having the derivative

$$g'(\epsilon) = \frac{2\epsilon}{1-\epsilon^2} \left(2\epsilon + (1-\epsilon^2) \ln\left(\frac{1+\epsilon}{1-\epsilon} \right) \right) \; > \; 0 \qquad (0 < \epsilon < 1).$$

Thus, g is strictly increasing on $[0,1)$. Consequently, $g(\epsilon) > g(0) = 0$, i.e. $f'(\epsilon) > 0$, for $0 < \epsilon < 1$. Therefore, f is strictly increasing on $(0,1)$ too. Moreover, using (2.10)–(2.11) with $p = 1$, we have

$$f(\epsilon) = \tfrac{\epsilon^2-1}{\epsilon} \cdot 2 \left(\epsilon + \vartheta \cdot \tfrac{2\epsilon^3}{3(1-\epsilon^2)} \right) = 2(\epsilon^2 - 1) \left(1 + \vartheta \cdot \tfrac{2}{1-\epsilon^2} \cdot \tfrac{\epsilon^2}{3} \right),$$

for some $\vartheta = \vartheta(\epsilon) \in (0,1)$. Hence, $\lim\limits_{\epsilon\downarrow 0} f(\epsilon) = -2$, i.e. $\lim\limits_{\epsilon\downarrow 0} A(\epsilon) = \lim\limits_{\epsilon\uparrow 1} \frac{1}{2\pi}\big(f(\epsilon)+2\big) = 0$. In addition, $\lim\limits_{\epsilon\uparrow 1} f(\epsilon) = \lim\limits_{\epsilon\uparrow 1} \left[\frac{\epsilon^2-1}{\epsilon} \cdot 2\ln(1+\epsilon) \right] - \frac{1}{1} \cdot \lim\limits_{h\downarrow 0} \Big(-h\ln(h) \Big) = 0$, where $h = 1 - \epsilon^2$. Thus, $\lim\limits_{\epsilon\uparrow 1} A(\epsilon) = \lim\limits_{\epsilon\uparrow 1} \frac{1}{2\pi}\big(f(\epsilon)+2\big) = \frac{1}{\pi}$.

According to (3.5), all summands in $B_n(\epsilon)$ and $\delta_n(\epsilon)$ (see (3.12)) are positive, i.e. the sequences $n \mapsto B_n(\epsilon)$ and $n \mapsto \delta_n(\epsilon)$ are strictly increasing; consequently the sequence $n \mapsto M_n(\epsilon)$ is also strictly increasing, for every $\epsilon \in (0,1)$.

Since all coefficients of the power series $B_n(\epsilon)$ and $\delta_n(\epsilon)$ (see (3.9) and (3.12)) are positive, due to (3.5), the functions $\epsilon \mapsto M_n(\epsilon)$ and $\epsilon \mapsto \delta_n(\epsilon)$ are strictly increasing on the interval $(0,1)$, for any $n \in \mathbb{N}$.

According to (3.12) and (3.5), we estimate, for $\epsilon \in (0,1]$,

$$0 < \delta_n(\epsilon) < \sum_{i=n+1}^{\infty} \left(\frac{2}{\pi(2i-1)} - \frac{2}{\pi(2i+1)} \right) \frac{\epsilon^{2n+2}}{2n+1} = \frac{2\epsilon^{2n+2}}{\pi(2n+1)^2},$$

using the telescoping method of summation. $\square$

Example 3.3. *Theorem 3.2 is quite useful for an estimate of $\mu_\infty(\epsilon)$, consequently for an estimate of the perimeter of an ellipse. For example, for a very flattened ellipse with $q = 0.01$ we have $0.99994 < \epsilon(q) < 0.99995$ where $0.36315 < M_{20}(0.99995) < 0.36316\ldots$ and $\delta^*_{20}(0.99995) < 0.00038$. Therefore, $0.36315 < \mu_\infty(0.99995) < 0.36316+0.00038 = 0.36354$. Thus, to three decimal places, we have $\mu_\infty(0.99995) = 0.363\ldots$. Consequently, the perimeter $P(a,b)$ of the corresponding ellipse is given as $P(a,b) = 4a\cdot\frac{\pi}{2}\big(1-\mu_\infty(0.99995)\big) \approx 4a\cdot\frac{\pi}{2}\big(1-0.363\big) \approx 4.002\,a$ (compare with relations (1.4)).*

Remark 3.4. *Referring to Abel's theorem on the boundary behavior of a power series, if we continuously extend the domain of the function $A(\epsilon)$ to the closed interval $[0,1]$ by using limits, $A(0) := 0$ and $A(1) := \frac{1}{\pi}$, then (3.6), (3.7), (3.9) and (3.10) are all valid also for $\epsilon = 0$ and $\epsilon = 1$.*

Remark 3.5. *Alternatively, we can estimate the remainder $r^{**}_n(\epsilon) := \mu_\infty(\epsilon) - M_n(\epsilon)$ as follows:*

$$r^{**}_n(\epsilon) \le \sum_{i=n+1}^{\infty} \frac{w_i^2\,\epsilon^{2i}}{2i-1} \le \frac{w_{n+1}^2\epsilon^{2n+2}}{2n+1}\sum_{j=0}^{\infty}\epsilon^{2j}$$
$$= \frac{w_{n+1}^2\epsilon^{2n+2}}{(2n+1)(1-\epsilon^2)} \le \frac{1}{1-\epsilon^2}\cdot\frac{2\epsilon^{2n+2}}{\pi(2n+1)^2}\,.$$

This simple method works quite well for ϵ, which "differs enough from 1", but it is useless for ϵ, which is close to 1.

4 The main result

Theorem 4.1. *For every $n\in\mathbb{N}$, the perimeter $P(a,b)$ of an ellipse having semi-major and semi-minor axes, a and b, the aspect ratio $q = q(a,b) := b/a$, and the eccentricity $\epsilon = \epsilon(a,b) := \sqrt{1-q^2}$, the n-th approximation $Q_n(a,b) \approx P(a,b)$,*

$$Q_n(a,b) := 2\pi a\Big(1 - M_n(\epsilon)\Big) = 2\pi a\Big(1 - A(\epsilon) - B_n(\epsilon)\Big), \tag{4.1}$$

has the relative error,

$$\frac{P(a,b)-Q_n(a,b)}{P(a,b)} =: \rho_n(q) \qquad \Big(q = q(a,b) = \big(\tfrac{b}{a}\big)^2\Big)\ ,$$

estimated as

$$-\frac{1}{(2n+1)^2}\,e^{-q^2(n+1)} \le -\frac{\big(1-q^2\big)^{n+1}}{(2n+1)^2} =: \rho^*_n(q) \le \rho_n(q) \le 0\,.$$

Here, $A(\epsilon)$ and $B_n(\epsilon)$ are defined in Theorem 3.2 and we have $B_{n+1}(\epsilon) = B_n(\epsilon) + \Big(w_{n+1}^2 - \frac{2}{\pi(2n+3)}\Big)\frac{\epsilon^{2n+2}}{2n+1}$, for $n\in\mathbb{N}$ and $0\le\epsilon\le 1$.

Proof. Thanks to (1.1), (1.2), (1.4) and (3.3), we estimate

$$-\frac{P(a,b)-Q_n(a,b)}{P(a,b)} = -\frac{2\pi a\Big(1-M_n(\epsilon)-\delta_n(\epsilon)\Big)-2\pi a\Big(1-M_n(\epsilon)\Big)}{P(a,b)}$$

$$\overset{(1.4)}{<} \frac{2\pi a\,\delta_n(\epsilon)}{4a} \leq \frac{\pi\,\delta_n(\epsilon)}{2} < \frac{\pi}{2}\cdot\frac{2\epsilon^{2n+2}}{\pi(2n+1)^2} = \frac{\epsilon^{2n+2}}{(2n+1)^2},$$

where, considering the convexity of the exponential function or, referring to [16, (6a)] with $\varepsilon = q^2$ and $h = -q^2$, we have

$$\epsilon^{2n+2} = (1-q^2)^{n+1} \leq e^{-q^2(n+1)} \qquad (0 \leq q < 1). \qquad \square$$

Figures 2–3 show, for several values of n, the graphs of actual relative errors $-\rho_n(q) = \big[\mu_\infty\big(\epsilon(q)\big) - M_n\big(\epsilon(q)\big)\big]/\big[1-\mu_\infty\big(\epsilon(q)\big)\big]$ (left) together with their upper bounds $-\rho_n^*(q)$ (right).

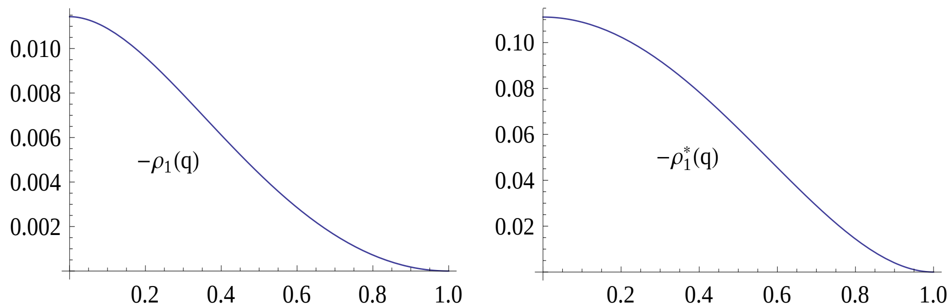

Figure 2: The graphs of the functions $q \mapsto -\rho_1(q)$ and $q \mapsto -\rho_1^*(q)$.

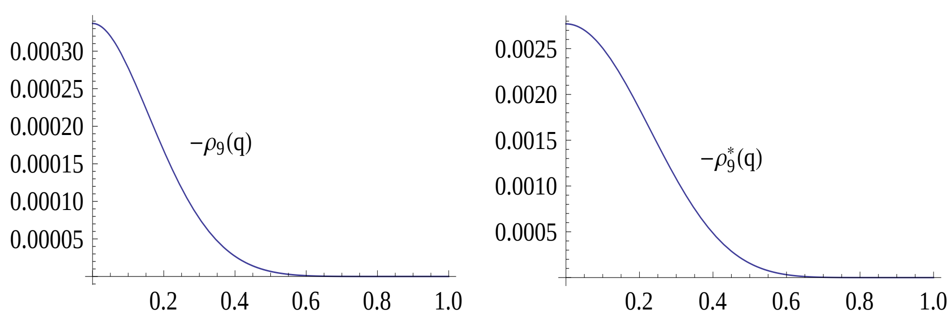

Figure 3: The graphs of the functions $q \mapsto -\rho_9(q)$ and $q \mapsto -\rho_9^*(q)$.

Table 1 additionally confirms the usefulness of the derived formula.

Conclusion. The article demonstrates that with the help of 277 years old Maclaurin series the perimeter of an ellipse can be accurately estimated, even if an ellipse flattens into a line segment. This is done only by elementary means, not using complex analysis or elliptical integral theory, neither arithmetic-geometric means nor hypergeometric functions.

q	0.00001	0.1	0.2	0.3	$0,4$	$0,5$
$-\rho_{20}(q)$	$< 8{\cdot}10^{-5}$	$< 6{\cdot}10^{-5}$	$< 2{\cdot}10^{-5}$	$< 5{\cdot}10^{-6}$	$< 6{\cdot}10^{-7}$	$< 4{\cdot}10^{-8}$
$-\rho^*_{20}(q)$	$< 6{\cdot}10^{-4}$	$< 5{\cdot}10^{-4}$	$< 3{\cdot}10^{-4}$	$< 9{\cdot}10^{-5}$	$< 2{\cdot}10^{-5}$	$< 2{\cdot}10^{-6}$

Table 1: The actual error $\rho_{20}(q)$ and the a priori estimated error $\rho^*_{20}(q)$.

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Generalized trace pseudo-spectrum of matrix pencils

Aymen Ammar, Aref Jeribi and Kamel Mahfoudhi
Department of Mathematics
Faculty of Sciences of Sfax, University of Sfax
Route de soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia
ammar_aymen84@yahoo.fr, aref.Jeribi@fss.rnu.tn, kamelmahfoudhi72@yahoo.com

ABSTRACT

The objective of the study was to investigate a new notion of generalized trace pseudo-spectrum for an matrix pencils. In particular, we prove many new interesting properties of the generalized trace pseudo-spectrum. In addition, we show an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

RESUMEN

El objetivo de este estudio es investigar una nueva noción de pseudo-espectro traza generalizado para pinceles de matrices. En particular, demostramos variadas propiedades nuevas e interesantes del pseudo-espectro traza generalizado. Adicionalmente, mostramos un análogo del teorema espectral de aplicaciones para el pseudo-espectro traza generalizado en el álgebra de matrices.

Keywords and Phrases: pseudo-spectrum, condition pseudo-spectrum, trace pseudo-spectrum.
2010 AMS Mathematics Subject Classification: 15A09, 15A86, 65F40, 15A60, 65F15.

1 Introduction

Let $\mathcal{M}_n(\mathbb{C})$ ($\mathcal{M}_n(\mathbb{R})$) denote the algebra of all $n \times n$ complex (real) matrices, $\mathcal{I}$ denotes the $n \times n$ identity matrix and the conjugate transpose of $\mathcal{U}$ is denoted by $\mathcal{U}^*$. We denote by Tr, (resp. Det) the trace (resp. determinant) map on $\mathcal{M}_n(\mathbb{C})$. In the present paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$\mathcal{U}x = \lambda\mathcal{V}x.$$

Next, let $\lambda \in \mathbb{C}$ and

$$s_n(\lambda\mathcal{V} - \mathcal{U}) \leq \ldots \leq s_2(\lambda\mathcal{V} - \mathcal{U}) \leq s_1(\lambda\mathcal{V} - \mathcal{U})$$

be the singular values of the matrix pencils $\lambda\mathcal{V} - \mathcal{U}$ where $s_1(\lambda\mathcal{V} - \mathcal{U})$ is the smallest and $s_n(\lambda\mathcal{V} - \mathcal{U})$ is largest singular values of the matrix pencil. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, then the set of all eigenvalues of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\sigma(\mathcal{U}, \mathcal{V})$ and is defined as

$$\sigma(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : \ \lambda\mathcal{V} - \mathcal{U} \text{ is not invertible}\Big\},$$

and its spectral radius by

$$r(\mathcal{U}, \mathcal{V}) = \sup\Big\{|\lambda| : \lambda \in \sigma(\mathcal{U}, \mathcal{V})\Big\}.$$

For an $n \times n$ complex matrices $\mathcal{U}$ and $\mathcal{V}$ and a non-negative real number ε, the pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is defined as the following closed set in the complex plane

$$\sigma_\varepsilon(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : \ s_n(\lambda\mathcal{V} - \mathcal{U}) \leq \varepsilon\Big\}.$$

Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $0 < \varepsilon < 1$. The condition pseudo-spectrum of the matrix pencils $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\Sigma_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\Sigma_\varepsilon(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : s_n(\lambda\mathcal{V} - \mathcal{U}) \leq \varepsilon\ s_1(\lambda\mathcal{V} - \mathcal{U})\Big\}.$$

Let ε be a small positive number. For an operator $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, recall that the determinant spectrum of matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is the set $d_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$d_\varepsilon(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : |\det(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon\Big\}.$$

The analysis of eigenvalues and eigenvectors has had a great effect on mathematics, science, engineering, and many other fields. Then, there are countless applications for this type of analysis. The study of matrix pencils is by now a very thoughtful subject, with the notion of pseudospectrum playing a key role in the theory. However, matrix pencils play an important role in numerical linear algebra, perturbation theory, generalized eigenvalue problems. In this paper, we interest by a generalization of eigenvalues called generalized trace pseudo-spectrum for an element in the matrix algebra to give more information about the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$. For more information on various details on the above concepts, properties and applications of pseudo-spectrum [2, 3, 6, 7, 9], condition spectrum [1, 4, 5] and determinant spectrum [8]. Now, we introduce the new concept of the generalized trace pseudo-spectrum in the following definition.

Definition 1.1. *For $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U} \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$ and is defined as*

$$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}) = \sigma(\mathcal{U},\mathcal{V}) \bigcup \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})| \leq \varepsilon\Big\}.$$

The generalized trace pseudoresolvent of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is denoted by $\mathrm{Tr}\rho_\varepsilon(\mathcal{U},\mathcal{V})$ and is defined as

$$\mathrm{Tr}\rho_\varepsilon(\mathcal{U},\mathcal{V}) = \rho(\mathcal{U},\mathcal{V}) \bigcap \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})| > \varepsilon\Big\}.$$

The singular values of a the matrix pencil are important not only for their role in diagonalization but also for their utility in a variety of applications. Since $\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$ use all the singular values of $\lambda\mathcal{V}-\mathcal{U}$ to get defined, it is expected to give more information about $\mathcal{U},\mathcal{V}$ than pseudo-spectrum and condition spectrum. Since the definition use idea of "Trace" the generalization of eigenvalues defined above is named as generalized trace pseudo-spectrum. It is easily seen that the map

$$\mathcal{U} \to \mathrm{Tr}(\mathcal{U})$$

is continuous linear functional. Here, some important properties of the trace of $\mathcal{U},\mathcal{B} \in \mathcal{M}_n(\mathbb{C})$ are

$$\mathrm{Tr}(\mathcal{U}\mathcal{B}) = \mathrm{Tr}(\mathcal{B}\mathcal{U}),$$
$$\mathrm{Tr}(\alpha\mathcal{U}) = \alpha\mathrm{Tr}(\mathcal{U}) \quad \text{with } \alpha \in \mathbb{C},$$
$$\mathrm{Tr}(\mathcal{U}+\mathcal{B}) = \mathrm{Tr}(\mathcal{U}) + \mathrm{Tr}(\mathcal{B}).$$

An outline of this paper is the following. In Section 2, we focuses on a new description of the generalized trace pseudo-spectra. Not only do we give a characterization of the generalized trace pseudo-spectrum in the matrix algebra. but also we investigate the connection between generalized trace pseudo-spectrum and algebraic multiplicity of the eigenvalues. In Section 3, we give an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

2 Generalized trace pseudo-spectrum.

In this section, some relevant properties of the generalized trace pseudo-spectrum are discussed in detail. For $\mathcal{U},\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is denoted by $\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$ and is defined as

$$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}) = \sigma(\mathcal{U},\mathcal{V}) \bigcup \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})| \leq \varepsilon\big\}.$$

The generalized trace pseudo-resolvent of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\mathrm{Tr}\rho_\varepsilon(\mathcal{U},\mathcal{V})$ and is defined as

$$\mathrm{Tr}\rho_\varepsilon(\mathcal{U},\mathcal{V}) = \rho(\mathcal{U},\mathcal{V}) \bigcap \{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| > \varepsilon\}$$

while the generalized trace pseudo-spectral radius of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is defined as

$$\mathrm{Trr}_\varepsilon(\mathcal{U},\mathcal{V}) := \sup\Big\{|\lambda| : \lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})\Big\}.$$

Remark 2.1. *Let $\mathcal{U},\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$. Then, if $\mathcal{V}$ is nonsingular, then it is possible to reduce the generalized trace pseudo-spectrum to a standard trace pseudo-spectrum for the matrices $\mathcal{V}^{-1}\mathcal{U}$ or $\mathcal{U}\mathcal{V}^{-1}$. i.e.*

$$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}) = \sigma(\mathcal{V}^{-1}\mathcal{U},\mathcal{I}) \bigcup \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda - \mathcal{V}^{-1}\mathcal{U})| \le \varepsilon\Big\},$$

or

$$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}) = \sigma(\mathcal{U}\mathcal{V}^{-1},\mathcal{I}) \bigcup \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda - \mathcal{U}\mathcal{V}^{-1})| \le \varepsilon\Big\}.$$

The following theorem gives some properties of the generalized trace pseudo-spectrum that follow in a straightforward manner from the definition of the generalized trace pseudo-spectrum.

Theorem 2.1. *Let $\mathcal{U},\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,*

(i) $\mathrm{Tr}_0(\mathcal{U},\mathcal{V}) = \bigcap\limits_{\varepsilon>0} \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$.

(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\mathrm{Tr}_{\varepsilon_1}(\mathcal{U},\mathcal{V}) \subset \mathrm{Tr}_{\varepsilon_2}(\mathcal{U},\mathcal{V})$.*

(iii) *$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$ is a non-empty compact subset of $\mathbb{C}$.*

(iv) *If $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\{0\}$, then $\mathrm{Tr}_\varepsilon(\beta\mathcal{U} + \alpha\mathcal{V},\mathcal{V}) = \beta\mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U},\mathcal{V}) + \alpha$.*

(v) $\mathrm{Tr}_\varepsilon(\alpha\mathcal{V},\mathcal{V}) = \left\{\lambda \in \mathbb{C} : |\lambda - \alpha| \le \dfrac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|}\right\}$ *for all $\lambda,\alpha \in \mathbb{C}$.*

Proof. The proofs of items (i) and (ii) are clear from the definition of generalized trace pseudo-spectrum.

(iii) Using the continuity from $\mathbb{C}$ to $[0,\infty[$ of the map

$$\lambda \to |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})|,$$

we get that $\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$ is a compact set in the complex plane containing the eigenvalues of the matrix pencils $\lambda\mathcal{V} - \mathcal{U}$.

(iv) In fact, it is well know

$$\begin{aligned} \mathrm{Tr}_\varepsilon(\beta\mathcal{U}+\alpha\mathcal{V},\mathcal{V}) &= \left\{\lambda\in\mathbb{C}:|\mathrm{Tr}(\lambda\mathcal{V}-\beta\mathcal{U}-\alpha\mathcal{V})|\le\varepsilon\right\} \\ &= \left\{\lambda\in\mathbb{C}:|\beta|\left|\mathrm{Tr}\left(\frac{\lambda-\alpha}{\beta}\mathcal{V}-\mathcal{U}\right)\right|\le\varepsilon\right\} \\ &= \left\{\lambda\in\mathbb{C}:\left|\mathrm{Tr}\left(\frac{\lambda-\alpha}{\beta}\mathcal{V}-\mathcal{U}\right)\right|\le\frac{\varepsilon}{|\beta|}\right\}. \end{aligned}$$

Then, $\lambda\in\mathrm{Tr}_\varepsilon(\beta\mathcal{U}+\alpha\mathcal{V},\mathcal{V})$. Thus, $\dfrac{\lambda-\alpha}{\beta}\in\mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U},\mathcal{V})$. Hence, $\lambda\in\beta\mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U},\mathcal{V})+\alpha$.

(v) Let $\lambda\in\mathrm{Tr}_\varepsilon(\alpha\mathcal{V},\mathcal{V})$, then

$$|\mathrm{Tr}(\lambda\mathcal{V}-\alpha\mathcal{V})| = |\lambda-\alpha||\mathrm{Tr}(\mathcal{V})|\le\varepsilon.$$

This means that $\mathrm{Tr}_\varepsilon(\alpha\mathcal{V},\mathcal{V})=\left\{\lambda\in\mathbb{C}:|\lambda-\alpha|\le\dfrac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|}\right\}$ for all $\lambda,\alpha\in\mathbb{C}$. Q.E.D.

Theorem 2.2. *Let $\mathcal{U},\mathcal{V}\in\mathcal{M}_n(\mathbb{C})$ and $\varepsilon>0$. Then,*

(i) *If $\mathcal{U}=\mathcal{Z}\mathcal{B}\mathcal{Z}^{-1}$ and $\mathcal{Z}\mathcal{V}=\mathcal{V}\mathcal{Z}$ for all nonsingular matrix $\mathcal{Z}\in\mathcal{M}_n(\mathbb{C})$ we have,*

$$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})=\mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{V}).$$

(ii) *If $\mathcal{U}=\mathcal{Z}\mathcal{B}\mathcal{Z}^{-1}$ and $\mathcal{V}=\mathcal{Z}\mathcal{K}\mathcal{Z}^{-1}$ for all nonsingular matrix $\mathcal{Z}\in\mathcal{M}_n(\mathbb{C})$ we have,*

$$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})=\mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{K}).$$

(iii) *The map $T\to\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$ is an upper semi continuous function from $\mathcal{M}_n(\mathbb{C})$ to compact subsets of $\mathbb{C}$.* ♢

Proof. (i) Let $\lambda\in\mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{V})$, then

$$\begin{aligned} |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{B})| &= |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{Z}^{-1}\mathcal{U}\mathcal{Z})|, \\ &= |\mathrm{Tr}(\lambda\mathcal{Z}^{-1}\mathcal{Z}\mathcal{V}-\mathcal{Z}^{-1}\mathcal{U}\mathcal{Z})| \\ &= |\mathrm{Tr}(\mathcal{Z}^{-1}(\lambda\mathcal{Z}\mathcal{V}-\mathcal{U}\mathcal{Z})| \\ &= |\mathrm{Tr}(\mathcal{Z}^{-1}(\lambda\mathcal{V}-\mathcal{U})\mathcal{Z}|=|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})|\le\varepsilon. \end{aligned}$$

It follows that, $\lambda\in\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$.

The proofs of items (ii) and (iii) follows immediately from Definition 1.1. Q.E.D.

The following example shows that the converse of the assertion (i) is not true.

Example 2.1. *Let* $\mathcal{U} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\mathcal{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Then,* $\mathcal{U}$ *and* $\mathcal{B}$ *are not similar and for* $\varepsilon > 0$, *we have*

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{B}, \mathcal{V}) = \{\lambda \in \mathbb{C} : |\lambda - 2| \le \varepsilon\}.$$

In the following, we obtain additional results on $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ that are useful in our analysis.

Theorem 2.3. *Let* $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, *and* $\varepsilon > 0$. *Then, there is* $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ *such that* $|\mathrm{Tr}(\mathcal{D})| \le \varepsilon$ *and* $\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = 0$ *if, and only if,* $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. $\diamondsuit$

Proof. To see this, we suppose that there exists $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(\mathcal{D})| \le \varepsilon$ and

$$\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = 0.$$

Then,

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| = |\mathrm{Tr}(\mathcal{D})| \le \varepsilon.$$

Thus, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. Conversely, let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. Then, we will discuss these two cases:

$\underline{1^{st}\ case}$: If $\lambda \in \mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$, then it is sufficient to take $(\mathcal{D} = 0_{n\times n})$.

$\underline{2^{nd}\ case}$: $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) \backslash \mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. Then,

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| \le \varepsilon.$$

Now, we consider

$$\mathcal{D} = \frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n}\, \mathcal{I}.$$

It is easy to verify that, $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ and

$$|\mathrm{Tr}(\mathcal{D})| = \left|\mathrm{Tr}\left(\frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n}\, \mathcal{I}\right)\right| = \frac{|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})|}{n}\, \mathrm{Tr}(\mathcal{I}) \le \varepsilon.$$

Also, we have

$$\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = \mathrm{Tr}\left(\lambda\mathcal{V} - \mathcal{U} - \frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n}\, \mathcal{I}\right) = 0.$$

Q.E.D.

Theorem 2.4. *Let* $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ *and* $\varepsilon > 0$. *Then,*

$$\mathrm{Tr}_\delta(\mathcal{U}, \mathcal{V}) + \Theta_\varepsilon \subseteq \mathrm{Tr}_{\varepsilon+\delta}(\mathcal{U}, \mathcal{V}), \tag{1}$$

holds for $\delta, \varepsilon > 0$ *with* Θ_ε, *denoting the closed disk in the complex plane centered at the origin with radius* $\frac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|}$. *If we take* $\delta = 0$, *we obtain an inner bound for* $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$, *namely*

$$\mathrm{Tr}_0(\mathcal{U}, \mathcal{V}) + \Theta_\varepsilon \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}). \tag{2}$$

Proof. Let $\lambda \in \mathrm{Tr}_\delta(\mathcal{U},\mathcal{V})+\Theta_\varepsilon$. Then, there exists there exists $\lambda_1 \in \mathrm{Tr}_\delta(\mathcal{U},\mathcal{V})$ and $\lambda_2 \in \Theta_\varepsilon$ such that $\lambda = \lambda_1+\lambda_2$. Therefore,

$$|\mathrm{Tr}(\lambda_1\mathcal{V}-\mathcal{U})| \le \delta$$

and

$$\begin{aligned}
|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})| &= |\mathrm{Tr}((\lambda_1+\lambda_2)\mathcal{V}-\mathcal{U})| \\
&= |\mathrm{Tr}(\lambda_2\mathcal{V})+\mathrm{Tr}(\lambda_1\mathcal{V}-\mathcal{U})| \\
&\le |\lambda_2||\mathrm{Tr}(\mathcal{V})|+|\mathrm{Tr}(\lambda_1\mathcal{V}-\mathcal{U})| \\
&\le |\mathrm{Tr}(\mathcal{V})||\lambda_2|+|\mathrm{Tr}(\lambda_1\mathcal{V}-\mathcal{U})| \le \varepsilon+\delta,
\end{aligned}$$

so that (1) holds. Finally, let $\delta = 0$, then the desired inclusion (2) is obtained. Q.E.D.

Theorem 2.5. *Let $\mathcal{U},\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ such that $\mathcal{U}\mathcal{B}=\mathcal{B}\mathcal{U}$ and $\varepsilon > 0$. If $\mathcal{U}$ is normal, then*

$$\mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{B},\mathcal{V}) \subseteq \sigma(\mathcal{U},\mathcal{V})+\mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{V}).$$

Proof. We assume that $\mathcal{U}$ is normal, so there exists a unitary matrix $\mathcal{Z} \in \mathcal{M}_n(\mathbb{C})$ such that

$$\mathcal{Z}^*\mathcal{U}\mathcal{Z} = \lambda_1\mathcal{I}_{n_1} \oplus \lambda_2\mathcal{I}_{n_2} \oplus \ldots \oplus \lambda_k\mathcal{I}_{n_k}.$$

The condition $\mathcal{U}\mathcal{B}=\mathcal{B}\mathcal{U}$ implies that

$$\mathcal{Z}^*\mathcal{B}\mathcal{Z} = \mathcal{U}_1 \oplus \mathcal{U}_2 \ldots \oplus \mathcal{U}_k$$

where, $\mathcal{U}_i \in \mathcal{M}_{n_k}(\mathbb{C})$, $i = 1,\ldots,k$. Then,

$$\begin{aligned}
\mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{B},\mathcal{V}) &= \mathrm{Tr}_\varepsilon(\mathcal{Z}^*\mathcal{U}\mathcal{Z}+\mathcal{Z}^*\mathcal{B}\mathcal{Z},\mathcal{V}) \\
&= \mathrm{Tr}_\varepsilon((\lambda_1\mathcal{I}_{n_1}+\mathcal{U}_1)\oplus\ldots\oplus(\lambda_k\mathcal{I}_{n_k}+\mathcal{U}_k),\mathcal{V}) \\
&= \bigcup_{i=1}^{k} \mathrm{Tr}_\varepsilon(\lambda_i\mathcal{I}_{n_i}+\mathcal{U}_i,\mathcal{V}) \\
&= \bigcup_{i=1}^{k} \lambda_i+\mathrm{Tr}_\varepsilon(\mathcal{U}_i,\mathcal{V}) \\
&\subseteq \sigma(\mathcal{U},\mathcal{V})+\mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{V}).
\end{aligned}$$

The proof is thus complete. Q.E.D.

Remark 2.2. *Let $\mathcal{U},\mathcal{B}$ and $\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then, using Theorem 2.5, we obtain the following inequality,*

$$\mathrm{Trr}_\varepsilon(\mathcal{U}+\mathcal{B},\mathcal{V}) \subseteq r(\mathcal{U},\mathcal{V})+\mathrm{Trr}_\varepsilon(\mathcal{B},\mathcal{V}).$$

♢

Theorem 2.6. *Let* $\mathcal{U}, \mathcal{B}$ *and* $\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ *and* $\varepsilon > 0$. *Then,*

(i) $\mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V})$.

(ii) $\mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) + \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{B}, \mathcal{V})$.

Proof. (i) Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V})$, then

$$\begin{aligned} \varepsilon \geq |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U}\mathcal{B})| &= |\mathrm{Tr}(\lambda\mathcal{V}) + \mathrm{Tr}(-\mathcal{U}\mathcal{B})| \\ &= |\mathrm{Tr}(\lambda\mathcal{V}) + \mathrm{Tr}(-\mathcal{B}\mathcal{U})| \\ &= |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{B}\mathcal{U})|. \end{aligned}$$

Hence, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V})$. Thus,

$$\mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V}).$$

The conclusion can be obtained similarly to the first inclusion, then we deduce that

$$\mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}).$$

(ii) Let $\lambda \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) + \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V})$. Then, there exists

$$\lambda_1 \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) \text{ and } \lambda_1 \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V})$$

such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U}) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \mathrm{Tr}(\lambda_2\mathcal{V} - \mathcal{B}) \leq \frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned} |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{B})| &= |\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U} + \lambda_2\mathcal{V} - \mathcal{B})| \\ &\leq |\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U})| + |\mathrm{Tr}(\lambda_2\mathcal{V} - \mathcal{B})| \\ &\leq \varepsilon \end{aligned}$$

Then, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{B}, \mathcal{V})$. Q.E.D.

Theorem 2.7. *Let* $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ *and* $\mathcal{N} \in \mathcal{M}_n(\mathbb{C})$ *is a nilpotent matrix and* $\varepsilon > 0$. *Then,*

$$\mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{N}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

♢

Proof. "$\subseteq$" Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{N}, \mathcal{V})$, then $|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N})| \leq \varepsilon$. Since

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U}) - \mathrm{Tr}(\mathcal{N})| \leq \varepsilon.$$

Using the fact that the matrix trace vanishes on nilpotent matrices, therefore

$$\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

Hence,

$$\mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

" $\supseteq$ " Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$, then $|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon$. Now, we can write for any $\lambda \in \mathbb{C}$

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| = |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N} + \mathcal{N})| = |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N}) + \mathrm{Tr}(\mathcal{N})|.$$

Because, $\mathrm{Tr}(\mathcal{N}) = 0$, it follows that $|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N})| \leq \varepsilon$. Consequently,

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V}).$$

Q.E.D.

3 Trace pseudospectral mapping Theorem

Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and f be an analytic function defined on D, an open set containing $\mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. For each $\varepsilon > 0$, we define

$$\varphi(\varepsilon) = \sup_{\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})} |\mathrm{Tr}\big(f(\lambda)\mathcal{V} - f(\mathcal{U})\big)|$$

and suppose there exists $\varepsilon_0 > 0$ such that $\mathrm{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D)$. Then, for $0 < \varepsilon < \varepsilon_0$ we define

$$\phi(\varepsilon) = \sup_{\mu \in f^{-1}(\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \cap D} |\mathrm{Tr}(\mu\mathcal{V} - \mathcal{U})|.$$

Lemma 3.1. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, then $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are well defined, $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$.*

Proof. In the order to prove that $\varphi(\varepsilon)$ is well defined, we define $h : \mathbb{C} \to \mathbb{R}_+$

$$h(\lambda) = |\mathrm{Tr}\big(f(\lambda)\mathcal{V} - f(\mathcal{U})\big)|$$

Since $h(\lambda)$ is continuous and $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ is a compact subset of $\mathbb{C}$, then it is clear that

$$\varphi(\varepsilon) = \sup\big\{h(\lambda) : \lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})\big\}.$$

We conclude, $\varphi(\varepsilon)$ is well defined. Now, let assume that there exists $\varepsilon_0 > 0$ such that

$$\mathrm{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D).$$

We show that for $0 < \varepsilon < \varepsilon_0$, $\phi(\varepsilon)$ is well defined. Define $g : \mathbb{C} \to \mathbb{R}_+$,

$$g(\mu) = |\mathrm{Tr}(\mu\mathcal{V} - \mathcal{U})|.$$

Since g is continuous for all $\mu \in \mathbb{C}$, then $\phi(\varepsilon)$ is well defined. It is also clear that $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are a monotonically non-decreasing function, $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ goes to zero as ε goes to zero. Q.E.D.

Theorem 3.1. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on D, an open set containing $\mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. Then, for each*

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V}),$$

where $\varphi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. Then, using Lemma 3.1 we obtain that $\varphi(\varepsilon)$ is well defined and $\lim\limits_{\varepsilon \to 0} \varphi(\varepsilon) = 0$. Therefore, $h(\lambda) \leq \varphi(\varepsilon)$. Hence

$$|\mathrm{Tr}\big(f(\lambda)\mathcal{V} - f(\mathcal{U})\big)| := h(\lambda) \leq \varphi(\varepsilon).$$

Thus, $f(\lambda) \in \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V})$. This means that

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V}).$$

Q.E.D.

Theorem 3.2. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on D, an open set containing $\mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. Then, for each*

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})).$$

where $\phi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V})$. Then, using Lemma 3.1 we obtain the existence of $\varepsilon_0 > 0$ such that

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq \mathrm{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D).$$

Consider $\mu \in D$ such that $\lambda = f(\mu)$. Then $\mu \in f^{-1}(\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}))$, hence

$$g(\mu) \leq \phi(\varepsilon).$$

Therefore,

$$|\mathrm{Tr}\big(\mu\mathcal{V} - \mathcal{U}\big)| := g(\mu) \leq \phi(\varepsilon)$$

Thus, $\mu \in \mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})$. Then, $\lambda = f(\mu) \in f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V}))$. This means that

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})).$$

Q.E.D.

Corollary 3.1. *Combining the two inclusions in Theorems 3.1 and 3.2, we get*

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}),\mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varphi(\varepsilon))}(\mathcal{U},\mathcal{V})$$

and

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}),\mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U},\mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\phi(\varepsilon))}(f(\mathcal{U}),\mathcal{V}).$$

Here are some remarks.

Remark 3.1. (i) *It will be clear from the proofs of Theorems 3.1 and 3.2 that the the functions φ and ϕ measure the sizes of the trace pseudo-spectra are optimal.*

(ii) *From the definitions of φ and ϕ, the set inclusions are sharp in the sense that the functions cannot be replaced by smaller functions.*

(iii) *In general, the spectral mapping theorem is not true for generalized trace pseudo-spectrum.*

Example 3.1. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta \neq 0$ and let $\mathcal{U} = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$, $\mathcal{V} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $f(\lambda) = \lambda^2$. Then $f(\mathcal{U}) = \begin{pmatrix} \alpha^2 & \alpha+\beta \\ 0 & \beta^2 \end{pmatrix}$. A direct computation shows that*

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}),\mathcal{V}) = \left\{\lambda \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2\right\},$$

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})) = \left\{\lambda^2 \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2\right\}.$$

We can see for all $\varepsilon > 0$ that $\mathrm{Tr}_\varepsilon(f(\mathcal{U}),\mathcal{V}) \neq f(\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}))$.

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Certain results for η-Ricci Solitons and Yamabe Solitons on quasi-Sasakian 3-Manifolds

Sunil Kumar Yadav[1], Abhishek Kushwaha[2] and Dhruwa Narain[3]
[1] *Department of Mathematics,*
Poornima College of Engineering,
ISI-6,RIICO Institutional Area, Sitapura, Jaipur- 302022, Rajasthan,India
`prof_sky16@yahoo.com`

[2,3] *Department of Mathematics & Statistics*
D.D.U.Gorakhpur University,
Gorakhpur-273009, Uttar Pradesh, India
`ab11sk1991@gmail.com, dhruwanarain.dubey@gmail.com`

ABSTRACT

We classify quasi-Sasakian 3-manifold with proper η-Ricci soliton and investigate its geometrical properties. Certain results of Yamabe soliton on such manifold are also presented. Finally, we construct an example of non-existence of proper η-Ricci soliton on 3-dimensional quasi-Sasakian manifold to illustrate the results obtained in previous section of the paper.

RESUMEN

Clasificamos 3-variedades cuasi-Sasakianas con solitones η-Ricci propios e investigamos sus propiedades geométricas. Ciertos resultados sobre el solitón de Yamabe en dichas variedades también se presentan. Finalmente, construimos un ejemplo de la no existencia de solitones η-Ricci propios en una 3-variedad cuasi-Sasakiana para ilustrar los resultados contenidos en el artículo.

Keywords and Phrases: Quasi-Sasakian 3-manifold, infinitesimal contact transformation, η-Ricci soliton, Yamabe soliton.

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1 Introduction

In 1982, Hamilton [17] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metric g_{ij} on a Riemannian manifold defined as follows:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \tag{1.1}$$

where R_{ij} denotes the Ricci tensor of a Riemannian manifold and t is the time. Ricci soliton are special solution of the Ricci flow equation (1.1) of the form $g_{ij} = \sigma(t)\Psi_t g_{ij}$ with the initial condition $g_{ij}(0) = g_{ij}$, where Ψ_t is the diffeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a natural generalization of an Einstein metric. We recall the notion of Ricci soliton according to [9]. On a Riemannian manifold M, a Ricci soliton is a triple (g, V, μ) with the Riemannian metric g, a vector field V, called potential vector field, μ a real scalar and S is the Ricci tensor such that

$$(\mathfrak{L}_V g)(X, Y) + 2S(X, Y) + 2\mu g(X, Y) = 0, \tag{1.2}$$

where $\mathfrak{L}_V$ is the Lie-derivative along the vector field V on M. It is clear that a Ricci soliton with V zero or a Killing vector field reduces to an Einstein metric. A Ricci soliton is said to be shrinking, steady and expanding according as μ is negative, zero and positive, respectively. The Ricci soliton have been studied by several authors such as ([12],[18],[20],[28],[36]).

As a generalization of a Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [10]. This notion has also been studied in [10] for Hopf hypersurfaces in complex-space-forms. An η-Ricci soliton is a 4-tuple (g, V, μ, α), where V is a vector field on M, μ and α are real constants and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$(\mathfrak{L}_V g)(X, Y) + 2S(X, Y) + 2\mu g(X, Y) + 2\alpha\eta(X)\eta(Y) = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g. In particular, if $\alpha = 0$ then the notion of an η-Ricci soliton (g, V, μ, α) reduces to the notion of a Ricci soliton (g, V, μ). If $\alpha \neq 0$, then the η-Ricci soliton are known as the proper η-Ricci soliton.Thus the notion of η-Ricci soliton have been studied by many authors like ([7],[8],[31],[32],[33]). The notion of Yamabe flow was introduced by Richard Hamilton at the same time as the Ricci flow [17], as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on $(M^n, g)(n \geq 3)$. A time-dependent metric $g(\cdot, t)$ on a Riemannian or, pseudo Riemannian manifold M is said to evolve by the Yamabe flow if the metric g satisfies

$$\frac{\partial g(t)}{\partial t} = -\kappa g(t), \quad g(0) = g_0, \tag{1.4}$$

on M, where κ is the scalar curvature correspond to g. Ye [35] has found that a point-wise elliptic gradient estimate for the Yamabe flow on a locally conformally flat compact Riemannian manifold. In case of Ricci flow, Yamabe soliton or the singularities of the Yamabe flow appear naturally.

The significance of Yamabe flow lies in the fact that it is a natural geometric deformation to metric of constant scalar curvature. One notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t}g(t) = -2\alpha(t)$, where α stands for the Ricci tensor). However in dimension $n > 2$, the Yamabe and Ricci flow do not agree, since the first one preserves the conformal class of metric but the Ricci flow does not in general. Just as Ricci soliton is a special solution of the Ricci flow, a Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphism ϕ_t generated by a fixed vector field V on M , and homotheries, that is, $g(., t) = \sigma(t)\phi_*(t)g_0$.

A Yamabe soliton is defined on a Riemannian or, pseudo-Riemannian manifold (M, g) by a vector field V satisfying the equation [6]:

$$\frac{1}{2}(\mathfrak{L}_V g)(X, Y) = (\kappa - \lambda)g(X, Y), \tag{1.5}$$

where $\mathfrak{L}_V$ denotes the Lie-derivative of the metric g along the vector field V, κ stands for the scalar curvature, while λ is a soliton constant. A Yamabe soliton is said to be expanding, steady, or shrinking, respectively, if $\lambda < 0, \lambda = 0$ or $\lambda > 0$. Otherwise, it will be called indefinite. Given a Yamabe soliton, if $V = Df$ holds for a smooth function $f : M \to \Re$ on M, the equation (1.5) becomes Hess $f = (r - \lambda)g$, where Hess f denotes the Hessian of f and D denotes the gradient operator of g on M^n. In this case f is called the potential function of the Yamabe soliton and g is said to be a gradient Yamabe soliton.

The notion of quasi-Sasakian structure was introduced by Blair [4] to unify Sasakian and cosymplectic structures. Tanno [30] also added some remarks on quasi-Sasakian structure.The properties of such manifold have been studied by several authors,viz., Gonzalez and Chinea [16], Kanemaki [21] and Oubina [26]. Kim [22] studied quasi-Sasakian manifold and proved that fibred Riemannian spaces with invariant fibers normal to the structure vector field do not admit nearly Sasakian or contact structure but a Sasakian or cosymplectic structure. Recently, quasi-Sasakian manifold have been the subject of growing interest in view of finding the significant of applications to physics, in particular to supargravity and magmatic theory ([3],[1]). Quasi-Sasakian structure have wide application in the mathematical analysis of string theory ([2],[14]). On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Olszak [27] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [25]. Next he has proved that if the manifold is additionally conformally flat with β=constant, then (a) it is locally a product of R and a two-dimensional Kaehlerian space of constant Gauss curvature (the cosymplectic case), or, (b) it is constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure).

Now, we give some necessary definition and proposition that are uses in latter section.

Definition1.1[6] *A vector field V is said to be conformal for Yamabe soliton if it satisfying the* *equation*

$$\mathfrak{L}_V g = 2\omega g, \tag{1.6}$$

where ω *is called the conformal coefficient, that is,* $\omega = (\kappa - \lambda)$. *Moreover, if* $\omega = 0$, *is equivalent to* V *being Killing.*

Definition1.2[8] *A vector field* X *on an almost contact Riemannian manifold* M *is said to be infinitesimal transformation if there exists a smooth function* υ *on* M *such that*

$$(\mathfrak{L}_X \eta)(Y) = \upsilon\eta(Y), \tag{1.7}$$

for every smooth vector field X *and* Y. *If* $\upsilon = 0$ *then* X *is called a strict infinitesimal transformation.*

Proposition1.1[34] *On an* n*-dimensional Riemannian or, pseudo Riemannian manifold* (M^n, g) *endowed with a conformal vector field* V, *we have*

$$(\mathfrak{L}_V S)(X, Y) = -(n-2)g(\nabla_X D\omega, Y) + (\Delta\omega)g(X, Y),$$

$$(\mathfrak{L}_V \kappa) = -2\omega\kappa + 2(n-1)\Delta\omega,$$

for any vector fields X *and* Y, *where* D *denotes the gradient operator and* $\Delta = -\mathrm{div} D$ *denotes the Laplacan operator of* g.

The outline of this paper is to consider 3-dimensional quasi-Sasakian manifold with the structure function β is constant. In Section 2, we recall the basic results of η-Ricci soliton on quasi-Sasakian 3-manifold. In Section 3 and Section 4, we examine the η-Ricci soliton on quasi-Sasakian 3-manifold admitting codazzi type and cyclic parallel Ricci tensor, respectively. Further, the Section 5, Section 6 and Section 7, deals with an almost pseudo Ricci symmetric, φ-Ricci symmetric and conformally flat with η-Ricci soliton on quasi-Ssakian 3- manifold respectively. The geometrical properties of a special weakly Ricci symmetric and η-recurrent on quasi-Sasakian 3-manifold are studied in Section 8 and Section 9, respectively. In Section 10, we deals quasi-Sasakian 3-manifolds with $Q \cdot R = 0$ and obtain new results for η-Ricci soliton on such manifold. In Section 11, we deduce some results related to Yamabe soliton on quasi-Sasakian 3-manifold. At last, we construct an example of non-existence of proper η-Ricci soliton on quasi-Sasakian 3-manifold.

2 Preliminaries

Let M be a $(2n+1)$-dimensional an almost contact metric manifold equipped with an almost contact metric structure $(\varphi, \zeta, \eta, g)$ consisting of a $(1,1)$ tensor field φ, a vector field ζ, a 1-form η and a Riemannian metric g, which satisfies

$$\varphi^2 = -I + \eta \otimes \zeta, \tag{2.1}$$

$$\eta(\zeta) = 1, \quad \eta \circ \zeta = 0, \quad \varphi\zeta = 0, \tag{2.2}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \zeta), \tag{2.3}$$

for all $U, V \in \chi(M)$, where$\chi(M)$ is the Lie-algebra of the vector fields of M^{2n+1}. Let Φ be the fundamental 2-form of M^{2n+1} defined by

$$\Phi(U, V) = g(U, \varphi V), \tag{2.4}$$

for all $U, V \in \chi(M)$. Then $\Phi(U, \zeta) = 0$, $U \in \chi(M)$. M^{2n+1} is said to be quasi-Sasakian if the almost contact structure $(\varphi, \zeta, \eta, g)$ is normal and the fundamental 2-form Φ is closed, that is, every $U, V \in \mathfrak{I}^{2n+1}$, where $\mathfrak{I}^{2n+1}$ denotes the modulus of vector fields on M^{2n+1}.

$$(i)\ \ [\varphi, \varphi](U, V) + d\eta(U, V)\zeta = 0, \quad (ii)\ \ d\Phi = 0. \tag{2.5}$$

There are many types of quasi-Sasakian structures ranging from the cosymplectic case, $d\eta = 0 (rank\eta = 1)$, to the Sasakian case, $\eta \wedge (d\eta)^n \neq 0$ $(rank\eta = 2n + 1, \Phi = d\eta)$. The 1-form η has rank $\bar{r} = 2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p$=0, and has rank $\bar{r}$=$2p+1$ if $(d\eta)^{p+1}$=0 and $\eta \wedge (d\eta)^p \neq 0$. We also say that $\bar{r}$ is the rank of the quasi-Sasakian structure. Blair [7], proved that there are no quasi-Sasakian structure of even rank, some theorems regarding Kaehlerian manifolds and existence of quasi-Sasakian manifold.

An almost contact metric manifold M^{2n+1} is a 3-dimensional quasi-Sasakian manifold if and only if [27]

$$\nabla_U \zeta = -\beta\varphi U, \ \ U \in \chi(M), \tag{2.6}$$

for a certain function β on M, such that $\zeta\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection on M. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. Here we have shown that the assumption $\zeta\beta = 0$ is not necessary.

As per consequence (2.6), we find that [27]

$$(\nabla_U \varphi)(V) = \beta[g(U, V)\zeta - \eta(V)U]. \tag{2.7}$$

In view of (2.6) and (2.7), we obtain

$$\nabla_U(\nabla_V \zeta) = -(U\beta)\varphi V - \beta^2[g(U, V)\zeta - \eta(V)U] - \beta\varphi\nabla_U V. \tag{2.8}$$

This implies that

$$R(U, V)\zeta = -(U\beta)\varphi V + (V\beta)\varphi U + \beta^2[\eta(V)U - \eta(U)V]. \tag{2.9}$$

Thus from (2.9), we get

$$R(U,V,W,\zeta) = (U\beta)g(\varphi V,W) - (V\beta)g(\varphi U,W) - \beta^2[\eta(V)g(U,W) - \eta(U)g(V,W)]. \tag{2.10}$$

Substituting $U = \zeta$ in (2.10), we have

$$R(\zeta,V,W,\zeta) = \beta^2[g(V,W) - \eta(V)\eta(W)] + g(\varphi V,W)\zeta\beta. \tag{2.11}$$

Interchanging V and W of (2.11) it yields

$$R(\zeta,W,V,\zeta) = \beta^2[g(W,V) - \eta(W)\eta(V)] + g(\varphi W,V)\zeta\beta. \tag{2.12}$$

Since $R(\zeta,V,W,\zeta) = R(W,\zeta,\zeta,V) = R(\zeta,W,V,\zeta)$. Then from (2.11) and(2.12), we obtain

$$[g(\varphi V,W) - g(\varphi W,V)]\zeta\beta = 0. \tag{2.13}$$

Therefore, we can easily verify that $\zeta\beta = 0$.

In a 3-dimensional Riemannian manifold we have

$$\begin{aligned} R(U,V)W &= \{S(V,W)U - S(U,W)V + g(V,W)QU - g(U,W)QV\} \\ &\quad -\frac{\kappa}{2}\{g(V,W)U - g(U,W)V\}, \end{aligned} \tag{2.14}$$

where S and κ are the Ricci tensor and the scalar curvature, respectively, and Q denotes the Ricci operator defined by $g(QU,V) = S(U,V)$.

It is well known that the Ricci tensor S of a quasi-Sasakian 3-manifold is given by [28]

$$S(U,V) = \left[\frac{\kappa}{2} - \beta^2\right] g(U,V) + \left[3\beta^2 - \frac{\kappa}{2}\right] \eta(U)\eta(V). \tag{2.15}$$

As a consequence of (2.15), we find the Ricci operator Q

$$QU = \left[\frac{\kappa}{2} - \beta^2\right] U + \left[3\beta^2 - \frac{\kappa}{2}\right] \eta(U)\zeta. \tag{2.16}$$

From (2.15), we obtain

$$S(U,\zeta) = 2\beta^2\eta(U). \tag{2.17}$$

Keeping in mind the Equ.(2.12),(2.13),(2.14) and (2.15), we have

$$R(U,V)\zeta = \beta^2[\eta(V)U - \eta(U)V], \tag{2.18}$$

for all $U,V,\in \chi(M)$. Also from (2.6), we have

$$(\nabla_U\eta)V = g(\nabla_U\zeta,V) = -\beta g(\varphi U,V). \tag{2.19}$$

Again from (2.15), it follows that

$$S(\varphi U,\varphi V) = S(U,V) - 2\beta^2\eta(U)\eta(V). \tag{2.20}$$

Proposition 2.1. *A 3-dimensional non-cosymplectic quasi-Sasakian manifold with* η*-Ricci soliton is an* η*-Einstein manifold.*

Proof. Assume that the quasi-Sasakian 3-manifold admits a proper η-Ricci soliton (g, ζ, μ, α). Then from (1.3), we have

$$2S(U, V) = -(L_{\zeta} g)(U, V) - 2\mu g(U, V) - 2\alpha\eta(U)\eta(V), \tag{2.21}$$

for all smooth vector fields $U, V \in \chi(M)$. Of the two natural situations regarding the vector field $V : V \in \mathrm{span}\{\zeta\}$ and $V \perp \zeta$, we investigate only the case $V = \zeta$. Our interest is in the expression for $L_{\zeta} g + 2S + 2\mu g + 2\alpha\eta \otimes \eta$.

A straight forward computations give

$$\begin{aligned}(L_{\zeta} g)(U, V) = & \ g(\nabla_U \zeta, V) + g(U, \nabla_V \zeta) \ , \\ & = -\beta \ [g(\varphi U, V) + g(U, \varphi V)] = 0.\end{aligned} \tag{2.22}$$

Using (2.22) in (2.21), we get

$$S(U, V) = -\mu g(U, V) - \alpha \eta(U) \eta(V). \tag{2.23}$$

From the last equation,the proof ends. □

Proposition 2.2. *If a 3-dimensional non-cosymplectic quasi-Sasakian manifold admits* η*-Ricci soliton, then* $\mu + \alpha = -2\beta^2$.

Proof. From (2.15), we have

$$S(U, V) = \left[\frac{\kappa}{2} - \beta^2\right] g(U, V) + \left[3\beta^2 - \frac{\kappa}{2}\right] \eta(U)\eta(V). \tag{2.24}$$

Comparing (2.24) with (2.23), we get $\mu = \frac{1}{2}[2\beta^2 - \kappa]$ and $\alpha = \frac{1}{2}[\kappa - 6\beta^2]$. In continuation we get $\mu + \alpha = -2\beta^2$. From the last equation, the proof ends. □

3 Ricci tensor of Codazzi type

Gray [15] introduced the notion of cyclic parallel and Codazzi type Ricci tensors. A Riemannian manifold is said to possesses a cyclic parallel Ricci tensor if its non-vanishing Ricci tensor S of type $(0, 2)$ satisfies the condition

$$(\nabla_U S)(V, W) + (\nabla_V S)(W, U) + (\nabla_W S)(U, V) = 0, \tag{3.1}$$

for arbitrary vector fields U, V, W on M. Again a Riemannian manifold is said to have a Ricci tensor of Codazzi type if S is non-zero and satisfies

$$(\nabla_U S)(V, W) = (\nabla_V S)(U, W), \tag{3.2}$$

for all the vector fields U, V, W on M.

We consider proper η-Ricci soliton on quasi-Sasakian 3-manifold with Ricci tensor of Codazzi type. Taking covariant derivative of (2.23) along W and using (2.19), we have

$$\begin{aligned}(\nabla_W S)(U, V) &= -\alpha\,[(\nabla_W \eta)(U)\eta(V) + \eta(U)(\nabla_W \eta)(V)] \\ &= \alpha\beta[g(U, \varphi W)\eta(V) + g(V, \varphi W)\eta(U)].\end{aligned} \tag{3.3}$$

Since the Ricci tensor S of M is of Codazzi type. Then

$$(\nabla_W S)(U, V) = (\nabla_U S)(W, V). \tag{3.4}$$

Making use of (3.3) in (3.4), we yields

$$\alpha\beta[g(U, \varphi W)\eta(V) + g(V, \varphi W)\eta(U)] = \alpha\beta[g(W, \varphi U)\eta(V) + g(V, \varphi U)\eta(W)]. \tag{3.5}$$

Setting $W = \zeta$ in (3.5), we theorize $\beta \neq 0, \alpha = 0$, which is a refutation. Thus a non-cosymlectic quasi-Sasakian 3-manifold with a Ricci tensor of Codazzi type does not admits a proper η-Ricci soliton. In this way we terminate the following result:

Theorem 3.1. *A non-cosymplectic quasi-Sasakian* 3*-manifold accompanied by Ricci tensor of Codazzi type does not possess a proper* η*-Ricci soliton.*

Corollary 3.2. *For a proper* η*-Ricci soliton on a non-cosymplectic quasi-Sasakian* 3*-manifold, the scalar curvature is constant if and only if the vector field* ζ *is harmonic.*

Corollary 3.3. *There exists no constant scalar curvature for a proper* η*-Ricci soliton of non-cosymplectic quasi-Sasakian* 3*-manifold, provided the vector field* ζ *is non-harmonic.*

4 Cyclic parallel Ricci tensor

This section is affectionate to the study of proper η-Ricci soliton on quasi-Sasakian 3-manifold bearing cyclic parallel Ricci tensor. On that account

$$(\nabla_U S)(V, W) + (\nabla_V S)(W, U) + (\nabla_W S)(U, V) = 0. \tag{4.1}$$

On the other hand, we have (3.3) and left hand side of (4.1), we have

$$\begin{aligned}(\nabla_U S)(V, W) + (\nabla_V S)(W, U) + (\nabla_W S)(U, V) = \alpha\beta\,[&g(V, \varphi U)\eta(W) + g(W, \varphi U)\eta(V) + g(W, \varphi V)\eta(U) \\ &+ g(U, \varphi V)\eta(W) + g(U, \varphi W)\eta(V) + g(V, \varphi W)\eta(U)].\end{aligned} \tag{4.2}$$

Taking in hand (2.3) and (4.2), we reached

$$(\nabla_U S)(V, W) + (\nabla_V S)(W, U) + (\nabla_W S)(U, V) = 0. \tag{4.3}$$

Thus we are in condition to plight the following result:

Theorem 4.1. *A quasi-Sasakian* 3*-manifold bearing proper* η*-Ricci soliton always satisfies cyclic parallel Ricci tensor.*

5 Almost pseudo Ricci symmetric

Chaki and Kawaguchi [11] introduced the concept of almost pseudo Ricci symmetric manifolds as an extended class of pseudo symmetric manifolds. A Riemannian manifold (M, g) is called an almost pseudo Ricci symmetric manifold $(APRS)_n$, if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfying the following condition:

$$(\nabla_U S)(V, W) = [A(U) + B(U)]S(V, W) + A(V)S(U, W) + A(W)S(U, V), \tag{5.1}$$

where A and B are two non-zero 1-forms defined by

$$A(U) = g(U, \rho_1), \;\; B(U) = g(U, \rho_2). \tag{5.2}$$

By taking cyclic sum of (5.1), we see that

$$\begin{aligned}(\nabla_U S)(V, W) + (\nabla_V S)(W, U) + (\nabla_W S)(U, V) = [3A(U) + B(U)]S(V, W)\\ +[3A(V) + B(V)]S(U, W) + [3A(W) + B(W)]S(U, V).\end{aligned} \tag{5.3}$$

Let M admits a cyclic Ricci tensor, then (5.3) reduces

$$\begin{aligned}[3A(U) + B(U)]S(V, W) + [3A(V) + B(V)]S(U, W)\\ +[3A(W) + B(W)]S(U, V) = 0.\end{aligned} \tag{5.4}$$

Replacing W by ζ in (5.4) and using (2.23) and (5.2), we get

$$\begin{aligned}-(\mu + \alpha)[3A(U) + B(U)]\eta(V) - (\mu + \alpha)[3A(V) + B(V)]\eta(U)\\ +[3\eta(\rho_1) + \eta(\rho_2)]S(U, V) = 0.\end{aligned} \tag{5.5}$$

In (5.5), substituting $V = \zeta$ and using (2.2), (2.23) and (5.2), we yield

$$-(\mu + \alpha)[3A(U) + B(U)] - 2(\mu + \alpha)[3\eta(\rho_1) + \eta(\rho_2)]\eta(U) = 0. \tag{5.6}$$

Again replacing U by ζ and using (5.2) in (5.6), we obtain

$$-(\mu + \alpha)\,[3\eta\,(\rho_1) + \eta\,(\rho_2)] = 0, \tag{5.7}$$

which implies

$$3\eta(\rho_1) + \eta(\rho_2) = 0, \tag{5.8}$$

since $(\mu + \alpha) \neq 0$. In view of (5.6) and (5.8), it follows that $3A(U) + B(U) = 0$. Thus we can state the following result.

Theorem 5.1. *There is no almost pseudo Ricci symmetric proper η-Ricci soliton on non-cosymplectic quasi-Sasakian 3-manifold admitting cyclic Ricci tensor, unless $3A + B$ vanishes everywhere on M.*

Consequently, if we keep in mind from (5.7) that $3\eta(\rho_1) + \eta(\rho_2) \neq 0$, in this case $\mu + \alpha = 0$, but for η-Ricci soliton on non-cosymplectic quasi-Sasakian 3-manifold $\alpha + \mu = -2\beta^2$. Therefore for this condition $\alpha = -\beta^2$ and $\mu = -\beta^2$. Thus we state the following result.

Corollary 5.2. *A proper η-Ricci soliton on almost pseudo Ricci symmetric non-cosymplectic quasi-Sasakian 3-manifold with cyclic Ricci tensor is of type $(g, V, -\beta^2, -\beta^2)$.*

6 φ-Ricci Symmetric

This segment is affectionate to the study of φ-Ricci Symmetric proper η-Ricci soliton on a quasi-Sasakian 3-manifold and deduce some result. A quasi-Sasakian 3-manifold is said to be φ-Ricci symmetric if the Ricci operator Q satisfies

$$\varphi^2(\nabla_U Q)V = 0, \tag{6.1}$$

for all smooth vector fields $U, V \in \chi(M)$. If X, Y are orthogonal to ζ, then the manifold is said to be locally φ-Ricci symmetric. It is well-known that φ-symmetric implies φ-Ricci symmetric, but the converse, is not, in general true. φ-Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [13].

From (2.23), it follows that

$$QU = -\mu U - \alpha\eta(U)\zeta, \tag{6.2}$$

for all smooth vector fields U. Proceeding covariant derivative of (6.2), we acquire

$$\begin{aligned}(\nabla_U Q)V &= -\alpha\beta[g(U, \varphi V)\zeta - \eta(V)\varphi U] \\ &= \alpha\beta[g(\varphi U, V)\zeta + \eta(V)\varphi U].\end{aligned} \tag{6.3}$$

Applying φ^2 on both sides of (6.3), we get

$$\varphi^2(\nabla_U Q)V = \alpha\beta\,\eta(V)\,\varphi^3 U. \tag{6.4}$$

Making use of (6.1), from (6.4), it walk behind that $\beta \neq 0, \alpha = 0$, which is a counter statement. Thus we are in a condition to plight the following result:

Theorem 6.1. *A φ-Ricci symmetric non-cosymplectic quasi-Sasakian 3-manifold does not admits a proper η-Ricci soliton.*

In [24], authors prove that in a 3-dimensional non-cosymplectic quasi-Sasakian mnaifold φ-Ricci symmetric and φ-symmetric are equivalent provided β is a constant. Thus using this facts we state the following result.

Corollary 6.2. *A φ-symmetric non-cosymplectic quasi-Sasakian 3-manifold does not possess a proper η-Ricci soliton.*

Differentiating (2.16) covariantly along W and applying φ^2 both side, we get

$$\varphi^2(\nabla_W Q)V = \frac{1}{2}[d\kappa(W)(-V + \eta(V)\zeta) + (6\beta^2 - \kappa)\eta(V)\varphi^2(\nabla_W \zeta)]. \tag{6.5}$$

If V is orthogonal to ζ, then from (6.4) and (6.5), we have

$$\frac{1}{2}d\kappa(W)V = 0. \tag{6.6}$$

It implies that $d\kappa = 0$. Hence the scalar curvature κ is constant.Thus we state the following result.

Corollary 6.3. *A non-cosymplectic quasi-Sasakian 3-manifold bearing proper η-Ricci soliton is locally φ-Ricci symmetric if and only if the scalar curvature κ is constant.*

7 Conformally flat

In this constituent we review conformally flat quasi-Sasakian 3-manifolds with a proper η-Ricci soliton. Then we have [25].

$$(\nabla_U S)(V, W) - (\nabla_V S)(U, W) = \frac{1}{4}[g(V, W)d\kappa(U) - g(U, W)d\kappa(V)]. \tag{7.1}$$

Making use of (2.23) in (7.1), we have

$$\begin{aligned} \alpha\beta[g(V, \varphi U)\eta(W) + g(W, \varphi U)\eta(V) - g(U, \varphi V)\eta(W) - g(W, \varphi V)\eta(U)] \\ = \tfrac{1}{4}[g(V, W)d\kappa(U) - g(U, W)d\kappa(V)]. \end{aligned} \tag{7.2}$$

Substituting $U = \zeta$ in (7.2), we obtain

$$4\alpha\beta g(\varphi W, V) = -\eta(W)d\kappa(V). \tag{7.3}$$

That restricted to

$$4\alpha\beta\,\varphi\,V = -d\kappa(V)\zeta.$$

From the above equation it walk behind that $4\alpha\beta\,\varphi^2\,V = 0$ and hence $\beta \neq 0, \alpha = 0$, which is a counter statement. Therefore we state the following result:

Theorem 7.1. *A conformally flat non-cosymplectic quasi-Sasakian* 3*-manifold does not possess a proper* η*-Ricci soliton.*

8 Special weakly Ricci symmetric

The notion of a special weakly Ricci symmetric manifold was introduced and studied by Singh and Quddus [29]. An n-dimensional Riemannian manifold (M, g) is called a special weakly Ricci symmetric manifold $(SWRS)_n$ if

$$(\nabla_X S)(Y, Z) = 2\varepsilon(X)S(Y, Z) + \varepsilon(Y)S(X, Z) + \varepsilon(Z)S(Y, X), \tag{8.1}$$

where ε is a 1-form and is defined by

$$\varepsilon(X) = g(X, \rho), \tag{8.2}$$

where ρ is the associated vector field. Let the Eq.(8.1) and (8.2) hold on quasi-Sasakian 3-manifold. Taking cyclic sum of (8.1), we get

$$\begin{aligned}(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y)\\ = 4[\varepsilon(X)S(Y, Z) + \varepsilon(Y)S(Z, X) + \varepsilon(Z)S(X, Y)].\end{aligned} \tag{8.3}$$

Let M admits a cyclic parallel Ricci tensor. Then (8.3) reduces to

$$\varepsilon(X)S(Y, Z) + \varepsilon(Y)S(Z, X) + \varepsilon(Z)S(X, Y) = 0. \tag{8.4}$$

Taking $Z = \zeta$ in (8.4) and using (2.23) and (8.2), we have

$$-(\alpha + \mu)[\varepsilon(X)\eta(Y) + \varepsilon(Y)\eta(X)] + \eta(\rho)S(X, Y) = 0. \tag{8.5}$$

Again, taking $Y = \zeta$ in (8.5) and then using (2.23) and (8.2), we get

$$-(\alpha + \mu)[\varepsilon(X) + 2\eta(\rho)\eta(X)] = 0. \tag{8.6}$$

Taking $X = \zeta$ in (8.6) and using (8.2), we obtain

$$-3(\alpha + \mu)\eta\,(\rho) = 0. \tag{8.7}$$

In this case if $\eta\,(\rho) = 0$ and $\alpha + \mu \neq 0$, then from (8.6) we have $\varepsilon(X) = 0$, $\forall X \in \chi(M)$. Again if $\eta\,(\rho) \neq 0, \alpha + \mu = 0$, in this case $\alpha = -\beta^2, \mu = -\beta^2$. It leads to the following result:

Theorem 8.1. *If a special weakly Ricci symmetric non-cosymplectic quasi-Sasakian* 3*-manifold with a proper* η*-Ricci soliton admits a cyclic parallel Ricci tensor, then the* 1*-form* ε *is vanish identically on* M*.*

Corollary 8.2. *A proper* η*-Ricci soliton on a special weakly Ricci symmetric non-cosymplectic quasi-Sasakian* 3*-manifold admits cyclic Ricci tensor is of type* $(g, V, -\beta^2, -\beta^2)$ *if the* 1*-form* $\varepsilon \neq 0$*.*

Again, if a complete Einstein quasi-Sasakian 3-manifold is compact. Then we have [19].

$$S(X,Y) = \vartheta g(X,Y), \quad \vartheta = 2\beta^2.$$

It is well-known that for complete Einstein quasi-Sasakian 3-manifold, $(\nabla_X S)(Y,Z) = 0$ and $S(X,Y) = \vartheta g(X,Y)$. Then (8.1) gives

$$2\varepsilon(X)g(Y,Z) + \varepsilon(Y)g(X,Z) + \varepsilon(Z)g(Y,X) = 0. \tag{8.8}$$

Taking $Z = \zeta$ in (8.8) and then using (8.2), we get

$$2\varepsilon(X)\eta(Y) + \varepsilon(Y)\eta(X) + \eta(\rho)g(Y,X) = 0. \tag{8.9}$$

Again taking $X = \zeta$ in (8.9) and then using (8.2), we get

$$3\eta(\rho)\eta(Y) + \varepsilon(Y) = 0. \tag{8.10}$$

Taking $Y = \zeta$ in (8.10) and using (8.2), we obtain

$$\eta(\rho) = 0. \tag{8.11}$$

Making use of (8.11) in (8.10), we get $\varepsilon(Y) = 0, \ \forall \ Y \in \chi(M)$. Finally we have the following result:

Theorem 8.3. *A special weakly Ricci symmetric non-cosymplectic quasi-Sasakian* 3*-manifold can not be compact if the* 1 *-form* $\varepsilon \neq 0$*.*

9 η-recurrent

A quasi-Sasakian manifold is said to be η-recurrent if its non-vanishing Ricci tensor S satisfies the following condition

$$(\nabla_U S)(\varphi V, \varphi W) = A(U)S(\varphi V, \varphi W), \tag{9.1}$$

for all $U, V, W \in \chi(M)$, where $A(U) = g(U,\rho)$, ρ is the associated vector field of the 1-form A. In particular, if the 1-form A vanishes identically on M, then it is said to be η-parallel. This notion for Sasakian manifold was first introduced by Kon [21]. In view of (2.3), (2.19) and (2.23), we have

$$\begin{aligned}(\nabla_U S)(\varphi V, \varphi W) &= \mu\beta[g(U,\varphi V)\eta(W) + g(U,\varphi W)\eta(V) \\ &\quad -g(U,\varphi W)\eta(V) - g(\varphi V, U)\eta(W)] = 0.\end{aligned} \tag{9.2}$$

Making use of (9.2) in (9.1), we get

$$A(U)S(\varphi V, \varphi W) = 0. \tag{9.3}$$

Again using (2.23) in (9.3), we obtain

$$-\mu A(U)g(\varphi V, \varphi W) = 0. \tag{9.4}$$

This implies that $A(U) \neq 0$, $g(\varphi V, \varphi W) \neq 0$. Therefore, we conclude that $\mu = 0$, that is, the Ricci soliton is always steady. So we have the following result.

Theorem 9.1. *If a non-cosymplectic quasi-Sasakian 3-manifold with proper η-Ricci soliton satisfying η-recurrent, then the Ricci soliton is always steady.*

Corollary 9.2. *The necessary condition for a non-cosymplectic quasi-Sasakian 3-manifold with proper η-Ricci soliton to be η-parallel, the Ricci soliton is always shrinking.*

10 The curvature condition $Q \cdot R = 0$.

In this section we are going to study, a proper η-Ricci soliton on a quasi-Sasakian 3-manifold that satisfying the curvature condition $Q \cdot R = 0$. Then

$$(Q \cdot R)(U, V)W = 0, \tag{10.1}$$

for all smooth vector fields $U, V, W \in \chi(M)$. From (10.1), it is obvious that

$$Q(R(U, V)W) - R(QU, V)W - R(U, QV)W - R(U, V)QW = 0. \tag{10.2}$$

Making use of (2.14) and (2.23), Eq. (10.2) reduces to

$$\begin{aligned}
&4\alpha\mu\eta(U)\eta(W)V + 2\mu^2\eta(U)\eta(W)V - 4\alpha\mu\eta(V)\eta(W)U + \alpha\kappa\eta(U)\eta(W)V \\
&\quad -\alpha\kappa\eta(V)\eta(W)U + 2\mu[\{-\mu g(V, W) - \alpha\eta(V)\eta(W)\}U \\
&\quad -\{-\mu g(U, W) - \alpha\eta(U)\eta(W)\}V + g(V, W)\{(-\mu U - \alpha\eta(U)\zeta\} \\
&\quad -g(U, W)\{(-\mu V - \alpha\eta(V)\zeta\} - \frac{\kappa}{2}\{g(V, W)U - g(U, W)V\}] \\
&\quad -2\alpha^2\eta(V)\eta(W)U + \alpha\mu g(V, W)\eta(U)\zeta + \alpha^2 g(V, W)\eta(U)\zeta = 0.
\end{aligned} \tag{10.3}$$

Putting $U = W = \zeta$ in (10.3), we get

$$\begin{aligned}
&4\alpha\mu\, V + 2\mu^2 V - 4\alpha\mu\eta\,(V)\zeta + \alpha\kappa V - \alpha\kappa\eta\,(V)\zeta \\
&\quad +2\mu[-\mu\eta\,(V)\zeta - \alpha\eta\,(V)\zeta + \mu V + \alpha V - \mu\eta(V)\zeta \\
&\quad -\alpha\eta(V)\zeta + \mu V + \alpha\eta\,(V)\zeta - \tfrac{\kappa}{2}\eta(V)\zeta + \tfrac{\kappa}{2}(V)] \\
&\quad -2\alpha^2\eta\,(V)\zeta + \mu\alpha\eta\,(V)\zeta + \alpha^2\eta\,(V)\zeta = 0.
\end{aligned}$$

Applying the inner product of the above equation with ζ, we obtain

$$\alpha(\mu+\alpha)\eta(V)=0. \tag{10.4}$$

It walk behind that $\alpha \neq 0$, which is a counter statement. Thus $\alpha+\mu=0$. On other hand for a η-Ricci soliton on a quasi-Sasakian 3-manifold, $\alpha+\mu=-2\beta^2$. Therefore for this condition $\alpha=-\beta^2$ and $\mu=-\beta^2$. Thus we sate the following result:

Theorem 10.1. *A proper* η*-Ricci soliton on a non-cosymplectic quasi-Sasakian* 3*-manifold satisfying the curvature condition* $Q\cdot R=0$ *is of type* $(g, V, -\beta^2, -\beta^2)$.

As the dissertation of our work, we keep in mind the Corollary 5.2, Corollary 8.2 and Theorem 10.1, we state the following result.

Theorem 10.2. *If a proper* η*-Ricci soliton on a non-cosymplectic quasi-Sasakian* 3*-manifold* M *is of type* $(g, V, -\beta^2, -\beta^2)$, *then the following conditions are equivalent:*

i) M^n *is almost pseudo Ricci symmetric with cyclic Ricci tensor,*

ii) M^n *is special weakly Ricci symmetric and its Ricci tensor is cyclic parallel,*

iii) $Q\cdot R=0$ *holds on* M^n.

11 Yamabe solitons

In this section we find some results related to Yamabe soliton on quasi-Sasakian 3-manifolds. We consider a Yamabe soliton (g, ζ). From (1.5) we have

$$\frac{1}{2}(\mathfrak{L}_V g)(X,Y)=(\kappa-\lambda)g(X,Y), \tag{11.1}$$

which implies that

$$g(\nabla_X\zeta,Y)+g(X,\nabla_Y\zeta)=2(\kappa-\lambda)g(X,Y). \tag{11.2}$$

Keeping in mind (2.6), Equ.(11.2) reduces to

$$2(\kappa-\lambda)g(X,Y)=0 \tag{11.3}$$

Taking $X=\zeta$ in (11.3), we get $\lambda=\kappa$. Then equation (1.5) reduces to $\mathfrak{L}_V g=0$, that is, V is Killing vector field. Moreover, λ is constant then the scalar curvature κ is also constant. Thus we state the following result.

Theorem 11.1. *If the metric of a* 3*-dimensional non-cosymplectic quasi-Sasakian* 3*-manifold is a Yamabe soliton then the manifold is space of constant curvature.*

Besides it, from (1.5), we have $\mathfrak{L}_V g = 0$, thus V is Killing. Differentiating covariently along an arbitrary vector field X, we have $\nabla_X \mathfrak{L}_V g = 0$.

The identity
$$(\nabla_X \mathfrak{L}_V g)(U, W) = g((\mathfrak{L}_V \nabla)(X, U), W) + g((\mathfrak{L}_V \nabla)(X, W), U), \tag{11.4}$$
can be reduced from the formula [34].
$$(\mathfrak{L}_V \nabla_X\, g - \nabla_X \mathfrak{L}_V g - \nabla_{[V,X]} g)(U, W) = -g((\mathfrak{L}_V \nabla)(X, U), W) - g((\mathfrak{L}_V \nabla)(X, W), U).$$
This implies that
$$g((\mathfrak{L}_V \nabla)(W, X), U) + g((\mathfrak{L}_V \nabla)(W, U), X) = 0. \tag{11.5}$$

According to equation (11.4) and (11.5), the skew-symmetric property of ϕ, we get $(\mathfrak{L}_V \nabla)(U, W) = 0$, which implies that $(\mathfrak{L}_V \nabla)(\zeta, \zeta) = 0$. Also, using geodesic properties of ζ, we have
$$(\mathfrak{L}_V \nabla)(X, U) = -\nabla_X \nabla_U V - \nabla_{\nabla_X U} V + R(V, X)U,$$
which yields $\nabla_\zeta \nabla_\zeta V + R(V, \zeta)\zeta = 0$. This means that V is Jacobi along the direction of ζ. So we have the following result.

Theorem 11.2. *If the metric of a non-cosymplectic quasi-Sasakian 3-manifold is a Yamabe soliton, then the flow vector field V is Killing and is Jacobi along the direction of ζ.*

It is well-known that the Reeb vector field ζ is a unit vector field, that is, $g(\zeta, \zeta) = 1$.Taking Lie-derivative of it along the vector fled V and using (1.5), we get
$$\eta(\mathfrak{L}_V \zeta) = -(\mathfrak{L}_V \eta)(\zeta) = (\lambda - \kappa). \tag{11.6}$$

Moreover, in view of $\omega = (\kappa - \lambda)$, $(n = 3)$ and Proposition 1.1, we obtain

(i) $(\mathfrak{L}_V S)(X, Y) = -g(\nabla_X D\kappa, Y) + \Delta\kappa g(X, Y)$.

(ii) $(\mathfrak{L}_V \kappa) = -2\kappa(\kappa - \lambda) + 4\Delta\kappa$.

Since g is a Yamabe soliton, then taking the Lie-derivative of (2.15), and using the above equation, we get
$$\begin{aligned} -g(\Delta_X D\kappa, Y) =& \frac{1}{2}(\mathfrak{L}_V \kappa)[g(X, Y) - \eta(X)\eta(Y)] \\ &+ [2(\frac{\kappa}{2} - \beta^2)(\kappa - \lambda)]g(X, Y) \\ &+ (3\beta^2 - \frac{\kappa}{2})[(\mathfrak{L}_V \eta)(X)\eta(Y) + (\mathfrak{L}_V \eta)(Y)\eta(X)]. \end{aligned}$$

Since ζ is Killing, therefore $\zeta\kappa = 0$. Differentiating covariantly along the direction of an arbitrary vector field X, we have $g(\Delta_X D\kappa, \zeta)$=$(\beta\varphi X)\kappa$. Substituting $Y = \zeta$ in above equation, we have

$$-2\beta(\varphi X)\kappa = [2(\kappa - 2\beta^2)(\kappa - \lambda) - 2\Delta\kappa]\eta(X) + (6\beta^2 - \kappa)[(\mathfrak{L}_V\eta)X + (\kappa - \lambda)\eta(X)]. \tag{11.7}$$

Taking $X = \zeta$ in (11.9), using (11.6) and Proposition 1.1, we obtain

$$\Delta\kappa = 4\beta^2(\kappa - \lambda) \tag{11.8}$$

In view of (11.9) and(11.10), we yields

$$(6\beta^2 - \kappa)(\mathfrak{L}_V\eta)X = -2\beta(\varphi X)\kappa - [(\kappa - \lambda)(\kappa - 6\beta^2]\eta(X) \tag{11.9}$$

Since κ is constant then from (11.11) one can say that either $\kappa = 6\beta^2$ or $\kappa \neq 6\beta^2$. In particular if $\kappa = 6\beta^2$ then from (2.15) we have $S = 2\beta^2 g$, that is, M is an Einstein manifold of constant curvature β^2. Thus as per above consequences, we state the following result.

Corollary 11.3. *If the metric of a 3-dimensional non-cosymplectic quasi-Sasakian manifold admits a Yamabe soliton and $\kappa = 6\beta^2$ then the manifold is an Einstein.*

Corollary 11.4. *For a 3-dimensional cosymplectic manifold which admits a Yamabe soliton always has constant harmonic scalar curvature, that is $\Delta\kappa = 0$.*

Corollary 11.5. *If a 3-dimensional non-cosymplectic quasi-Sasakian manifold with constant harmonic scalar curvature admitting Yamabe soliton then the manifold is space of constant curvature.*

On the other hand, if $\kappa \neq 6\beta^2$ then from (11.8), we get $\mathfrak{L}_V\eta = 0$. Then the equation (1.7) implies that $\upsilon = 0$. Thus we state the following result.

Theorem 11.6. *If the metric of a 3-dimensional non-cosymplectic quasi-Sasakian manifold is a Yamabe soliton, then the conformal contact transformation of the conformal vector field is strict.*

12 An Example

We consider a 3-dimensional manifold $M^3 = \{(u, v, w) \in \mathfrak{R}^3, (u, v, w) \neq (0, 0, 0)\}$, where (u, v, w) is the standard coordinate in $\mathfrak{R}^3$. Let (e_1, e_2, e_3) be linearly independent vector fields at each point of M, identify by

$$e_1 = \frac{\partial}{\partial v}, \quad e_2 = \frac{\partial}{\partial w}, \quad e_3 = \beta\left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial w} - w\frac{\partial}{\partial v}\right)$$

and

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \beta e_2, \quad [e_2, e_3] = -\beta e_1.$$

Let the Riemannian metric g on M^3 is defined as

$$g(e_1,e_2)=g(e_2,e_3)=g(e_1,e_3)=0,\ g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.$$

and given by

$$g=\frac{1}{\beta^2}\left[(1-\beta^2v^2-\beta^2w^2)du\otimes du+\beta^2 dv\otimes dv+\beta^2 dw\otimes dw\right].$$

Let η be the 1-form has the significance

$$\eta(U)=g(U,e_3)$$

for any $U\in\chi(M^3)$ and φ be the $(1,1)$ tensor field defined by

$$\varphi e_1=-e_2,\quad \varphi e_2=e_1,\quad \varphi e_3=0.$$

Making use of the linearity of φ and g, we have

$$\eta(e_3)=1,$$

$$\varphi^2(U)=-U+\eta(U)e_3$$

and

$$g(\varphi U,\varphi V)=g(U,V)-\eta(U)\eta(V),$$

for any $U,W\in\chi(M^3)$. Thus for $e_3=\zeta$, the structure $(M^3,\eta,\zeta,\varphi,g)$ leads to a contact metric structure on M^3. The Riemannian connection ∇ of metric tensor g is given by the beauty of Koszul's formula

$$\begin{aligned}2g(\nabla_U V,W)&=U(g(V,W))+V(g(W,X))-W(g(U,V))\\&\quad -g(U,[V,W])-g(V,[U,W])+g(W,[U,V]).\end{aligned}$$

Making use of the Koszul's formula, we get

$$\begin{cases}\nabla_{e_2}e_3=-\beta e_1, & \nabla_{e_2}e_2=0, & \nabla_{e_2}e_1=-\beta e_3,\\ \nabla_{e_3}e_3=0, & \nabla_{e_3}e_2=0, & \nabla_{e_3}e_1=-\beta e_3,\\ \nabla_{e_1}e_3=\beta e_2, & \nabla_{e_1}e_2=-\beta e_3, & \nabla_{e_1}e_1=0.\end{cases}$$

Consequently $(M^3,\eta,\zeta,\varphi,g)$ is an quasi-Sasakian structure that satisfies,

$$(\nabla_U\varphi)V=\beta(g(U,V)\zeta-\eta(V)U),\quad \nabla_U\zeta=-\beta\varphi U,$$

where $\beta\neq 0$. Hence $(M^3,\eta,\zeta,\varphi,g)$ define non-cosymplectic quasi-Sasakian 3-manifold. Therefore, we find the components of curvature tensor as follows:

$$\begin{cases}R(e_2,e_3)e_3=\beta^2e_2, & R(e_2,e_3)e_1=-\beta^2e_3, & R(e_3,e_2)e_2=\beta^2e_3,\\ R(e_1,e_3)e_3=\beta^2e_1, & R(e_3,e_1)e_1=\beta^2e_3, & R(e_2,e_1)e_1=\beta^2e_2-\beta^2e_1,\\ R(e_1,e_2)e_2=\beta^2e_1, & R(e_1,e_2)e_3=\beta^2e_3, & R(e_3,e_1)e_2=0.\end{cases}$$

From the above we can easily evaluate the value of the Ricci tensor as follows:

$$\begin{cases} S(e_1, e_1) = 2\beta^2, \quad S(e_2, e_2) = 2\beta^2, \quad S(e_3, e_3) = 2\beta^2 \\ S(e_1, e_2) = 0 \qquad S(e_2, e_3) = 0, \qquad S(e_2, e_3) = 0. \end{cases}$$

Also, the scalar curvature κ is given by:

$$\kappa = \sum_{i=1}^{3} g(e_i e_i) S(e_i, e_i) = 6\beta^2$$

Also from the Equ.(2.23), we get

$$S(e_1, e_1) = S(e_2, e_2) = -\mu, \;\; S(e_3, e_3) = -\mu - \alpha.$$

It is clear that $\mu = -2\beta^2$ and $\alpha = 0$. Thus the manifold does not admits proper η-Ricci soliton. Hence the Theorem 3.1, Theorem 4.1, Theorem 6.1 and Theorem 7.1 are verified.

Let $\{e_1, e_2, e_3\}$ be a basis of the tangent space at any point. For any vector $X, Y \in \chi(M^{2n+1})$, we have

$$X = a_1 e_1 + b_1 e_2 + c_1 e_3 \;\; , Y = a_2 e_1 + b_2 e_2 + c_2 e_3,$$

where $a_i, b_i, c_i \in \Re \backslash \{0\}$, for all $i = 1, 2, 3$.

Thus $g(X, Y) = a_1 a_2 + b_1 b_2 + c_1 c_2$, and $S(X, Y) = 2\beta^2 \{a_1 a_2 + b_1 b_2 + c_1 c_2\}$. Then we obtain $S(X, Y) = 2\beta^2 g(X, Y)$, that is, the manifold M is an Einstein manifold. Hence Corollary 11.3 are hold.

Remark 12.1. *In this example $\beta \neq 0$ and $\mu < 0$. Thus the Ricci soliton in a 3-dimensional non-cosymplectic quasi-Sasakian manifold is always shrinking.*

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Press requirement: The abstract should be no longer than 250 words. CUBO strongly encourages the use of LaTeX for manuscript preparation. References should be headed numerically, alphabetically organized and complete in all its details. Authors' initials should precede their names; journal title abbreviations should follow the style of Mathematical Reviews.

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