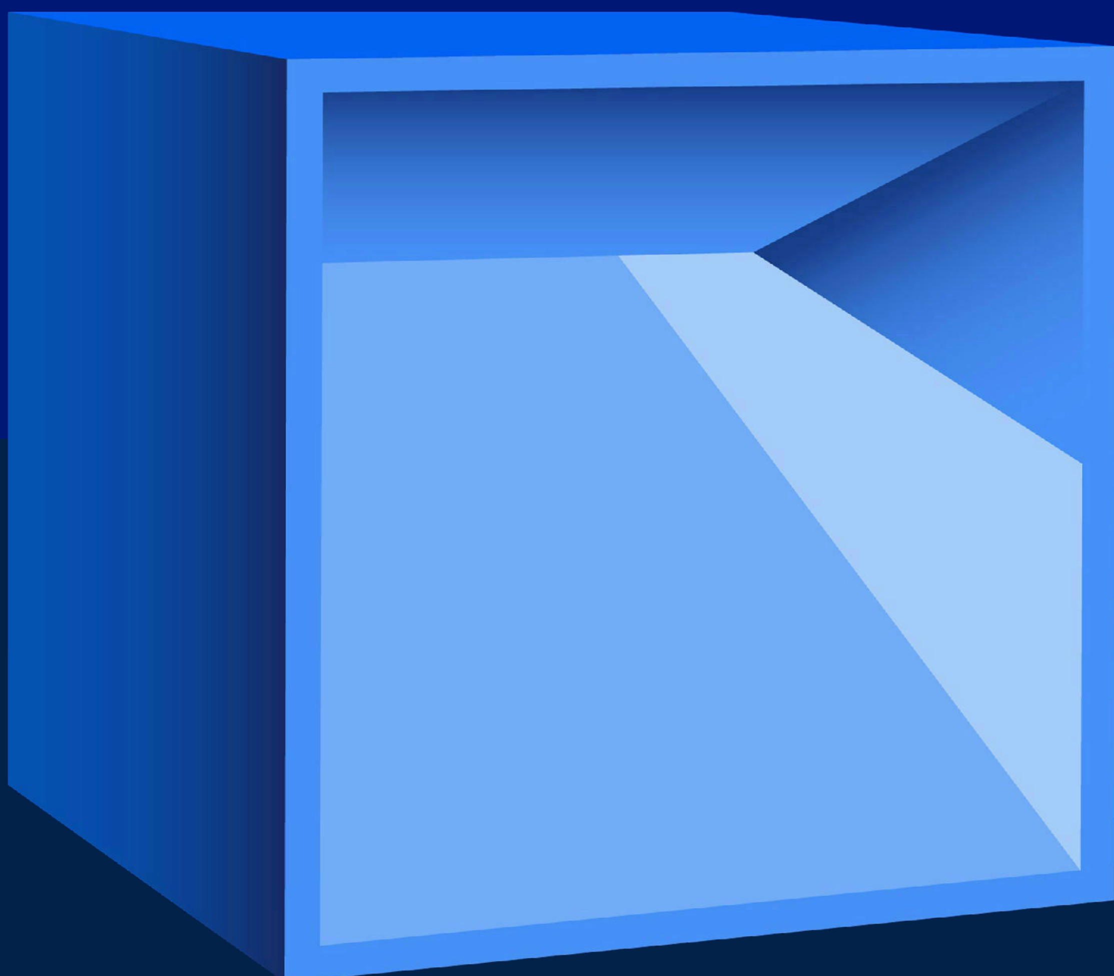
ISSN 0719-0646
ONLINE VERSION

UNIVERSIDAD DE LA FRONTERA

VOLUME 21 · ISSUE 3
2019

Cubo

A Mathematical Journal



Facultad de Ingeniería y Ciencias - Departamento de Matemática y Estadística

Temuco - Chile

CUBO
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CUBO
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Universidad de La Frontera
Volume 21/N$^{\underline{o}}$03 – DECEMBER 2019

SUMMARY

CUBO A Mathematical Journal
Vol.21, Nº03, (01–08). December 2019
http://dx.doi.org/10.4067/S0719-06462019000300001

The K-theory ranks for crossed products of C^*-algebras by the group of integers

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ABSTRACT

We study the K-theory ranks for crossed products of C^*-algebras by the group of integers. As an application, we obtain certain estimates for the K-theory ranks of the group C^*-algebras of torsion free, finitely generated, nilpotent or solvable discrete groups, written as successive semi-direct products.

RESUMEN

Estudiamos los rangos de K-teoría para productos cruzados de C^*-álgebras por el grupo de los enteros. Como aplicación, obtenemos ciertas estimaciones para los rangos de K-teoría de las C^*-álgebras de grupos libres de torsión, finitamente generados, nilpotentes o solubles, escritos como productos semidirectos sucesivos.

Keywords and Phrases: K-theory, C*-algebra, crossed product, Betti number, discrete group.

2010 AMS Mathematics Subject Classification: 46L05, 46L55, 46L80

1 Introduction

In this paper we study the (free or $\mathbb{Z}$) ranks of the K-theory groups for crossed products of C^*-algebras by $\mathbb{Z}$ the group of integers. Such C^*-algebras and their K-theory play fundamental roles in the theory of C^*-algebras and K-theory (cf. Blackadar [1], Pedersen [2], Tomiyama [10], Wegge-Olsen [11]). By using the Pimsner-Voiculescu six-term exact sequence (PV) of the K-theory groups of the crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$ by automorphisms (Pimsner and Voiculescu [3], cf. [1]), in Section 2 we estimate the K-theory group ranks of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ in terms of those of $\mathfrak{A}$. This simple result should be new in some insight and interesting in some sense, as another introductory step in this developed research area. As an easy, direct application of PV, in Section 3 we obtain certain estimates for the K-theory ranks of the group C^*-algebras of torsion free, finitely generated, nilpotent or solvable discrete groups, written as successive semi-direct products by torsion free, abelian groups. There may be more other applications left to be considered, but not so many probably. May as well refer to [5], [6], [7], [8], [9] for some related details. In particular, in [5], [7], and [9], the K-theory groups of the C^*-algebras of the generalized Heisenberg discrete nilpotent groups as typical examples of non-type I discrete amenable groups are computed by some methods of determining K-theory class generators as projections or unitaries, of the K-theory groups, but it seems that still, the K-theory groups of the C^*-algebras of general (torsion free, finitely generated) nilpotent (or solvable) discrete groups are not yet done completely, because of some difficulties involving successive unknown group actions. However, this time, without determining their K-theory groups as groups, the K-theory group rank estimates are obtained by us in such a way mentioned above, as the motivated examples, as given in Section 3.

2 The K-theory ranks for crossed product C^*-algebras by $\mathbb{Z}$

Let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product C^*-algebra of $\mathfrak{A}$ by an action α of $\mathbb{Z}$ on $\mathfrak{A}$ by automorphisms, where $\alpha_n = \alpha^n = \alpha \circ \cdots \circ \alpha$ as the n-fold composition of $\alpha = \alpha_1 : \mathfrak{A} \to \mathfrak{A}$ for $n \in \mathbb{Z}$ (cf. Blackadar [1], Pedersen [2], Tomiyama [10]). There is the following **Pimsner-Voiculescu** six-term exact sequence of the K-theory abelian groups (K_0 additive and K_1 multiplicative) (Pimsener and Voiculescu [3], cf. [1]):

$$\begin{tikzcd}
K_0(\mathfrak{A}) \arrow[r, "(\mathrm{id}-\alpha)_*"] & K_0(\mathfrak{A}) \arrow[r, "i_*"] & K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \arrow[d, "\exp", "\partial"'] \\
K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \arrow[u, "\partial", "\mathrm{ind}"'] & K_1(\mathfrak{A}) \arrow[l, "i_*"'] & K_1(\mathfrak{A}), \arrow[l, "(\mathrm{id}-\alpha)_*"']
\end{tikzcd}$$

where $\mathrm{id} : \mathfrak{A} \to \mathfrak{A}$ is the identity map and $i : \mathfrak{A} \to \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is the canonical inclusion map and the K-theory group maps $(\mathrm{id} - \alpha)_*$ and i_* are induced by $\mathrm{id} - \alpha$ and i, respectively, and the upward and downward arrows as the boundary maps ∂ are the index map as ind and the exponential map as exp, respectively.

It follows from exactness of the PV diagram above that

Lemma 2.1. *For any* C^**-algebra* $\mathfrak{A}$ *and any* $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$*, we have the following short exact sequences: for* $j = 0, 1$*,*

$$\begin{aligned} 0 \longrightarrow K_j(\mathfrak{A})/(\mathrm{id}-\alpha)_* K_j(\mathfrak{A}) = K_j(\mathfrak{A})/\ker(i_*) \\ \xrightarrow{i_*} K_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \xrightarrow{\partial} \mathrm{im}(\partial) = \ker(\mathrm{id}-\alpha)_* \to 0 \end{aligned}$$

with $(\mathrm{id}-\alpha)_* K_j(\mathfrak{A}) = \ker(i_*) \subset K_j(\mathfrak{A})$*, where* $(\mathrm{id}-\alpha)_* K_j(\mathfrak{A})$ *is the image of* $K_j(\mathfrak{A})$ *under* $(\mathrm{id}-\alpha)_*$ *and* $\ker(\mathrm{id}-\alpha)_*$ *is the kernel of* $(\mathrm{id}-\alpha)_*$ *on* K_0 *or* K_1*, and* $\mathrm{im}(\partial)$ *is the image of the boundary map* ∂ *equal to* exp *or* ind*.*

Let G be an abelian group. We denote by $\mathrm{rank}_{\mathbb{Z}}\, G$ the $\mathbb{Z}$-rank (or free rank) of G, which is also called the **Betti** number of G, denoted as $b(G)$. For a C^*-algebra $\mathfrak{A}$, set $b_j(\mathfrak{A}) = b(K_j(\mathfrak{A}))$ for $j = 0, 1$, each of which we call the j-th **Betti** number of $\mathfrak{A}$ (cf. [6]). We denote by $t(G)$ the **torsion** rank of G, which is defined to be the number of direct sum components of indecomposable, finite cyclic groups in G. Set $t_j(\mathfrak{A}) = t(K_j(\mathfrak{A}))$ for $j = 0, 1$, each of which we may call the j-th **torsion** rank of $\mathfrak{A}$.

Recall as a fundament fact in group theory that a finitely generated abelian group H has the following direct product decomposition:

$$H \cong \mathbb{Z}^{b(H)} \times \mathbb{Z}_{p_1^{n_1}} \times \cdots \mathbb{Z}_{p_{t(H)}^{n_{t(H)}}},$$

where $p_1, \cdots p_{t(H)}$ are primes and $n_1, \cdots, n_{t(H)}$ are some positive integers and each $\mathbb{Z}_{p_j^{n_j}} = \mathbb{Z}/p_j^{n_j}\mathbb{Z}$ for $1 \le j \le t(H)$ is the finite cyclic group of order $p_j^{n_j}$, that is indecomposable, and these powers of primes are distinct.

Lemma 2.2. *For a short exact sequence* $1 \to H \to G \to G/H \to 1$ *of finitely generated, abelian groups, we have* $b(H) \le b(G)$ *and* $b(G/H) \le b(G)$ *and* $b(G) = b(H) + b(G/H)$*.*

Proof. Note that there is no homomorphism from a finite torsion group to a torsion free group. Hence $b(H) \le b(G)$, and $b(G/H) = b(G) - b(H) \le b(G)$. $\square$

Proposition 1. *For any* $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$*, we have that for* $j = 0, 1$*,*

$$b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \le b_0(\mathfrak{A}) + b_1(\mathfrak{A})$$

and $b(K_j(\mathfrak{A})/(\mathrm{id}-\alpha)_* K_j(\mathfrak{A})) \le b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$*.*

Proof. By using the Lemmas 2.1 and 2.2 above, we obtain

$$\begin{aligned} b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) &= b_j(K_j(\mathfrak{A})/\ker(i_*)) + b_{j+1}(\ker(\mathrm{id} - \alpha)_*) \\ &\leq b_j(K_j(\mathfrak{A})) + b_{j+1}(K_{j+1}(\mathfrak{A})) \end{aligned}$$

for $j = 0, 1$ and $j + 1$ (mod 2), and $b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \geq b_j(K_j(\mathfrak{A})/\ker(i_*))$. □

Let G be an abelian group. Let G_f and G_t denote the free and torsion parts of G respectively, so that $G \cong G_f \times G_t$ with $b(G) = b(G_f)$ and $t(G) = t(G_t)$.

Lemma 2.3. *Let G be a finitely generated, abelian group and H a subgroup. Then there is the following short exact sequence of groups, preserving the free and torsion parts of H and G :*

$$0 \to H = H_f \times H_t \to G = G_f \times G_t \to G/H = (G_f/H_f) \times (G_t/H_t) \to 0$$

with $G_t \cong H_t \times (G_t/H_t)$ and $(G_f/H_f)_t \times (G_t/H_t) \cong (G/H)_t$ and $(G_f/H_f)_f = (G/H)_f$. It then follows that

$$t(H) \leq t(G) \leq t(H) + t(G/H).$$

Proof. Note that there are injective maps from $\mathbb{Z}$ to $\mathbb{Z}$ and from $\mathbb{Z}^k$ to $\mathbb{Z}^l$ with $k \leq l$, but there is no injective map from $\mathbb{Z}$ to a finite cyclic group. It follows that an injective map from H to G preserves their free and torsion parts. Note also that G_t/H_t is a torsion group, but G_f/H_f may have its free part $(G_f/H_f)_f$ and torsion part $(G_f/H_f)_t$. □

Remark. The inequality $t(G/H) \leq t(G)$ does not hold in general. For instance, there is a quotient map from $G = \mathbb{Z}$ to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, with $H = 2\mathbb{Z}$, so that $t(H) = t(G) = 0 < 1 = t(G/H) = t(H) + t(G/H)$.

Proposition 2. *It then follows that for $j = 0, 1 \in \mathbb{Z}_2$,*

$$t_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq t(K_j(\mathfrak{A})/(\mathrm{id} - \alpha)_* K_j(\mathfrak{A})) + t(\ker(\mathrm{id} - \alpha)_*)$$

with $\ker(\mathrm{id} - \alpha)_* \subset K_{j+1}(\mathfrak{A})$ *as a subgroup, and*

$$t(K_j(\mathfrak{A})/(\mathrm{id} - \alpha)_* K_j(\mathfrak{A})) \leq t_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}).$$

Remark. Let $\mathfrak{A}$ be a C^*-algebra. Set $\chi(\mathfrak{A}) = b_0(\mathfrak{A}) - b_1(\mathfrak{A})$, which is called the **Euler** characteristic of $\mathfrak{A}$, where we assume that it is defined to be an integer or $\pm\infty$ (or formally $\infty - \infty$). If $\chi(\mathfrak{A})$ and $\chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ are finite, then it holds that $\chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0$ by using the PV diagram (see [6] or [8]).

Let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$ the n-fold successive crossed product C^*-algebra of $\mathfrak{A}$ by successive actions $\alpha(j)$ of $\mathbb{Z}$ $(1 \leq j \leq n)$. It then follows that

Theorem 2.1. *For such an* n*-fold successive crossed product* C^**-algebra of a* C^**-algebra* $\mathfrak{A}$ *by* n *successive actions of* $\mathbb{Z}$ *as above or below, we have*

$$b_j(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}) \leq 2^{n-1}(b_0(\mathfrak{A}) + b_1(\mathfrak{A}))$$

for $j = 0, 1$.

Proof. When $n = 2$, we have

$$\begin{aligned} b_j(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z}) &\leq b_0(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z}) + b_1(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z}) \\ &\leq 2(b_0(\mathfrak{A}) + b_1(\mathfrak{A})). \end{aligned}$$

When $n = 3$, we have

$$\begin{aligned} b_j(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z} \rtimes_{\alpha(3)} \mathbb{Z}) &\leq b_0(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z}) + b_1(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z}) \\ &\leq 2[b_0(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z}) + b_1(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z})] \\ &\leq 2^2(b_0(\mathfrak{A}) + b_1(\mathfrak{A})). \end{aligned}$$

The general case follows by induction with respect to n. □

3 Examples and more

Example 1. Let $C(\mathbb{T}^n)$ be the C^*-algebra of all continuous, complex-valued functions on the n-dimensional torus $\mathbb{T}^n$, which is also the univesal C^*-algebra generated by mutually commuting n unitaries. The C^*-algebra is regarded as the successive crossed product C^*-algebra of $\mathbb{C}$ by trivial actions id of $\mathbb{Z}$:

$$C(\mathbb{T}^n) \cong C^*(\mathbb{Z}^n) \cong \mathbb{C} \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$$

with $\alpha(j) = \mathrm{id}$ for $1 \leq j \leq n$, via the Fourier transform from $C^*(\mathbb{Z}^n)$ to $C(\mathbb{T}^n)$, with $\mathbb{T}^n$ as the dual group of $\mathbb{Z}^n$. It then follows that

$$b_j(C(\mathbb{T}^n)) \leq 2^{n-1}(b_0(\mathbb{C}) + b_1(\mathbb{C})) = 2^{n-1}(1+0) = 2^{n-1}$$

for $j = 0, 1$. Moreover, the estimate equality holds. Because $K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$ (cf. [11]), which is also deduced by using the Pimsner-Voiculescu six-term exact sequence repeatedly.

Example 2. Let $\mathbb{T}^n_\Theta$ denote the n-dimensional noncommutative torus, which is the C^*-algebra generated by n unitaries u_j such that $u_j u_k = e^{2\pi i \theta_{j,k}} u_k u_j$ for $1 \leq j, k \leq n$, where $i = \sqrt{-1}$ and $\Theta = (\theta_{j,k})$ is a $n \times n$ skew adjoint matrix over $\mathbb{R}$ of reals so that $-\Theta = \Theta^t$ the transpose of Θ (cf. [1], [11]). The C^*-algebra is regarded as the successive crossed product C^*-algebra of $\mathbb{C}$ by id of $\mathbb{Z}$:

$$\mathbb{T}^n_\Theta \cong \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$$

and by successive actions $\alpha(j)$ for $2 \leq j \leq n$ given by

$$\alpha(j)u_k = \mathrm{Ad}(u_j)u_k = u_j u_k u_j^* = e^{2\pi i\theta_{j,k}} u_k$$

for $1 \leq k \leq j-1$. It then follows that

$$b_j(\mathbb{T}^n_\Theta) \leq 2^{n-1}(b_0(\mathbb{C}) + b_1(\mathbb{C})) = 2^{n-1}(1+0) = 2^{n-1}$$

for $j = 0, 1$. Moreover, the estimate equality holds. b Because $K_j(\mathbb{T}^n_\Theta) \cong \mathbb{Z}^{2^{n-1}}$, which is deduced by using the Pimsner-Voiculescu six-term exact sequence repeatedly. Note that Example 3.1 is just the case where Θ is the zero matrix.

Example 3. Let H_{2n+1} be the discrete Heisenberg nilpotent group of rank $2n+1$, consisting of the following $(n+2) \times (n+2)$ invertible matrices:

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & a & c \\ 0_{n,1} & 1_n & b^t \\ 0 & 0_{1,n} & 1 \end{pmatrix} \in GL_{n+2}(\mathbb{R}) \,|\, a, b \in \mathbb{Z}^n, c \in \mathbb{Z} \right\}$$

where 1_n is the $n \times n$ identity matrix and $0_{j,k}$ is the $j \times k$ zero matrix, and with $a, b \in \mathbb{Z}^n$ as row vectors and b^t the transpose of b. The group H_{2n+1} is viewed as the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ of tuples (c, b, a) identified with the matrices above, where the action α is defined by matrix multiplication as

$$\alpha_a(c, b) = a(c, b)a^{-1} = (c + \sum_{j=1}^{n} a_j b_j, b) \in \mathbb{Z}^{n+1},$$

where $a = (a_1, \cdots, a_n) = (0, 0_n, a)$ and $(c, b) = (c, b_1, \cdots, b_n) = (c, b, 0_n)$, with $0_n = (0, \cdots, 0)$ the zero of $\mathbb{Z}^n$. Then the group C^*-algebra $C^*(H_{2n+1}) = C^*(\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n)$ is regarded as the crossed product C^*-algebra $C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n$, where the action α of the semi-direct product group is extended and identified with that of the crossed product C^*-algebra, by the same symbol as α (also in what follows). Note that each element of an amenable (such as nilpotent or solvable) discrete group Γ is identified with the corresponding unitary under the left regular representation λ on $l^2(\Gamma)$ the Hilbert space of all square summable, complex-valued functions on Γ (cf. [2]). Let e_j $(1 \leq j \leq 2n+1)$ be the canonical basis for $\mathbb{Z}^{n+1}$ and $\mathbb{Z}^n$ in $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ and let $u_j = \lambda_{e_j}$ $(1 \leq j \leq 2n+1)$ be the corresponding unitaries in $C^*(\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n)$. Then we have that

$$\begin{aligned} \alpha_a(u_1) &= \lambda_{\alpha_a(e_1)} = \lambda_{e_1} = u_1, \\ \alpha_a(u_j) &= \lambda_{\alpha_a(e_j)} = \lambda_{a_{j-1}e_1 + e_j} = u_1^{a_{j-1}} u_j \end{aligned}$$

for $2 \leq j \leq n+1$. It then follows that

$$b_j(C^*(H_{2n+1})) \leq 2^{n-1}(b_0(C(\mathbb{T}^{n+1})) + b_1(C(\mathbb{T}^{n+1})) = 2^{n-1}(2^n + 2^n) = 2^{2n}$$

for $j = 0, 1$. In fact, it is computed in [9, Theorem 4.7] that $K_j(C^*(H_{2n+1})) \cong \mathbb{Z}^{2^n(2^n-1)+1}$ for $j = 0, 1$, with $2^n(2^n - 1) + 1 \leq 2^{2n}$ for $n \geq 1$ (cf. [5], [7]).

Theorem 3.1. *Let* G *be a successive semi-direct product of torsion free, finitely generated discrete group, written as* $G = \mathbb{Z}^{n_0} \rtimes_{\alpha(1)} \mathbb{Z}^{n_1} \cdots \rtimes_{\alpha(k)} \mathbb{Z}^{n_k}$ *for some* $n_0, \cdots, n_k \geq 1$, $k \geq 1$. *Let* $C^*(G)$ *be the group* C^**-algebra of* G. *Then* $b_j(C^*(G)) \leq 2^{n_0+n_1+\cdots+n_k-1}$ *for* $j = 0, 1$.

Proof. Note that

$$C^*(G) \cong C^*(\mathbb{Z}^{n_0}) \rtimes_{\alpha(1)} \mathbb{Z}^{n_1} \cdots \rtimes_{\alpha(k)} \mathbb{Z}^{n_k}$$

with $C^*(\mathbb{Z}^{n_0}) \cong C(\mathbb{T}^{n_0})$, where the right hand side above is viewed as an $n_1 + \cdots + n_k$ fold, crossed product C^*-algebra by the successive actions of $\mathbb{Z}$. □

Theorem 3.2. *Let* G *be a torsion free, finitely generated nilpotent discrete group, with* $b(G) = n$. *Then* $b_j(C^*(G)) \leq 2^{n-1}$ *for* $j = 0, 1$.

Proof. It is well known that such a nilpotent discrete group can be written as such a successive semi-direct product as in the theorem above. □

Remark. These theorems above partially answer to a question as given in the Remark of [9, Theorem 4.7]. Note that any torsion free, finitely generated solvable discrete group may be not be written as such a successive semi-direct product as above, in the sense as neither always being split nor being supper-solvable with such a normal series (cf. [4]).

Acknowledgement. The author would like to thank the referee for several critical comments and suggestions for some improvement as in the introduction.

References

[1] B. Blackadar, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

[2] G. K. Pedersen, C^*-*Algebras and their Automorphism Groups*, Academic Press (1979).

[3] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product* C^**-algebras*, J. Operator Theory **4** (1) (1980), 93-118.

[4] Derek J. S. Robinson, *A Course in the Theory of Groups*, Second Edition, Graduate Texts in Math., **80**, Springer (1996).

[5] T. Sudo, *K-theory of continuous fields of quantum tori*, Nihonkai Math. J. **15** (2004), 141-152.

[6] T. Sudo, *K-theory ranks and index for* C^**-algebras*, Ryukyu Math. J. **20** (2007), 43-123.

[7] T. Sudo, *K-theory for the group* C^**-algebras of nilpotent discrete groups*, Cubo A Math. J. **15** (2013), no. 3, 123-132.

[8] T. Sudo, *The Euler characteristic and the Euler-Poincaré formula for* C^**-algebras*, Sci. Math. Japon. **77**, No. 1 (2014), 119-138 :e-2013, 621-640.

[9] T. Sudo, *The K-theory for the* C^**-algebras of nilpotent discrete groups by examples*, Bulletin of the Faculty of Science, University of the Ryukyus, No. 104, (September 2017), 1-40.

[10] J. Tomiyama, *Invitation to* C^**-algebras and topological dynamics*, World Scientific (1987).

[11] N. E. Wegge-Olsen, *K-theory and* C^**-algebras*, Oxford Univ. Press (1993).

CUBO A Mathematical Journal
Vol.21, N$^{\underline{o}}$03, (09–27). December 2019
http://dx.doi.org/10.4067/S0719-06462019000300009

Naturality and definability II

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ABSTRACT

We regard an algebraic construction as a set-theoretically defined map taking structures A to structures B which have A as a distinguished part, in such a way that any isomorphism from A to A' lifts to an isomorphism from B to B'. In general the construction defines B up to isomorphism over A. A construction is *uniformisable* if the set-theoretic definition can be given in a form such that for each A the corresponding B is determined uniquely. A construction is *natural* if restriction from B to its part A always determines a map from the automorphism group of B to that of A which is a split surjective group homomorphism. We prove that there is no transitive model of ZFC (Zermelo-Fraenkel set theory with Choice) in which the uniformisable constructions are exactly the natural ones. We construct a transitive model of ZFC in which every uniformisable construction (with a restriction on the parameters in the formulas defining the construction) is 'weakly' natural. Corollaries are that the construction of algebraic closures of fields and the construction of divisible hulls of abelian groups have no uniformisations definable in ZFC without parameters.

RESUMEN

Consideramos una construcción algebraica como una aplicación conjuntista tomando estructuras A a estructuras B que tienen a A como parte distinguida, de manera tal que cualquier isomorfismo de A a A' se levanta a un isomorfismo de B a B'. En general la construcción define B salvo isomorfismo sobre A. Una construcción es *uniformizable* si la definición conjuntista puede darse de forma tal que para cada A el B correspondiente está determinado únicamente. Una construcción es *natural* si la restricción de B a su parte A siempre determina una aplicación desde el grupo de automorfismos de B al correspondiente de A que es un homomorfismo de grupos sobreyectivo que escinde. Probamos que no existe un modelo transitivo de ZFC (teoría de conjuntos de Zermelo-Fraenkel con Axioma de Elección) en el cual las construcciones uniformizables sean exactamente las naturales. Construimos un modelo transitivo de ZFC en el cual toda construcción uniformizable (con una restricción en los parámetros de las fórmulas definiendo la construcción) es 'débilmente' natural. Como corolarios obtenemos que la construcción de clausuras algebraicas de cuerpos y la construcción de cápsulas divisibles de grupos abelianos no tienen uniformizaciones definibles en ZFC sin parámetros.

Keywords and Phrases: Naturality, uniformisability, transitive models, ZFC set theory

2010 AMS Mathematics Subject Classification: 08A35, 03E35

1 Introduction

In two papers [4] and [6] we noted that in common practice many algebraic constructions are defined only 'up to isomorphism' rather than explicitly. We mentioned some questions raised by this fact, and we gave some partial answers. The present paper provides much fuller answers, though some questions remain open. Our main result, Theorem 5.1, implies at once that there is a transitive model of Zermelo-Fraenkel set theory with Choice (ZFC) in which every construction explicitly definable without parameters is 'weakly natural' (a weakening of the notion of a natural transformation). A corollary is that there are models of ZFC in which some well-known constructions, such as algebraic closure of fields, are not explicitly definable without parameters; some of these consequences were reported in [5]. We also show (Theorem 4.3) that there is no transitive model of ZFC in which the constructions explicitly definable (with parameters) are precisely the natural ones. The main questions left open are to extend Theorem 5.1 to constructions definable with parameters, and to determine whether Theorem 5.1 holds without the word 'weakly'.

Most of this work was done when the second author visited the first at Queen Mary, London University under SERC Visiting Fellowship grant GR/E9/639 in summer 1989, and later when the two authors took part in the Mathematical Logic year at the Mittag-Leffler Institute in Djursholm in September 2000. The first author had made a conjecture relating uniformisability to naturality. The second author proposed the approach of section 4 on the first occasion and the idea behind the proof of Theorem 5.1 on the second. Between 1975 and 2000 the authors (separately or together) had given some six or seven false proofs of versions of Theorem 5.1 or its negation. The authors thank Ian Hodkinson for his invaluable help (while research assistant to Hodges under SERC grant GR/D/33298) in unpicking some of the earlier false proofs. The first author also thanks the second author for his willingness to persist for several decades with these highly elusive problems.

2 Constructions up to isomorphism

To make this paper self-contained, we repeat or paraphrase some definitions from [6].

Definition 2.1. *Let M be a transitive model of ZFC. By a* construction *(in M) we mean a triple $\boldsymbol{C} = \langle \phi_1, \phi_2, \phi_3 \rangle$ where*

(1) $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are formulas of the language of set theory, possibly with parameters from M;

(2) ϕ_1 and ϕ_2 respectively define first-order languages L and L^- in M; every symbol of L^- is a symbol of L, and the symbols of $L \setminus L^-$ include a 1-ary relation symbol P;

(3) the class $\{a : M \models \phi_3(a)\}$ is in M a class of L-structures, called the graph *of $\boldsymbol{C}$;*

(4) if B is in the graph of $\boldsymbol{C}$ then P^B, the set of elements of B satisfying Px, forms the domain of an L^--structure B^- inside B; thus if Q is a relation symbol of L^- then $Q^{B^-} = Q^B \restriction P^B$, and similarly for function symbols; the class of all structures B^- as B ranges over the graph of $\boldsymbol{C}$ is called the domain *of $\boldsymbol{C}$;*

(5) the domain of $\boldsymbol{C}$ is closed under isomorphism; and if A, B are in the graph of $\boldsymbol{C}$ then every isomorphism from A^- onto B^- extends to an isomorphism from A onto B.

A typical example is the construction whose domain is the class of fields, and the structures B in the graph are the algebraic closures of B^-, with B^- picked out by the relation symbol P. The algebraic closure of a field is determined only up to isomorphism over the field; in the terminology below, algebraic closures are 'representable' but not known to be 'uniformisable'. (What we called 'definable' in [6], and 'explicitly definable' in the introduction above, we now call 'uniformisable'; the new term agrees better with the common mathematical use of these words.)

Definition 2.2. *(1) We say that the construction $\boldsymbol{C}$ is X-*representable *(in M) if X is a set in M and all the parameters of ϕ_1, ϕ_2, ϕ_3 lie in X. We say that $\boldsymbol{C}$ is* small *if the domain of $\boldsymbol{C}$ (and hence also its graph) contains only a set of isomorphism types of structures.*

(2) An important special case is where the domain of $\boldsymbol{C}$ contains exactly one isomorphism type of structure; in this case we say $\boldsymbol{C}$ is unitype.

The map $B^- \mapsto B$ on the domain of a construction **C** is in general not single-valued; but by clause (5) it is single-valued up to isomorphism over B^-.

Definition 2.3. *(1) We say that $\boldsymbol{C}$ is* uniformisable *(in M) if its graph can be uniformised, i.e. there is a formula $\phi_4(x, y)$ of set theory (the* uniformising formula*) such that*

> *for each A in the domain of $\boldsymbol{C}$ there is a unique B such that $M \models \phi_4(A, B)$, and this B is an L-structure in the graph of $\boldsymbol{C}$ with $A = B^-$.*

*(2) We say that $\boldsymbol{C}$ is X-*uniformisable *(in M) if there is such a ϕ_4 whose parameters lie in the set X.*

3 Splitting, naturality and weak naturality

Definition 3.1. *Let $\nu : G \to H$ be a surjective group homomorphism.*

(i) A splitting *of ν is a group homomorphism $s : H \to G$ such that νs is the identity on H. We say that ν* splits *if it has a splitting.*

(ii) By a weak splitting *of ν we mean a set-theoretic map $s : H \to G$ such that*

(a) νs *is the identity on* H*;*

(b) The composite map

$$H \xrightarrow{s} G \xrightarrow{nat} G/\mathcal{Z}(G)$$

is a homomorphism, where $\mathcal{Z}(G)$ *is the centre of* G *and nat is the natural homomorphism.*

In particular every splitting is a weak splitting.

(iii) We say that ν weakly splits *if it has a weak splitting.*

Definition 3.2. *Let* $\boldsymbol{C}$ *be a construction. If* B *is in the graph of* $\boldsymbol{C}$ *and* $A = B^-$*, then by (4) in section 2, restriction from* B *to* A *induces a homomorphism* $\nu : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$*, and by (5) this homomorphism is surjective. We say that* $\boldsymbol{C}$ *is* natural *if for every such* B *the homomorphism* ν *splits. We say that* $\boldsymbol{C}$ *is* weakly natural *if for every such* B *the homomorphism* ν *weakly splits.*

Note that if $\mathbf{C}$ is not (weakly) natural, then some structure B in the graph of $\mathbf{C}$ witnesses this, so by restricting $\mathbf{C}$ to the isomorphism type of B we get a unitype construction which is not (weakly) natural.

Example One. The construction of algebraic closures of fields is not weakly natural. The construction of divisible hulls of abelian groups is not weakly natural. Both these facts are proved in [5], using cohomology of finite abelian groups and (for the fields) some Galois theory. So they hold in any model of ZFC.

Example Two. There are constructions that are weakly natural but not natural. The simplest is as follows. The structures B in the graph have six elements a, b, c, d, e, f and the positive diagram

$$Pa, Pb, Rac, Rae, Rbd, Rbf, Scd, Sde, Sef, Sfc.$$

The signature of B consists of the relation symbols P, R, S, and the signature of $A = B^-$ is empty. Then $\mathrm{Aut}(B) = \mathbb{Z}/4\mathbb{Z}$, $\mathrm{Aut}(A) = \mathbb{Z}/2\mathbb{Z}$ and $\nu : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$ is the natural surjection. There is no splitting, because the automorphism of A of order 2 lifts only to automorphisms of B of order 4. But the construction is weakly natural because $\mathrm{Aut}(A)$ is abelian and hence is its own centre.

In [6] we conjectured that there are models of set theory in which each representable construction is uniformisable if and only if it is natural. Section 4 will show that no reasonable version of this conjecture is true. Sections 5 and 6 will show that there are models in which uniformisability implies weak naturality. Section 7 solves some of the problems raised in [4] and [6], and notes some connections with other things in the literature.

4 Uniformisability

Definition 4.1. *A structure B is said to be* rigid *if it has no nontrivial automorphisms. We will say that a construction $\boldsymbol{C}$ is* rigid-based *if for every structure B in the graph of $\boldsymbol{C}$, B^- has no nontrivial automorphisms.*

A rigid-based construction is trivially natural.

Let M be a transitive model of set theory. We will use a device that takes any construction $\mathbf{C}$ in M to a construction $\mathbf{C}^r$, called its *rigidification*. The device exploits the fact that if X is any nonempty set and $TC(X)$ is the transitive closure of X, then the structure $(TC(X), \epsilon)$ is rigid, thanks to the axiom of Foundation.

Suppose B is in the graph of $\mathbf{C}$. Then without affecting any of the relevant isomorphisms, we can assume that none of the elements of B outside P^B lie in $TC(P^B)$. For example we can make a set-theoretic replacement of each element b outside P^B by the ordered pair $\langle b, TC(P^B)\rangle$.

To form $\mathbf{C}^r$, each structure B^- in the domain of $\mathbf{C}$ is replaced by a two-part structure B^{r-}, where the first part is B^- and the second part consists of the set $TC(P^B)$ with a membership relation ε copying that in M. Now the structure B^r is defined to be the amalgam of B and B^{r-}, so that B^{r-} is $(B^r)^-$. Then $\mathbf{C}^r$ is the closure of the class

$$\{B^r : B \text{ in the graph of } \mathbf{C}\}$$

under isomorphism in M. It is clear that $\mathbf{C}^r$ and the map $B \mapsto B^r$ are definable in M using no parameters beyond those in the formulas representing $\mathbf{C}$.

Lemma 4.2. *If $\boldsymbol{C}$ is any construction, then $\boldsymbol{C}^r$ is rigid-based, natural and not small.*

Proof. If B^- is in the domain of $\mathbf{C}$, then B^{r-} is rigid because its set of elements is transitively closed; so $\mathbf{C}^r$ is rigid-based. Naturality follows at once. Since the domain of $\mathbf{C}$ is closed under isomorphism, the relevant transitive closures are arbitrarily large. □

Theorem 4.3. *There is no transitive model M of ZFC in which both the following are true:*

(a) Every rigid-based construction in M is uniformisable.

(b) Every unitype uniformisable construction in M is weakly natural.

In particular there is no transitive model of ZFC in which the natural constructions are exactly the uniformisable ones.

Proof. Suppose M is a counterexample. By Example One in section 3 there are some non-weakly-natural constructions in M. So by restricting to a single isomorphism type we can find a unitype non-weakly-natural construction $\mathbf{C}$ in M. Then $\mathbf{C}^r$ is rigid-based and hence uniformisable by assumption. But we can use the uniformising formula of $\mathbf{C}^r$ to uniformise $\mathbf{C}$ with the same parameters. So by the assumption on M again, $\mathbf{C}$ is weakly natural; contradiction. □

The next result gives some finer information about small constructions.

Theorem 4.4. *Let M be a transitive model of ZFC, Y a set in M and $\bar{c}$ a well-ordering of Y in M. Assume:*

> *In M, if X is any set, then every unitype X-representable rigid-based construction is $X \cup Y$-uniformisable.*

Then

> *In M, every small $\emptyset$-representable construction is $\{\bar{c}\}$-uniformisable,*

and hence there are unitype $\{\bar{c}\}$-uniformisable constructions that are not weakly natural.

Proof. Let γ be the length of $\bar{c}$. Write $\bar{v}$ for the sequence of variables $(v_i : i < \gamma)$. In M we can well-order (definably, with no parameters) the class of pairs $\langle j, \psi \rangle$ where j is an ordinal and $\psi(x, y, z, \bar{v})$ is a formula of set theory. We write H_j for the set of sets hereditarily of cardinality less than $\aleph_j$ in M.

Let $\mathbf{C}$ be a small $\emptyset$-representable construction in M. Then $\mathbf{C}^r$ is an $\emptyset$-representable rigid-based construction. It is not small; but if B is any structure in the graph of $\mathbf{C}$, let $\mathbf{C}_B$ be the construction got from $\mathbf{C}^r$ by restricting the graph to structures isomorphic to B^r. Then $\mathbf{C}_B$ is a unitype and $\{B\}$-representable rigid-based construction, so by assumption it is $\{B\} \cup Y$-uniformisable, say by a formula $\psi_B(-, -, B, \bar{c})$ where $B, \bar{c}$ are the parameters.

By the reflection principle in M there is an ordinal j such that

$$M \models \exists C(C \in \mathbf{C}_B \wedge C^- = B^{r-} \wedge C \text{ is the unique set such that “}H_j \models \psi_B(B^{r-}, C, B, \bar{c})\text{”}).$$

Hence in M there is a first pair $\langle j_B, \psi_B \rangle$, definable from B, such that

$$M \models \exists C(C \in \mathbf{C}_B \wedge C^- = B^{r-} \wedge C \text{ is the unique set such that “}H_{j_B} \models \psi_B(B^{r-}, C, B, \bar{c})\text{”}).$$

Since all of this is uniform in B, it follows that the construction $\mathbf{C}$ is $\{\bar{c}\}$-uniformisable in M by the formula $\phi(x, y, \bar{c})$ which says

$$y = C|L \text{ where } H_{j_x} \models \psi_x(x^{r-}, C, x, \bar{c}).$$

The last clause of the theorem follows by choosing $\mathbf{C}$ suitably, for example using Example One of section 3. □

5 The set theory

Theorem 5.1. *Let M be a countable transitive model of ZFC and GCH, and λ a transfinite cardinal in M. Then there is a forcing extension N of M with the following property. If $\boldsymbol{C}$ is a uniformisable unitype construction defined in N with parameters in M, whose graph contains a structure B in M with B and* $\mathrm{Aut}(B)$ *both of cardinality $\leqslant \lambda$, then $\boldsymbol{C}$ is weakly natural in N.*

The proof of this theorem will occupy this and the next section. The idea is to consider any unitype construction $\mathbf{C}$ whose parameters lie in M, and introduce a very homogeneous generic structure $B^\star$ into the graph of $\mathbf{C}$. The homogeneity of $B^\star$ will make it impossible to uniformise without some form of naturality. This is a novel argument. At present we can apply it simultaneously for all unitype constructions satisfying the stated restriction to a fixed λ. We expect that a similar proof by class forcing will eliminate this restriction, but this is delayed.

Our notation for forcing mainly follows Jech [7]. We define $\mathbb{P}$ to be the notion of forcing in M that consists of all partial maps from $\lambda^{++} \times \lambda^{++} \times \lambda^{++}$ to 2 which have domain of cardinality at most λ. We abbreviate $\lambda^{++} \times \lambda^{++} \times \lambda^{++}$ to $(\lambda^{++})^3$.

Lemma 5.2. *The notion of forcing $\mathbb{P}$ is λ^+-closed and satisfies the λ^{++}-chain condition.* $\square$

For definiteness we take $M^{\mathbb{P}}$, the class of $\mathbb{P}$-names, to be the smallest class of elements of M such that if X is any subset of $M^{\mathbb{P}}$ and for each $y \in X$, I_y is a non-empty antichain in $\mathbb{P}$, then $\{(p, y) : y \in X, p \in I_y\}$ is a $\mathbb{P}$-name in $M^{\mathbb{P}}$; the *domain* of this $\mathbb{P}$-name is X. Then for every $\mathbb{P}$-generic G the interpretation of the name $\dot{x} = \{(p, y) : y \in X, p \in I_y\}$ is the set $\dot{x}[G] = \{y[G] : \exists p \in G, (p, y) \in \dot{x}\}$. We write $\dot{x}$ for $\mathbb{P}$-names, and $\check{x}$ for the canonical $\mathbb{P}$-name of the element $x \in M$.

We take a $\mathbb{P}$-generic set G over M and we put $N = M[G]$. We will prove Theorem 5.1 for this N. In M we fix a unitype construction $\mathbf{C}$, a structure B in the graph of $\mathbf{C}$, and a uniformising formula $\phi(x, y)$. We write A for B^-.

Definition 5.3. *In M we define two homomorphisms, I from the group of permutations of $(\lambda^{++})^3$ to the group of automorphisms of $\mathbb{P}$ as ordered set; and J from the group of automorphisms of $\mathbb{P}$ to the group of permutations of $M^{\mathbb{P}}$. Thus:*

(a) Let α be a permutation of $(\lambda^{++})^3$ and $p \in \mathbb{P}$. Then we define $\alpha^I(p)$ by

$$(\alpha^I(p))(\alpha(i, j, k)) = p(i, j, k) \text{ for all } i, j, k < \lambda^{++}.$$

(b) Let γ be an automorphism of $\mathbb{P}$. Then γ^J is defined on $M^{\mathbb{P}}$ by induction on rank:

$$\gamma^J \dot{x} = \{(\gamma p, \gamma^J \dot{y}) : (p, \dot{y}) \in \dot{x}\}.$$

The maps I and J are clearly homomorphisms.

Lemma 5.4. *Let γ be an automorphism of $\mathbb{P}$ which is in M. Then:*

(a) If G is a $\mathbb{P}$-generic set over M, then γG is $\mathbb{P}$-generic over M, and for every $\mathbb{P}$-name $\dot{x}$ we have

$$(\gamma^J \dot{x})[\gamma G] = \dot{x}[G]$$

(where $\gamma G = \{\gamma p : p \in G\}$).

(b) If $\dot{x}$ is a $\mathbb{P}$-name then $(\alpha) \Rightarrow (\beta)$, where we write

(α): for every pair $(p, \dot{y})$, $(p, \dot{y}) \in \dot{x}$ if and only if $(\gamma p, \gamma^J \dot{y}) \in \dot{x}$.

(β): $\gamma^J(\dot{x}) = \dot{x}$.

Proof. . For (a), by induction on the rank of $\dot{x}$,

$$\begin{aligned} \dot{x}[G] &= \{\dot{y}[G] : \exists p \in G, (p, \dot{y}) \in \dot{x}\} \\ &= \{\gamma^J \dot{y}[\gamma G] : \exists \gamma p \in \gamma G, (\gamma p, \gamma^J \dot{y}) \in \gamma^J \dot{x}\} \\ &= \{\dot{z}[\gamma G] : \exists q \in \gamma G, (q, \dot{z}) \in \gamma^J \dot{x}\} \\ &= (\gamma^J \dot{x})[\gamma G]. \end{aligned}$$

Part (b) is immediate from the definition of γ^J. □

Since G is $\mathbb{P}$-generic, $\bigcup G$ is a total map from $(\lambda^{++})^3$ to 2. For each $i < \lambda^{++}$ and $j < \lambda^{++}$, we define $a_{ij} = \{k < \lambda^{++} : \bigcup G(i, j, k) = 1\}$ and $a'_i = \{a_{ij} : j < \lambda^+\}$, so that a'_i is a set of λ^{++} independently generic subsets of λ^{++}. If a and b are (in N) sets of subsets of λ^{++}, we put $a \equiv b$ iff the symmetric difference of a and b has cardinality $\leqslant \lambda$. We write a_i for the equivalence class $(a'_i)^{\equiv}$. The $\mathbb{P}$-names $\dot{a}_{ij}, \dot{a}'_i, \dot{a}_i$ can be chosen in $M^{\mathbb{P}}$ independently of the choice of G.

Consider again the structures A and B in M. Without loss we can suppose that $\mathrm{dom}(A)$ is an initial segment of λ. In $M[G]$ there is a map e which takes each element i of A to the corresponding set $a_i = \dot{a}_i[G]$; by means of e we can define a copy $A^\star$ of A whose elements are the sets a_i ($i \in \mathrm{dom}(A)$).

Lemma 5.5. *The $\mathbb{P}$-names $\dot{A}^\star, \dot{e}$ can be chosen to be independent of the choice of G. Also we can take the boolean names $\dot{a}_{ij}$ and $\dot{a}'_i$ to be*

$$\begin{aligned} \dot{a}_{ij} &= \{(((i, j, k) \mapsto 1), \check{k}) : (i, j, k) \in (\lambda^{++})^3\}, \\ \dot{a}'_i &= \{(1, \dot{a}_{ij}) : j < \lambda^{++}\}. \end{aligned}$$

□

A notion of forcing $\mathbb{Q}$ in M is said to be *homogeneous* if for any two conditions $p, q \in \mathbb{P}$ there is an automorphism α of $\mathbb{Q}$ in M such that p and αq are compatible.

Lemma 5.6. $\mathbb{P}$ *is homogeneous.* □

By this lemma and the fact that A, B and the parameters of the uniformising formula ϕ lie in M, the statement "ϕ uniformises a construction on the class of structures isomorphic to A, which takes A to B" is true in N independently of the choice of G. In particular there are $\mathbb{P}$-names $\dot{B}^\star, \dot{\varepsilon}$ such that

$$\begin{aligned} &||\dot{B} \text{ is the unique structure such that } \phi(\dot{A}^\star, \dot{B}^*) \text{ holds,} \\ &\dot{e} : \check{A} \to \dot{A}^* \text{ is the isomorphism such that } \dot{e}(\check{\imath}) = \dot{a}_i \text{ for} \\ &\text{each } i \in \mathrm{dom}(\check{A}), \text{ and } \dot{\varepsilon} : \check{B} \to \dot{B}^* \text{ is an isomorphism} \\ &\text{which extends } \dot{e}||_{\mathbb{P}} = 1. \end{aligned} \tag{5.1}$$

Lemma 5.7. *Let G be $\mathbb{P}$-generic over M. Then:*

(a) $\mathrm{Aut}(A)^M = \mathrm{Aut}(A)^{M[G]}$.

(b) $\mathrm{Aut}(B)^M = \mathrm{Aut}(B)^{M[G]}$.

(c) The set of maps from $\mathrm{Aut}(A)$ *to* $\mathrm{Aut}(B)$ *is the same in M as it is in $M[G]$.*

Proof. . $\mathbb{P}$ is λ^+-closed by Lemma 5.2. Hence no new permutations of A or B are added since $|A| \leq |B| \leq \lambda$ in M; this proves (a), (b). Likewise (c) holds since $|\mathrm{Aut}(A)| \leq |\mathrm{Aut}(B)| \leq \lambda$ in M. □

We regard $\mathrm{Aut}(A)$ as a permutation group on λ^{++} by letting it fix all the elements of λ^{++} which are not in $\mathrm{dom}(A)$.

We write Π for the cartesian product $\prod_{\lambda^{++}} \mathrm{Aut}(A)$ of λ^{++} copies of the group $\mathrm{Aut}(A)$, in the sense of M. Then each element α of Π can be regarded as a map $\alpha : \lambda^{++} \to \mathrm{Aut}(A)$ in M. We write $\mathcal{N}$ for the subgroup of Π consisting of those α such that for some finite sequence of ordinals

$$0 = i_0 < i_1 < \ldots < i_n < i_{n+1} = \lambda^{++}$$

the map α is constant on each interval $[i_k, i_{k+1})$ $(0 \leqslant k \leqslant n)$. The elements of $\mathcal{N}$ will be called *neat maps*. We write π for the map from $\mathcal{N}$ to $\mathrm{Aut}(A)$ which takes each neat map to its eventual value. We write $\mathcal{N}^-$ for the set of all neat maps α with $\pi(\alpha) = 1$. For each ordinal $i < \lambda^{++}$ we write $\mathcal{N}_i$ for the set of neat maps α such that $\alpha(j) = 1$ for all $j < i$. We write $\mathcal{N}_i^-$ for $\mathcal{N}^- \cap \mathcal{N}_i$.

Lemma 5.8. *As a subset of the group* Π, $\mathcal{N}$ *forms a group with subgroups* $\mathcal{N}^-$, $\mathcal{N}_i$ $(i < \lambda^{++})$. *The map* $\pi : \mathcal{N} \to \mathrm{Aut}(A)$ *is a surjective group homomorphism.*

Proof. . From the definitions. $\square$

The neat map $\alpha \in \Pi$ determines a permutation α^K of the set $(\lambda^{++})^3$ by

$$\alpha^K(i,j,k) = (\alpha(j)(i), j, k).$$

Hence α induces an automorphism α^{KIJ} of $M^{\mathbb{P}}$.

Lemma 5.9. *Suppose* $\alpha : \lambda^{++} \to \mathrm{Aut}(A)$ *is neat. Then* α^{KIJ} *setwise fixes the set* $\{\dot{a}_i : i \in \mathrm{dom}(A)\}$ *of canonical names of the elements of* $\dot{A}^*[G]$, *and it acts on this set in the way induced by* $\pi(\alpha)$ *and the map* $i \mapsto \dot{a}_i$. *Thus* $\alpha^{KIJ}(\dot{a}_i) = \dot{a}_{\pi(\alpha)(i)}$.

Proof. . We use the boolean names in Lemma 5.5. For $\dot{a}_{ij}$,

$$\begin{aligned} \alpha^{KIJ}\dot{a}_{ij} &= \{(\alpha^{KI}((i,j,k) \mapsto 1), \alpha^{KIJ}(\check{k})) : (i,j,k) \in (\lambda^{++})^3\} \\ &= \{((\alpha^K(i,j,k) \mapsto 1), \check{k}) : (i,j,k) \in (\lambda^{++})^3\} \\ &= \{(\alpha(j)(i), j, k) \mapsto 1), \check{k}) : (i,j,k) \in (\lambda^{++})^3\} \\ &= \dot{a}_{\alpha(j)i,j}. \end{aligned}$$

Then for $\dot{a}'_i$,

$$\begin{aligned} \alpha^{KIJ}\dot{a}'_i &= \{(\alpha^{KI}(1, \dot{a}_{ij}) : j < \lambda^{++}\} \\ &= \{(1, \dot{a}_{\alpha(j)i,j}) : j < \lambda^{++}\}. \end{aligned}$$

We claim that with boolean value 1, $\{(1, \dot{a}_{\alpha(j)i,j}) : j < \lambda^{++}\} \equiv \dot{a}'_{\pi\alpha(i)}$. For this, first note that

$$\dot{a}'_{\pi\alpha(i)} = \{(1, \dot{a}_{\pi(\alpha)i,j}) : j < \lambda^{++}\}.$$

Since α is neat, there is $j_0 < \lambda^{++}$ such that $\alpha(j) = \pi\alpha$ whenever $j \geqslant j_0$. So for any generic G, $\{(1, \dot{a}_{\alpha(j)i,j}) : j < \lambda^{++}\}[G]$ and $\dot{a}'_{\pi\alpha(i)}[G]$ differ in at most $|j_0|$ elements. The lemma follows. $\square$

Lemma 5.10. *For each element* i *of* A *and each neat map* α, $\dot{a}_{\pi(\alpha)(i)}[\alpha G] = \dot{a}_i[G]$. *In particular* $\dot{A}^\star[\alpha G] = \dot{A}^\star[G]$.

Proof. By Lemma 5.9, $\dot{a}_{\pi(\alpha)(i)}[\alpha G] = (\alpha\dot{a}_i)[\alpha G]$. Then by Lemma 5.4 and the fact that $\alpha\dot{a}_i$ lies in $M^{\mathbb{P}}$,

$$(\alpha\dot{a}_i)[\alpha G] = \dot{a}_i[G].$$

This shows that $\dot{A}^\star[\alpha G] = \dot{A}^\star[G]$. $\square$

We write $\dot{\varepsilon}^{-1}$ for a $\mathbb{P}$-name such that $\dot{\varepsilon}^{-1}[G] = (\dot{\varepsilon}[G])^{-1}$ for all generic G.

Lemma 5.11. *Suppose α is a neat map and G is $\mathbb{P}$-generic over M. Then $\dot{B}^*[\alpha^{-1}G] = \dot{B}^*[G]$, and the map $(\dot{\varepsilon}^{-1} \circ \alpha\dot{\varepsilon})[G]$ is an automorphism of B which extends $\pi(\alpha)$.*

Proof. Since $M[\alpha^{-1}G] = M[G]$ and $\dot{A}^*[\alpha^{-1}G] = \dot{A}^*[G]$, statement (5.1) (before Lemma 5.7) tells us that $\dot{e}[\alpha^{-1}G](i) = \dot{a}_i[\alpha^{-1}G]$ for each $i \in \mathrm{dom}(A)$, and that $\dot{B}^*[\alpha^{-1}G] = \dot{B}^*[G]$ and $\dot{\varepsilon}[G]^{-1} \circ \dot{\varepsilon}[\alpha^{-1}G]$ extends $\dot{e}[G]^{-1} \circ \dot{e}[\alpha^{-1}G]$. Now using Lemma 5.10,

$$\dot{e}[G]^{-1} \circ \dot{e}[\alpha^{-1}G](i) = \dot{e}[G]^{-1}(\dot{a}_i[\alpha^{-1}G])$$

$$= \dot{e}[G]^{-1}(\dot{a}_{\pi(\alpha)(i)}[G]) = \pi(\alpha)(i).$$

□

Lemma 5.12. *For every neat map α and all $p \in \mathbb{P}$ there are $p' \leqslant p$ and $g \in \mathrm{Aut}B$ extending $\pi(\alpha)$, such that*

$$p' \Vdash_{\mathbb{P}} \dot{\varepsilon}^{-1} \circ \alpha(\dot{\varepsilon}) = \check{g}.$$

Proof. Let f be $\pi(\alpha)$. By Lemma 5.11 we have

$$\begin{aligned} 1 \quad &= ||\dot{\varepsilon}^{-1} \circ \alpha\dot{\varepsilon} \text{ is an automorphism of } B \text{ extending } \check{f}||_{\mathbb{P}} \\ &= \textstyle\sum_g ||\dot{\varepsilon}^{-1} \circ \alpha\dot{\varepsilon} = \check{g}||_{\mathbb{P}} \end{aligned}$$

where g ranges over the automorphisms of B that extend f. □

Definition 5.13. *(a) For each $p \in \mathbb{P}$ and each $i < \lambda^{++}$, define $t_{p,i}$ to be the set of all pairs (f, g), with $f \in \mathrm{Aut}(A)$ and $g \in \mathrm{Aut}(B)$, such that for some $\alpha \in \mathcal{N}_i$, $\pi(\alpha) = f$ and*

$$p \Vdash_{\mathbb{P}} \dot{\varepsilon}^{-1} \circ \alpha\dot{\varepsilon} = \check{g}.$$

(b) Clearly if $p' \leqslant p$ then $t_{p',i} \supseteq t_{p,i}$. The number of possible values for f and g is $\leqslant \lambda$ by choice of λ, and $\mathbb{P}$ is λ^+-closed; so there is p_i such that for all $p' \leqslant p_i$,

$$t_{p',i} = t_{p_i,i}.$$

We fix a choice of p_i for each i, and we write t_i for the resulting value $t_{p_i,i}$.

(c) For each i and each (f, g) in t_i we choose α in $\mathcal{N}_i$ with $\pi(\alpha) = f$ so that

$$p_i \Vdash_{\mathbb{P}} \dot{\varepsilon}^{-1} \circ \alpha\dot{\varepsilon} = \check{g}.$$

We write $\alpha^i_{f,g}$ for this α.

Lemma 5.14. *For each $i < \lambda^{++}$, t_i is a subset of* $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ *such that*

(a) for each (f, g) in t_i, $g|A = f$;

(b) for each f in $\mathrm{Aut}(A)$ *there is g with (f, g) in t_i.*

(So $t_i(-, -)$ is a first attempt at a lifting of the restriction map from $\mathrm{Aut}(B)$ *to* $\mathrm{Aut}(A)$.*)*

Proof. By Lemma 5.12 and the surjectivity of π. □

Lemma 5.15. *There is a stationary subset S of λ^{++} such that:*

(a) for each $i \in S$ and $j < i$, the domain of p_j is a subset of $i \times i \times i$;

(b) for each $i \in S$ and $j < i$, every map $\alpha^j_{f,g} : \lambda^{++} \to$ $\mathrm{Aut}(A)$ *is constant on $[i, \lambda^{++})$;*

(c) for all $i, j \in S$, $t_i = t_j$;

(d) there is a condition $p^\star \in \mathbb{P}$ such that for all $i \in S$, $p_i \restriction (i \times i \times i) = p^\star$.

Proof. First, there is a club $C \subseteq \lambda^{++}$ on which (a) and (b) hold. Let S_η be $\{\delta < \lambda^{++} : \mathrm{cf}(\delta) = \lambda^+\}$. Clearly $S_\nu = S_\eta \cap C$ is stationary; and for each $i \in S_\nu$, $p_i \restriction (i \times i \times i)$ has domain $\subseteq j \times j \times j$ for some $j = j_i < i$. Then by Fődor's lemma there is a stationary subset S of S_ν on which (c) and (d) hold. □

6 The weak lifting

Continuing Section 5, we use the notation S, $p^\star$ from Lemma 5.15. We write t for the constant value of t_i ($i \in S$) from clause (c) of Lemma 5.15, and t^- for the set of all g such that $(1, g) \in t$. We write $\nu : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$ for the restriction map. If X is a subset of $\mathrm{Aut}(B)$, we write $\langle X \rangle$ for the subgroup of $\mathrm{Aut}(B)$ generated by X.

Lemma 6.1. *The relation t is a subset of* $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ *that projects onto* $\mathrm{Aut}(A)$, *and if (f, g) is in t then $\nu(g) = f$.*

Proof. This repeats Lemma 5.14 (a) and (b). □

Lemma 6.2. *If (f_1, g_1) and (f_2, g_2) are both in t then $(f_1 f_2, g_1 g_2)$ is in t.*

Proof. Take any $i, j \in S$ with $i < j$. Put $\alpha_1 = \alpha^j_{f_1,g_1}$, $\alpha_2 = \alpha^i_{f_2,g_2}$ and $\alpha_3 = \alpha_1\alpha_2$. Note that $\alpha_1\alpha_2$ is in $\mathcal{N}_i$ since $i < j$.

Trivially we have

$$p_j \Vdash \dot{\varepsilon}^{-1} \circ \alpha_3(\dot{\varepsilon}) = \dot{\varepsilon}^{-1} \circ \alpha_1(\dot{\varepsilon}) \circ (\alpha_1(\dot{\varepsilon}))^{-1} \circ \alpha_3(\dot{\varepsilon})$$

and by assumption

$$p_j \Vdash \dot{\varepsilon}^{-1} \circ \alpha_1(\dot{\varepsilon}) = \check{g}_1.$$

So

$$p_j \Vdash \dot{\varepsilon}^{-1} \circ \alpha_3(\dot{\varepsilon}) = \check{g}_1 \circ (\alpha_1(\dot{\varepsilon}))^{-1} \circ \alpha_1(\alpha_2\dot{\varepsilon}).$$

Also by assumption

$$p_i \Vdash \dot{\varepsilon}^{-1} \circ \alpha_2(\dot{\varepsilon}) = \check{g}_2.$$

Acting on this last formula by α_1 gives

$$\alpha_1 p_i \Vdash \alpha_1\dot{\varepsilon}^{-1} \circ \alpha_1\alpha_2\dot{\varepsilon} = \alpha_1\check{g}_2.$$

Now $\alpha_1\check{g}_2 = \check{g}_2$. Also $\alpha_1 p_i = p_i$ since the support of p_i lies entirely below j (by Lemma 5.15(a)), and $\alpha_1 = \alpha^j_{f_1,g_1}$ is the identity in this region since it lies in $\mathcal{N}_j$. So we have shown that

$$p_i \Vdash \alpha_1\dot{\varepsilon}^{-1} \circ \alpha_1\alpha_2\dot{\varepsilon} = \check{g}_2.$$

Now we note that $p_i \cup p_j$ is a condition in $\mathbb{P}$, by (a), (d) of Lemma 5.15. Hence we have that

$$p_i \cup p_j \Vdash \dot{\varepsilon}^{-1} \circ \alpha_3\dot{\varepsilon} = \check{g}_1\check{g}_2.$$

Since α_3 is in $\mathcal{N}_i$, this shows that

$$(f_1f_2, g_1g_2) \in t_{p_i \cup p_j, i}.$$

Then by the maximality property of p_i,

$$(f_1f_2, g_1g_2) \in t_{p_i,i}$$

so that (f_1f_2, g_1g_2) is in t.

□

Corollary 6.3. *If (f, g_1) and (f, g_2) are in t then $g_1g_2^{-1}$ is in $\langle t^- \rangle$.*

Proof. By Lemma 6.1 there is some $g' \in \mathrm{Aut}(B)$ such that (f^{-1}, g') is in t. Then by Lemma 6.2, $(1, g_1 g')$ and $(1, g_2 g')$ are in t and so $g_1 g'$, $g_2 g'$ are in t^-. Hence the element

$$g_1 g_2^{-1} = (g_1 g')(g_2 g')^{-1}$$

lies in $\langle t^- \rangle$.

$\square$

Lemma 6.4. *Every element of* t^- *is central in* $\mathrm{Aut}(B)$*.*

Proof. Suppose $g_2 \in t^-$, so that $(1, g_2) \in t$. Consider $(f_1, g_2) \in t$, and apply the notation of the proof of Lemma 6.2 with $f_2 = 1$. In that notation, α_1 is the identity below j and α_2 is the identity below i (since $i, j \in S$). But also g_2 lies in t^-, so α_2 is the identity on $[j, \lambda^+)$. In particular α_1 commutes with α_2.

As in the proof of Lemma 6.2 we have

$$p_i \Vdash \dot{\varepsilon}^{-1} \circ \alpha_3 \dot{\varepsilon} = \dot{\varepsilon}^{-1} \circ \alpha_2 \dot{\varepsilon} \circ \alpha_2 \dot{\varepsilon}^{-1} \circ \alpha_3 \dot{\varepsilon}.$$

As before, we have that

$$p_i \Vdash \dot{\varepsilon}^{-1} \circ \alpha_2 \dot{\varepsilon} = \check{g}_2$$

and

$$\alpha_2 p_j \Vdash \alpha_2 \dot{\varepsilon}^{-1} \circ \alpha_2 \alpha_1 \dot{\varepsilon} = \alpha_2 \check{g}_1.$$

Now the support of p_j lies below i or within $[j, \lambda^+) \times \mathrm{dom} A$, and α_2 is the identity in both these regions, and so $\alpha_2(p_j) = p_j$. Thus, since α_1 commutes with α_2,

$$p_j \Vdash \alpha_2 \dot{\varepsilon}^{-1} \circ \alpha_3 \dot{\varepsilon} = \check{g}_1.$$

So as before,

$$p_i \cup p_j \Vdash \dot{\varepsilon}^{-1} \circ \alpha_3 \dot{\varepsilon} = \check{g}_2 \check{g}_1.$$

Recalling that in the proof of Lemma 6.2 we showed that

$$p_i \cup p_j \Vdash \dot{\varepsilon}^{-1} \circ \alpha_3(\dot{\varepsilon}) = \check{g}_1 \check{g}_2,$$

we deduce that

$$p_i \cup p_j \Vdash \check{g}_1 \check{g}_2 = \check{g}_2 \check{g}_1.$$

But the equation $g_1 g_2 = g_2 g_1$ is about the ground model, and hence it is true.

$\square$

Now in M choose a map $s : \mathrm{Aut}(A) \to \mathrm{Aut}(B)$ so that for each $f \in \mathrm{Aut}(A)$, $s(f)$ is some g with $(f, g) \in t$. This is possible by Lemma 6.1.

Lemma 6.5. *In M the map s is a weak splitting of $\nu : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$.*

Proof. Trivially νs is the identity on $\mathrm{Aut}(A)$. Write $s' : \mathrm{Aut}(A) \to \mathcal{Z}(\mathrm{Aut}(B))$ for the composite of s and $nat : \mathrm{Aut}(B) \to \mathcal{Z}(\mathrm{Aut}(B))$. We show that s' is a homomorphism as follows. Suppose $f_1 f_2 = f_3$ in $\mathrm{Aut}(A)$. Put $g_i = s(f_i)$ for each i $(1 \leqslant i \leqslant 3)$. Then by Lemma 6.2, $(f_3, g_1 g_2)$ is in t, so by Corollary 6.3 and Lemma 6.4, $g_1 g_2 g_3^{-1}$ is in $\langle t^- \rangle \subseteq \mathcal{Z}(\mathrm{Aut}(B))$. Then

$$\begin{aligned} s'(f_1)\sigma'(f_2) &= g_1 \mathcal{Z}(\mathrm{Aut}(B)).g_2 \mathcal{Z}(\mathrm{Aut}(B)) \\ &= g_1 g_2.\mathcal{Z}(\mathrm{Aut}(B)) \\ &= g_3 \mathcal{Z}(\mathrm{Aut}(B)) = s'(f_3) \end{aligned}$$

as required. □

This completes the proof of Theorem 5.1. □

7 Answers to questions

The results above answer most of the problems stated in [6]. In that paper we showed:

> **Theorem 3 of [6]** If $\mathbf{C}$ is a small natural construction in a model of ZFC, then $\mathbf{C}$ is uniformisable with parameters.

We asked (Problem A) whether it is possible to remove the restriction that $\mathbf{C}$ is small. The answer is No:

Theorem 7.1. *There is a transitive model of ZFC in which some $\emptyset$-representable construction is natural but not uniformisable (even with parameters).*

Proof. Let N be the model of Theorem 5.1. Let $\mathbf{C}$ be some construction $\emptyset$-representable in N which is not weakly natural (cf. Example One in section 3). Then by Theorem 5.1, $\mathbf{C}$ is not uniformisable. The rigidifying construction $\mathbf{C}^r$ of section 3 is $\emptyset$-representable, natural and not uniformisable. □

Problem B asked whether in Theorem 3 of [6] the formula defining $\mathbf{C}$ can be chosen so that it has only the same parameters as the formulas chosen to represent $\mathbf{C}$. The answer is No:

Theorem 7.2. *There is a transitive model N of ZFC with the following property:*

For every set Y there are a set X and a unitype rigid-based (hence small natural) X-representable construction that is not $X \cup Y$-uniformisable.

Proof. Take N to be the model given by Theorem 5.1. Let Y be any set in N. If N and Y are not as stated above, then for every set X and every unitype rigid-based X-representable construction in N, X is $X \cup Y$-uniformisable. So the hypothesis of Theorem 4.4 holds, and by that theorem there is in N a small $\{\bar{c}\}$-uniformisable construction that is not weakly natural. But this contradicts the choice of N. □

Problem C asked whether there are transitive models of ZFC in which every uniformisable construction is natural. Theorem 5.1 is the best answer we have for this; the problem remains open.

In [4] one of us asked whether there can be models of ZFC in which the algebraic closure construction on fields is not uniformisable.

Theorem 7.3. *There are transitive models of ZFC in which:*

(a) no formula without parameters defines for each field a specific algebraic closure for that field, and

(b) no formula without parameters defines for each abelian group a specific divisible hull of that group.

Proof. Let the model N be as in Theorem 5.1. In N the constructions of Example One in section 3 are not uniformisable, since they are not weakly natural. So these two examples prove (a) and (b) respectively. □

We close with some remarks on related notions in other papers.

One result in [4] was that there is no primitive recursive set function which takes each field to an algebraic closure of that field. This is an absolute result which applies to every transitive model of ZFC, and so it is not strictly comparable with the consistency results proved above. In this context we note that Garvin Melles showed [8] that there is no "recursive set-function" (he gives his own definition for this notion) which finds a representative for each isomorphism type of countable torsion-free abelian group.

The paper [1] of Adámek et al. gives a simple universal algebraic sufficient condition for injective hull constructions not to be natural, and notes that two of their examples are also in

[6]. The comparison between our notions and theirs is a little tricky. For both Adámek et al. and us, 'natural' is as in 'natural transformation' in the categorical sense. But we work in different categories. In this paper and [6], the relevant morphisms are isomorphisms; but for [1] they include embeddings. Hence the notion of naturality in [1] is stricter than ours. For example their condition implies that the MacNeille completion of posets, which embeds every poset in a lattice, is not natural. But it is natural in our sense, since isomorphisms between posets lift functorially to isomorphisms between their MacNeille completions. In fact this is clear from the standard definition of MacNeille completions ([2] p. 40ff), which also provides a uniformisation of this construction in any model of ZFC. It seems very unlikely that the condition in [1] adapts to give a sufficient condition for failure of weak naturality in the sense above.

In a related context Harvey Friedman [3] used the term 'naturalness' in a weaker sense than ours.

References

[1] Jiří Adámek, Horst Herrlich, Jiří Rosický and Walter Tholen, 'Injective hulls are not natural', *Algebra Universalis* 48 (2002) 379–388.

[2] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge 1990.

[3] H. Friedman, 'On the naturalness of definable operations', *Houston J. Math.* 5 (1979) 325–330.

[4] W. Hodges, 'On the effectivity of some field constructions', *Proc. London Math. Soc.* (3) 32 (1976) 133–162.

[5] W. Hodges, 'Definability and automorphism groups', in Proceedings of International Congress in Logic, Methodology and Philosophy of Science, Oviedo 2003, ed. Petr Hájek et al., King's College Publications, London 2005, pp. 107–120; ISBN 1-904987-21-4.

[6] W. Hodges and S. Shelah, 'Naturality and definability I', *J. London Math. Soc.* 33 (1986) 1–12.

[7] T. Jech, *Set theory* (Academic Press, New York, 1978).

[8] G. Melles, 'Classification theory and generalized recursive functions', D.Phil. dissertation, University of California at Irvine, 1989.

CUBO A Mathematical Journal
Vol.21, N^{o}03, (29–38). December 2019
`http://dx.doi.org/10.4067/S0719-06462019000300029`

Ostrowski-Sugeno fuzzy inequalities

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ABSTRACT

We present Ostrowski-Sugeno fuzzy type inequalities. These are Ostrowski-like inequalities in the context of Sugeno fuzzy integral and its special properties are investigated. Tight upper bounds to the deviation of a function from its Sugeno-fuzzy averages are given. This work is greatly inspired by [3] and [1].

RESUMEN

Presentamos desigualdades de Ostrowski-Sugeno de tipo fuzzy. Estas son desigualdades de tipo Ostrowski en el contexto de integrales fuzzy de Sugeno y se investigan sus propiedades especiales. Se entregan cotas superiores ajustadas para la desviación de una función de sus promedios fuzzy de Sugeno. Este trabajo está inspirado principalmente por [3] y [1].

Keywords and Phrases: Sugeno fuzzy, integral, function fuzzy average, deviation from fuzzy mean, fuzzy Ostrowski inequality.

2010 AMS Mathematics Subject Classification: Primary: 26D07, 26D10, 26D15, 41A44, Secondary: 26A24, 26D20, 28A25.

1 Introduction

The famous Ostrowski ([3]) inequality motivates this work and has as follows:

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\left\|f'\right\|_\infty,$$

where $f \in C'([a,b])$, $x \in [a,b]$, and it is a sharp inequality. One can easily notice that

$$\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)=\frac{(x-a)^2+(b-x)^2}{2(b-a)}.$$

Another motivation is author's article [1].

First we give a survey about Sugeno fuzzy integral and its basic properties. Then we derive a series of Ostrowski-like inequalities to all directions in the context of Sugeno integral and its basic important particular properties. We also give applications to special cases of our problem we deal with.

2 Background

In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented.

Definition 2.1. *(Fuzzy measure [5, 7]) Let* Σ *be a* σ*-algebra of subsets of* X*, and let* $\mu : \Sigma \to [0,+\infty]$ *be a non-negative extended real-valued set function. We say that* μ *is a fuzzy measure iff:*

(1) $\mu(\varnothing) = 0$,

(2) $E, F \in \Sigma : E \subseteq F$ *imply* $\mu(E) \leq \mu(F)$ *(monotonicity),*

(3) $E_n \in \Sigma$ *(*$n \in \mathbb{N}$*),* $E_1 \subset E_2 \subset \ldots$*, imply* $\lim\limits_{n\to\infty}\mu(E_n) = \mu(\cup_{n=1}^\infty E_n)$ *(continuity from below);*

(4) $E_n \in \Sigma$ *(*$n \in \mathbb{N}$*),* $E_1 \supset E_2 \supset \ldots$*,* $\mu(E_1) < \infty$*, imply* $\lim\limits_{n\to\infty}\mu(E_n) = \mu(\cap_{n=1}^\infty E_n)$ *(continuity from above).*

Let (X, Σ, μ) be a fuzzy measure space and f be a non-negative real-valued function on X. We denote by $\mathcal{F}_+$ the set of all non-negative real valued measurable functions, and by $L_\alpha f$ the set: $L_\alpha f := \{x \in X : f(x) \geq \alpha\}$, the α-level of f for $\alpha \geq 0$.

Definition 2.2. *Let* (X, Σ, μ) *be a fuzzy measure space. If* $f \in \mathcal{F}_+$ *and* $A \in \Sigma$*, then the Sugeno integral (fuzzy integral) [6] of* f *on* A *with respect to the fuzzy measure* μ *is defined by*

$$(S)\int_A f d\mu := \vee_{\alpha\geq 0}\left(\alpha \wedge \mu(A \cap L_\alpha f)\right), \tag{1}$$

where $\vee$ *and* $\wedge$ *denote the sup and inf on* $[0,\infty]$*, respectively.*

The basic properties of Sugeno integral follow:

Theorem 2.3. *([4, 7]) Let (X, Σ, μ) be a fuzzy measure space with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. Then*

1) $(S)\int_A f d\mu \le \mu(A)$;

2) $(S)\int_A k d\mu = k \wedge \mu(A)$ *for a non-negative constant* k*;*

3) if $f \le g$ *on* A*, then* $(S)\int_A f d\mu \le (S)\int_A g d\mu$;

4) if $A \subset B$*, then* $(S)\int_A f d\mu \le (S)\int_B f d\mu$;

5) $\mu(A \cap L_\alpha f) \le \alpha \Rightarrow (S)\int_A f d\mu \le \alpha$;

6) if $\mu(A) < \infty$*, then* $\mu(A \cap L_\alpha f) \ge \alpha \Leftrightarrow (S)\int_A f d\mu \ge \alpha$;

7) when $A = X$*,* $(S)\int_A f d\mu = \vee_{\alpha \ge 0}(\alpha \wedge \mu(L_\alpha f))$;

8) if $\alpha \le \beta$*, then* $L_\beta f \subseteq L_\alpha f$;

9) $(S)\int_A f d\mu \ge 0$.

Theorem 2.4. *([7, p. 135]) Let* $f \in \mathcal{F}_+$*, the class of all finite nonnegative measurable functions on* (X, Σ, μ)*. Then*

1) if $\mu(A) = 0$*, then* $(S)\int_A f d\mu = 0$*, for any* $f \in \mathcal{F}_+$*;*

2) if $(S)\int_A f d\mu = 0$*, then* $\mu(A \cap \{x | f(x) > 0\}) = 0$;

3) $(S)\int_A f d\mu = (S)\int_A f \cdot \chi_A d\mu$*, where* χ_A *is the characteristic function of* A*;*

4) $(S)\int_A (f + a) d\mu \le (S)\int_A f d\mu + (S)\int_A a d\mu$*, for any constant* $a \in [0, \infty)$.

Corollary 2.5. *([7, p. 136]) Let* $f, f_1, f_2 \in \mathcal{F}_+$*. Then*

1) $(S)\int_A (f_1 \vee f_2) d\mu \ge (S)\int_A f_1 d\mu \vee (S)\int_A f_2 d\mu$;

2) $(S)\int_A (f_1 \wedge f_2) d\mu \le (S)\int_A f_1 d\mu \wedge (S)\int_A f_2 d\mu$;

3) $(S)\int_{A\cup B} f d\mu \ge (S)\int_A f d\mu \vee (S)\int_B f d\mu$;

4) $(S)\int_{A\cap B} f d\mu \le (S)\int_A f d\mu \wedge (S)\int_B f d\mu$.

In general we have

$$(S)\int_A (f_1 + f_2) d\mu \ne (S)\int_A f_1 d\mu + (S)\int_A f_2 d\mu,$$

and

$$(S)\int_A a f d\mu \ne a (S)\int_A f d\mu, \text{ where } a \in \mathbb{R},$$

see [7, p. 137].

Lemma 2.6. *([7, p. 138])* $(S)\int_A f d\mu = \infty$ *if and only if* $\mu(A \cap L_\alpha f) = \infty$ *for any* $\alpha \in [0, \infty)$.

We need

Definition 2.7. *([2]) A fuzzy measure* μ *is subadditive iff* $\mu(A \cup B) \leq \mu(A) + \mu(B)$*, for all* $A, B \in \Sigma$.

We mention the following result

Theorem 2.8. *([2]) If* μ *is subadditive, then*

$$(S)\int_X (f+g)\, d\mu \leq (S)\int_X fd\mu + (S)\int_X gd\mu, \tag{2}$$

for all measurable functions $f, g : X \to [0, \infty)$.

Moreover, if (2) holds for all measurable functions $f, g : X \to [0, \infty)$ *and* $\mu(X) < \infty$*, then* μ *is subadditive.*

Notice here in (1) we have that $\alpha \in [0, \infty)$.

We have the following corollary.

Corollary 2.9. *If* μ *is aubadditive,* $n \in \mathbb{N}$*, and* $f : X \to [0, \infty)$ *is a measurable function, then*

$$(S)\int_X nfd\mu \leq n\,(S)\int_X fd\mu, \tag{3}$$

in particular it holds

$$(S)\int_A nfd\mu \leq n\,(S)\int_A fd\mu, \tag{4}$$

for any $A \in \Sigma$.

Proof. By inequality (2). □

A very important property of Sugeno integral follows.

Theorem 2.10. *If* μ *is subadditive measure, and* $f : X \to [0, \infty)$ *is a measurable function, and* $c > 0$*, then*

$$(S)\int_A cfd\mu \leq (c+1)\,(S)\int_A fd\mu, \tag{5}$$

for any $A \in \Sigma$.

Proof. Let the ceiling $\lceil c \rceil = m \in \mathbb{N}$, then by Theorem 2.3 (3) and (4) we get

$$(S)\int_A cfd\mu \leq (S)\int_A mfd\mu \leq m\,(S)\int_A fd\mu \leq (c+1)\,(S)\int_A fd\mu,$$

proving (5). □

3 Main Results

From now on in this article we work on the fuzzy measure space $([a,b], \mathcal{B}, \mu)$, where $[a,b] \subset \mathbb{R}$, $\mathcal{B}$ is the Borel σ-algebra on $[a,b]$, and μ is a finite fuzzy measure on $\mathcal{B}$. Typically we take it to be subadditive.

The functions f we deal with here are continuous from $[a,b]$ into $\mathbb{R}_+$.

We make the following remark

Remark 3.1. *Let* $f \in C^1([a,b], \mathbb{R}_+)$, *and* μ *is a subadditive fuzzy measure such that* $\mu([a,b]) > 0$, $x \in [a,b]$. *We will estimate*

$$E := \left| (S) \int_{[a,b]} f(x)\, d\mu(t) - \mu([a,b]) \wedge f(x) \right| \tag{6}$$

(by Theorem 2.3 (2))

$$= \left| (S) \int_{[a,b]} f(t)\, d\mu(t) - (S) \int_{[a,b]} f(x)\, d\mu(t) \right|.$$

We notice that

$$f(t) = f(t) - f(x) + f(x) \leq |f(t) - f(x)| + f(x),$$

then (by Theorem 2.3 (3) and Theorem 2.4 (4))

$$(S) \int_{[a,b]} f(t)\, d\mu(t) \leq (S) \int_{[a,b]} |f(t) - f(x)|\, d\mu(t) + (S) \int_{[a,b]} f(x)\, d\mu(t), \tag{7}$$

that is

$$(S) \int_{[a,b]} f(t)\, d\mu(t) - (S) \int_{[a,b]} f(x)\, d\mu(t) \leq (S) \int_{[a,b]} |f(t) - f(x)|\, d\mu(t). \tag{8}$$

Similarly, we have

$$f(x) = f(x) - f(t) + f(t) \leq |f(t) - f(x)| + f(t),$$

then (by Theorem 2.3 (3) and Theorem 2.8)

$$(S) \int_{[a,b]} f(x)\, d\mu(t) \leq (S) \int_{[a,b]} |f(t) - f(x)|\, d\mu(t) + (S) \int_{[a,b]} f(t)\, d\mu(t),$$

that is

$$(S) \int_{[a,b]} f(x)\, d\mu(t) - (S) \int_{[a,b]} f(t)\, d\mu(t) \leq (S) \int_{[a,b]} |f(t) - f(x)|\, d\mu(t). \tag{9}$$

By (8) and (9) we derive that

$$\left| (S) \int_{[a,b]} f(t)\, d\mu(t) - (S) \int_{[a,b]} f(x)\, d\mu(t) \right| \leq (S) \int_{[a,b]} |f(t) - f(x)|\, d\mu(t). \tag{10}$$

Consequently it holds

$$E \overset{\text{(by (6), (10))}}{\leq} (S)\int_{[a,b]} |f(t) - f(x)|\, d\mu(t)$$

(and by $|f(t) - f(x)| \leq \|f'\|_\infty |t - x|$*)*

$$\leq (S)\int_{[a,b]} \|f'\|_\infty |t - x|\, d\mu(t) \overset{\text{(by (5))}}{\leq} (\|f'\|_\infty + 1)(S)\int_{[a,b]} |t - x|\, d\mu(t). \tag{11}$$

We have proved the following Ostrowski-like inequality

$$\left| \frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \frac{\mu([a,b] \wedge f(x))}{\mu([a,b])} \right| \leq \tag{12}$$

$$\frac{(\|f'\|_\infty + 1)}{\mu([a,b])}(S)\int_{[a,b]} |t - x|\, d\mu(t).$$

The last inequality can be better written as follows:

$$\left| \frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{(\|f'\|_\infty + 1)}{\mu([a,b])}(S)\int_{[a,b]} |t - x|\, d\mu(t). \tag{13}$$

Notice here that $\left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \leq 1$*, and* $\frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) \leq \frac{\mu([a,b])}{\mu([a,b])} = 1$*, where* $(S)\int_{[a,b]} f(t)\, d\mu(t) \geq 0$.

I.e. If $f : [a,b] \to \mathbb{R}_+$ *is a Lipschitz function of order* $0 < \alpha \leq 1$*, i.e.* $|f(x) - f(y)| \leq K|x - y|^\alpha$*,* $\forall\, x, y \in [a,b]$*, where* $K > 0$*, denoted by* $f \in \mathrm{Lip}_{\alpha,K}([a,b], \mathbb{R}_+)$*, then we get similarly the following Ostrowski-like inequality:*

$$\left| \frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{(K + 1)}{\mu([a,b])}(S)\int_{[a,b]} |t - x|^\alpha\, d\mu(t). \tag{14}$$

We have proved the following Ostrowski-Sugeno inequalities:

Theorem 3.2. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$*,* $x \in [a,b]$.

1) Let $f \in C^1([a,b], \mathbb{R}_+)$*, then*

$$\left| \frac{1}{\mu([a,b])}(S)\int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{(\|f'\|_\infty + 1)}{\mu([a,b])}(S)\int_{[a,b]} |t - x|\, d\mu(t). \tag{15}$$

2) Let $f \in \mathrm{Lip}_{\alpha,K}([a,b],\mathbb{R}_+)$, $0 < \alpha \leq 1$, *then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$
$$\frac{(K+1)}{\mu([a,b])} (S) \int_{[a,b]} |t-x|^{\alpha}\, d\mu(t). \tag{16}$$

We make the following remark

Remark 3.3. *Let* $f \in C^1([a,b],\mathbb{R}_+)$ *and* $g \in C^1([a,b])$, *by Cauchy's mean value theorem we get that*

$$(f(t) - f(x))\, g'(c) = (g(t) - g(x))\, f'(c),$$

for some c *between* t *and* x; *for any* $t, x \in [a,b]$.

If $g'(c) \neq 0$, *we have*

$$(f(t) - f(x)) = \left(\frac{f'(c)}{g'(c)}\right)(g(t) - g(x)).$$

Here we assume that $g'(t) \neq 0$, $\forall\ t \in [a,b]$. *Hence it holds*

$$|f(t) - f(x)| \leq \left\| \frac{f'}{g'} \right\|_{\infty} |g(t) - g(x)|, \tag{17}$$

for all $t, x \in [a,b]$.

We have again as before (see (11))

$$E \leq (S) \int_{[a,b]} |f(t) - f(x)|\, d\mu(t) \overset{\text{(by (17))}}{\leq}$$
$$(S) \int_{[a,b]} \left\| \frac{f'}{g'} \right\|_{\infty} |g(t) - g(x)|\, d\mu(t) \overset{\text{(by (5))}}{\leq}$$
$$\left(\left\| \frac{f'}{g'} \right\|_{\infty} + 1 \right) (S) \int_{[a,b]} |g(t) - g(x)|\, d\mu(t). \tag{18}$$

We have established the following general Ostrowski-Sugeno inequality:

Theorem 3.4. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$, $x \in [a,b]$. *Let* $f \in C^1([a,b],\mathbb{R}_+)$ *and* $g \in C^1([a,b])$ *with* $g'(t) \neq 0$, $\forall\ t \in [a,b]$. *Then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$
$$\frac{\left(\left\| \frac{f'}{g'} \right\|_{\infty} + 1 \right)}{\mu([a,b])} (S) \int_{[a,b]} |g(t) - g(x)|\, d\mu(t). \tag{19}$$

We give for $g(t) = e^t$ the next result

Corollary 3.5. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$, $x \in [a,b]$. *Let* $f \in C^1([a,b], \mathbb{R}_+)$, *then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{\left(\left\| \frac{f'}{e^t} \right\|_\infty + 1\right)}{\mu([a,b])} (S) \int_{[a,b]} \left| e^t - e^x \right| d\mu(t). \tag{20}$$

When $g(t) = \ln t$ we get the following corollary.

Corollary 3.6. *Suppose that* μ *is a fuzzy subadditive measure with* $\mu([a,b]) > 0$, $x \in [a,b]$ *and* $a > 0$. *Let* $f \in C^1([a,b], \mathbb{R}_+)$. *Then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])}\right) \right| \leq$$

$$\frac{(\|tf'(t)\|_\infty + 1)}{\mu([a,b])} (S) \int_{[a,b]} \left| \ln \frac{t}{x} \right| d\mu(t). \tag{21}$$

Many other applications of Theorem 3.4 could follow but we stop it here.

We make the following remark.

Remark 3.7. *Let* $f \in \left[C([a,b], \mathbb{R}_+) \cap C^{n+1}([a,b])\right]$, $n \in \mathbb{N}$, $x \in [a,b]$. *Then by Taylor's theorem we get*

$$f(y) - f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} (y-x)^k + R_n(x,y), \tag{22}$$

where the remainder

$$R_n(x,y) := \int_x^y \left(f^{(n)}(t) - f^{(n)}(x)\right) \frac{(y-t)^{n-1}}{(n-1)!} dt; \tag{23}$$

here y *can be* $\geq x$ *or* $\leq x$.

By [1] we get that

$$|R_n(x,y)| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!} |y-x|^{n+1}, \quad \text{for all } x, y \in [a,b]. \tag{24}$$

Here we assume $f^{(k)}(x) = 0$, *for all* $k = 1, ..., n$.

Therefore it holds

$$|f(t) - f(x)| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!} |t-x|^{n+1}, \text{ for all } t, x \in [a,b]. \tag{25}$$

Here we have again

$$E \leq (S) \int_{[a,b]} |f(t) - f(x)| \, d\mu(t) \overset{\text{(by Theorem 2.3 (3) and (25))}}{\leq}$$

$$(S) \int_{[a,b]} \frac{\left\| f^{(n+1)} \right\|_{\infty}}{(n+1)!} |t - x|^{n+1} \, d\mu(t) \overset{\text{(by (5))}}{\leq}$$

$$\left(\frac{\left\| f^{(n+1)} \right\|_{\infty}}{(n+1)!} + 1 \right) (S) \int_{[a,b]} |t - x|^{n+1} \, d\mu(t). \tag{26}$$

We have derived the following high order Ostrowski-Sugeno inequality:

Theorem 3.8. *Let* $f \in \left[C([a,b], \mathbb{R}_+) \cap C^{n+1}([a,b]) \right]$, $n \in \mathbb{N}$, $x \in [a,b]$. *We assume that* $f^{(k)}(x) = 0$, *all* $k = 1, ..., n$. *Here* μ *is subadditive with* $\mu([a,b]) > 0$. *Then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])} \right) \right| \leq$$

$$\frac{\left(\frac{\left\| f^{(n+1)} \right\|_{\infty}}{(n+1)!} + 1 \right)}{\mu([a,b])} (S) \int_{[a,b]} |t - x|^{n+1} \, d\mu(t), \tag{27}$$

which generalizes (15).

When $x = \frac{a+b}{2}$ we get the following corollary

Corollary 3.9. *Let* $f \in \left[C([a,b], \mathbb{R}_+) \cap C^{n+1}([a,b]) \right]$, $n \in \mathbb{N}$. *Assume that* $f^{(k)}\left(\frac{a+b}{2}\right) = 0$, $k = 1, ..., n$. *Here* μ *is subadditive with* $\mu([a,b]) > 0$. *Then*

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) \, d\mu(t) - \left(1 \wedge \frac{f\left(\frac{a+b}{2}\right)}{\mu([a,b])} \right) \right| \leq$$

$$\frac{\left(\frac{\left\| f^{(n+1)} \right\|_{\infty}}{(n+1)!} + 1 \right)}{\mu([a,b])} (S) \int_{[a,b]} \left| t - \frac{a+b}{2} \right|^{n+1} \, d\mu(t). \tag{28}$$

References

[1] G.A. Anastassiou, *Ostrowski type inequalities*, Proc. Amer. Math. Soc. 123(1995), 3775-3781.

[2] M. Boczek, M. Kaluszka, *On the Minkowaki-Hölder type inequalities for generalized Sugeno integrals with an application*, Kybernetica, 52(3) (2016), 329-347.

[3] A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

[4] E. Pap, *Null-Additive Set functions*, Kluwer Academic, Dordrecht, 1995.

[5] D. Ralescu, G. Adams, *The fuzzy integral*, J. Math. Anal. Appl., 75 (1980), 562-570.

[6] M. Sugeno, *Theory of fuzzy integrals and its applications*, PhD thesis, Tokyo Institute of Technology (1974).

[7] Z. Wang, G.J. Klir, *Fuzzy Measure Theory*, Plenum, New York, 1992.

CUBO A Mathematical Journal
Vol.21, $N^{\underline{o}}$ 03, (39–61). December 2019
http://dx.doi.org/10.4067/S0719-06462019000300039

Stability And Boundedness In Nonlinear And Neutral Difference Equations using New Variation of Parameters Formula And Fixed Point Theory

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ABSTRACT

In the case of nonlinear problems, whether in differential or difference equations, it is difficult and in some cases impossible to invert the problem and obtain a suitable mapping that can be effectively used in fixed point theory to qualitatively analyze its solutions. In this paper we consider the existence of a positive sequence and utilize it in the capacity of integrating factor to obtain a new variation of parameters formula. Then, we will use the obtained new variation of parameters formula and revert to the contraction principle to arrive at results concerning, boundedness, periodicity and stability. The author is working on parallel results for the continuous case.

RESUMEN

En el caso de problemas no-lineales, ya sea en ecuaciones diferenciales o en diferencias, es difícil y en algunos casos imposible invertir el problema y obtener una aplicación apropiada que pueda ser efectivamente usada en teoría de punto fijo para analizar quantitativamente sus soluciones. En este paper consideramos la existencia de una sucesión positiva y la usamos en la capacidad del factor de integración para obtener una nueva fórmula de variación de parámetros. Luego, usaremos esta nueva fórmula de variación de parámetros y volver al principio de contracción para obtener resultados que involucran acotamiento, periodicidad y estabilidad. El autor se encuentra trabajando en resultados paralelos para el caso continuo

Keywords and Phrases: New variation of parameters, Difference, Neutral, Sability, Boundedness, Fixed point theorems, Contraction mapping, equi-boundedness.
2010 AMS Mathematics Subject Classification: 39A11.

1 Introduction

For motivational purpose we consider the linear difference equation

$$x(t+1) = a(t)x(t),\ x(t_0) = x_0,\ t \geq t_0 \geq 0. \tag{1.1}$$

It is clear that the solution of (1.1) is given by

$$x(t) = x_0 \prod_{s=t_0}^{t-1} a(s), \tag{1.2}$$

provided that $a(t) \neq 0$ for all $t \in \mathbb{Z}^+$. Throughout this paper we adopt the convention that for any sequence $x(k)$

$$\sum_{k=a}^{b} x(k) = 0 \text{ and } \prod_{k=a}^{b} x(k) = 1 \text{ whenever } a > b.$$

For more on the calculus of difference equations we refer to [6]- [8] and [13].

Let $v(t)$ be a sequence such that $v : \mathbb{Z}^+ \to \mathbb{R}$ with $v(t) \neq 0$ for all $t \in \mathbb{Z}^+$. Multiply both sides of (1.1) by $\prod_{s=t_0}^{t} v^{-1}(s)$ to obtain

$$x(t+1) \prod_{s=t_0}^{t} v^{-1}(s) = a(t)x(t) \prod_{s=t_0}^{t} v^{-1}(s),$$

Thus the above expression can be written in the compact form

$$\Delta\Big[x(t) \prod_{s=t_0}^{t-1} v^{-1}(s)\Big] = \Big[\big(a(t) - v(t)\big)x(t)\Big] \prod_{s=t_0}^{t} a^{-1}(s). \tag{1.3}$$

Summing equation (1.3) from t_0 to t-1 gives

$$x(t) = x_0 \prod_{s=t_0}^{t-1} v(s) + \sum_{r=t_0}^{t-1} \big(a(r) - v(r)\big)x(r) \prod_{s=r+1}^{t-1} v(s). \tag{1.4}$$

Note that (1.4) reduces to (1.2) if we set $v(t) = a(t)$ in (1.4). To obtain asymptotic stability of the zero solution of (1.1) using (1.2) one would have to assume that

$$\prod_{s=t_0}^{t} a(s) \to 0,\ \text{as } t \to \infty.$$

On the hand, if we use (1.4) instead, then such requirement is not necessary. But instead, we would have to ask that

$$\prod_{s=t_0}^{t} v(s) \to 0,\ \text{as } t \to \infty.$$

Such technique of inversion is of more importance when the right hand of (1.1) is either totally nonlinear or totally delayed. To see this, we consider the nonlinear difference equation

$$x(t+1) = f(t, x(t)),\ x(t_0) = x_0, \tag{1.5}$$

where the function $f : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous. The subject of stability and boundedness in difference equations is vast and we refer to [16] and [17]. We begin by stating some definitions .

Definition 1.1. *We say $x(t) := x(t, t_0, x_0)$ is a solution of* (1.5) *if $x(t_0) = x_0$ and satisfies* (1.5) *for $t \geq t_0 \geq 0$.*

Definition 1.2. *The zero solution of* (1.5) *is stable if for any $\epsilon > 0$ and any integer $t_0 \geq 0$ there exists a $\delta > 0$ such that $|x_0| \leq \delta$ implies $|x(t, t_0, x_0)| \leq \epsilon$ for $t \geq t_0$.*

Definition 1.3. *The zero solution of* (1.5) *is asymptotically stable if it is stable and $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$.*

For more on stability we refer to [9] and [11]. We begin with the following Lemma. Its proof follows along the lines of the derivation of (1.4).

Lemma 1.4. *If $x(t)$ is a solution of* (1.5) *on an interval $\mathbb{Z}^+ \cap [0, T]$ and satisfies the initial condition $x(t_0) = x_0,\ t_0 \geq 0$, then $x(t)$ is a solution of the summation equation if and only if*

$$x(t) = x_0 \prod_{s=t_0}^{t-1} v(s) + \sum_{r=t_0}^{t-1} \big(f(r, x(r)) - v(r)x(r)\big) \prod_{s=r+1}^{t-1} v(s), \tag{1.6}$$

where $v : \mathbb{Z}^+ \to \mathbb{R}$ with $v(t) \neq 0$ for all $t \in \mathbb{Z}^+$.

Next, we will use (1.6) to define a mapping on the proper space and show the zero solution is (AS). Let $\mathcal{C}$ be the set of all real-valued bounded sequences. Define the space

$$\mathcal{S} = \{\phi : [0, \infty) \to \mathbb{R}/\phi \in \mathcal{C}, |\phi(t)| \leq L, \phi(t) \to 0,\ \text{as } t \to \infty\}.$$

Then

$$(\mathcal{S}, ||\cdot||)$$

is a complete metric space under the uniform metric

$$\rho(\phi_1, \phi_2) = ||\phi_1 - \phi_2||,$$

where

$$||\phi|| = \sup_{t \in \mathbb{Z}^+} \{|\phi(t)|\}.$$

Assume

$$f(t, 0) = 0. \tag{1.7}$$

We assume the function f is locally Lipschitz on the set $\mathcal{S}$.
That is, for any ϕ_1 and $\phi_2 \in \mathcal{S}$, we have

$$|f(t,\phi_1) - f(t,\phi_2)| \leq \lambda(t)||\phi_1 - \phi_2||, \tag{1.8}$$

for $\lambda : [0,\infty) \to (0,\infty)$. Assume for $\phi \in \mathcal{S}$ and positve constant L, we have that

$$\Big|x_0 \prod_{s=t_0}^{t-1} v(s)\Big| + L\sum_{r=t_0}^{t-1}\big(|v(r)| + \lambda(r)\big)\Big|\prod_{s=r+1}^{t-1} v(s)\Big| \leq L. \tag{1.9}$$

Note that (1.9) implies that

$$\sum_{r=t_0}^{t-1}\big(|v(r)| + \lambda(r)\big)\Big|\prod_{s=r+1}^{t-1} v(s)\Big| \leq \alpha < 1.$$

The next theorem offers results about stability and boundedness. For more results on the stability and boundedness using fixed point theory, we refer the interest reader to the book [18] and to the paper [19].

Theorem 1.5. *Assume* (1.7)-(1.9). *Suppose there exists a positive constant k such that*

$$\Big|\prod_{s=t_0}^{t-1} v(s)\Big| \leq k, \tag{1.10}$$

then the unique solution of (1.5) *is bounded and its zero solution is stable.*
If, in addition,

$$\prod_{s=t_0}^{t-1} v(s) \to 0, \tag{1.11}$$

then the zero solution of (1.5) *is asymptotically stable.*

Proof. For $\phi \in \mathcal{S}$, define the mapping $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$ by

$$(\mathfrak{P}\phi)(t) = x_0\prod_{s=t_0}^{t-1} v(s) + \sum_{r=t_0}^{t-1}\big(f(r,\phi(r)) - \phi(r)v(r)\prod_{s=r+1}^{t} v(s) \tag{1.12}$$

It is clear that $(\mathfrak{P}\phi)(t_0) = x_0$. Now for $\phi \in \mathcal{S}$, we have that

$$\big|(\mathfrak{P}\phi)(t)\big| \leq |x_0|k + \sum_{r=t_0}^{t-1}\Big(\lambda(r)|\phi(r)| + |\phi(r)||v(r)\Big)\Big|\prod_{s=r+1}^{t} v(s)\Big|.$$

Consequently,

$$||\mathfrak{P}\phi|| \leq |x_0|k + \sum_{r=t_0}^{t-1}\big(|v(r)| + \lambda(r)\big)\Big|\prod_{s=r+1}^{t} v(s)\Big|||\phi||$$

Or,

$$\|\mathfrak{P}\phi\| \leq |x_0|k + \alpha\|\phi\| \leq L. \tag{1.13}$$

Since $\mathfrak{P}$ is continuous we have that $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$. Next we show that $\mathfrak{P}$ is a contraction.
For $\phi_1, \phi_2 \in \mathcal{S}$, we have from (1.12) that

$$\begin{aligned}\big|(\mathfrak{P}\phi_1)(t) - (\mathfrak{P}\phi_2)(t)\big| &\leq \sum_{r=t_0}^{t-1} \big(|v(r)| + \lambda(r)\big)\Big|\prod_{s=r+1}^{t} v(s)\Big| ||\phi_1 - \phi_2|| \\ &\leq \alpha||\phi_1 - \phi_2||.\end{aligned}$$

This shows that $\mathfrak{P}$ is a contraction. By Banach's contraction mapping principle,$\mathfrak{P}$ has a unique fixed point $x \in \mathcal{S}$ which is bounded. Moreover, the unique fixed point is a solution of (1.5) on $[0, \infty)$. Next we show the zero solution is stable. Let x be the unique solution. Let $\varepsilon > 0$ be given and chose $\delta = \varepsilon\frac{1-\alpha}{k}$. Thus for $|x_0| < \delta$, we have by (1.13) that

$$(1 - \alpha)||x|| \leq |x_0|k < \delta k.$$

Or

$$||x|| \leq \varepsilon.$$

Left to prove that

$$(\mathfrak{P}\varphi)(\mathrm{t}) \to 0, \text{ as } t \to \infty.$$

We have already proved that the zero solution of (1.5) is stable. Let δ be the one from stability such that $|x_0| < \delta$ and define

$$\mathcal{S}^* = \Big\{\varphi : \mathbb{Z}^+ \to \mathbb{R}|\ \varphi(t_0) = x_0\ , ||\varphi|| \leq \epsilon \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\Big\}. \tag{1.14}$$

Let $\mathfrak{P}$ be given by (1.12) and define $\mathfrak{P} : S^* \to S^*$. The map $\mathfrak{P}$ is contraction and it maps from S^* into itself.
We next show that $(\mathfrak{P}\varphi)(t)$ goes to zero as t goes to infinity.
The first term on the right of (1.12) goes to zero due to condition (1.11). Left to show that

$$|\sum_{r=t_0}^{t-1} \Big(f(r, x(r)) - v(r)\phi(r)\Big) \prod_{s=r+1}^{t-1} v(s)| \to 0, \text{ as } t \to \infty.$$

Let $\varphi \in S^*$ then $|\varphi(t)| \leq \epsilon$. Also, since $\varphi(t) \to 0$ as $t \to \infty$, there exists a $t_1 > 0$ such that for $t > t_1$, $|\varphi(t)| < \epsilon_1$ for $\epsilon_1 > 0$. Due to condition (1.11) there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$\Big|\prod_{s=t_1}^{t} v(s)\Big| < \frac{\epsilon_1}{\alpha\epsilon}.$$

Thus for $t > t_2$, we have

$$
\begin{aligned}
&\Big|\sum_{r=t_0}^{t-1}\Big(f(r,x(r)) - v(r)\phi(r)\Big)\prod_{s=r+1}^{t-1} v(s)|\Big| \le \sum_{r=t_0}^{t-1}\Big(\lambda(r)+v(r)\Big)|\phi(r)|\Big|\prod_{s=r+1}^{t-1} v(s)\Big| \\
\le\ & \sum_{r=t_0}^{t_1-1}\Big(\lambda(r)+v(r)\Big)|\phi(r)|\prod_{s=r+1}^{t-1} v(s)\Big| \\
&+\sum_{r=t_1}^{t-1}\Big(\lambda(r)+v(r)\Big)|\phi(r)|\Big|\prod_{s=r+1}^{t-1} v(s)\Big| \\
\le\ & \epsilon\sum_{r=t_0}^{t_1-1}\Big(\lambda(r)+v(r)\Big)|\phi(r)|\Big|\prod_{s=r+1}^{t-1} v(s)\Big| + \epsilon_1\alpha \\
\le\ & \epsilon\sum_{r=t_0}^{t_1-1}\Big(\lambda(r)+v(r)\Big)|\prod_{s=r+1}^{t_1-1} v(s)\Big|\prod_{s=t_1}^{t-1} v(s)\Big| + \epsilon_1\alpha \\
\le\ & \epsilon\Big|\prod_{s=t_1}^{t-1} v(s)\Big|\sum_{r=t_0}^{t_1-1}\Big(\lambda(r)+v(r)\Big)\Big|\prod_{s=r+1}^{t_1-1} v(s)\Big| + \epsilon_1\alpha \\
\le\ & \epsilon\alpha|\prod_{s=t_1}^{t-1} v(s)| + \epsilon_1\alpha \\
\le\ & \epsilon_1 + \epsilon_1\alpha.
\end{aligned}
$$

Since ϵ_1 is arbitrary small, this shows that $(\mathfrak{P}\varphi)(t) \to 0$ as $t \to \infty$. As $\mathfrak{P}$ has a unique fixed point, say x it implies the asymptotic stability of the zero solution of (1.11). This completes the proof. □

2 Contraction Versus Large Contraction

Now we consider particular nonlinear equation and rewrite so we can invert the usual way. Consequently, contraction mapping principle can no longer be useful. Let

$$f(t,x) = -a(t)x^3 + l(t,x),$$

where $l(t,x)$ continuous and satisfies a smallness condition. Thus, We consider

$$x(t+1) = -a(t)x^3 + l(t,x). \tag{2.1}$$

mentioned paper as an example for illustrating the need for Large Contraction. In [4], the author put (1.12) in the form

$$x(t+1) = -a(t)x + a(t)(x - x^3) + l(t,x). \tag{2.2}$$

Then by the variation of parameters formula we have

$$x(t) = x_0 \prod_{s=t_0}^{t-1} a(s) + \sum_{r=t_0}^{t-1} \Big(a(r)\big(x(r) - x^3(r)\big) + l(t, x(r)) \Big) \prod_{s=r+1}^{t-1} a(s). \tag{2.3}$$

It is naive to believe that every map can be defined so that it is a contraction, even with the strictest conditions. To see this, we consider

$$g(x) = x - x^3$$

then for $x, y \in \mathbb{R}$ with $|x|, |y| \leq \frac{\sqrt{3}}{3}$ we have that

$$|g(x) - g(y)| = |x - x^3 - y + y^3| \leq |x - y| \left(1 - \frac{x^2 + y^2}{2}\right)$$

and the contraction constant tends to one as $x^2 + y^2 \to 0$. As a consequence, the regular contraction mapping principle failed to produce any results. For more on this and Large contraction, we refer to [18], P: 52. To get around it, we let $v(t)$ be a sequence such that $v : \mathbb{Z}^+ \to \mathbb{R}$ with $v(t) \neq 0$ for all $t \in \mathbb{Z}^+$. By similar steps as in the development of (1.4) we arrive at the variation of parameters formula

$$x(t) = x_0 \prod_{s=t_0}^{t-1} v(s) + \sum_{r=t_0}^{t-1} \Big(v(r)x(r) - a(t)x^3(r) + l(t, x(r)) \Big) \prod_{s=r+1}^{t-1} v(s) \tag{2.4}$$

Thus, one can show that the map given by

$$f(x) = v(r)x(r) - a(t)x^3(r),$$

is a contraction on some bounded and small set provided a and v have small magnitudes. To better illustrate our intention we set $l(t, x) = 0$, and consider (2.1). Then from the above variation of parameters formula, we have that

$$x(t) = x_0 \prod_{s=t_0}^{t-1} v(s) + \sum_{r=t_0}^{t-1} \Big(v(r)x(r) - a(t)x^3(r) \Big) \prod_{s=r+1}^{t-1} v(s). \tag{2.5}$$

Assume for $\phi \in \mathcal{S}$ and positve constant L, we have that

$$\Big| x_0 \prod_{s=t_0}^{t-1} v(s) \Big| + \sum_{r=t_0}^{t-1} \big(L|v(r)| + L^3|a(r)| \big) \Big| \prod_{s=r+1}^{t-1} v(s) \Big| \leq L, \tag{2.6}$$

and

$$\sum_{r=t_0}^{t-1} \big(|v(r)| + 3L^2|a(r)| \big) \Big| \prod_{s=r+1}^{t-1} v(s) \Big| \leq \alpha < 1. \tag{2.7}$$

The next theorem offers results about stability and boundedness. For more results on the stability and boundedness using fixed point theory, we refer the interest reader to the book [18] and to the paper [19].

Theorem 2.1. *Assume* (1.7), (1.10), (2.6) *and* (2.7). *Then the unique solution of* (2.1) *is bounded and its zero solution is stable.*
If, in addition, (1.11) *holds, then the zero solution of* (2.1) *is asymptotically stable.*

Proof. For $\phi \in \mathcal{S}$, define the mapping $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$, by

$$(\mathfrak{P}\phi)(t) = x_0 \prod_{s=t_0}^{t-1} v(s) + \sum_{r=t_0}^{t-1} \big(v(r)\phi(r) - \phi^3(r)a(r)\big) \prod_{s=r+1}^{t} v(s) \tag{2.8}$$

It is clear that $(\mathfrak{P}\phi)(t_0) = x_0$. Now for $\phi \in \mathcal{S}$, we have that

$$\begin{aligned} \big|(\mathfrak{P}\phi)(t)\big| &\leq |x_0|k + \sum_{r=t_0}^{t-1} \Big(|v(r)||\phi(r)| + |\phi^3(r)||a(r)|\Big)\Big|\prod_{s=r+1}^{t} v(s)\Big| \\ &\leq \Big|x_0 \prod_{s=t_0}^{t-1} v(s)\Big| + \sum_{r=t_0}^{t-1} \big(L|v(r)| + L^3|a(r)|\big)\Big|\prod_{s=r+1}^{t-1} v(s)\Big|. \end{aligned}$$

Thus,

$$\|\mathfrak{P}\phi\| \leq L.$$

Since $\mathfrak{P}$ is continuous we have that $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$. Next we show that $\mathfrak{P}$ is a contraction.
For $\phi_1, \phi_2 \in \mathcal{S}$, we have from (1.12) that

$$\begin{aligned} \big|(\mathfrak{P}\phi_1)(t) - (\mathfrak{P}\phi_2)(t)\big| &\leq \sum_{r=t_0}^{t-1} \big(|v(r)||\phi_1(r) - \phi_2(r)| \\ &+ \sum_{r=t_0}^{t-1} |a(r)||\phi_1(r) - \phi_2(r)|(\phi_1^2(r) + |\phi_1(r)\phi_2(r)| + \phi_1^2(r))\Big|\prod_{s=r+1}^{t} v(s)\Big| \\ &\leq \sum_{r=t_0}^{t-1} \big(|v(r)| + 3L^2|a(r)|\big)\Big|\prod_{s=r+1}^{t-1} v(s)\Big|||\phi_1 - \phi_2|| \\ &\leq \alpha||\phi_1 - \phi_2||. \end{aligned}$$

This shows that $\mathfrak{P}$ is a contraction. By Banach's contraction mapping principle,$\mathfrak{P}$ has a unique fixed point $x \in \mathcal{S}$ which is bounded. The proof for stability and asymptotic stability follow along the lines of the proof of Theorem 1.5. □

For the rest of this section we set $l(t,x) = 0$ in (2.3) and use large contraction and prove parallel theorem to Theorem 2.1. We saw before that the function or map, $g(x) = x - x^3$ does not define a contraction. To get around it we use the notion of large contraction that was introduced by Burton in [5] . We will restate the contraction mapping principle and Krasnoselskii's fixed point theorems in which the regular contraction is replaced with large contraction. Then based on the notion of large contraction, we introduce a theorem to obtain boundedness results in which large contraction is substituted for regular contraction.

Definition 2.2. *Let* $(\mathcal{M},d)$ *be a metric space and* B: $\mathcal{M} \to \mathcal{M}$. *The map* B *is said to be large contraction if* $\phi, \varphi \in \mathcal{M}$, *with* $\phi \neq \varphi$ *then* $d(B\phi, B\varphi) \leq d(\phi, \varphi)$ *and if for all* $\varepsilon > 0$, *there exists a* $\delta \in (0,1)$ *such that*

$$[\phi, \varphi \in \mathcal{M}, d(\phi, \varphi) \geq \varepsilon] \Rightarrow d(B\phi, B\varphi) \leq \delta d(\phi, \varphi).$$

The next theorems are alternative to the regular contraction mapping principle and Krasnoselskii's fixed point theorem in which we substitute Large Contraction for regular contraction. The proofs of the two theorems and the statement of Definition 2.2 can be found in [5].

Theorem 2.3. *Let* $(\mathcal{M}, \rho)$ *be a complete metric space and* B *be a large contraction. Suppose there are an* $x \in \mathcal{M}$ *and an* $L > 0$ *such that* $\rho(x, B^n x) \leq L$ *for all* $n \geq 1$. *Then* B *has a unique fixed point in* $\mathcal{M}$.

Next we state and prove a remarkable theorem by Adivar, Raffoul and Islam that generalizes the concept of Large Contraction. Its proof can be found in [18]. The theorem provides easily checked sufficient conditions under which a mapping is a large contraction. Several authors have published it in their work without the proper citations.
Consider the mapping H defined by

$$H(x(u)) = x(u) - h(x(u)). \tag{2.9}$$

Let $\alpha \in (0,1]$ be a fixed real number and define the set $\mathbb{M}_\alpha$ by

$$\mathbb{M}_\alpha = \{\phi : \phi \in C(\mathbb{R}, \mathbb{R}) \text{ and } \|\phi\| \leq \alpha\}. \tag{2.10}$$

H.1. $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-\alpha, \alpha]$ and differentiable on $(-\alpha, \alpha)$,

H.2. The function h is strictly increasing on $[-\alpha, \alpha]$,

H.3. $\sup\limits_{t \in (-\alpha, \alpha)} h'(t) \leq 1$.

Theorem 2.4. *([1])[Adivar-Raffoul-Islam] (Classifications of Large Contraction Theorem) Let* $h : \mathbb{R} \to \mathbb{R}$ *be a function satisfying (H.1-H.3). Then the mapping* H *in (2.9) is a large contraction on the set* $\mathbb{M}_\alpha$.

Example 2.5. *Let* $\alpha \in (0,1)$ *and* $k \in \mathbb{N}$ *be fixed elements and* $u \in (-1,1)$.

1. *The condition (H.2) is not satisfied for the function* $h_1(u) = \frac{1}{2k} u^{2k}$.

2. *The function* $h_2(u) = \frac{1}{2k+1} u^{2k+1}$ *satisfies (H.1-H.3).*

Proof. Since $h_1'(u) = u^{2k-1} < 0$ for $-1 < u < 0$, the condition (H.2) is not satisfied for h_1. Evidently, (H.1-H.2) hold for h_2. (H.3) follows from the fact that $h_2'(u) \leq \alpha^{2k}$ and $\alpha \in (0,1)$. $\square$

We have the following lemma. Define the mapping

$$H(x) = x - x^3. \tag{2.11}$$

Lemma 2.6. *Let $\|\cdot\|$ denote the supremum norm. If*

$$\mathbb{M} = \left\{ \phi : \mathbb{Z} \to \mathbb{R} \mid \phi(0) = \phi_0, \quad and \ \|\phi\| \le \frac{\sqrt{3}}{3} \right\},$$

then the mapping H defined by (2.11) is a large contraction on the set $\mathbb{M}$.

Proof. Let $\alpha = \frac{\sqrt{3}}{3}$ and $h(x) = x^3$. Then, clearly h satisfies (H.1-H.2). Moreover, $\sup\limits_{x \in (-\alpha,\alpha)} h'(x) = 1$, which satisfies H.3. Hence by Theorem 2.4 defines a large contraction. $\square$

For $\psi \in \mathbb{M}$, we define the map $B : \mathbb{M} \to \mathbb{M}$ by

$$(B\psi)(t) = \psi_0 \prod_{s=0}^{t-1} a(s) + \sum_{s=0}^{t-1} \Big(a(s) H(\psi(s)) \prod_{u=s+1}^{t-1} a(u) \Big) \tag{2.12}$$

Lemma 2.7. *Assume for all $t \in \mathbb{Z}$*

$$|\psi_0| \Big| \prod_{s=0}^{t-1} a(s) \Big| + \frac{2\sqrt{3}}{9} \sum_{s=0}^{t-1} \Big| \prod_{u=s}^{t-1} a(u) \Big| \le \frac{\sqrt{3}}{3}. \tag{2.13}$$

If H is a large contraction on $\mathbb{M}$, then so is the mapping B.

Proof. It is easy to see that

$$|H(x(t))| = |x(t) - x(t)^3| \le \frac{2\sqrt{3}}{9} \text{ for all } x \in \mathbb{M}.$$

By Lemma 2.6, H is a large contraction on $\mathbb{M}$. Hence, for $x, y \in \mathbb{M}$ with $x \ne y$, we have $\|Hx - Hy\| \le \|x - y\|$. Hence,

$$\begin{aligned} |Bx(t) - By(t)| &\le \sum_{s=0}^{t-1} |H(x(s)) - H(y(s))| \Big| \prod_{u=s}^{t-1} a(u) \Big| \\ &\le \frac{2\sqrt{3}}{9} \sum_{s=0}^{t-1} \Big| \prod_{u=s}^{t-1} a(u) \Big| \|x - y\| \\ &= \|x - y\|. \end{aligned}$$

Taking supremum norm over the set $[0, \infty)$, we get that $\|Bx - By\| \le \|x - y\|$. For a given $\varepsilon \in (0, 1)$, suppose $x, y \in \mathbb{M}$ with $\|x - y\| \ge \varepsilon$. Then for $\delta = \min\left\{1 - \varepsilon^2/16, 1/2\right\}$, which implies that $0 < \delta < 1$. Hence, for all such $\varepsilon > 0$ we know that

$$[x, y \in \mathbb{M}, \|x - y\| \ge \varepsilon] \Rightarrow \|Hx - Hy\| \le \delta \|x - y\|.$$

Therefore, using (2.13), one easily verify that

$$\|Bx - By\| \leq \delta\|x - y\|.$$

The proof is complete. □

We arrive at the following theorem in which we prove boundedness.

Theorem 2.8. *Assume* (2.13). *Then* (2.1) *has a unique solution in* $\mathbb{M}$ *which is bounded.*

Proof. $(\mathbb{M}, \|\cdot\|)$ is a complete metric space of bounded sequences. For $\psi \in \mathbb{M}$ we must show that $(B\psi)(t) \in \mathbb{M}$. From (2.12) and the fact that

$$|H(x(t))| = |x(t) - x(t)^3| \leq \frac{2\sqrt{3}}{9} \quad \text{for all } x \in \mathbb{M},$$

we have

$$\begin{aligned} |(B\psi)(t)| &\leq |\psi_0|\Big|\prod_{s=0}^{t-1} a(s)\Big| + \frac{2\sqrt{3}}{9}\sum_{s=0}^{t-1}\Big|\prod_{u=s}^{t-1} a(u)\Big| \\ &\leq \frac{\sqrt{3}}{3}. \end{aligned}$$

This shows that $(B\psi)(t) \in \mathbb{M}$. Lemma 2.6 implies the map B is a large contraction and hence by Theorem 2.3, the map B has a unique fixed point in $\mathbb{M}$ which is a solution of (2.1). This completes the proof. □

3 Periodic Solutions

In this section we apply our new method to linear or nonlinear difference equations to show the existence of periodic solutions without the requirement of some classic conditions. To better illustrate our approach, we consider the nonlinear difference equation

$$x(t+1) = a(t)x(t) + f(t, x(t)) \tag{3.1}$$

where f is continuous in x. Let T be an integer such that $T \geq 1$. We assume the periodicity condition

$$a(t+T) = a(t), \text{ and } f(t+T, \cdot) = f(t, \cdot). \tag{3.2}$$

Let BC is the space of bounded sequences $\phi : \mathbb{Z} \to \mathbb{R}$ with the maximum norm $||\cdot||$. Define $P_T = \{\phi \in BC, \phi(t+T) = \phi(t)\}$. Then P_T is a Banach space when it is endowed with the maximum norm

$$\|x\| = \max_{t \in [0, T-1]} |x(t)|.$$

Also, we assume that

$$\prod_{s=t-T}^{t-1} a(s) \neq 1. \tag{3.3}$$

Throughout this section we assume that $a(t) \neq 0$ for all $t \in [0, T-1]$. Let $x \in P_T$. Then Eqn. (3.1) is equivalent to

$$\Delta\Big[x(t) \prod_{s=t_0}^{t-1} a^{-1}(s)\Big] = f(t, x(t)) \prod_{s=t_0}^{t} a^{-1}(s). \tag{3.4}$$

Summing equation (3.4) from $t-T$ to $t-1$ and using the fact that $x(t-T) = x(t)$, gives

$$x(t) = \Big(1 - \prod_{s=t-T}^{t-1} a(s)\Big)^{-1} \sum_{r=t-T}^{t-1} f(r, x(r)) \prod_{s=r+1}^{t-1} a(s). \tag{3.5}$$

Define the mapping $\mathfrak{P}$ on P_T by

$$(\mathfrak{P}\phi)(t) = \Big(1 - \prod_{s=t-T}^{t-1} a(s)\Big)^{-1} \sum_{r=t-T}^{t-1} f(r, \phi(r)) \prod_{s=r+1}^{t-1} a(s). \tag{3.6}$$

One can easily verify that $(\mathfrak{P}\phi)(t+T) = (\mathfrak{P}\phi)(t)$, and hence $\mathfrak{P} : P_T \to P_T$.

Theorem 3.1. *Suppose $a(t) \neq 0$ for all $t \in [0, T-1]$ and assume (3.3). Suppose the function f is Lipschitz continuous with Lipschitz constant k. If*

$$k\Big|\Big(1 - \prod_{s=t-T}^{t-1} a(s)\Big)^{-1}\Big| \sum_{r=t-T}^{t-1} \Big|\prod_{s=r+1}^{t-1} a(s)\Big| \leq \alpha,$$

for $\alpha \in (0,1)$, then Eqn. (3.1) has a unique periodic solution.

Proof. The proof is easily obtained by direct application of contraction mapping principle on the set P_T. ☐

Next, we use our new technique to avoid the requirement that $a(t) \neq 0$ for all $t \in [0, T-1]$ along with condition (3.3). Let $v(t)$ be a sequence such that $v : \mathbb{Z}^+ \to \mathbb{R}$ with $v(t) \neq 0$ for all $t \in [0, T-1$. Assume (3.2) and for $v \in P_T$, multiply both sides of (3.1) by $\prod_{s=t_0}^{t} v^{-1}(s)$ to obtain

$$\Delta\Big[x(t) \prod_{s=t_0}^{t-1} v^{-1}(s)\Big] = \Big[\big(a(t)x(t) - v(t)x(t) + f(t, x(t))\Big] \prod_{s=t_0}^{t} v^{-1}(s). \tag{3.7}$$

Summing equation (3.7) from $t-T$ to t-1 gives and using the fact that $x(t-T) = x(t)$, gives

$$x(t) = \Big(1 - \prod_{s=t-T}^{t-1} v(s)\Big)^{-1} \sum_{r=t-T}^{t-1} [a(r)x(r) - v(r)x(r) + f(r, x(r))] \prod_{s=r+1}^{t-1} v(s). \tag{3.8}$$

Define the mapping $\mathfrak{P}$ on P_T by

$$(\mathfrak{P}\phi)(t) = \Big(1 - \prod_{s=t-T}^{t-1} v(s)\Big)^{-1} \sum_{r=t-T}^{t-1} [a(r)x(r) - v(r)x(r) + f(r, \phi(r)) \prod_{s=r+1}^{t-1} v(s). \tag{3.9}$$

One can easily verify that $(\mathfrak{P}\phi)(t+T) = (\mathfrak{P}\phi)(t)$, and hence $\mathfrak{P} : P_T \to P_T$.

Theorem 3.2. *Suppose $v(t) \neq 0$ for all $t \in [0, T-1]$ and assume*

$$\prod_{s=t-T}^{t-1} v(s) \neq 1. \tag{3.10}$$

Suppose the function f is Lipschitz continuous with Lipschitz constant k,. If

$$\Big|\Big(1 - \prod_{s=t-T}^{t-1} v(s)\Big)^{-1}\Big| \sum_{r=t-T}^{t-1} [|a(r)| + |v(r)| + k]\Big| \prod_{s=r+1}^{t-1} v(s)\Big| \leq \alpha,$$

for $\alpha \in (0,1)$, then Eqn. (3.1) *has a unique periodic solution.*

Proof. The proof is easily obtained by direct application of the contraction mapping principle on the set P_T. ☐

Next we display an example.

Example 3.3. *For positive constant k, we consider the difference equation*

$$x(t+1) = \big(1 - (-1)^t\big)x(t) + \frac{kx}{1+x^2} \tag{3.11}$$

It is clear that $a(t) = \big(1 - (-1)^t\big)$ is periodic of period $T = 2$ and $a(0) = 0$. Hence Theorem 3.1 can not be applied. On the other hand we may apply Theorem 3.2 by taking $v(t) = \frac{(-1)^t}{2}$, for sufficiently small k.

4 Neutral Difference Equations

We extend the results of the previous sections to the neutral difference equation with functional delay

$$x(t+1) = a(t)x(t) + b(t)x(t - g(t)) + c(t)\Delta x(t - g(t)) \tag{4.1}$$

where where $a, b, c : \mathbb{Z} \to \mathbb{R}$, and $g : \mathbb{Z} \to \mathbb{Z}^+$. Moreover, we will discuss the concept of equi-boundedness.

If for some positive constant k, $|g| \leq k$ then for any integer $t_0 \geq 0$, we define $\mathbb{Z}_0$ to be the set of integers in $[t_0 - k, t_0]$. If g is unbounded then $\mathbb{Z}_0$ will be the set of integers in $(-\infty, t_0]$. We assume the existence of a given bounded initial sequence $\psi(t) : \mathbb{Z}_0 \to \mathbb{R}$. We will use the summation by parts formula

$$\sum \Big(Ex(t)\Delta z(t)\Big) = x(t)z(t) - \sum z(t)\Delta x(t)$$

where E is defined as $Ex(t) = x(t+1)$.

Definition 4.1. *We say $x(t) := x(t, t_0, \psi)$ is a solution of (4.1) if $x(t) = \psi(t)$ on $\mathbb{Z}_0$ and satisfies (4.1) for $t \geq t_0$.*

Definition 4.2. *The zero solution of (4.1) is stable if for any $\epsilon > 0$ and any integer $t_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi(t)| \leq \delta$ on $\mathbb{Z}_0$ implies $|x(t, t_0, \psi)| \leq \epsilon$ for $t \geq t_0$.*

Definition 4.3. *The zero solution of (4.1) is asymptotically stable if it is stable and if for any integer $t_0 \geq 0$ there exists $r(t_0) > 0$ such that $|\psi(t)| \leq r(t_0)$ on $\mathbb{Z}_0$ implies $|x(t, t_0, \psi)| \to 0$ as $t \to \infty$.*

Definition 4.4. *A solution $x(t, t_0, \psi)$ of (4.1) is said to be bounded if there exist a $B(t_0, \psi) > 0$ such that $|x(t, t_0, \psi)| \leq B(t_0, \psi)$ for $t \geq t_0$.*

Definition 4.5. *The solutions of (4.1) are said to be equi-bounded if for any t_0 and any $B_1 > 0$, there exists a $B_2 = B_2(t_0, B_1) > 0$ such that $|\psi(t)| \leq B_1$ on $\mathbb{Z}_0$ implies $|x(t, t_0, \psi)| \leq B_2$ for $t \geq t_0$.*

For the remaining of the section we assume that there is a positive constant k, $|g| \leq k$.

Lemma 4.6. *If $x(t)$ is a solution of (4.1) and satisfies the initial condition $x(t) = \psi(t)$ for $t \in \mathbb{Z}_0$, then $x(t)$ is a solution of the summation equation if and only if*

$$\begin{aligned} x(t) &= \big[x(t_0) - c(t_0-1)x(t_0 - g(t_0))\big] \prod_{s=t_0}^{t-1} v(s) + c(t-1)x(t-g(t)) \\ &+ \sum_{r=t_0}^{t-1} \Big[(a(r) - v(r))x(r) \prod_{s=r+1}^{t-1} v(s)\Big] \\ &+ \sum_{r=t_0}^{t-1} \Big([b(r) - \phi(r)]x(r-g(r)) \prod_{s=r+1}^{t-1} v(s)\Big),\ t \geq t_0 \end{aligned} \tag{4.2}$$

where

$$\phi(r) = c(r) - c(r-1)v(r).$$

$[0, T]$ *where $v : \mathbb{Z} \cap [-k, \infty) \to \mathbb{R}$ with $v(t) \neq 0$.*

Multiply both sides of (4.1) by $\prod_{s=t_0}^{t} v^{-1}(s)$ and then notice the resulting expression is equivalent to

$$\begin{aligned}
\Delta\Big[x(t)\prod_{s=t_0}^{t-1}v^{-1}(s)\Big] &= \Big[(a(t)-v(t))x(t)+b(t)x(t-g(t))\\
&+ c(t)\Delta x(t-g(t))\Big]\prod_{s=t_0}^{t}v^{-1}(s).
\end{aligned}$$

Summing the above expression from t_0 to t-1 gives

$$\begin{aligned}
x(t)\prod_{s=t_0}^{t-1}v^{-1}(s)-x(t_0) &= \sum_{r=t_0}^{t-1}\Big[(a(r)-v(r))x(r)\\
&+ b(r)x(r-g(r))+c(r)\Delta x(r-g(r))\Big]\prod_{s=t_0}^{r}v^{-1}(s)
\end{aligned}$$

Dividing both sides by $\prod_{s=t_0}^{t-1}v^{-1}(s)$, gives

$$\begin{aligned}
x(t) &= x(t_0)\prod_{s=t_0}^{t-1}v(s)+\sum_{r=t_0}^{t-1}\Big[(a(r)-v(r))x(r)\prod_{s=r+1}^{t-1}v(s)\Big]\\
&+ \sum_{r=t_0}^{t-1}\Big[b(r)x(r-g(r))\\
&\quad +c(r)\Delta x(r-g(r))\Big]\prod_{s=t_0}^{r}v^{-1}(s)\prod_{s=t_0}^{t-1}v(s)\\
&\quad = x(t_0)\prod_{s=t_0}^{t-1}v(s)+\sum_{r=t_0}^{t-1}\Big[(a(r)-v(r))x(r)\prod_{s=r+1}^{t-1}v(s)\Big]\\
&+ \sum_{r=t_0}^{t-1}\Big[b(r)x(r-g(r))\Big]\prod_{s=r+1}^{t-1}v(s)\\
&\quad +\sum_{r=t_0}^{t-1}\Big[c(r)\Delta x(r-g(r))\Big]\prod_{s=r+1}^{t-1}v(s)
\end{aligned}$$

Using summation by parts and after some calculations and simplification we arrive at (4.2).

Theorem 4.7. *Suppose $v(t)\neq 0$ for $t\geq t_0$ and $v(t)$ satisfies*

$$\Big|\prod_{s=t_0}^{t-1}v(s)\Big|\leq M$$

for $M > 0$. Also, suppose that there is an $\alpha \in (0,1)$ such that

$$\begin{aligned}|c(t-1)| \quad &+ \quad \sum_{r=t_0}^{t-1} \Big|a(r) - v(r\Big| \Big|\prod_{s=r+1}^{t-1} v(s)\Big| \\ &+ \quad \sum_{r=t_0}^{t-1} \Big[|b(r) - \phi(r)|\Big] \Big|\prod_{s=r+1}^{t-1} a(s)\Big| \le \alpha,\ t \ge t_0. \end{aligned} \tag{4.3}$$

Then solutions of (4.1) *are equi-bounded.*

Proof. Let B_1 and B_2 be two positive constants to be defined later in the proof and let $\psi(t)$ be a bounded initial function satisfying $|\psi(t)| \le B_1$ on $\mathbb{Z}_0$. Define

$$S = \Big\{\varphi : \mathbb{Z} \to \mathbb{R} |\ \varphi(t) = \psi(t) \text{ on } \mathbb{Z}_0 \text{ and } ||\varphi|| \le B_2\Big\},$$

where

$$||\varphi|| \quad = \quad \sup_{t \in \mathbb{Z}} |\varphi(t)|.$$

Then $\Big(S, ||\cdot||\Big)$ is a complete metric space.
Define mapping $P : S \to S$ by

$$\Big(P\varphi\Big)(t) = \psi(t) \text{ on } \mathbb{Z}_0$$

and

$$\begin{aligned}\Big(P\varphi\Big)(t) \quad &= \quad \Big[\psi(t_0) - c(t_0-1)\psi(t_0 - g(t_0))\Big] \prod_{s=t_0}^{t-1} v(s) + c(t-1)\varphi(t-g(t)) \\ &+ \sum_{r=t_0}^{t-1} \Big[(a(r) - v(r))\varphi(r) \prod_{s=r+1}^{t-1} v(s)\Big] \\ &+ \sum_{r=t_0}^{t-1} \Big[\Big(b(r) - \phi(r)\Big)\varphi(r - g(r)) \prod_{s=r+1}^{t-1} v(s)\Big],\ t \ge t_0. \end{aligned} \tag{4.4}$$

Let $B_1 > 0$ be given. Choose B_2 such that

$$|1 - c(t_0 - 1)| M B_1 + \alpha B_2 \le B_2 \tag{4.5}$$

We first show that P maps from S to S. By (4.5)

$$\begin{aligned}|(P\varphi)(t)| &\le |1 - c(t_0-1)| M B_1 + \alpha B_2 \\ &\le B_2 \qquad \text{for } t \ge t_0\end{aligned}$$

Thus P maps from S into itself. We next show that P is a contraction under the supremum norm. Let $\zeta, \eta \in S$. Then

$$\begin{aligned} |(P\zeta)(t) - (P\eta)(t)| &\leq \Big(|c(t-1)| + \sum_{r=t_0}^{t-1} \Big[|b(r) - \phi(r)|\Big] \Big| \prod_{s=r+1}^{t-1} v(s)\Big|\Big) ||\zeta - \eta|| \\ &+ \sum_{r=t_0}^{t-1} \Big|a(r) - v(r\Big| \prod_{s=r+1}^{t-1} v(s) ||\zeta - \eta|| \\ &\leq \alpha ||\zeta - \eta||. \end{aligned}$$

This shows that P is a contraction. Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (4.1). Hence solutions of (4.1) are equi-bounded. □

Theorem 4.8. *Assume that the hypotheses of Theorem 4.7 hold. Then the zero solution of* (4.1) *is stable.*

Proof. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$|1 - c(t_0 - 1)| M\delta + \alpha\epsilon \leq \epsilon. \tag{4.6}$$

Let $\psi(t)$ be a bounded initial function satisfying $|\psi(t)| \leq \delta$. Define the complete metric space S by

$$S = \Big\{\varphi : \mathbb{Z} \to \mathbb{R} |\ \varphi(t) = \psi(t) \text{ on } \mathbb{Z}_0 \text{ and } ||\varphi|| \leq \epsilon\Big\}.$$

Let $P : S \to S$ be defined by (4.4). Then, from the proof of Theorem 4.8 we have that P is a contraction map and for any $\varphi \in S, ||P\varphi|| \leq \epsilon$.

Hence the zero solution of (4.1) is stable.

Theorem 4.9. *Assume that the hypotheses of Theorem 4.7 hold. Also assume that*

$$\prod_{s=t_0}^{t-1} v(s) \to 0 \textit{ as } t \to \infty, \tag{4.7}$$

Then the zero solution of (4.1) *is asymptotically stable.*

Proof. We have already shown that the zero solution of (4.1) is stable. Let $r(t_0)$ be the δ of stability of the zero solution.
Let $\psi(t)$ be any initial discrete function satisfying $|\psi(t)| \leq r(t_0)$. Define

$$S^* = \Big\{\varphi : \mathbb{Z} \to \mathbb{R} |\ \varphi(t) = \psi(t) \text{ on } \mathbb{Z}_0,\ ||\varphi|| \leq \epsilon \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\Big\}.$$

Define $P : S^* \to S^*$ by (4.4). The from Theorem 4.7, the map P is a contraction and it maps from S^* into itself.
Left to show that $(P\varphi)(t) \to 0$ as $t \to \infty$.

Let $\varphi \in S^*$. Then the first first term on the right of (4.4) goes to zero. The second term on the right side of (4.4) goes to zero due condition (4.7) and the fact that $\varphi \in S^*$.
Now we show that the second term on the right side of (4.7) goes to zero as $t \to \infty$. Let $\varphi \in S^*$ then $|\varphi(t)| \leq \epsilon$. Also, since $\varphi(t) \to 0$ as $t \to \infty$, there exists a $t_1 > 0$ such that for $t > t_1$, $|\varphi(t)| < \epsilon_1$ for $\epsilon_1 > 0$. Due to condition (4.7) there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$\Big|\prod_{s=t_1}^{t} v(s)\Big| < \frac{\epsilon_1}{\alpha\epsilon}.$$

Thus for $t > t_2$, we have

$$\begin{aligned}
&\Big|\sum_{r=t_0}^{t-1}\Big[a(r)-v(r)\Big]\varphi(r)\prod_{s=r+1}^{t-1}v(s)\Big| \leq \sum_{r=t_0}^{t-1}\Big|(a(r)-v(r))\varphi(r)\prod_{s=r+1}^{t-1}v(s)\Big| \\
\leq\ & \sum_{r=t_0}^{t_1-1}\Big|(a(r)-v(r))\varphi(r)\prod_{s=r+1}^{t-1}v(s)\Big| \\
& + \sum_{r=t_1}^{t-1}\Big|(a(r)-v(r))\varphi(r)\prod_{s=r+1}^{t-1}v(s)\Big| \\
\leq\ & \epsilon\sum_{r=t_0}^{t_1-1}\Big|(a(r)-v(r))\prod_{s=r+1}^{t-1}v(s)\Big| + \epsilon_1\alpha \\
\leq\ & \epsilon\sum_{r=t_0}^{t_1-1}\Big|[a(r)-v(r)]\prod_{s=r+1}^{t_1-1}v(s)\prod_{s=t_1}^{t-1}v(s)\Big| + \epsilon_1\alpha \\
\leq\ & \epsilon\Big|\prod_{s=t_1}^{t-1}v(s)\Big|\sum_{r=t_0}^{t_1-1}\Big|[a(r)-v(r)]\prod_{s=r+1}^{t_1-1}v(s)\Big| + \epsilon_1\alpha \\
\leq\ & \epsilon\alpha|\prod_{s=t_1}^{t-1}v(s)| + \epsilon_1\alpha \\
\leq\ & \epsilon_1 + \epsilon_1\alpha.
\end{aligned}$$

This shows that the second term of (4.4) goes to zero as t goes to infinity. Showing that the last term on the right side of (4.7) goes to zero as $t \to \infty$ is similar, and hence we omit. This implies that $(P\varphi)(t) \to 0$ as $t \to \infty$.
By the contraction mapping principle, P has a unique fixed point that solves (4.1) and goes to zero as t goes to infinity. This concludes that the zero solution of (4.1) is asymptotically stable. □

Remark 4.10. *If the delay function $g(t)$ is unbounded, then we may prove a similar theorem to Theorem 4.9 by making the additional requirement that $t-g(t)\to 0,\ $ as $t\to\infty$.*

5 Example

Example 5.1. *Solutions of the linear neutral difference equation*

$$x((t+1)=\frac{2^{t+1}}{8(1+t)!}x(t-2)+\frac{2^{t+1}}{8(1+t)!}\Delta x(t-2),\ t\geq 0 \tag{5.1}$$

are equi-bounded and the zero solution is asymptotically stable.

Proof. Let $v(t)=\dfrac{1}{3(1+t)}$. Comparing terms, we see that $a(t)=0,\ b(t)=c(t)=\dfrac{2^{t+1}}{8(1+t)!}$. Set $t_0=0$. Then (4.3) is equivalent to

$$\begin{aligned}|c(t-1)| &+ \sum_{r=0}^{t-1}|v(r)|\Big|\prod_{s=r+1}^{t-1}v(s)\Big| \\ &+ \sum_{r=0}^{t-1}\Big[|b(r)-\phi(r)|\Big]\Big|\prod_{s=r+1}^{t-1}v(s)\Big| \\ &\leq \frac{2^t}{8(t)!}+\sum_{r=0}^{t-1}\prod_{s=r}^{t-1}\frac{1}{3(1+s)}+\sum_{r=0}^{t-1}\frac{2^r}{8(1+r)!}\prod_{s=r+1}^{t-1}\frac{1}{3(1+s)}.\end{aligned}$$

Now,

$$\begin{aligned}\sum_{r=0}^{t-1}\frac{2^r}{8(1+r)!}\prod_{s=r+1}^{t-1}\frac{1}{3(1+s)} &\leq 1/3\sum_{r=0}^{t-1}\frac{2^r}{8(1+r)!}\frac{1}{(r+2)(r+3)...(t)} \\ &\leq 1/3\sum_{r=0}^{t-1}\frac{2^r}{8t!} \\ &\leq \frac{1}{24t!}(2^t-1)\leq\frac{2^t}{24t!}.\end{aligned}$$

Similarly, by estimating $\frac{1}{1+s}\leq 1$, for $s\geq 0$, we have that

$$\begin{aligned}\sum_{r=0}^{t-1}\prod_{s=r}^{t-1}\frac{1}{3(1+s)} &\leq \sum_{r=0}^{t-1}(\frac{1}{3})^{t-r} \\ &\leq (\frac{1}{3})^t\sum_{r=0}^{t-1}3^r=(\frac{1}{3})^t[\frac{3^r}{2}]|_0^{t-1} \\ &\leq \frac{1}{6}[1-2^{1-t}]\leq 1/6\end{aligned}$$

Combining the two inequalities we end up with

$$\begin{aligned}
|c(t-1)| &+ \sum_{r=0}^{t-1} |v(r)| \Big| \prod_{s=r+1}^{t-1} v(s) \Big| \\
&+ \sum_{r=0}^{t-1} \Big[|b(r) - \phi(r)| \Big] \Big| \prod_{s=r+1}^{t-1} v(s) \Big| \\
&\leq \frac{2^t}{8(t)!} + \frac{1}{3} + \frac{2^t}{24t!} \\
&\leq \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2} < 1.
\end{aligned}$$

Hence (4.3) is satisfied. It is clear that condition (4.7) is satisfied for the specified value of v. This implies the zero solution is asymptotically stable, by Theorem 4.9. Left to show solutions are equi-bounded.

Since $t_0 = 0$, we have that $\mathbb{Z}_0 = [-2, 0]$.
Let $B_1 > 0$ be given and $\psi(t) : \mathbb{Z}_0 \to \mathbb{R}$ be a given initial function with $|\psi(t)| \leq B_1$. We need to choose B_2 so that (4.5) is satisfied. It is clear that $c(t_0 - 1) = c(-1) = \dfrac{1}{8}$, and hence $|1 - c(t_0 - 1)| = 1 - \dfrac{1}{8} = \dfrac{7}{8}$. In addition

$$\Big| \prod_{s=0}^{t-1} v(s) \Big| \leq M$$

is satisfied for $M = \frac{1}{3}$. From the above calculation for asymptotic stability, we see that $\alpha = \frac{1}{2}$. Now we choose B_2 such that

$$\frac{7}{24} B_1 \leq \frac{B_2}{2}.$$

Then, in our case, inequality (4.5) corresponds to

$$|1 - c(t_0 - 1)| M B_1 + \alpha B_2 \leq B_2.$$

Or equivalently,

$$\frac{7}{24} B_1 + \frac{B_2}{2} \leq B_2,$$

is satisfied. $\square$

Remark 5.2. *We mention that the work of Islam-Yankson in [12] can not be applied to our example due to the absence of the linear term $a(t)x(t)$.*

It is worth mentioning that the results of Section 4 can be easily extended to the nonlinear neutral difference equation

$$x(t+1) = a(t)x(t) + c(t)\Delta x(t - g(t)) + q(t, x(t), x(t - g(t))) \tag{5.2}$$

where $a(t), c(t)$ and $g(t)$ are defined as before. We assume that, $q(t,0,0) = 0$ for the stability and q is locally Lipschitz in x and y. That is, there is a $K > 0$ so that if $|x|, |y|, |z|$ and $|w| \leq K$ then

$$|q(x,y) - q(z,w)| \leq L|x-z| + E|y-w|$$

for some positive constants L and E.
Note that

$$\begin{aligned}
|q(x,y)| &= |q(x,y) - q(0,0) + q(0,0)| \\
&\leq |q(x,y) - q(0,0)| + |q(0,0)| \\
&\leq L|x| + E|y|.
\end{aligned}$$

Remark 5.3. *The method of Section 4 can be easily used to extend the existence of periodic solutions to systems of the form of* (4.1) *and* (5.2)*, see [15].*

References

[1] Adivar, M,. Islam, M. and Raffoul, Y., *Separate contraction and existence of periodic solutions in totally nonlinear delay differential equations,* Hacettepe Journal of Mathematics and Statistics, 41 (1) (2012), 1-13.

[2] T. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[3] T. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, New York, 1985.

[4] T. Burton and T. Furumochi, *Fixed points and problems in stability theory*, Dynam. Systems Appl. 10(2001), 89-116.

[5] Burton, T.A. *Integral equations, implicit functions, and fixed points*, Proc. Amer. Math. Soc. 124 (1996), 2383-2390.

[6] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 1999.

[7] S. Elaydi, *Periodicity and stability of linear Volterra difference systems,* J. Math. Anal. Appl. ,181(1994), 483-492.

[8] S. Elaydi and S. Murakami, *Uniform asymptotic stability in linear Volterra difference equations,* J. Differ. Equations Appl., 3(1998), 203-218.

[9] P. Eloe, M. Islam and Y. Raffoul, *Uniform asymptotic stability in nonlinear Volterra discrete systems*, Special Issue on Advances in Difference Equations IV, Computers Math. Appl., 45(2003), 1033-1039.

[10] Y. Hino and S. Murakami, *Total stability and uniform asymptotic stability for linear Volterra equations,* J. London Math. Soc., 43(1991), 305-312.

[11] M. Islam and Y. Raffoul, *Exponential stability in nonlinear difference equations* , J. Differ. Equations Appl., 9(2003), 819-825.

[12] M. Islam and E. Yankson *Boundedness and stability in nonlinear delay difference equations employing fixed point theory*, Electronic Journal of Qualitative Theory of Differential Equations 2005, No. 26, 1-18.

[13] W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications*, Harcourt Academic Press, San Diego, 2001.

[14] J. Liu, *A First Course In The Qualitative Theory of Differential Equations*,Pearson Education, Inc., Upper Saddle River, New Jersey 07458, 2003.

[15] M. Maroun and Y. Raffoul , *Periodic solutions in nonlinear neutral difference equations with functional delay,* J. Korean Math. Soc. 42 (2005), no. 2, 255-268.

[16] R. Medina, *Asymptotic behavior of Volterra difference equations,* Computers Math. Appl. 41(2001), 679-687.

[17] R. Medina, *The asymptotic behavior of the solutions of a Volterra difference equation,* Computers Math. Appl. 181(1994), 19-26.

[18] Y. Raffoul, *Qualitative Theory of Volterra Difference Equations*, Springer, New York, 2018.

[19] Y. Raffoul, *Stability and periodicity in discrete delay equations, J. Math. Anal. Appl.* 324 (2006) 1356-1362.

[20] Y. Raffoul, *Stability in neutral nonlinear differential equations with functional delays using fixed point theory,* Mathematical and Computer Modelling, 40(2004), 691-700.

[21] Y. Raffoul, *General theorems for stability and boundedness for nonlinear functional discrete systems,* J. Math. Anal. Appl. ,279(2003), 639-650.

CUBO A Mathematical Journal
Vol.21, N^{o}03, (63–74). December 2019
http://dx.doi.org/10.4067/S0719-06462019000300063

Beta-almost Ricci solitons on Sasakian 3-manifolds

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ABSTRACT

In this paper we characterize the Sasakian 3-manifolds admitting β-almost Ricci solitons whose potential vector field is a contact vector field. Among others we prove that a β-almost Ricci soliton whose potential vector field is a contact vector field on a Sasakian 3-manifold is shrinking, Einstein and non-trivial. Moreover, we prove that this type of manifolds are isometric to a sphere of radius $\sqrt{7}$.

RESUMEN

En este artículo caracterizamos las 3-variedades Sasakianas que admiten solitones β-casi Ricci cuyo campo de vectores potencial es un campo de vectores de contacto. Entre otros, probamos que un solitón β-casi Ricci cuyo campo de vectores potencial es un campo de vectores de contacto en una 3-variedad Sasakiana se contrae, es Einstein y no trivial. Más aún, probamos que este tipo de variedades son isométricas a una esfera de radio $\sqrt{7}$.

Keywords and Phrases: Ricci soliton, β-almost Ricci soliton, Sasakian 3-manifolds, Einstein.
2010 AMS Mathematics Subject Classification: 53C15, 53C25.

1 Introduction

In 1982, R. S. Hamilton [17] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g = -2S, \tag{1}$$

where S denotes the Ricci tensor of g. Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [5]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (2) are interesting and useful in physics and are often referred as quasi-Einstein ([6],[7]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [14] who discusses some aspects of it. Recently, the notion of almost Ricci soliton has been introduced in [24] by Piagoli, Riogoli, Rimoldi and Setti.

The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive respectively. Ricci solitons have been studied by several authors ([8], [9], [18], [19], [20], [27], [28], and many others).

Recently, Gomes, Wang and Xia [26] generalized almost Ricci soliton to h -almost Ricci soliton as follows:

Definition 1.1. *A complete connected Riemannian manifold* (M^{2n+1}, g) *is said to be a* β*-almost Ricci soliton, denoted by* $(M^{2n+1}, g, V, \beta, \lambda)$*, if there exist a smooth vector field* V *on* M^{2n+1} *such that*

$$S + \frac{\beta}{2}\pounds_V g + \lambda g = 0, \tag{3}$$

where λ *and* β *are smooth functions on* M^{2n+1}*.* λ *is called soliton function and* V *is called the potential vector field.*

A β-almost Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive respectively. A β-almost Ricci soliton is called β-Ricci soliton if λ is constant. A β-almost Ricci soliton is said to be trivial, that is, Einstein if the flow vector field V is homothetic, that is, $\mathcal{L}_V g = cg$, for some constant c. Otherwise, it is called non-trivial. A β-almost Ricci soliton is said to be β-almost gradient Ricci soliton if the potential vector field V is the gradient of a smooth function f on M^{2n+1}, that is, $V = Df$, where D is the gradient operator of g on M^{2n+1}. For convenience, we denote $(M^{2n+1}, g, Df, \beta, \lambda)$ as a β-almost gradient Ricci soliton with potential function f.

In particular, a Ricci soliton is a 1-almost Ricci soliton with constant soliton λ and an almost Ricci soliton is nothing but a 1-almost Ricci soliton. Recently, Ghosh and Patra studied [16] the k-almost Ricci solitons on contact geometry. In [1], Barros and Ribeiro proved that a compact almost Ricci soliton with constant scalar curvature is isometric to an Euclidean sphere. In this connection, a theorem has also been proved by Gomes, Wang and Xia in [26] for β-almost Ricci soliton which is given as follows:

Theorem 1.1. *[26] Let $(M^n, g, V, \beta, \lambda), n \geq 3$ be a non-trivial β-almost Ricci soliton with constant scalar curvature r. If M^n is compact, then it is isometric to a standard sphere $S^n(c)$ of radius $c = \sqrt{\frac{2n(2n+1)}{r}}$.*

The above Theorem will be used in later to prove our results.

The paper is organized as follows:

After preliminaries in Section 2, we study β-almost Ricci solitons on a Sasakian 3-manifold. Among others we prove that β-almost Ricci solitons whose potential vector field is a contact vector field on Sasakian 3-manifolds are shrinking and Einstein. Beside these, we prove that this type of manifolds are isometric to a sphere of radius $\sqrt{7}$. Also we prove that a β-almost Ricci soliton whose potential vector field is a contact vector field on a Sasakian 3-manifold is non-trivial.

2 Preliminaries

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([2], [3])

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{4}$$

The first and one of the remaining three relations in (4) imply the other two relations in (4). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{5}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with the (ϕ, ξ, η)-structure, that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{6}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{7}$$

and

$$g(X, \xi) = \eta(X), \tag{8}$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).
An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{9}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.
Given the contact metric manifold (M, η, ξ, ϕ, g), we define a symmetric (1,1)-tensor field h as $h = \frac{1}{2}L_\xi \phi$, where $L_\xi \phi$ denotes Lie differentiation in the direction of ξ. We have the following identities ([2], [3]):

$$h\xi = 0, \qquad h\phi + \phi h = 0, \tag{10}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{11}$$

$$\nabla_\xi \phi = 0, \tag{12}$$

$$R(\xi, X)\xi - \phi R(\xi, \phi X)\xi = 2(h^2 + \phi^2)X, \tag{13}$$

$$(\nabla_\xi h)X = \phi X - h^2 \phi X + \phi R(\xi, X)\xi, \tag{14}$$

$$S(\xi, \xi) = 2n - \mathrm{tr}h^2, \tag{15}$$

$$R(X, Y)\xi = -(\nabla_X \phi)Y + (\nabla_Y \phi)X - (\nabla_X \phi h)Y + (\nabla_Y \phi h)X. \tag{16}$$

Here, ∇ is the Levi-Civita connection and R is the Riemannian curvature tensor of (M, g) with the sign convention defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{17}$$

for vector fields X, Y, Z on M. The tensor $l = R(.,\xi)\xi$ is the Jacobi operator with respect to the characteristic field ξ.
If the characteristic vector field ξ is a Killing vector field, the contact metric manifold (M, η, ξ, ϕ, g) is called K-contact manifold. This is the case if and only if $h = 0$. The contact structure on M is said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by (5), is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian metrices are K-contact and K-contact metrics on 3-manifolds are Sasakian. For a Sasakian manifold, the following hold ([2], [3]):

$$\nabla_X \xi = -\phi X, \tag{18}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{19}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{20}$$

$$Q\xi = 2n\xi, \tag{21}$$

where Q denotes the $(1,1)$-tensor metrically equivalent to the Ricci tensor of g. The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z &= [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{22}$$

where S and r are the Ricci tensor and scalar curvature respectively and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.
It is known that the Ricci tensor of a Sasakian 3-manifold is given by [4]

$$S(X,Y) = \frac{1}{2}\{(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)\} \tag{23}$$

where r is the scalar curvature which need not be constant, in general. So, g is Einstein (hence has constant curvature 1) if and only if $r = 6$.
As a consequence of (23), we have

$$S(X,\xi) = 2\eta(X). \tag{24}$$

Contact metric manifolds have also been studied by several authors ([4], [10], [11], [12], [13], [21], [22], [23], [25], and many others).

Definition 2.1. *([16]) A vector field* V *on a contact manifold is said to be a contact vector field if it preserves the contact form* η, *that is*

$$\pounds_V \eta = \psi \eta, \tag{25}$$

for some smooth function ψ *on* M. *When* $\psi = 0$ *on* M, *the vector field* V *is called a strict contact vector field.*

Lemma 2.1. *([15]) If a vector field* X *leaves the structure tensor* ϕ *of the contact metric manifold* M *invariant, then there exists a constant* c *such that* $\pounds_X g = c(g + \eta \otimes \eta)$.

3 β-almost Ricci solitons on Sasakian 3-manifolds

In this section we characterize Sasakian 3-manifolds M^3 admitting β-almost Ricci solitons whose potential vector field V is a contact vector field. Then the equations (3) and (25) hold good. The equation (3) can be exhibited as

$$S(X, Y) + \frac{\beta}{2}\{g(\nabla_X V, Y) + g(X, \nabla_Y V)\} + \lambda g(X, Y) = 0. \tag{26}$$

Using (23) in the above equation we get

$$\begin{aligned} \beta\{g(\nabla_X V, Y) + g(X, \nabla_Y V)\} &= -(r + 2\lambda - 2)g(X, Y) \\ &\quad + (r - 6)\eta(X)\eta(Y). \end{aligned} \tag{27}$$

Tracing the equation (27) we obtain

$$\beta \operatorname{div} V = -(r + 3\lambda). \tag{28}$$

With the help of (25) we have

$$\pounds_V d\eta = d\pounds_V \eta = (d\psi) \wedge \eta + \psi(d\eta). \tag{29}$$

Let us consider ω as the volume form of the manifold M^3, that is,

$$\omega = \eta \wedge d\eta \neq 0. \tag{30}$$

Taking Lie derivative of the preceding equation along the potential vector field V and using (25) and (29) we have $\pounds_V \omega = 2\psi\omega$, and hence

$$\operatorname{div} V = 2\psi. \tag{31}$$

Using the foregoing equation in (28) we infer

$$r = -2\psi\beta - 3\lambda. \tag{32}$$

The soliton equation (3) also can be represented as

$$S(X, Y) + \frac{\beta}{2}(\pounds_V g)(X, Y) + \lambda g(X, Y) = 0. \tag{33}$$

Substituting $X = Y = \xi$ in (33) we get

$$\beta g(\pounds_V \xi, \xi) = \lambda + 2. \tag{34}$$

Putting $Y = \xi$ in (33) and using (24),

$$\frac{\beta}{2}(\pounds_V \eta)(X) - \frac{\beta}{2} g(X, \pounds_V \xi) + (\lambda + 2)\eta(X) = 0. \tag{35}$$

Making use of (25) we obtain

$$\beta \pounds_V \xi = (\psi\beta + 2\lambda + 4)\xi. \tag{36}$$

By the virtue of (34) and (36) we have

$$\psi\beta = -\lambda - 2. \tag{37}$$

Using (37), (36) entails

$$\beta \pounds_V \xi = (\lambda + 2)\xi. \tag{38}$$

From (9) we deduce that

$$(\pounds_V d\eta)(X, Y) = (\pounds_V g)(X, \phi Y) + g(X, (\pounds_V \phi)Y). \tag{39}$$

Multiplying both sides of (39) by β and then using (33) we infer

$$\beta(\pounds_V d\eta)(X, Y) = -2S(X, \phi Y) - 2\lambda g(X, \phi Y) + \beta g(X, (\pounds_V \phi)Y). \tag{40}$$

In view of (23) and (40) we get

$$\beta(\pounds_V d\eta)(X, Y) = -(r + 2\lambda - 2)g(X, \phi Y) + \beta g(X, (\pounds_V \phi)Y). \tag{41}$$

From (29) we derive

$$(\pounds_V d\eta)(X, Y) = \frac{1}{2}\{d\psi(X)\eta(Y) - d\psi(Y)\eta(X)\} + \psi g(X, \phi Y). \tag{42}$$

Comparing (41) and (42), after simplification we obtain

$$2\beta(\pounds_V \phi)Y = 2(r + 2\lambda - 2)\phi Y + \beta\eta(Y)D\psi - \beta(Y\psi)\xi + 2\psi\beta\phi Y. \tag{43}$$

Replacing Y by ξ, we get

$$2\beta(\mathcal{L}_V\phi)\xi = \beta D\psi - \beta(\xi\psi)\xi. \tag{44}$$

With the help of (4) and (36) we find that

$$\beta(\mathcal{L}_V\phi)\xi = 0. \tag{45}$$

Applying (45) on (44) we have

$$D\psi = (\xi\psi)\xi. \tag{46}$$

Taking inner product of (46) with X gives

$$d\psi(X) = (\xi\psi)\eta(X), \tag{47}$$

or equivalently,

$$d\psi = (\xi\psi)\eta. \tag{48}$$

Taking exterior derivative we get

$$d(\xi\psi)\wedge\eta + (\xi\psi)d\eta = 0. \tag{49}$$

Taking wedge product of (49) with η we have

$$(\xi\psi)\eta\wedge d\eta = 0, \tag{50}$$

from which it follows that

$$\xi\psi = 0, \tag{51}$$

Since $\eta\wedge d\eta \neq 0$, and by (48),

$$d\psi = 0 \tag{52}$$

and hence ψ is constant. Integrating (31) and then using **Divergence Theorem** we infer

$$\psi = 0. \tag{53}$$

Thus the potential vector field V becomes a strict contact vector field and hence we have the following:

Theorem 3.1. *Let* $(M^3, g, V, \beta, \lambda)$ *be a non-trivial* β*-almost Ricci soliton whose potential vector field is a contact vector field on a Sasakian* 3*-manifold. Then the potential vector field is a strict* *contact vector field.*

By the virtue of (37) and (53) we find

$$\lambda = -2. \tag{54}$$

Therefore, the β-almost Ricci soliton is shrinking. Thus we are in a position to state that

Theorem 3.2. *A non-trivial β-almost Ricci soliton $(M^3, g, V, \beta, \lambda)$ whose potential vector field is a contact vector field on a Sasakian 3-manifold is shrinking.*

Making use of (53) and (54), from (32) we get

$$r = 6. \tag{55}$$

Then we can conclude that

Theorem 3.3. *The scalar curvature of a non-trivial β-almost Ricci soliton $(M^3, g, V, \beta, \lambda)$ whose potential vector field is a contact vector field on a Sasakian 3-manifold is 6.*

With the help of (55) from (23) we have

$$S(X, Y) = 2g(X, Y). \tag{56}$$

Hence we can state the following:

Theorem 3.4. *A non-trivial β-almost Ricci soliton $(M^3, g, V, \beta, \lambda)$ whose potential vector field is a contact vector field on Sasakian 3-manifold is Einstein.*

From (55) we can say that r is constant. Then in view of Theorem 1.1 we can conclude the following:

Theorem 3.5. *Let $(M^3, g, V, \beta, \lambda)$ be a non-trivial β-almost Ricci soliton whose potential vector field is a contact vector field on a Sasakian 3-manifolds. Then $(M^3, g, V, \beta, \lambda)$ is isometric to a sphere $S^3(c)$ of radius $c = \sqrt{7}$.*

Using (53), (54) and (55) in (43) we infer

$$(\pounds_V \phi)Y = 0, \tag{57}$$

as we have considered β as positive, that is, V leaves the structure tensor ϕ of the Sasakian 3-manifold invariant. Then, by Lemma 2.2, exists a constant a such that

$$\pounds_V g = a(g + \eta \otimes \eta), \tag{58}$$

which shows that the β-almost Ricci solitons are non-trivial. Thus our next theorem can be stated as follows:

Theorem 3.6. *Let* $(M^3, g, V, \beta, \lambda)$ *be a non-trivial* β*-almost Ricci soliton whose potential vector field is a contact vector field on a Sasakian* 3*-manifolds. Then the* β*-almost Ricci solitons are non-trivial.*

Acknowledgement: The authors are thankful to the referee for his/her valuable suggestions and comments towards the improvement of the paper. The author Debabrata Kar is supported by the Council of Scientific and Industrial Research, India (File no : 09/028(1007)/2017-EMR-1).

References

[1] Barros, A. and Ribeiro, Jr., *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc., **140**(3) (2012), 1033-1040.

[2] Blair, D. E., *Lecture notes in Mathematics*, **509**, Springer-Verlag Berlin(1976).

[3] Blair, D. E., *Riemannian Geometry of contact and symplectic manifolds*, Birkhäuser, Boston, 2002.

[4] Blair, D. E., Koufogiorgos, T. and Sharma, R., *A classification of* 3*-dimensional contact metric manifolds with* $Q\phi = \phi Q$, Kodai Math. J., **13**(1990), 391-401.

[5] Calin, C. and Crasmareanu, M., From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds, Bull. Malays. Math. Soc., **33**(3) (2010), 361-368.

[6] Chave, T. and Valent, G., Quasi-Einstein metrics and their renoirmalizability properties, Helv. Phys. Acta., **69** (1996) 344-347.

[7] Chave, T. and Valent, G., On a class of compact and non-compact quasi-Einstein metrics and their renoirmalizability properties, Nuclear Phys. B., **478** (1996), 758-778.

[8] Deshmukh, S., *Jacobi-type vector fields on Ricci solitons*, Bull. Math. Soc. Sci. Math. Roumanie, **55**(103)1 (2012), 41-50.

[9] Deshmukh, S., Alodan, H. and Al-Sodais, H., *A Note on Ricci Soliton*, Balkan J. Geom. Appl. **16**(1) (2011), 48-55.

[10] De, U. C. and Biswas, S., *A note on* ξ*-conformally flat contact manifolds*, Bull. Malays. Math. Soc., **29** (2006), 51-57.

[11] De, U. C. and Mondal, A. K., *Three dimensional Quasi-Sasakian manifolds and Ricci solitons*, SUT J. Math., **48**(1) (2012), 71-81.

[12] De, U. C. and Mondal, A. K., *The structure of some classes of* 3*-dimensional normal almost contact metric manifolds*, Bull. Malays. Math. Soc., **36**(2) (2013), 501-509.

[13] De, Avik and Jun, J. B., *On N(k)-contact metric manifolds satisfying certain curvature conditions*, Kyungpook Math. J., **51**(4) (2011), 457-468.

[14] Friedan, D., Nonlinear models in $2+\epsilon$ dimensions, Ann. Phys., **163** (1985), 318-419.

[15] Ghosh, A. and Sharma, R., Sasakian metric as a Ricci soliton and related results, J. Geom. Phys., **75** (2014), 1-6.

[16] Ghosh, A. and Patra, D. S., The k-almost Ricci solitons and contact geometry, J. Korean. Math., **55**(1) (2018), 161-174.

[17] Hamilton, R. S., The Ricci flow on surfaces, Mathematics and general relativity, (Santa Cruz, CA, 1986), 237-262. Contemp. Math., **71**, American Math. Soc., 1988.

[18] Hamilton, R. S., Three manifolds with positive Ricci curvature, J. Differential Geom., **17** (1982), 255-306

[19] Ivey, T., *Ricci solitons on compact 3-manifolds*, Diff. Geom. Appl., **3** (1993),301-307.

[20] Kim, B.H., *Fibered Riemannian spaces with quasi-Sasakian structures*, Hiroshima Math. J., **20** (1990), 477-513.

[21] Majhi, P., and De, U. C., *Classifications of N(k)-contact metric manifolds satisfying certain curvature conditions*, Acta Math. Univ. Commenianae, **LXXXIV**(1) (2015), 167-178.

[22] Özgür, C., *Contact metric manifolds with cyclic-parallel Ricci tensor*, Diff. Geom. Dynamical systems, **4** (2002), 21-25.

[23] Özgür, C. and Sular, S., *On N(k)-contact metric manifolds satisfying certain conditions*, SUT J. Math., **44**(1) (2008), 89-99.

[24] Pigola, S., Rigoli, M., Rimoldi, M. and Setti, A., *Ricci almost solitons*, Ann. Sc. Norm. Super. Pisa Cl. Sci, (5)**10**(4) (2011), 757-799.

[25] Taleshian, A. and Hosseinzabeh, A. A., *Investigation of Some Conditions on* $N(k)$-*Quasi Einstein Manifolds*, Bull. Malays. Math. Soc. **34**(3)(2011), 455-464.

[26] Gomes, J. N., Wang, Q. and Xia, C., *On the* h-*almost Ricci soliton*, J. Geom. Phys., **114** (2017), 216-222.

[27] Wang, Y. and Liu, X., *Ricci solitons on three-dimensional* η-*Einstein almost Kenmotsu manifolds*, Taiwanese Journal of Mathematics, **19**(1) (2015), 91-100.

[28] Wang, Y., *Ricci solitons on 3-dimensional cosympletic manifolds*, Math. Slovaca, **67**(4) (2017), 979-984.

CUBO A Mathematical Journal
Vol.21, Nº 03, (75–91). December 2019
http://dx.doi.org/10.4067/S0719-06462019000300075

Weak solutions to Neumann discrete nonlinear system of Kirchhoff type

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ABSTRACT

We prove the existence of weak solutions for discrete nonlinear system of Kirchhoff type. We build some Hilbert spaces with suitable norms. We define the notion of weak solution corresponding to the problem (1.1). The proof of the main result is based on a minimization method of an energy functional J.

RESUMEN

Probamos la existencia de soluciones débiles para sistemas discretos no-lineales de tipo Kirchhoff. Construimos algunos espacios de Hilbert con normas apropiadas. Definimos la noción de solución débil correspondiente al problema (1.1). La demostración del resultado principal se basa en un método de minimización de un funcional de energía J.

Keywords and Phrases: Nonlinear difference equations, anisotropic nonlinear discrete systems, minimization methods, weak solutions.

2010 AMS Mathematics Subject Classification: 47A75; 35B38; 35P30; 34L05; 34L30.

1 Introduction

In this paper, we are going to investigate the existence of weak solutions for the following anisotropic nonlinear discrete system.

For $i = 1, \cdots, n$

$$\begin{cases} -M\left(A(k-1, \Delta u_i(k-1))\right)\Delta(a(k-1, \Delta u_i(k-1))) = f_i(k, u(k)), \ k \in \mathbb{Z}[1,T] \\ \\ \Delta u_i(0) = \Delta u_i(T) = 0 \end{cases} \tag{1.1}$$

where $\Delta u_i(k) = u_i(k+1) - u_i(k)$ is the forward difference operator for any $i = 1, \cdots, n$; $\mathbb{Z}[1,T] = \{1, \ldots, T\}$ for $T \geq 2$ and a, f_i are functions to be defined later.

In the last few years, great attention has been paid to the study of fourth-order nonlinear difference equations. These equations have been widely used to study discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. For background and recent results, we refer the reader to [2]-[12], [14] and the references therein.
Note that in recent years, much attention has been paid to problems not local since they appear in physical phenomena like the theory of nonlinear elasticity, heat diffusion, etc. Among this problems, we find Kirchhoff type problems, which are known by the presence of the term $M(\int_\Omega |\nabla u|^2)\Delta u$ in the continuous case. As far as we know, the first study which deals with anisotropic discrete boundary value problems of $p(.)$-Kirchhoff type difference equation was done by Yucedag (see [11]). The function $M(A(k-1, \Delta u(k-1)))$ which appear in the left-hand side of problem (1.1) is more general.
The main operator $\Delta(a(k-1, \Delta u(k-1)))$ in problem (1.1) can be seen as a discrete counterpart of the anisotropic operator $\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a\left(x, \frac{\partial}{\partial x_i} u\right)$. The functional a derives from a potential with $a(k, \xi) = \frac{\partial}{\partial \xi} A(k, \xi)$.
Our goal is to use a minimization method in order to establish some existence results of solutions of (1.1). The idea of the proof is to transfer the problem of the existence of solutions for (1.1) into the problem of existence of a minimizer for some associated energy functional. This method was successfully used by Bonanno et al. [1] for the study of an eigenvalue nonhomogeneous Neumann problem, where, under an appropriate oscillating behaviour of the nonlinear term, they proved the existence of a determined open interval of positive parameters for which the problem considered admits infinitely many weak solutions that strongly converge to zero, in an appropriate Orlicz Sobolev space.
Motivated by the work of [13] where J. Zhao proved the existence of positive solutions, the approach presented in this article is different than the one given in the papers mentioned above. To the best of

our knowledge , results on existence of weak solutions of system (1.1), using minimization method, have not been found in the literature.
The remaining part of this paper is organized as follows. Section 2 is devoted to mathematical preliminaries. The main existence result is proved in Section 3. In the Section 4, we give an extension of our system.

2 Mathematical background

In the T-dimensional Hilbert space

$$H = \{u : \mathbb{Z}[0, T+1] \longrightarrow \mathbb{R}^n \quad \text{such that} \quad \Delta u(0) = \Delta u(T) = 0\},$$

with the inner product

$$\langle u, v\rangle = \sum_{i=1}^{n} \sum_{k=1}^{T+1} \Delta u_i(k-1)\Delta v_i(k-1), \qquad \forall\, u, v \in H,$$

we consider the norm

$$\|u\| = \left(\sum_{i=1}^{n} \sum_{k=1}^{T+1} |\Delta u_i(k-1)|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

We denote

$$H_i = \{u_i : \mathbb{Z}[0, T+1] \longrightarrow \mathbb{R} \quad \text{such that} \quad \Delta u_i(0) = \Delta u_i(T) = 0\}, \quad \text{for} \quad i = 1, \cdots, n$$

with the norm

$$|u_i|_h = \left(\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^2\right)^{\frac{1}{2}} \quad \forall\, u_i \in H_i \quad \text{for} \quad i = 1, \cdots, n. \tag{2.2}$$

Moreover, we may consider H_i with the following norm

$$|u_i|_m = \left(\sum_{k=1}^{T} |u_i(k)|^m\right)^{\frac{1}{m}} \quad \forall\, u_i \in H_i, \quad m \geq 2 \quad \text{for} \quad i = 1, \cdots, n. \tag{2.3}$$

We have the following inequalities (see [2])

$$T^{(2-m)/(2m)}|u_i|_2 \leq |u_i|_m \leq T^{1/m}|u_i|_2, \quad \forall\, u_i \in H_i, \quad m \geq 2 \quad \text{for} \quad i = 1, \cdots, n. \tag{2.4}$$

Let the function

$$p : \mathbb{Z}[0, T] \longrightarrow (2, +\infty) \tag{2.5}$$

denoted by

$$p^- = \min_{k\in\mathbb{Z}[0,T]} p(k) \qquad \text{and} \qquad p^+ = \max_{k\in\mathbb{Z}[0,T]} p(k).$$

For the data a and f_i, we assume the following.

$$(H_1). \quad \begin{cases} a(k,.) : \mathbb{R} \to \mathbb{R}, \quad k \in \mathbb{Z}[0,T] \text{ and there exists } \ A(.,.) : \mathbb{Z}[0,T] \times \mathbb{R} \to \mathbb{R} \\ \text{which satisfies } \ a(k,\xi) = \frac{\partial}{\partial \xi} A(k,\xi) \ \text{ and } \ A(k,0) = 0, \quad \text{for all } k \in \mathbb{Z}[0,T]. \end{cases}$$

(H_2). For all $k \in \mathbb{Z}[0,T]$ and $\xi \neq \eta$

$$(a(k,\xi) - a(k,\eta)) .(\xi - \eta) > 0. \tag{2.6}$$

(H_3). For any $k \in \mathbb{Z}[0,T]$, $\xi \in \mathbb{R}$, we have

$$A(k,\xi) \geq \frac{1}{p(k)} |\xi|^{p(k)}. \tag{2.7}$$

(H_4). For each $k \in \mathbb{Z}[0,T]$, the function $f_i(k,.) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is jointly continuous and there exists $(\alpha_i(.))_{1 \leq i \leq n} : \mathbb{Z}[0,T] \longrightarrow (0,+\infty)$ and a function $(r_i(.))_{1 \leq i \leq n} : \mathbb{Z}[0,T] \longrightarrow [2,+\infty)$ such that

$$|f_i(k,u)| \leq \alpha_i(k) \left(1 + |u_i(k)|^{r_i(k)-1}\right) \tag{2.8}$$

where $\quad 2 \leq r_i(k) < p^- \quad \text{for} \quad i = 1, \cdots, n.$
In what follows, we denote by :

$$r^- = \min_{\{(k,i) \in \mathbb{Z}[0,T] \times \mathbb{Z}[1,n]\}} r_i(k) \qquad \text{and} \qquad r^+ = \max_{\{(k,i) \in \mathbb{Z}[0,T] \times \mathbb{Z}[1,n]\}} r_i(k).$$

For each $i = 1, \cdots, n$, there exists $h_i \in \mathbb{R}^n$ such that

$$\nabla F_i(k,u)(h_i) = f_i(k,u) \quad \forall u \in H \quad \text{for} \quad i = 1, \cdots, n. \tag{2.9}$$

By (2.8) there exists $(\beta_i(.))_{1 \leq i \leq n} : \mathbb{Z}[0,T] \longrightarrow (0,+\infty)$ such that

$$|F_i(k,u)| \leq \beta_i(k) \left(1 + |u_i(k)|^{r_i(k)}\right) \quad \text{for} \quad i = 1, \cdots, n \tag{2.10}$$

where

$$0 < \underline{\beta} = \inf_{\{(k,i) \in \mathbb{Z}[0,T] \times \mathbb{Z}[1,n]\}} \beta_i(k) \leq \sup_{\{(k,i) \in \mathbb{Z}[0,T] \times \mathbb{Z}[1,n]\}} \beta_i(k) = \overline{\beta} < +\infty. \tag{2.11}$$

(H_5). We also assume that the function $M : (0,+\infty) \longrightarrow (0,+\infty)$ is continuous and non-decreasing and there exist positive numbers B_1, B_2 with $B_1 \leq B_2$ and $\alpha > 1$ such that

$$B_1 t^{\alpha - 1} \leq M(t) \leq B_2 t^{\alpha - 1} \quad \text{for} \quad t > t^* > 0. \tag{2.12}$$

Example 2.1.
There are many functions satisfying both $(H_1) - (H_4)$. *Let us mention the following.*

- $A(k,\xi) = \dfrac{1}{p(k)}\left(\left(1+|\xi|^2\right)^{p(k)/2} - 1\right)$, *where* $a(k,\xi) = \left(1+|\xi|^2\right)^{(p(k)-2)/2}\xi$, $\forall\ k \in \mathbb{Z}[0,T],\ \xi \in \mathbb{R}$,
- $f_i(k,\xi) = 1 + \left|\xi_i\right|^{p(k)-1}, \quad \forall\ (k,i) \in \mathbb{Z}[0,T] \times \mathbb{Z}[1,n]$ *and* $\xi = (\xi_1, \cdots, \xi_n)$,
- $M(t) = 1, \quad \forall\ t \in (0,+\infty)$.

Moreover, we may consider H with the following norm

$$\|u\|_m = \sum_{i=1}^{n}\left(\sum_{k=1}^{T}|u_i(k)|^m\right)^{\frac{1}{m}}, \quad \forall\ u \in H \quad \text{and} \quad m \geq 2. \tag{2.13}$$

Using the relation (2.4) we can prove the following lemma.

Lemma 2.2. *We have the following inequalities*

$$T^{(2-m)/(2m)}\|u\|_2 \leq \|u\|_m \leq T^{1/m}\|u\|_2, \quad \forall\ u \in H \quad \textit{and} \quad m \geq 2. \tag{2.14}$$

We need the following auxiliary results throughout our paper.

Lemma 2.3.

(1) *There exist two positive constant* $C_1,\ C_2$ *such that*

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq C_1\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - C_2, \tag{2.15}$$

for all $u \in H$ *with* $|u_i|_h > 1$.

(2) *For any* $m \geq 2$ *there exists a positive constant* c_m *such that*

$$\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^m \leq c_m\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^m, \quad \forall u \in H. \tag{2.16}$$

Indeed,

(1) By [6], there exists the positive constants λ_i and μ_i for $i = 1, \cdots n$

$$\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq \lambda_i\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - \mu_i \ \ \forall\ u_i \in H_i \ \text{and}\ |u_i|_h > 1.$$

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq \min_{1\leq i\leq n}(\lambda_i)\sum_{i=1}^{n}\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - \max_{1\leq i\leq n}(\mu_i)\,n.$$

Since the function $x \longmapsto x^{\frac{p^-}{2}}$ is convex because $p^- > 2$, then we have

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq \min_{1\leq i\leq n}(\lambda_i)\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - \max_{1\leq i\leq n}(\mu_i)\, n.$$

We deduce that

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq C_1\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - C_2.$$

(2) By [8], for any $m \geq 2$ there exists a positive constant c_m such that for $i = 1, \cdots, n$

$$\sum_{k=1}^{T}|u_i(k)|^m \leq c_m \sum_{k=1}^{T+1}|\Delta u_i(k-1)|^m \quad \forall\, u_i \in H_i.$$

Therefore

$$\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^m \leq c_m \sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^m \quad \forall\, u \in H.$$

3 Existence of weak solutions

In this section, we study the existence of weak solution of problem (1.1).

Definition 3.1. *A weak solutions of problem (1.1) is* $u \in H$ *such that*

$$\begin{aligned}&\sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1))\right)\sum_{k=1}^{T+1}a(k-1,\Delta u_i(k-1))\Delta v_i(k-1)\right]\\&=\sum_{i=1}^{n}\sum_{k=1}^{T}f_i(k,u(k))v_i(k)\end{aligned} \tag{3.1}$$

for all $v \in H$.

Note that, since H is a finite dimensional space, the weak solutions coincide with the classical solution the problem (1.1).

Theorem 3.2. *Assume that* $(H_1)-(H_5)$ *holds. Then, there exists a weak solution of the problem (1.1).*

To prove this, we define the energy functional $J : H \longrightarrow \mathbb{R}$ by

$$J(u) = \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) - \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big) \tag{3.2}$$

where $\widehat{M(t)} = \int_0^t M(s)ds.$

Lemma 3.3. *The functional* J *is well defined on* H *and is of class* $C^1\big(H, \mathbb{R}\big)$ *with the derivative given by*

$$\begin{aligned} \langle J'(u), v\rangle \ &= \sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1)\right)\sum_{k=1}^{T+1} a(k-1, \Delta u_i(k-1))\Delta v_i(k-1)\right] \\ &- \ \sum_{i=1}^{n}\sum_{k=1}^{T} f_i(k, u(k))v_i(k), \end{aligned} \tag{3.3}$$

for all $u, v \in H$.

Indeed, let's

$$I(u) = \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) \text{ and } \Lambda(u) = \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big).$$

Since $\widehat{M}(.)$, $A(k, .)$ and $F(k, .)$ are continuous for all $k \in \mathbb{Z}[0, T]$, then

$$|I(u)| = \left|\sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right)\right| < +\infty,$$

$$|\Lambda(u)| \ = \ \left|\sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big)\right| < +\infty.$$

The energy functional J is well defined on H.

It is not difficult to see that the functional I derivative are give by

$$\langle I'(u), v\rangle = \sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1)\right)\sum_{k=1}^{T+1} a(k-1, \Delta u_i(k-1))\Delta v_i(k-1)\right] \tag{3.4}$$

On the other hand, for all $u, v \in H$, there exists $h_i \in \mathbb{R}^n$ such that

$$\begin{aligned}
\langle \Lambda'(u), v\rangle &= \lim_{t\to 0^+} \frac{\Lambda(u+tv)-\Lambda(u)}{t}\\
&= \lim_{t\to 0^+} \sum_{i=1}^{n}\sum_{k=1}^{T} \frac{F_i(k,u(k)+tv(k))-F_i(k,u(k))}{t}\\
&= \sum_{i=1}^{n}\sum_{k=1}^{T} \lim_{t\to 0^+} \frac{F_i(k,u(k)+tv(k))-F_i(k,u(k))}{t}\\
&= \sum_{i=1}^{n}\sum_{k=1}^{T} \nabla F_i(k,u(k))(h_i)v_i(k)\\
&= \sum_{i=1}^{n}\sum_{k=1}^{T} f_i(k,u(k))v_i(k).
\end{aligned}$$

The functional J is clearly of class C^1 □

Lemma 3.4. *The functional* J *is lower semi-continuous.*

Indeed since the functional Λ is completely continuous and weakly lower semi-continuous, we have to prove the semi-continuity of I.

A is convex with respect to the second variable according (H_1) and (H_2). With the assumption (H_5) we conclude that I is convex. Thus, it is enough to show that I is lower semi-continuous. For this, we fix $u \in H$ and $\varepsilon > 0$. Since I is convex, we deduce that, for any $v \in H$.

$$\begin{aligned}
I(v) &\geq I(u) + \langle I'(u), v-u\rangle\\
&\geq I(u) - \sum_{i=1}^{n}\Bigg[M\left(\sum_{k=1}^{T+1} A(k-1,\Delta u_i(k-1))\right)\\
&\qquad\qquad \times \sum_{k=1}^{T+1} |a(k-1,\Delta u_i(k-1))||\Delta v_i(k-1)-\Delta u_i(k-1)|\Bigg]\\
&\geq I(u) - C_M\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1} |a(k-1,\Delta u_i(k-1))||\Delta v_i(k-1)-\Delta u_i(k-1)|\right),
\end{aligned}$$

where $C_M = \left(\sum_{i=1}^{n} M\left(\sum_{k=1}^{T+1} A(k-1,\Delta u_i(k-1)\right)\right)$

By using Schwartz inequality, we get :

$$\begin{aligned} I(v) &\geq I(u) - C_M \sum_{i=1}^{n} \Bigg[\left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \\ &\qquad\qquad \times \left(\sum_{k=1}^{T+1} |\Delta v_i(k-1) - \Delta u_i(k-1)|^2 \right)^{\frac{1}{2}} \Bigg] \\ &\geq I(u) - C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \\ &\qquad\qquad \times \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |\Delta v_i(k-1) - \Delta u_i(k-1)|^2 \right)^{\frac{1}{2}} \right] \end{aligned}$$

By (2.2)

$$I(v) \geq I(u) - C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \left[\sum_{i=1}^{n} |v_i - u_i|_h \right].$$

Since H_i is finite dimensional, there exist the positive constants θ_i for $i = 1, \cdots, n$ such that

$$|v_i|_h \leq \theta_i |v_i|_2 \quad \forall\, v_i \in H_i. \tag{3.5}$$

Then,

$$\begin{aligned} I(v) &\geq I(u) - C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \left[\sum_{i=1}^{n} \theta_i |v_i - u_i|_2 \right] \\ &\geq I(u) - \max_{1 \leq i \leq n} (\theta_i)\, C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \left[\sum_{i=1}^{n} |v_i - u_i|_2 \right]. \end{aligned}$$

Also, the space H is finite dimensional, there exists a positive constant γ such that:

$$\|u\|_2 \leq \gamma \|u\| \quad \forall\, u \in H.$$

From this, we have

$$\begin{aligned} I(v) &\geq I(u) - \gamma \max_{1 \leq i \leq n} (\theta_i)\, C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \|v - u\| \\ &\geq I(u) - \left[1 + \gamma \max_{1 \leq i \leq n} (\theta_i)\, C_M \sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \|v - u\| \end{aligned}$$

Finally

$$I(v) \geq I(u) - S(T,u)\|v-u\| \geq I(u) - \varepsilon, \tag{3.6}$$

for all $v \in H$ with $\|v-u\| < \delta = \frac{\varepsilon}{S(T,u)}$, where

$$S(T,u) = 1 + \gamma \max_{1\leq i\leq n}(\theta_i)\, C_M \sum_{i=1}^{n}\left(\sum_{k=1}^{T+1} |a(k-1,\Delta u_i(k-1))|^2\right)^{\frac{1}{2}}.$$

We conclude that J is weakly lower semi-continuous.

Proposition 3.5. *The functional* J *is coercive and bounded from below.*

Indeed, according to (2.7), (2.10)-(2.12) we have

$$\begin{aligned}
J(u) &= \sum_{i=1}^{n}\widehat{M}\left(\sum_{k=1}^{T+1} A(k-1,\Delta u_i(k-1))\right) - \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k,u(k)\big)\\
&\geq \frac{B_1}{\alpha(p^+)^{\alpha}}\left[\sum_{i=1}^{n}\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha}\right] - \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k,u(k)\big)\\
&\geq \frac{B_1}{\alpha(p^+)^{\alpha}}\left[\sum_{i=1}^{n}\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha}\right] - \sum_{i=1}^{n}\sum_{k=1}^{T} \beta_i(k)\left(1+|u_i(k)|^{r_i(k)}\right)\\
&\geq \frac{B_1}{\alpha(p^+)^{\alpha}}\left[\sum_{i=1}^{n}\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha}\right] - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T} \left(1+|u_i(k)|^{r_i(k)}\right)\\
&\geq \frac{B_1}{\alpha(p^+)^{\alpha}}\left[\sum_{i=1}^{n}\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha}\right] - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T} |u_i(k)|^{r_i(k)} - \overline{\beta}nT.
\end{aligned}$$

There exist η_i and ν_i such that

$$\begin{aligned}
J(u) \geq\ & \frac{B_1}{\alpha(p^+)^{\alpha}}\left[\min_{1\leq i\leq n}(\eta_i)\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha} - \max_{1\leq i\leq n}(\nu_i)\right]\\
&- \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \overline{\beta}nT.
\end{aligned} \tag{3.7}$$

To prove the coerciveness of the functional J, we may assume that $\|u\| > 1$ and we deduce from the above inequality (2.15) that

$$\begin{aligned} J(u) \geq & \frac{B_1}{\alpha(p^+)^\alpha}\left[\min_{1\leq i\leq n}(\eta_i)\left(C_1\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - C_2\right)^\alpha - \max_{1\leq i\leq n}(\nu_i)\right] \\ & -\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \overline{\beta}nT. \end{aligned}$$

There exist a function $K(\alpha, C)$ such that

$$\begin{aligned} J(u) \geq & \frac{B_1}{\alpha(p^+)^\alpha}\left(\min_{1\leq i\leq n}(\eta_i)C_1^\alpha\|u\|^{\alpha p^-} - \min_{1\leq i\leq n}(\eta_i)K(\alpha, C)C_2^\alpha - \max_{1\leq i\leq n}(\nu_i)\right) \\ - & \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \overline{\beta}nT. \end{aligned}$$

Namely

$$J(u) \geq A_1\|u\|^{\alpha p^-} - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - A_2,$$

where

$$A_1 = \frac{B_1}{\alpha(p^+)^\alpha}\min_{1\leq i\leq n}(\eta_i)C_1^\alpha$$

and

$$A_2 = \frac{B_1}{\alpha(p^+)^\alpha}\left(\min_{1\leq i\leq n}(\eta_i)K(\alpha, C)C_2^\alpha + \max_{1\leq i\leq n}(\nu_i)\right) + \overline{\beta}nT.$$

So

$$\begin{aligned} J(u) & \geq A_1\|u\|^{\alpha p^-} - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - A_2 \\ & \geq A_1\|u\|^{\alpha p^-} - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r^+} - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r^-} - A_2. \end{aligned}$$

Using (2.16)

$$J(u) \geq A_1\|u\|^{\alpha p^-} - (C_{r^-})\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^{r^-} - (C_{r^+})\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^{r^+} - A_2$$

By using (2.4) there exists the positive constants K_1 and K_2 such that

$$J(u) \geq A_1\|u\|^{\alpha p^-} - K_1\sum_{i=1}^{n}\left(\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^-}{2}} - K_2\sum_{i=1}^{n}\left(\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^+}{2}} - A_2.$$

There exist the positive constants A_3, A_4, A_5 and A_6 such that

$$\begin{aligned} J(u) &\geq A_1 \|u\|^{\alpha p^-} - K_1 A_3 \left(\sum_{i=1}^{n} \sum_{k=1}^{T} |\Delta u_i(k)|^2 \right)^{\frac{r^-}{2}} \\ &- K_1\, A_4 - A_5\, K_2 \left(\sum_{i=1}^{n} \sum_{k=1}^{T} |\Delta u_i(k)|^2 \right)^{\frac{r^+}{2}} - K_2\, A_6 - A_2. \end{aligned}$$

Consequently, there exist the positive constants A_7 , A_8 and A_9 such that

$$J(u) \geq A_1 \|u\|^{\alpha p^-} - A_7 \|u\|^{r^-} - A_8 \|u\|^{r^+} - A_9. \tag{3.8}$$

Recall that $p^- > \dfrac{r^+}{\alpha} \geq \dfrac{r^-}{\alpha}$. Then J is coercive.

Besides, for $\|u\| \leq 1$, we have with (3.7)

$$\begin{aligned} J(u) &\geq \frac{B_1}{\alpha (p^+)^\alpha} \left[\min_{1 \leq i \leq n} (\eta_i) \left(\sum_{i=1}^{n} \sum_{k=1}^{T+1} |\Delta u_i(k-1)|^{p(k-1)} \right)^{\alpha} - \max_{1 \leq i \leq n} (\nu_i) \right] \\ &- \overline{\beta} \sum_{i=1}^{n} \sum_{k=1}^{T} |u_i(k)|^{r_i(k)} - \overline{\beta} n T \\ &\geq -\frac{B_1}{\alpha (p^+)^\alpha} \max_{1 \leq i \leq n} (\nu_i) - \overline{\beta} \sum_{i=1}^{n} \sum_{k=1}^{T} |u_i(k)|^{r_i(k)} - \overline{\beta} n T \\ &\geq -\frac{B_1}{\alpha (p^+)^\alpha} \max_{1 \leq i \leq n} (\nu_i) - \overline{\beta} \sum_{i=1}^{n} \sum_{k=1}^{T} |u_i(k)|^{r^-} - \overline{\beta} \sum_{i=1}^{n} \sum_{k=1}^{T} |u_i(k)|^{r^+} - \overline{\beta} n T. \end{aligned}$$

Using (2.16)

$$J(u) \geq -\tfrac{B_1}{\alpha (p^+)^\alpha} \max_{1 \leq i \leq n} (\nu_i) - (K_{r^-}) \overline{\beta} \sum_{i=1}^{n} \sum_{k=1}^{T} |\Delta u_i(k)|^{r^-} - (K_{r^+}) \overline{\beta} \sum_{i=1}^{n} \sum_{k=1}^{T} |\Delta u_i(k)|^{r^+} - \overline{\beta} n T.$$

By using (2.14) there exists the positives constants K_1' and K_2' such that

$$J(u) \geq -\frac{B_1}{\alpha (p^+)^\alpha} \max_{1 \leq i \leq n} (\nu_i) - K_1' \sum_{i=1}^{n} \left(\sum_{k=1}^{T} |\Delta u_i(k)|^2 \right)^{\frac{r^-}{2}} - K_2' \sum_{i=1}^{n} \left(\sum_{k=1}^{T} |\Delta u_i(k)|^2 \right)^{\frac{r^+}{2}} - \overline{\beta} n T.$$

There exist the positive constants C'_3, C'_4, C'_5 and C'_6 such that

$$\begin{aligned} J(u) \quad &\geq \quad -\frac{B_1}{\alpha(p^+)^\alpha} \max_{1\leq i\leq n}(\nu_i) - K'_1 C'_3 \left(\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^-}{2}} \\ &- \quad K'_1\ C'_4 - C'_5\ K'_2 \left(\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^+}{2}} - K'_2\ C'_6 - \overline{\beta} nT. \end{aligned}$$

Consequently, there exist the positive constants C'_7 and C'_8 such that

$$\begin{aligned} J(u) \quad &\geq \quad -\frac{B_1}{\alpha(p^+)^\alpha} \max_{1\leq i\leq n}(\nu_i) - C'_7\|u\|^{r^-} - K'_1\ C'_4 - C'_8\|u\|^{r^+} - K'_2\ C'_6 - \overline{\beta} nT \\ &\geq \quad -\frac{B_1}{\alpha(p^+)^\alpha} \max_{1\leq i\leq n}(\nu_i) - C'_7 - K'_1\ C'_4 - C'_8 - K'_2\ C'_6 - \overline{\beta} nT. \end{aligned}$$

Thus, J is bounded from below □

Since J is weakly lower semi-continuous, bounded from below and coercive on H, using the relation between critical points of J and problem (1.1), we deduce that J has a minimizer which is a weak solution to problem (1.1).

4 An extension

In this section we are going to show that the existence result obtained for system (1.1) can be extended. Let's consider the following system.

For $i = 1, \cdots, n$

$$\begin{cases} -M\left(A(k-1, \Delta u_i(k-1))\right)\Delta(a(k-1, \Delta u_i(k-1))) + \sigma_i(k)\phi(k, u_i(k)) \\ \qquad\qquad\qquad\qquad = \delta_i(k) f_i(k, u(k)),\ \forall\ k \in \mathbb{Z}[1, T] \\ \Delta u_i(0) = \Delta u_i(T) = 0, \end{cases} \tag{4.1}$$

where $T \geq 2$ is a fixed integer, and we shall use the following assumption.

(H_6). $\sigma_i : \mathbb{Z}[1, T] \longrightarrow \mathbb{R}$ and $\delta_i : \mathbb{Z}[1, T] \longrightarrow \mathbb{R}$ are such that $\sigma_i(k) \geq \sigma_0 > 0$ for

$(k, i) \in \mathbb{Z}[1, T] \times \mathbb{Z}[1, n]$ and $0 < \delta_i(k) \leq \sup_{\{(k,i)\in\mathbb{Z}[1,T]\times\mathbb{Z}[1,n]\}} |\delta_i(k)| = \delta_0.$

(H_7). $\phi(k, t) = |t|^{p(k)-2}t$ for $(k, t) \in \mathbb{Z}[0, T] \times \mathbb{R}$.

In the $T-$dimensional Hilbert space H with the inner product

$$\langle u, v\rangle = \sum_{i=1}^{n}\sum_{k=1}^{T+1}\Delta u_i(k-1)\Delta v_i(k-1) + \sum_{i=1}^{n}\sum_{k=1}^{T+1} u_i(k)v_i(k),$$

we consider the norm

$$\|u\| = \sqrt{\langle u, u\rangle} = \left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2 + \sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^2\right)^{\frac{1}{2}}.$$

Definition 4.1. *A weak solution of problem (4.1) is a function* $u \in H$ *such that*

$$\sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1)\right)\sum_{k=1}^{T+1}a(k-1,\Delta u_i(k-1))\Delta v_i(k-1)\right]$$
$$+\sum_{i=1}^{n}\sum_{k=1}^{T}\sigma_i(k)|u_i(k)|^{p(k)-2}u_i(k)v_i(k) = \sum_{i=1}^{n}\sum_{k=1}^{T}\delta_i(k)f_i(k,u(k))v_i(k).$$

for all $v \in H$.

Theorem 4.2. *Under the assumptions* (H_1)- (H_6) *the problem (4.1) has a least weak solution in* H.

Indeed, for $u \in H$ we define the energy functional corresponding to system (4.1) by

$$J(u) = \sum_{i=1}^{n}\widehat{M}\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1))\right) + \sum_{i=1}^{n}\sum_{k=1}^{T}\frac{\sigma_i(k)}{p(k)}|u_i(k)|^{p(k)} - \sum_{i=1}^{n}\sum_{k=1}^{T}\delta_i(k)F_i\big(k,u(k)\big).$$

Obviously, J is class $C^1(H,\mathbb{R})$ and is weakly lower semicontinuous, and we show that

$$\begin{aligned}\langle J'(u), v\rangle \;&=\; \sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1)\right)\sum_{k=1}^{T+1}a(k-1,\Delta u_i(k-1))\Delta v_i(k-1)\right]\\ &+\;\sum_{i=1}^{n}\sum_{k=1}^{T}\sigma_i(k)|u_i(k)|^{p(k)-2}u_i(k)v_i(k) - \sum_{i=1}^{n}\sum_{k=1}^{T}\delta_i(k)f_i\big(k,u(k)\big)v_i(k).\end{aligned}$$

for all $u, v \in H$.
This implies that the weak solution of system(4.1) coincides with the critical points of the functional J. It suffices to prove that J is bounded below and coercive in order to complete the proof.

$$\begin{aligned}
J(u) &= \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) + \sum_{i=1}^{n}\sum_{k=1}^{T} \frac{\sigma_i(k)}{p(k)}|u_i(k)|^{p(k)} - \sum_{i=1}^{n}\sum_{k=1}^{T} \delta_i(k)F_i\big(k, u(k)\big) \\
&\geq \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) - \sum_{i=1}^{n}\sum_{k=1}^{T} \delta_i(k)F_i\big(k, u(k)\big) \\
&\geq \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) - \delta_0 \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big).
\end{aligned}$$

We obtain

$$\begin{aligned}
J(u) &\geq \tfrac{B_1}{\alpha(p^+)^\alpha}\left[\min_{1\leq i\leq n}(\eta_i)\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha} - \max_{1\leq i\leq n}(\nu_i)\right] \\
&- \delta_0\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \delta_0\overline{\beta}nT.
\end{aligned} \tag{4.2}$$

For $\|u\| > 1$, by the same procedure, we prove that

$$J(u) \geq A'_1\|u\|^{\alpha p^-} - A'_7\|u\|^{r^-} - A'_8\|u\|^{r^+} - A'_9,$$

where $A'_1,\ A'_7,\ A'_8$ and A'_9 are the positive constants.

Hence $p^- > \dfrac{r^+}{\alpha} \geq \dfrac{r^-}{\alpha}$, J is coercive.

If $\|u\| \leq 1$ by (4.2) we have

$$J(u) \geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - \delta_0\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \delta_0\overline{\beta}nT.$$

By the same reasoning

$$J(u) \geq -D_1 - \delta_0\overline{\beta}nT$$

where $D_1 > 0$.
Thus, J is bounded from below □
Since J is weakly lower semi-continuous, bounded from below and coercive on H, using the relation between critical points of J and problem (4.1), we deduce that J has a minimizer which is a weak solution to problem (4.1).

Competing interests

The authors declare that there is no conflict of interest regarding the publication of the paper.

Acknowledgment

The authors express their deepest thanks to the editor and anonymous referee for their comments and suggestions on the article.

References

[1] G. Bonanno, G. Molica Bisci and V. Radulescu; *Arbitrarity small weak solutions for nonlinear eigenvalue problem in Orlicz-Sobolev spaces,* Monatshefte fur Mathematik, vol. **165**, no. 3-4, pp. 305-318, 2012.

[2] X. Cai and J. Yu; *Existence theorems for second-order discrete boundary value problems*, J. Math. Anal. Appl. **320** (2006), 649-661.

[3] A. Castro and R. Shivaji; *Nonnegative solutions for a class of radically symmetric nonpositone problems*, Proceedings of the American Mathematical Society, vol **106**, pp. 735-740,1989.

[4] Y. Chen, S. Levine, and M.Rao; *Variable exponent, linear growth functionals in image restoration,* SIAM Journal on Applied Mathematics, vol.**66**, no.4, pp. 1383-1406, 2006.

[5] L. Diening; *Theoretical and numerical results for electrorheogica fluids*, [PhD. thesis], University of Freiburg, 2002.

[6] A. Guiro, I. Nyanquini and S. Ouaro; *On the solvability of discrete nonlinear Neumann problems involving the* $\mathtt{p(x)}$*-Laplacian*, Adv. Differ. equ. **32** (2011).

[7] B. Koné and S. Ouaro; *Weak solutions for anisotropic discrete boundary value problems*, J. Differ. Equ. Appl. **16**(2) (2010), 1-11.

[8] M. Mihailescu, V. Radulescu and S. Tersian; *Eigenvalue problems for anisotropic discrete boundary value problems*, J. Differ. Equ. Appl. **15** (2009), 557-567.

[9] K. R. Rajagopal and M. Ruzicka; *Mathematical modeling of electrorheological materials,* Continuum Mechanics and Thermodynamics, vol.**13**, pp.59-78, 2001.

[10] M. Ruzicka, Electrorheological Fluids; *Modeling and Mathematical Theory*, vol. **1748** of Lecture Notes in Mathematics, Springer,Berlin, Germany, 2000.

[11] Z. Yucedag; *Existence of solutions for anisotropic discrete boundary value problems of Kirchhoff type*, Int. J. Differ. Equ. Appl, Vol. **13**(1) (2014), 1-15.

[12] G. Zhang and S. Liu; *On a class of semipositone discrete boundary value problem*, J. Math. Anal. Appl. **325** (2007), 175-182.

[13] J. Zhao; *Positive solutions and eigenvalue intervals for a second order p-Laplacian discrete system*, Adv. Differ. equ. **2018** 2018:281.

[14] V. Zhikov; *Averaging of functionals in the calculus of variations and elasticity,* Mathematics of the USSR-Izvestiya, vol.**29** (1987), pp. 33-66.

CUBO A Mathematical Journal
Vol.21, Nº 03, (93–105). December 2019
http://dx.doi.org/10.4067/S0719-06462019000300093

Wave propagation through a gap in a thin vertical wall in deep water

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ABSTRACT

The problem of oblique scattering of surface water waves by a vertical wall with a gap submerged in infinitely deep water is re-investigated in this paper. It is formulated in terms of two first kind integral equations, one involving the difference of potential across the wetted part of the wall and the other involving the horizontal component of velocity across the gap. The integral equations are solved approximately using one-term Galerkin approximations involving constants multiplied by appropriate weight functions whose forms are dictated by the physics of the problem. This is in contrast with somewhat complicated but known solutions of corresponding deep water integral equations for the case of normal incidence, used earlier in the literature as one-term Galerkin approximation. Ultimately this leads to very closed (numerically) upper and lower bounds of the reflection and transmission coefficients so that their averages produce fairly accurate numerical estimates for these coefficients. Known numerical results for normal incidence and for a narrow gap obtained by other methods in the literature are recovered, thereby confirming the correctness of the method employed here.

RESUMEN

En este artículo re-investigamos el problema de dispersión oblicua de ondas superficiales de agua por una pared vertical con una abertura sumergida en agua infinitamente profunda. Se formula en términos de dos ecuaciones integrales de primera especie, una involucrando la diferencia de potencial a través de la parte mojada de la pared y la otra involucrando la componente horizontal de la velocidad a través de la apertura. Las ecuaciones integrales son resueltas aproximadamente usando aproximaciones de Galerkin de un término involucrando constantes multiplicadas por funciones peso apropiadas, cuyas formas son dictadas por la física del problema. Esto se contrapone con lo complicado de soluciones conocidas para las correspondientes ecuaciones integrales de agua profunda para el caso de incidencia normal, usadas anteriormente en la literatura como aproximaciones de Galerkin de un término. Últimamente esto lleva a cotas superiores e inferiores muy cercanas (numéricamente) para los coeficientes de reflexión y transmisión de tal suerte que sus promedios producen estimaciones numéricas razonablemente precisas para estos coeficientes. Se recuperan resultados numéricos conocidos en la literatura para la incidencia normal y para una apertura delgada, confirmando que los métodos empleados son correctos.

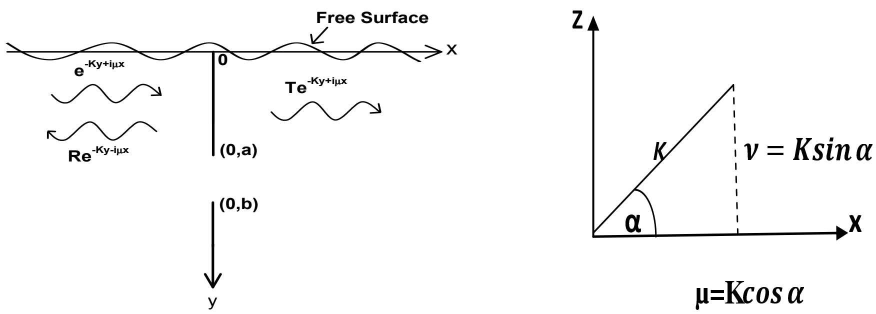
Keywords and Phrases: Thin vertical wall, submerged gap, integral equations, One-term Galerkin approximations, Constant as basis, Reflection and transmission coefficients.

2010 AMS Mathematics Subject Classification: 76B07, 76B15.

1 Introduction

The problem of oblique scattering of surface water waves by a thin vertical wall with a gap of arbitrary width submerged in infinitely deep water is re investigated here within the framework of linearized theory of water waves. Porter [10] investigated this problem for normal incidence of a surface wave train employing a reduction procedure and also an integral equation formulation, both leading to the same Riemann-Hilbert problem in the theory of complex variable, and the reflection and transmission coefficients are obtained in closed forms in terms of some definite integrals which could be computed numerically. When the gap is narrow, Tuck [12] earlier employed the method of matched asymptotic expansion to obtained the transmission coefficient approximately in terms of an analytical expression. Packham and Williams [9] employed an integral equation formulation based on Green's integral theorem to reduce the problem of narrow gap in uniform finite depth water to a first kind integral equation in horizontal component of velocity across the gap. They solved the integral equation approximately exploiting the concept of narrowness of the gap, and obtained an approximate analytical expression for the transmission coefficient. Mandal [7] employed an integral equation formulation based on Havelock's [6] expansion of water wave potential to solve the narrow gap problem in deep water for normal incidence, and obtained the transmission coefficients approximately by exploiting the concept of narrowness of the gap as has been done by Packham and Williams [9]. Chakrabarti et al [1] re-investigated Porter's problem by reducing it to a special logarithmic singular integral equation involving two unknown constants, one involving the unknown reflection coefficient, which were ultimately determined by two solvability criteria. Das et al [2] investigated the oblique scattering problem by formulating it in terms of two first kind integral equations after employing Havelock's [6] expansion of water wave potential, one involving the horizontal component of velocity across the gap and the other involving the difference of potential across the wetted parts of the wall. These were then solved approximately employing one-term Galerkin approximations involving somewhat complicated but exact solutions of the corresponding integral equations for the case of normal incidence as could be found from Porter [10]. Also, one-term Galerkin technique was employed recently by Roy et al [11] while studying the problem of water wave scattering by a pair of thin vertical barriers with unequal gaps submerged in deep water. However, it involves somewhat complicated but known exact solutions of the corresponding integral equations for a single barrier partially immersed in deep water and for normal incidence, as basis functions.

In the present paper, this problem is re-investigated employing one-term Galerkin approximation technique wherein the one-term approximations are taken to be simply constants multiplied by appropriate weight functions whose forms are dictated by the physics of the problem. This technique leads to very accurate close bounds(numerical) for the reflection and transmission coefficients so that their averages produce accurate numerical estimates for these coefficients. Known numerical results for normal incidence and also for a narrow gap obtained by other methods in the literature are recovered from the results obtained by the present method as special cases, thereby confirming the correctness of the method. Numerical results obtained by the present method are displayed graphically in a number of figures. It may be noted that this type of one-term Galerkin method to solve integral equations has not been employed in the literature on water waves earlier.



(a) Barrier configuration (b) Angle of incident wave

Figure 1: Sketch of the problem.

2 Mathematical formulation and solution

A Cartesian co-ordinate system is taken in which y-axis is chosen vertically downwards in the fluid region and the x, z-plane is taken as the rest position of the free surface. For a thin vertical wall with a gap submerged in deep water, its wetted parts are represented by $x = 0, y \in L = (0, a) \bigcup (b, \infty)$, wherein the gap is represented by $x = 0, y \in \bar{L} = (a, b)$. The problem is described in figure 1 wherein R and $|T|$ denote the reflection and transmission coefficient respectively. Full details of the problem is given in Das et al.[2] . For the problem of oblique scattering of surface water waves by the wall with a gap, let $f(y)(y \in \bar{L})$ denote the horizontal component of velocity across the gap, $g(y)(y \in \bar{L})$ denote the difference of potential function across the wetted parts of the wall, R and T denote the reflection and transmission coefficients respectively. Then the behaviors of $f(y)$ and $g(y)$ at the end points $y = a, y = b$ are given by

$$f(y) = \begin{cases} O\left((y-a)^{-\frac{1}{2}}\right) & \text{as } y \to a + 0, \\ O\left((b-y)^{-\frac{1}{2}}\right) & \text{as } y \to b - 0, \end{cases} \tag{2.1a}$$

and

$$g(y) = \begin{cases} O\left((a-y)^{\frac{1}{2}}\right) & \text{as } y \to a-0, \\ O\left((y-b)^{\frac{1}{2}}\right) & \text{as } y \to b+0. \end{cases} \tag{2.1b}$$

The relation between R, T and $f(y), g(y)$ are given by

$$T = 1 - R = -2i\sec\alpha \int_{\bar{L}} f(y)e^{-Ky}\,dy, \tag{2.2a}$$

$$R = -K\int_L g(y)e^{-Ky}\,dy, \tag{2.2b}$$

where α is the angle of incidence of train of surface water waves on the thin wall, $K = \frac{\sigma^2}{g}$, σ being the angular frequency and g is the gravity.

Let

$$F(y) = -\frac{2}{\pi R} f(y), y \in \bar{L}, \tag{2.3a}$$

$$G(y) = \frac{1}{\pi i K\cos\alpha(1-R)} g(y), y \in L, \tag{2.3b}$$

then it is easy to see that $G(y)$ and $F(y)$ satisfy the first kind integral equations (cf. Das et al [2], Mandal and Chakrabarti [8])

$$(\mathcal{M}G)(y) \equiv \int_L G(u)\mathcal{M}(y,u)du = e^{-Ky}, y \in L \tag{2.4a}$$

and

$$(\mathcal{N}F)(y) \equiv \int_L F(u)\mathcal{N}(y,u)du = e^{-Ky}, y \in \bar{L} \tag{2.4b}$$

where

$$\mathcal{M}(y,u) = \lim_{\epsilon\to+0}\int_0^\infty \frac{k_1 S(k,y)S(k,u)}{k^2+K^2}e^{-\epsilon k}dk, \tag{2.5a}$$

and

$$\mathcal{N}(y,u) = \int_0^\infty \frac{S(k,y)S(k,u)}{k_1(k^2+K^2)}dk, \tag{2.5b}$$

where $k_1 = \left(k^2+\nu^2\right)^{\frac{1}{2}}, \nu = K\sin\alpha, S(k,y) = k\cos ky - K\sin ky$ while $(2.2a)$ and $(2.2b)$ produce

$$\int_{\bar{L}} F(y)e^{-Ky}\,dy = C, \tag{2.6a}$$

$$\int_L G(y)e^{-Ky}\,dy = \frac{1}{\pi^2K^2C} \tag{2.6b}$$

where

$$C = \frac{1-R}{i\pi R}\cos\alpha. \tag{2.7}$$

(2.5a) and (2.5b) show that $\mathcal{M}(y,u)$ and $\mathcal{N}(y,u)$ are real and symmetric so that $G(u), F(u)$ satisfying (2.4a) and (2.4b) respectively are real and hence, C satisfying (2.6a) as well as (2.6b), is an unknown real quantity. Once C is found R and $T(= 1 - R)$ can be calculated using (2.7).

If $G(y)$ and $F(y)$ are chosen as one-term Galerkin approximations given by

$$G(y) \approx c_0 g_0(y), y \in L; F(y) \approx d_0 f_0(y), y \in \bar{L}, \tag{2.8}$$

then exploiting the properties of symmetry, self-adjointness and positive semi-definiteness of the integral operators $(\mathcal{M}G)(y)$ and $(\mathcal{N}f)(y)$ defined by (2.4) proceeding as in Evans and Morris [4] and Das et al [2], it can be shown that C has the bounds A, B

$$B \leq C \leq A \tag{2.9}$$

where A and B can be expressed in terms of integrals involving $g_0(y)$ and $f_0(y)$ respectively as given by

$$A = \frac{1}{\pi^2 K^2} \frac{\int_L g_0(y)(\mathcal{M}g_0)(y)dy}{(\int_L g_0(y)e^{-Ky}dy)^2}, \tag{2.10}$$

$$B = \frac{(\int_{\bar{L}} f_0(y)e^{-Ky}dy)^2}{\int_{\bar{L}} f_0(y)(\mathcal{N}f)(y)dy}. \tag{2.11}$$

It may be noted that A, B are independent of c_0, d_0 so that these can be chosen to be unity. The upper and lower bounds for $|R|$ and $|T|$ are now obtained as

$$R_1 \leq |R| \leq R_2, T_1 \leq |T| \leq T_2 \tag{2.12}$$

where

$$R_1 = \frac{1}{(1 + \pi^2 A^2 \sec^2 \alpha)^{\frac{1}{2}}}, R_2 = \frac{1}{(1 + \pi^2 B^2 \sec^2 \alpha)^{\frac{1}{2}}}, \tag{2.13a}$$

$$T_1 = \frac{\pi B \sec \alpha}{(1 + \pi^2 A^2 \sec^2 \alpha)^{\frac{1}{2}}}, T_2 = \frac{\pi A \sec \alpha}{(1 + \pi^2 B^2 \sec^2 \alpha)^{\frac{1}{2}}}. \tag{2.13b}$$

Das et al [2] chose $g_0(y)$ and $f_0(y)$ as the exact solutions of the integral equations (2.4a) and (2.4b) for the case of normal incidence($\alpha = 0^0$) and these involve quite complicated expressions (cf. Mandal and Chakrabarti [8]). Here we choose $g_0(y), f_0(y)$ as

$$g_0(y) = \begin{cases} \left(1 - \frac{y}{a}\right)^{\frac{1}{2}}, 0 < y < a, \\ \\ e^{-Ky}\left(\frac{y}{b} - 1\right)^{\frac{1}{2}}, b < y < \infty \end{cases} \tag{2.14a}$$

and

$$f_0(y) = \frac{a}{\{(y-a)(b-y)\}^{\frac{1}{2}}}, a < y < b. \tag{2.14b}$$

This choice of $f_0(y)$ and $g_0(y)$ is dictated by the behaviors of $f(y)$ and $g(y)$ at the end points $y = a$ and $y = b$.

Then, after substituting (2.14a) in (2.10), A is obtained as

$$A = \frac{\int_0^\infty \frac{k_1}{k^2+K^2}\left[kU(a,b,k,K) - KV(a,b,k,K)\right]^2 dk}{\pi^2 K^2 W^2(a,b,k,K)} \tag{2.15}$$

where

$$U(a,b,k,K) = \int_0^a \left(1-\frac{y}{a}\right)^{\frac{1}{2}} \cos ky\,dy + \int_b^\infty e^{-Ky}\left(\frac{y}{b}-1\right)^{\frac{1}{2}} \cos ky\,dy,$$

$$V(a,b,k,K) = \int_0^a \left(1-\frac{y}{a}\right)^{\frac{1}{2}} \sin ky\,dy + \int_b^\infty e^{-Ky}\left(\frac{y}{b}-1\right)^{\frac{1}{2}} \sin ky\,dy,$$

$$W(a,b,K) = \int_0^a e^{-Ky}\left(1-\frac{y}{a}\right)^{\frac{1}{2}} dy + \int_b^\infty e^{-2Ky}\left(\frac{y}{b}-1\right)^{\frac{1}{2}} dy.$$

$U(a,b,k,K), V(a,b,k,K)$ and $W(a,b,K)$ can be expressed analytically in terms of Young's and lower incomplete gamma functions(cf. Gradshteyn and Ryzhik [5]).

Similarly, after substituting (2.14b) in (2.11), B is obtained as

$$B = \frac{M_{0,0}^2(K(b-a))e^{-K(a+b)}}{K(b-a)\int_0^\infty \frac{J_0^2(\frac{k(b-a)}{2})}{k_1(k^2+K^2)}\left[k\cos k(\frac{a+b}{2}) - K\sin k(\frac{a+b}{2})\right]^2 dk} \tag{2.16}$$

where $M_{0,0}$ is the Whittaker function and J_0 is the Bessel function.

3 Numerical results

The lower and upper bounds of the reflection and transmission coefficients $|R|$ and $|T|$ respectively are evaluated numerically for various values of different parameters such as wavenumber Kb, angle of incidence α and $\frac{a}{b} = 0.5$. Only the lower and upper bounds R_1 and R_2 of $|R|$ are displayed in Table 1. Here we put $\alpha = 0^0$ in the expressions for R_1 and R_2 for obtaining numerical estimates for $|R|$ for the case of normal incidence and the bounds are also compared with exact values derived from Porter's [10] exact analytical results. Numerical values of upper and lower bounds of $|R|$ coincide within 3 to 4 decimal places and hence their averages provide very accurate estimates for the reflection coefficients. Similar computations have been carried out for the upper and lower bounds of $|T|$. However, these results are not displayed here. It has also been checked that these numerical estimates satisfy the energy identity $|R|^2 + |T|^2 = 1$, which provides a partial check on the correctness of the method. There are also other checks as described below. Also the numerical results presented in Table 1 are compared with those in Table 3 of Das et al [2]. Almost the same results are obtained. It may be noted that for the present method, the basis function $g_0(y)$ given by (2.14a) decays exponentially as $y \to \infty$ while for the method employed in Das et al [2] the basis function $f_1(y)$ given by (5.2) (and (5.3)) of Das et al [2] decays algebraically as $y \to \infty$. Because of this, the one-term Galerkin method with simplified basis functions employed here provides high accuracy in the numerical results.

	$\alpha = 0^0$			$\alpha = 30^0$		$\alpha = 60^0$		$\alpha = 85^0$	
Kb	R_1	R_2	$\|R\|$ Porter[1]	R_1	R_2	R_1	R_2	R_1	R_2
0.05	0.7251	0.7257	0.7251	0.6582	0.6587	0.4106	0.4109	0.0831	0.0831
0.4	0.4343	0.4344	0.4343	0.3605	0.3625	0.1823	0.1875	0.0306	0.0307
1.2	0.6500	0.6504	0.6502	0.5872	0.5877	0.3752	0.3755	0.0733	0.0772
2.0	0.9448	0.9472	0.9466	0.9236	0.9238	0.7950	0.7954	0.2092	0.2099
3.0	0.9960	0.9987	0.9960	0.9936	0.9937	0.9725	0.9771	0.6100	0.6107
4.0	0.9996	0.9999	0.9996	0.9993	0.9994	0.9967	0.9969	0.9206	0.9206

Table 1. Lower and upper bounds for the reflection coefficient of $|R|$ for various values of the parameters Kb, α and $\frac{a}{b} = 0.5$

As in Porter [10] and Tuck [12], let $a = h(1 - \frac{\mu}{2}), b = h(1 + \frac{\mu}{2}), \lambda = \frac{2\pi}{K}$ where h is the depth of the center of the gap below the free surface , μ is the ratio of the width of the gap to its mean depth and it lies between 0 to 2 and λ is the wavelength of the incident wave.

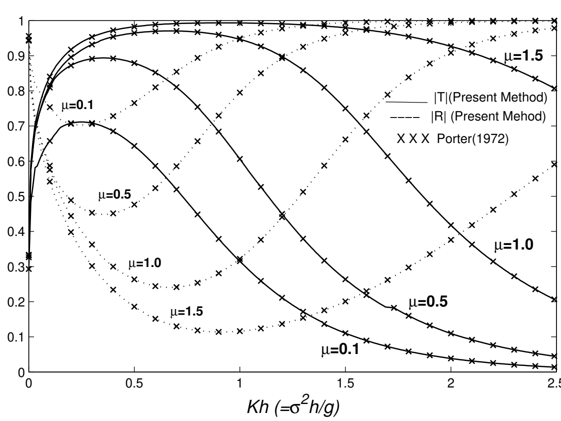


Figure 2: $|R|(...)$ and $|T|(-)$ against Kh for different values of μ, and $\alpha = 0^0$.

In figure 2, $|R|$ and $|T|$ are depicted against $Kh(= \frac{K(a+b)}{2})$ for different values of $\mu(= \frac{2(b-a)}{b+a})$

and for normal incidence($\alpha = 0^0$). Also $|R|$ and $|T|$ calculated from Porter's [1] exact expressions obtained by a completely different method are indicated in figure 2 by cross marks (x). From this figure it is observed that the curves of $|R|$ and $|T|$ plotted on the basis of the numerical results obtained by the present method and plotted on the basis of Porter's [10] exact results coincide. This gives another check on the correctness of the method.

In figure 3, $|T|^2$ is depicted against $\frac{h}{\lambda}(=\frac{K(a+b)}{4\pi})$ for different small values of $\mu = 0.05, 0.15, 0.4$ and for normal incidence($\alpha = 0^0$) so that the gap is narrow. Also $|T^2|$ calculated from Tuck's [12] result (expression given in (6.2) there) are indicated in figure 3 by cross marks (x). From this figure it is observed that the curves of $|T^2|$ plotted on the basis of the numerical results obtained by the present method and plotted on the basis of Tuck's [12] approximate result obtained by the method of matched asymptotic expansion coincide. This provides yet another check for the correctness of the results obtained by the present method.

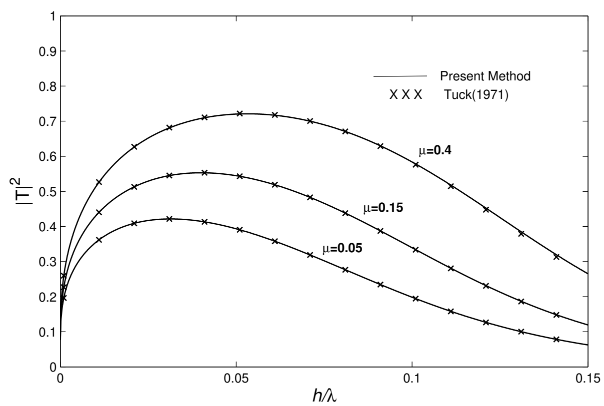


Figure 3: $|T|^2$ against $\frac{h}{\lambda}$ for different values of μ, and $\alpha = 0^0$.

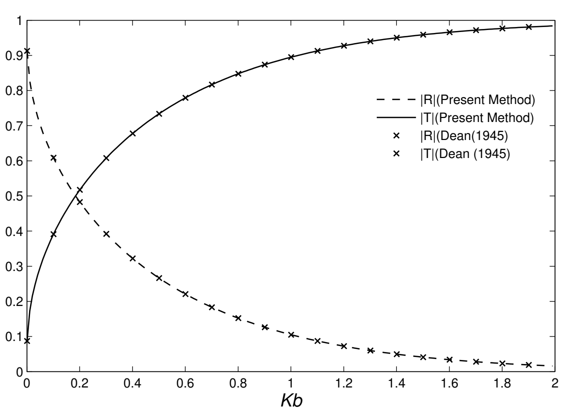


Figure 4: $|R|(...)$ and $|T|(-)$ against Kb for $\alpha = 0^0$.

In figure 4, $|R|$ and $|T|$ are depicted graphically against the wavenumber Kb for $\frac{a}{b} = 0$ so that the upper part of the wall is absent and the wall becomes a submerged barrier considered by Dean [3]. The curves of $|R|$ and $|T|$ almost coincide with the corresponding curves given by Dean [12] (indicated here by cross (x) marks). This produces a final check for the correctness of the results obtained by the present method.

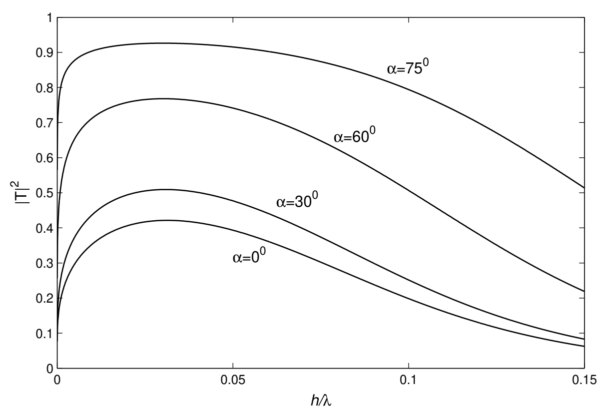


Figure 5: $|T|^2$ against $\frac{h}{\lambda}$ for different values of α, and $\mu = 0.05$

In figure 5, $|T|^2$ is depicted against $\frac{h}{\lambda}$ for different values of α with fixed $\mu = 0.05$(narrow gap). This is in fact an extension of Tuck's figure for a narrow gap and normal incidence to oblique incidence. All the conclusion drawn by Tuck [12] for normal incidence about the transmission of energy through a narrow gap can be extended for oblique incidence. For example, considerable transmission of energy occurs for long waves. From the figure 5 it is observed that transmission increases with the increase in the angle of incidence which is plausible. Also for a fixed angle of incidence, transmission first increases as the wavenumber increases and then it decreases steadily as the wavenumber further increases. This is due to the fact that for large wavenumber the waves are confined near the free surface so that most of these are reflected by the upper part of the thin wall.

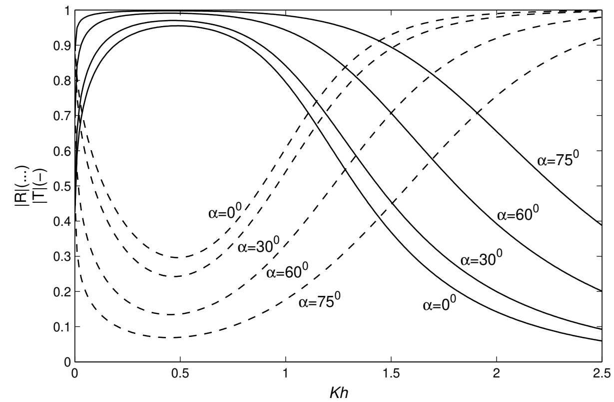


Figure 6: $|R|(\ldots)$ and $|T|(-)$ against Kh for different values of α, and $\mu = 0.5$

In figure 6, $|R|$ and $|T|$ are depicted against Kh for different values of α with fixed $\mu = 0.5$(moderate gap). This is again an extension of Porter's [10] curves for oblique incidence. This figure shows that for a wall with a moderate gap, as the angle of incidence increases, reflection coefficient decreases while transmission increases for fixed wavenumber. Incident waves are reflected by two parts of the wall. Obviously this reflection is maximum when waves are incident normally ($\alpha = 0^0$) on the wall and then reflection decreases gradually as α increases. This is plausible from physical considerations. Here however results for values of α from 0^0 to 75^0 are presented.

Again for fixed angle of incidence the reflection coefficient first decreases with increase of wavenumber and then increases asymptotically to unity as the wavenumber further increases. This is also plausible since for large wavenumber, the waves are confined near the free surface as mentioned earlier, so that most of the incident waves are reflected back. Reverse of this happens for the transmission coefficient i.e, transmission increases with the increase in the angle of incidence and for a fixed angle of incidence, transmission increases first with the increase of wavenumber and then decreases steadily to zero as the wavenumber further increases. It is interesting to note that for fixed angle of incidence, $\mu(0 < \mu < 2)$ is a crucial parameter in determining the transmission of wave energy through the gap at certain wavelengths. For $\mu = 1.0$, $|T|$ attains maximum near $Kh = 0.5$ corresponding to about more than 90 percent of wave energy transmission. For fixed α, as μ decreases i.e, as gap becomes smaller, $|T|$ decreases for all finite Kh which is shown in the figure 2. The curves in figures 5 and 6 may be regarded as new results.

4 Conclusion

The problem of water wave scattering by a thin vertical wall with a gap submerged in infinitely deep water is re-investigated by using integral equation formulations based on Havelock's expansion of water wave potential. Two first kind integral equations involving horizontal component of velocity across the gap and difference of velocity potential across the upper and lower parts of the wall are obtained. These are solved here approximately by using one-term Galerkin approximations involving constants multiplied by appropriate weight functions whose forms are dictated by the behaviour at the end points of the gap and at infinite depth. Exploitation of the symmetry and positive semi-definiteness of the operators of the integral equations lead to expressions for upper and lower bounds for the reflection and transmission coefficients. These bounds, when computed numerically, coincide upto 3-4 decimal places so that their averages produce very accurate numerical estimates for the reflection and transmission coefficients. Known numerical results(in the form of graphs) for the problem of water wave scattering by a thin wall with a gap, available in the literature by employing different methods, are recovered from the results obtained by the present method as special cases. The method employed here appears to be quite simple in comparison to other known methods employed for this problem. It is felt that this type of one-term Galerkin technique involving simple basis functions can be employed to study wave scattering by other types of obstacles with submerged edges such as multiple thin vertical barriers, thick rectangular barriers, wave scattering by step-type bottom topography etc.

5 Acknowledgments

The authors thank the referee for his comments and suggestions to improve the paper in the present form. B C Das thanks the UGC, India, for providing financial support (File no: 22/12/2013(ii)EU-V), as a research scholar of the University of Calcutta, India. This work is also supported by SERB through the research project no. EMR/2016/005315

References

[1] A. Chakrabrti, S.R. Manam, S. Banerjea, Scattering of surface water waves involving a vertical barrier with a gap, J. Eng. Math., 45 (2003), 183-194.

[2] P. Das, S. Banerjea, B. N. Mandal, Scattering of oblique waves by a thin vertical wall with a submerged gap, Arch. Mech.,, 48 (1996), 959-972.

[3] W. R. Dean, On the reflection of surface waves by a submerged plane barrier, Proc. Camb. Phil. Soc., 41 (1945), 231-238.

[4] D. V. Evans, C.A.N. Morris, The effect of a fixed vertical barrier on oblique incidence surface waves in deep water, J. Inst. Math. Applic., 9 (1972), 198-204.

[5] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products, London, Academic press, 1980.

[6] T. H. Havelock, Forced surface waves on water, Phil. Mag., 8 (1929), 569-576.

[7] B. N. Mandal, A note on the diffraction of water waves by a vertical wall with a narrow gap, Arch. Mech., 39 (1987), 269-273.

[8] B. N. Mandal, A. Chakrabarti, Water wave scattering by barrier, WIT Press, Southampton, UK, 2000.

[9] B. A. Packham, W. E. Williams, A note on the transmission of water waves through small apertures, J. Math. Anal. Appl., 10 (1972), 176-184.

[10] D. Porter, The transmission of surface waves through a gap in a vertical barrier, Proc. Camb. Phil. Soc., 71 (1972), 411-421.

[11] R. Roy, U. Basu, B. N. Mandal, Water wave scattering by a pair of thin vertical barriers with submerged gaps, J. Eng. Math., 105 (2017), 85-97.

[12] E. O. Tuck, Transmission of water waves through small apertures, J. Fluid Mech., 49 (1971), 65-74.

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VOLUME 21 • ISSUE 3
DECEMBER 2019

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