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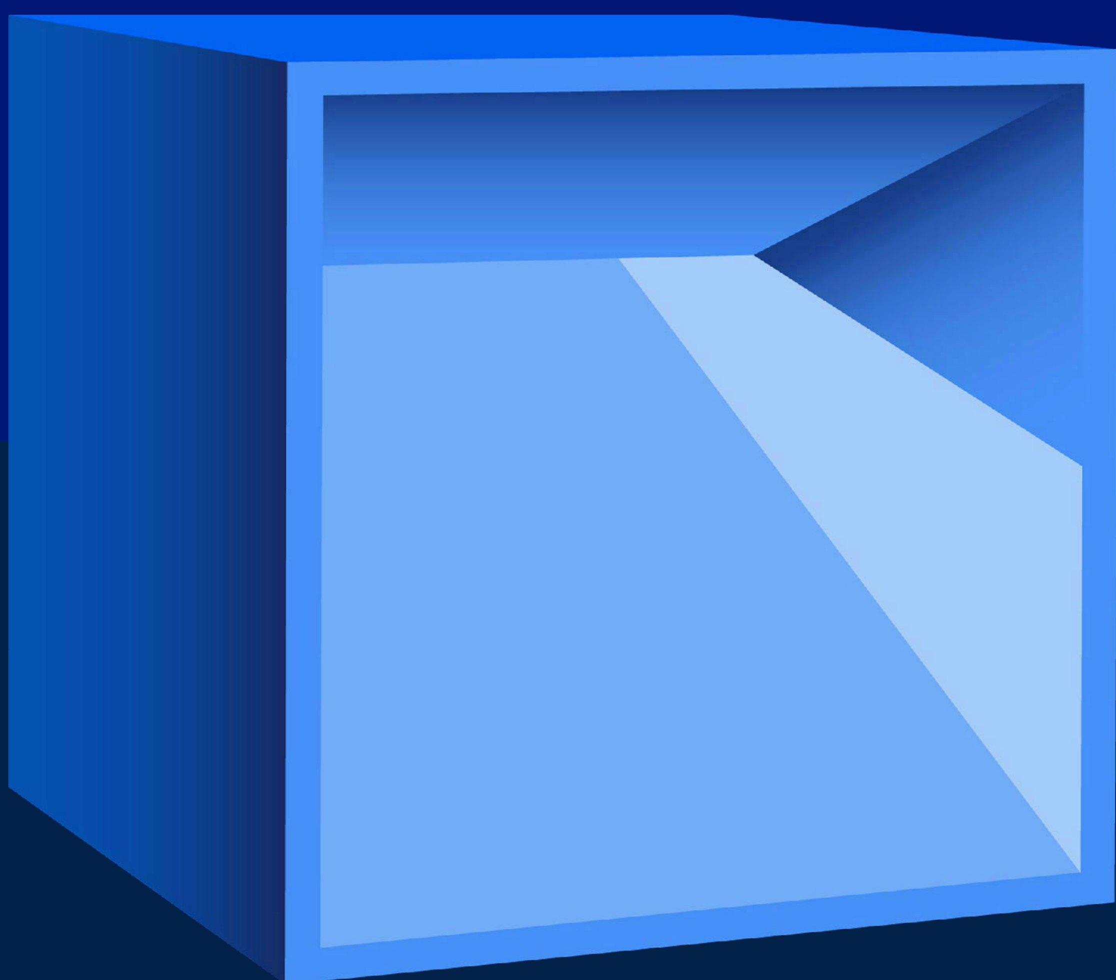

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SUMMARY

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Topological algebras with subadditive boundedness radius

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ABSTRACT

Let A be a topological algebra and β a subadditive boundedness radius on A. In this paper we show that β is, under certain conditions, automatically submultiplicative. Then we apply this fact to prove that the spectrum of any element of A is non-empty. Finally, in the case when A is a normed algebra, we compare the initial normed topology with the normed topology τ_β, induced by β on A, where $\beta^{-1}(0) = 0$.

RESUMEN

Sea A un álgebra topológica y β un radio de acotamiento subaditivo en A. En este artículo mostramos que β es, bajo ciertas condiciones, automáticamente submultiplicativo. Luego aplicamos este hecho para probar que el espectro de cualquier elemento de A es no-vacío. Finalmente, en el caso cuando A es una álgebra normada, comparamos la topología normada inicial con la topología normada τ_β, inducida por β en A, donde $\beta^{-1}(0) = 0$.

Keywords and Phrases: Topological algebra, strongly sequential algebra, boundedness radius.

2020 AMS Mathematics Subject Classification: 46H05; 46H20

1 Introduction and Preliminaries

A *topological algebra* is an algebra A which is a topological vector space in such a way that the ring multiplication in A is separately continuous. (i.e., continuous in each one of the two variables the latter operation being a map of $A \times A$ into A.)

If the multiplication of a given topological algebra A is in both variables (i.e., jointly) continuous, we say that A is a *topological algebra with a continuous multiplication*. (See e.g., [9, p. 4. Definition 1.1.])

In what follows the topological algebras cosidered are supposed with a continuous multiplication and to be Hausdorff.

Among topological algebras, the normed ones have been studied intensively by many mathematicians where the norm is used as a useful tool in measuring the distance between two elements. But, since 1940, there has been a considerable interest in investigating other topological algebras in absence of any norm and this led to the introduction of some famous topological algebras such as locally bounded algebras, locally convex algebras, locally multiplicatively convex algebras, et cetera.

The common idea to study a non-normed topological algebra, say A, is to substitute the role of a norm on A, which determines the topology, in an appropriate way. For instance, Aoki [2] proved that on a locally bounded algebra A, there is a p-norm generating the original topology on A. Later, Zelazko [12] applied this fact on locally bounded algebras, and obtained many classical results, known in the context of normed algebras. For example, in a complete locally bounded topological algebra A, the operator $x \mapsto x^{-1}$ on $Inv(A)$ (the group of all invertible elements of A) is continuous. Also, the Cohen factorization theorem holds whenever A has a bounded approximate identity. The role of norm in a locally convex topological algebra A is played by a separating family of submultiplicative seminorms generating the topology on A. Recall that a seminorm is a non-negative real-valued function p on A such that

(i) $p(x + y) \leq p(x) + p(y)$

(ii) $p(\alpha x) = |\alpha|\ p(x)$

for all x and y in A and α in $\mathbb{C}$.

We say that p is submultiplicative seminorm if in addition $p(xy) \leq p(x)p(y)$ for all x and y in A.

Hence, there is a good motivation for mathematicians to extend notions and theorems from normed algebras to other topological algebras. A *locally convex algebra* is a topological algebra A whose the underlying topological vector space is a locally convex space. The topology of a such topological algebra is determined by a family of (non-zero) seminorms. In the case when the seminorms are submultiplicative, A is called a locally m-convex algebra.

Let A be a unital topological algebra with the unit e and $x \in A$. The spectrum of x, denoted by $\sigma(x)$, is defined

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin Inv(A)\}.$$

The spectral radius of x is the defined as

$$\rho(x) = \sup\{|\lambda| \ : \ \lambda \in \sigma(x)\}.$$

It is well known that $\rho(x) = \inf\{\|x^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}$ for every element x in a Banach algebra. Using this fact, Allan [1], introduced the notion of boundedness radius in a topological algebra A as follows

$$\beta(x) = \inf\{r > 0 \ : \ ((\frac{x^n}{r^n}))_n \to 0\} \ , \ (\inf \phi = +\infty).$$

Allan attempted to compare $\beta(x)$ with $\rho(x)$ and in one of the obtained results, shows that $\beta(x)$ is equal to $\rho(x)$ in a complete locally convex algebra [1, Theorem 3.12], specially, in a Banach algebra we have

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf\{\|x^n\|^{\frac{1}{n}} : n \in \mathbb{N}\} = \beta(x).$$

Oubbi [11] investigated on ρ and β and compared them together. In a topological algebra A, he showed that $\rho \leq \beta$ if and only if $\sum x^n$ is convergent whenever $x \in A$ and $\beta(x) < 1$. Also, he showed $\beta(x^n) = \beta(x)^n$ for all $n \in \mathbb{N}$ and $\beta(xy) \leq \beta(x)\beta(y)$ whenver $x, y \in A, \ xy = yx$. In any topological algebra, it is clear that $\beta(\lambda x) = |\lambda|\beta(x) \quad (x \in A, \lambda \in \mathbb{C})$.

Let A b a topological algebra. The boundedness radius β is said to be subadditive, if for each $x, y \in A$, $\beta(x + y) \leq \beta(x) + \beta(y)$. Moreover, β is called submultiplicative whenever $\beta(xy) \leq \beta(x)\beta(y)$.

Kinani, Oubbi and Oudadess [8], proved that in a unital and commutative locally convex algebra, β is subadditive and submultiplicative. [8, Proposition II.9.]

In this paper, we show that in a topological algebra, if β is finite and subadditive, then it is submultiplicative. (See Corollary 2.6.) Also, we refer to topological algebras in which β defines a norm on them. (See Theorem 2.10 and Theorem 2.11.) Actually, we consider a topological algebra A with boundedness radius β such that β satisfies the following conditions:

(1) $\beta^{-1}(0) = 0$

(2) $\forall x \in A, \ \beta(x) < \infty$

(3) $\forall x, y, \ \beta(x + y) \leq \beta(x) + \beta(y)$.

2 The normed topology τ_β induced by the boundedness radius β

In a locally convex algebra, amongst the important results is the following.

Theorem 2.1. *Let A be a unital and commutative locally convex algebra, then β is subadditive.*

Proof. See [8, Proposition II.9]. □

In this section, we first give an example to show that we can not drop commutativity in Theorem 2.1 and we also prove that locally convexity of an algebra is not sufficient for β to be subadditive. Finally, we consider topological algebras in which β is subadditive and we show that in such algebras, β is automatically submultiplicative.

Example 2.2. *Let $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be elements of the noncommutative algebra $A = M_2(\mathbb{C})$. Since a and b are nilpotent elements of A, $\beta(a) = \beta(b) = 0$, on the other hand $\beta(a+b) = 1$ because $a + b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and so $(a+b)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, now we have*

$$(\beta(a+b))^2 = \beta((a+b)^2) = \beta(I) = 1.$$

Which implies that $\beta(a+b) = 1$. Hence $\beta(a+b) \nleq \beta(a) + \beta(b)$.

This shows that we can not drop commutativity in Theorem 2.1.

Let A be a topological algebra and $B = A \times \mathbb{C}$ is equipped with the following multiplication

$$(x_1, \alpha_1)(x_2, \alpha_2) = (\alpha_1 x_2 + \alpha_2 x_1, \alpha_1\alpha_2) \quad (x_1, x_2 \in A,\ \alpha_1, \alpha_2 \in \mathbb{C}).$$

The unitization B of A is a topological algebra under the product topology of $B = A \times \mathbb{C}$ and we have:

Theorem 2.3. *The boundedness radius β is subadditive in B.*

Proof. It is enough to show that for every $z = (x, \alpha) \in B$, $\beta(z) = |\alpha|$.

Let $z = (x, \alpha) \in B$. If $\alpha = 0$ then $(x, 0)$ is a nilpotent element of B and $\beta(z) = 0 = |\alpha|$. For $\alpha \neq 0$, suppose that $n \in \mathbb{N}$, $\varepsilon > 0$. Let $r \in (|\alpha|, |\alpha| + \varepsilon)$ then $\frac{1}{r^n}\alpha^{n-1} \to 0$ and so $\frac{1}{r^n}\alpha^{n-1}x \to 0$. On the other hand, $(x, \alpha)^n = (nx\alpha^{n-1}, \alpha^n)$. Therefore

$$\begin{aligned} \frac{1}{(r+\varepsilon)^n}(x,\alpha)^n &= \frac{1}{(r+\varepsilon)^n}(nx\alpha^{n-1}, \alpha^n) \\ &= (n(\frac{r}{r+\varepsilon})^n \frac{x\alpha^{n-1}}{r^n}, \frac{\alpha^n}{(r+\varepsilon)^n}) \to (0,0). \end{aligned}$$

This shows that $\beta(z) \leq r + \varepsilon < |\alpha| + 2\varepsilon$, since ε is arbitrary, $\beta(z) \leq |\alpha|$.
For the converse, note that there exists $r > 0$ such that $r < \beta(x, \alpha) + \varepsilon$ and

$$\frac{(x, \alpha)^n}{r^n} = (\frac{nx\alpha^{n-1}}{r^n}, \frac{\alpha^n}{r^n}) \to (0, 0).$$

Hence, $\frac{\alpha^n}{r^n}$ convergent to zero. So $|\alpha| < r < \beta(x, \alpha) + \varepsilon$, it follows that $\beta(\alpha) = |\alpha| < \beta(z) + \varepsilon$. Thus $|\alpha| \leq \beta(z)$. □

Definition 2.4. *Let A be an algebra. We say that a seminorm p on A has square property, if it is square-preserving, namely,*

$$p(x^2) = p(x)^2 \ for \ all \ x \in A.$$

Dedania in 1998 [6] proved the following theorem.

Theorem 2.5. *Let A be an algebra and a seminorm p on A which has the square property. Then p is submultiplicative.*

According to the terminology in [4, p. 437, (6)] and due to Theorem 2.5, a seminorm p. as in the latter theorem, is finally a $uniform\ seminorm$.

Corollary 2.6. *Let A be a topological algebra, such that $\beta(x) < \infty$ and $\beta(x + y) \leq \beta(x) + \beta(y)$ for all $x, y \in A$. Then β is submultiplicative.*

Proof. Since $\beta(x^2) = \beta(x)^2$, the assertion follows from Theorem 2.5. □

Let A be a topological algebra such that β satisfies conditions (1)-(3) then β is a norm on A and by Corollary 2.6, β is a submultiplicative norm. Through this section, τ_β denotes the topology, induced by β on A. Now we face the following questions.

Question 1. Is there any topological algebra for which β satisfies the conditions (1)-(3)?

Question 2. Let (A, τ) be a topological algebra for which β satisfies the conditions (1)-(3). What is then the relation between τ and τ_β?

Question 3. Let A be a complete topological algebra such that β satisfies the conditions (1)-(3). Does the normed algebra (A, τ_β) is a Banach algebra?

In what follows, we are going to answer to these questions.

Theorem 2.7. *Let A be a unital and commutative semisimple Banach algebra, then β satisfies $(1) - (3)$.*

Proof. Since A is a commutative locally convex algebra, then, by [8, Lemma 2.9]

$$\beta(x + y) \leq \beta(x) + \beta(y) \ for \ all \ x, y \ in \ A.$$

Since A is a Banach algebra then A is locally bounded and using [3, Lemma 3.4], $\beta(x) \leq \|x\|$ for all x in A, hence β satisfies condition (2).

A is a commutative semisimple Banach algebra, using Corollary 7.(iv) in [5] we have $\rho^{-1}(0) = 0$. Because A is a Banach algebra, $\beta(x) = \rho(x)$ for each $x \in A$. So $\beta^{-1}(0) = 0$. It follows that β satisfies (1)-(3). □

Theorem 2.7 gives a category of algebras satisfying conditions (1)-(3) and an affirmative answer to question 1.

Definition 2.8. *A topological algebra A is called strongly sequential if there is a neighborhood U of zero such that, for all $x \in U$, $(x^k)_{k\in\mathbb{N}}$ converges to zero.*

Lemma 2.9. *Let A be a topological algebra. Then β is continuous at zero if and only if A is strongly sequential.*

Proof. See [3, Proposition 3.1]. □

In order to answer Question 2, we give the following theorem.

Theorem 2.10. *Let (A, τ) be a topological algebra. Suppose that β satisfies (1)-(3). Then (A, τ) is strongly sequential if and only if $\tau_\beta \subseteq \tau$.*

Proof. Suppose (A, τ) is strongly sequential. By Lemma 2.9, β is continuous at zero. Since β is subadditive, it is continuous on A. On the other hand, $\{\beta^{-1}(0, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base for the normed topology τ_β. Therefore $\tau_\beta \subseteq \tau$.

Conversely, let $x_\alpha \to 0$ in τ. Since $\tau_\beta \subseteq \tau$, x_α converges to zero in τ_β which implies that $\beta(x_\alpha) \to 0$. Hence β is continuous at zero and, again by Lemma 2.9, A is strongly sequential. □

In order to answer Question 3, first we characterize a topological algebra for which (A, τ_β) is a Banach algebra and then we apply this characterization to give a negative answer to Question 3.

It is well known (See e.g., [10, p. 41]) that in a commutative C^*-algebra, the unique C^*-norm is the spectral norm i.e.

$$\|x\| = \rho(x) \quad (x \in A) \tag{2.1}$$

On the other hand, for each $x \in A$, $\rho(x) = \beta(x)$ and so β is indeed the C^*-norm on A. Thus (A, τ_β) is a complete normed algebra whenever A is a C^*-algebra. The following theorem gives a more general characterization of topological algebras for which (A, τ_β) is complete.

In the sequel, by an F-algebra we mean a completely metrizable topological algebra.

Theorem 2.11. *Let (A, τ) be a strongly sequential F-algebra. Suppose that β satisfies (1)-(3). Then (A, τ_β) is a Banach algebra if and only if $\tau_\beta = \tau$.*

Proof. If $\tau_\beta = \tau$, then (A, τ_β) is a Banach algebra, trivially. For the converse, suppose that (A, τ_β) is a Banach algebra. By the assumption, (A, τ) is a strongly sequential F-algebra, so, Theorem 2.10 implies, $\tau_\beta \subseteq \tau$. From the open mapping theorem, one immediately gets that $\tau_\beta = \tau$. □

The following example shows that the answer of Question 3 is not affirmative.

Example 2.12. *Let A be the set of C^1-functions on the interval $[0,1]$ and $f \in A$. Then A is a semisimple commutative Banach algebra where the norm on A is given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ (See [10, p. 10, Example 1.2.6]).*

Thus β satisfies (1)-(3) and so (A, τ_β) is normed algebra. If (A, τ_β) is complete, then A is a Banach algebra, also it is a strongly sequential algebra (See e.g., [7, p. 58, Example 3.26]). Now, by Theorem 2.11, $\tau_\beta = \tau$. Let x be the identity map on $[0,1]$. Since $\beta(x) = 1$, the sequence $(\frac{x^n}{n})_n$ convergence to zero in τ_β- topology. But $\frac{x^n}{n} \nrightarrow 0$ in the original topology τ. This is a contradiction and so (A, τ_β) is not complete.

As we mentioned, if β satisfies (1)-(3), then it is an algebraic norm on A. But in the absence of one of the properties (1)-(3), β is not a norm necessarily. In what follows, we concentrate to study topological algebras for which β is not a norm.

Lemma 2.13. *Let $(A, \| \ \|)$ be a normed algebra such that, $\|a\|^2 = \|a^2\|$ for all $a \in A$. Then A is commutative.*

Proof. See [5, p. 77, Corollary 8, see also the comments after it]. □

Theorem 2.14. *Let A be a topological algebra such that*

(1) $\beta(x) < \infty$ $(x \in A)$

(2) $\beta(x+y) \leq \beta(x) + \beta(y)$ $(x, y \in A)$.

Then $\beta(xy - yx) = 0$ for all x, y in A.

Proof. Let $N = \{x \in A, \beta(x) = 0\}$. It is clear that N is an ideal in A and A/N is a normed algebra with the norm $\|x + N\| = \beta(x)$. Since β has the square property, $\|(x+N)^2\| = \|x+N\|^2$ for all x in A. Now, according to Lemma 2.13, the normed algebra A/N is commutative. Hence, $(x+N)(y+N) = (y+N)(x+N)$ for all x, y in A which means that $\beta(xy - yx) = 0$. □

Corollary 2.15. *Let A be a topological algebra and β satisfies (1)-(3), then A is commutative.*

Proof. Since $\beta(x^2) = \beta(x)^2$, the assertion follows from Lemma 2.13 or Theorem 2.14. □

Lemma 2.16. *Let A be a Banach algebra and $x, y \in A$. If $xy = yx$, $xox = yoy$ ($xoy = x+y-xy$) and $\beta(x+y) < 2$, then $x = y$.*

Proof. See [5, p. 44, Lemma 12]. □

Lemma 2.17. *Let A be a normed algebra and $x, y \in A$. If $xy = yx$, $xox = yoy$ and $\beta(x+y) < 2$, then $x = y$.*

Proof. Let $\tilde{A}$ be the completion of A. Suppose that T is an isometric isomorphism of A onto $\tilde{A}$. Since $(Tx)(Ty) = (Ty)(Tx)$, $(Tx)o(Ty) = (Ty)o(Tx)$ and $\beta_{\tilde{A}}(Tx+Ty) = \beta_{\tilde{A}}(T(x+y)) = \beta_A(x+y) < 2$ by Lemma 2.16, $Tx = Ty$ and so $x = y$. □

Theorem 2.18. *Let A be a topological algebra and $x, y \in A$ such that $xox = yoy$ and $\beta(x+y) < 2$. If β is finite and subadditive, then $\beta(x-y) = 0$.*

Proof. Assume that $N = \{x \in A, \beta(x) = 0\}$. By the proof of Theorem 2.14, A/N is a normed algebra under the norm $\|x+N\| = \beta(x)$. Then

$$(x+N)o(x+N) = (y+N)o(y+N),$$

and,

$$\begin{aligned} \beta_{A/N}((x+N)+(y+N)) &= \beta_{A/N}((x+y)+N) \\ &\leq \|(x+y)+N\| = \beta(x+y) < 2. \end{aligned}$$

Applying Lemma 2.17, $x+N = y+N$ which means that $\beta(x-y) = 0$. □

Theorem 2.19. *Let A be a topological algebra such that*

(i) $\beta(x+y) \leq \beta(x) + \beta(y)$ $(x, y \in A)$

(ii) $\beta(x) < \infty$ $(x \in A)$.

Then $\sigma_A(x) \neq \emptyset$ for all $x \in A$.

Proof. Let $N = \{x \in A, \beta(x) = 0\}$. Since A/N is a normed algebra, the spectrum of any of its elements is non-empty. (See e.g., [5, p. 22. Theorem 7.]) On the other hand, the canonical map $\pi : A \to A/N$ is an algebraic homomorphism and so $\emptyset \neq \sigma_{A/N}(\pi(x)) \subseteq \sigma_A(x)$ for each $x \in A$. (See e.g., [5, p. 48. Proposition 9]) This completes the proof. □

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Odd Harmonious Labeling of Some Classes of Graphs

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ABSTRACT

A graph $G(p,q)$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. In this paper we prove that T_p- tree, $T\hat{o}P_m$, $T\hat{o}2P_m$, regular bamboo tree, $C_n\hat{o}P_m$, $C_n\hat{o}2P_m$ and subdivided grid graphs are odd harmonious.

RESUMEN

Un grafo $G(p,q)$ se dice impar armonioso si existe una inyección $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ tal que la función inducida $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ definida por $f^*(uv) = f(u) + f(v)$ es una biyección. En este artículo probamos que los grafos T_p-árboles, $T\hat{o}P_m$, $T\hat{o}2P_m$, árboles bambú regulares, $C_n\hat{o}P_m$, $C_n\hat{o}2P_m$ y cuadrículas subdivididas son impar armoniosos.

Keywords and Phrases: harmonious labeling, odd harmonious labeling, transformed tree, subdivided grid graph, regular bamboo tree.

2020 AMS Mathematics Subject Classification: 05C78.

1 Introduction

Throughout this paper by a graph is implied as a finite, simple and undirected. For standard terminology and notation we follow Harary [3]. A graph $G(V, E)$ with p vertices and q edges is called a (p, q) − graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [1]. Graham and Sloane [2] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \to Z_q$ such that the induced function $f^* : E(G) \to Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod q$ is a bijection and f is called harmonious labeling of G. The concept of an odd harmonious labeling was due to Liang and Bai [14]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q - 1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f : V(G) \to \{0, 1, 2, \cdots, q\}$ then f is called as strongly odd harmonious labeling and G is called a strongly odd harmonious graph. The odd harmoniousness of a graph is useful for the solution of undetermined equations. The following results have been proved in [14]:

1. If G is an odd harmonious graph, then G is a bipartite graph. Hence any graph that contains an odd cycle is not an odd harmonious.

2. If a (p, q) − graph G is odd harmonious, then $2\sqrt{q} \le p \le (2q - 1)$.

3. If G is an odd harmonious Eulerian graph with q edges, then $q \equiv 0, 2(mod\ 4)$.

Followed by this, Vaidya and Shah [18], [19] showed that shadow and splitting graphs are odd harmonious. Selvaraju et al. [17] established that some path related graphs are odd harmonious. Jeyanthi et al. proved that the following graphs are odd harmonious: double quadrilateral snake and banana tree [5], cycle related graphs [6], plus graphs [7], super subdivision graphs [8], subdivided shell graphs [9], spider and necklace graphs [10], m-shadow, m-splitting and m-mirror graphs [11] and [12], grid graphs [13].

We use the following definitions in the subsequent section.

Definition 1.1. *Let $G = (V, E)$ be a graph. G is called a path P_n if $V = \{v_1, v_2, \cdots, v_n\}$ such that $1 \le i \le n$, $(v_i, v_{i+1}) \in E$.*

Definition 1.2. *The Cartesian product of graphs G and H denoted as $G \Box H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) | u \in V(G)$ and $v \in V(H)\}$ and (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $(v, v') \in E(H)$ or $v = v'$ and $(u, u') \in E(G)$. The Cartesian product of two paths P_m and P_n denoted by $P_m \times P_n$ is known as a grid graph on mn vertices and $2mn - (m + n)$ edges.*

Definition 1.3. *Let G be a graph with p vertices and H be any graph and x be a vertex of H. A graph $G\hat{\circ}H$ is obtained from G and p copies of H by identifying vertex x of i^{th} copy of H with i^{th} vertex of G.*

Definition 1.4. [4] *Let T be a tree and u_0 and v_0 be the two adjacent vertices in T. Let u and v be the two pendant vertices of T such that the length of the path $u_0 - u$ is equal to the length of the path $v_0 - v$. If the edge u_0v_0 is deleted from T and u and v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an ept) and the edge u_0v_0 is called transformable edge. If by some sequence of ept's, T can be reduced to a path, then T is called a T_p- tree (transformed tree) and such sequence regarded as a composition of mappings (ept's) denoted by P is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$. A T_p- tree and the sequence of two ept's reducing it to a path are illustrated in Figure 1.*

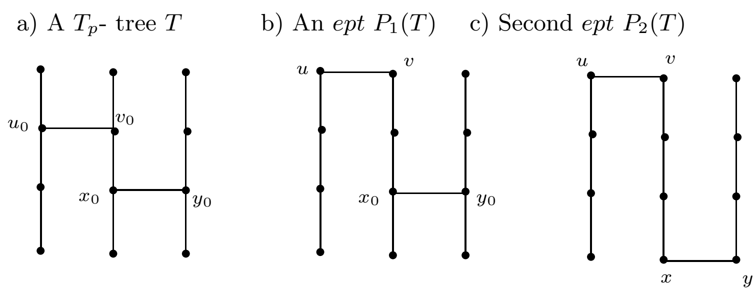

Figure 1: Transformed tree

Definition 1.5. [15] *Let T be a T_p-tree with n vertices $v_1, v_2, \cdots, v_n$. The graph $T\hat{\circ}P_m$ is obtained from T and n copies of P_m by identifying a pendant vertex of i^{th} copy of P_m with vertex v_i of T.*

Definition 1.6. [16] *Consider k copies of paths P_n of length $n-1$ and stars S_m with m pendant vertices. Identify one of the two pendant vertices of the j^{th} path with the centre of the j^{th} star. Identify the other pendant vertex of each path with a single vertex u_0 (u_0 is not in any of the star and path). The graph obtained is a regular bamboo tree.*

2 Main Results

In this section, we prove that T_p- tree, $T\hat{o}P_m$, $T\hat{o}2P_m$, regular bamboo tree, $C_n\hat{o}P_m$, $C_n\hat{o}2P_m$ and subdivided grid graphs are odd harmonious.

Theorem 2.1. *Every T_p- tree is strongly odd harmonious.*

Proof. Let T be a T_p-tree with n vertices. By definition, there exists a parallel transformation P of T, we have $V(P(T)) = V(T)$ and $E(P(T)) = (E(T) - E_d) \cup E_a$, where E_d is the set of deleted edges and E_a is the set of newly added edges through the sequence $P = (P_1, P_2, \cdots, P_l)$ of the *ept*'s used to obtain $P(T)$. Hence E_d and E_a have the same number of edges. Let $u_1, u_2, \cdots, u_n$ be the vertices of $P(T)$ successively, from one pendant vertex of $P(T)$ right up to the other. This T_p-tree has n vertices and $n-1$ edges.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q = n-1\}$ as follows:
$f(u_i) = i - 1, \ 1 \leq i \leq n.$

Let $(u_i u_j)$ be an edge of T, $1 \leq i<j \leq n$. Let the *ept* P_1 delete the edge $(u_i u_j)$ and adds the edge $(u_{i+t} u_{j-t})$ where t is the distance from u_i to u_{i+t} and also the distance from u_j to u_{j-t}. Let the parallel transformation P contain one of the constituent *ept*'s P_1. Since $(u_{i+t} u_{j-t})$ is an edge of $P(T)$, it follows that $i + t + 1 = j - t$, implies $j = i + 2t + 1$.

The induced edge label of $(u_i u_j)$ is given by
$f^*(u_i u_j) = f^*(u_i u_{i+2t+1}) = f(u_i) + f(u_{i+2t+1}) = 2(i + t) - 1,$
$f^*(u_{i+t} u_{j-t}) = f^*(u_{i+t} u_{i+t+1}) = f(u_{i+t}) + f(u_{i+t+1}) = 2(i + t) - 1,$
$f^*(u_i u_j) = f^*(u_{i+t} u_{j-t}).$

The induced edge label is
$f^*(u_i u_{i+1}) = 2i - 1, \ 1 \leq i \leq n - 1.$

Thus the induced edge labels are $1, 3, \cdots, 2n - 3$. Therefore every T_p-tree is strongly odd harmonious. □

A strongly odd harmonious labeling of a T_p- tree with 12 vertices is shown in Figure 2.

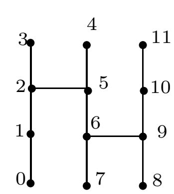

Figure 2: Strongly odd harmonious labeling of T_p- tree with 12 vertices

Theorem 2.2. *If T is a T_p-tree then the graph $T\hat{\circ}P_m$ is strongly odd harmonious.*

Proof. Let T be a T_p-tree with n vertices. By definition there exists parallel transformation $P(T)$, we have $V(P(T)) = V(T)$ and $E(P(T)) = (E(T) - E_d) \cup E_a$, where E_d is the set of deleted edges and E_a is the newly added edges through the sequence $P = (P_1, P_2, \cdots, P_l)$ of the *ept*'s used to obtain $P(T)$. Hence E_d and E_a have the same number of edges. Let $u_1, u_2, \cdots, u_n$ be the vertices of $P(T)$ successively, from one pendant vertex of $P(T)$ right up to the other. Let $x_0^j, x_1^j, \cdots, x_{m-1}^j$, $1 \le j \le n$ be the vertices of the j^{th} copy of P_m. Identify x_0^j with u_j, where $1 \le j \le n$. Then the graph $T\hat{\circ}P_m$ has nm vertices and $nm - 1$ edges.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q = nm - 1\}$ as follows:

$f(u_j) = mj - 1,$ $\quad j = 1, 3, \cdots, n-1,$

$f(u_j) = m(j-1),$ $\quad j = 2, 4, \cdots, n,$

For $1 \le i \le m-1$, $f(x_i^j) = mj - i - 1,$ $\quad j = 1, 3, \cdots, n-1,$

$f(x_i^j) = m(j-1) + i,$ $\quad j = 2, 4, \cdots, n.$

Let (u_iu_j) be an edge of T, $1 \le i < j \le n$. Let the *ept* P_1 delete the edge (u_iu_j) and add the edge $(u_{i+t}u_{j-t})$ where t is the distance from u_i to u_{i+t} and also the distance from u_j to u_{j-t}. Let the parallel transformation P contain one of the constituent *ept*'s P_1. Since $(u_{i+t}u_{j-t})$ is an edge of $P(T)$, it follows that $i + t + 1 = j - t$, implies $j = i + 2t + 1$. Therefore i and j are of opposite equivalence, that is, i is even and j is odd or vice-versa.

The induced edge label of (u_iu_j) is given by

$f^*(u_iu_j) = f^*(u_iu_{i+2t+1}) = f(u_i) + f(u_{i+2t+1}) = 2m(i+t) - 1,$

$f^*(u_{i+t}u_{j-t}) = f^*(u_{i+t}u_{i+t+1}) = f(u_{i+t}) + f(u_{i+t+1}) = 2m(i+t) - 1,$

$f^*(u_iu_j) = f^*(u_{i+t}u_{j-t}).$

The induced edge labels are

$f^*(u_ju_{j+1}) = 2mj - 1,$ $\quad 1 \le j \le n-1,$

For $1 \le i \le m-2$, $f^*(x_i^jx_{i+1}^j) = 2mj - 2i - 3,$ $\quad j = 1, 3, \cdots, n-1,$

$f^*(x_i^jx_{i+1}^j) = 2m(j-1) + 2i + 1,$ $\quad j = 2, 4, \cdots, n,$

$f^*(u_jx_1^j) = 2mj - 3,$ $\quad j = 1, 3, \cdots, n-1,$

$f^*(u_jx_1^j) = 2m(j-1) + i,$ $\quad j = 2, 4, \cdots, n.$

Thus the induced edge labels are $1, 3, \cdots, 2mn - 3$. Hence every $T\hat{\circ}P_m$ is strongly odd harmonious. □

A strongly odd harmonious labeling of $T\hat{\circ}P_4$ where T is a T_p-tree with 10 vertices is shown in Figure 3.

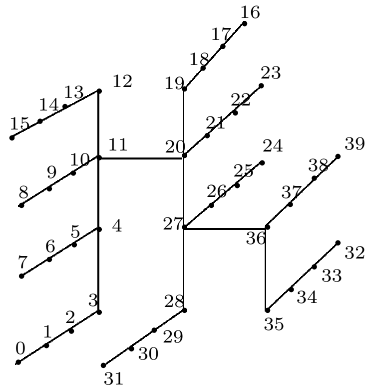

Figure 3: Strongly odd harmonious labeling of $T\hat{o}P_4$

Theorem 2.3. *If T is a T_p-tree then the graph $T\hat{o}2P_m$ is strongly odd harmonious.*

Proof. Let T be a T_p-tree with n vertices. By definition there exists a parallel transformation $P(T)$, we have $V(P(T)) = V(T)$ and $E(P(T)) = (E(T) - E_d) \cup E_a$, where E_d is the set of deleted edges and E_a is the set of newly added edges through the sequence $P = (P_1, P_2, \cdots, P_l)$ of the *ept*'s used to obtain $P(T)$. Hence E_d and E_a have the same number of edges. Let $x^j_{1,0}, x^j_{1,1}, x^j_{1,2}, \cdots, x^j_{1,m-1}$ and $x^j_{2,0}, x^j_{2,1}, x^j_{2,2}, \cdots, x^j_{2,m-1}$, $1 \le j \le n$ be the vertices of two disjoint paths P_m. Identify $x^j_{1,0}$ and $x^j_{2,0}$ with u_j, $1 \le j \le n$ to obtain $T\hat{o}2P_m$. Then the graph $T\hat{o}2P_m$ has $n(2m-1)$ vertices and $n(2m-1) - 1$ edges.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q = n(2m-1) - 1\}$ as follows:
$f(u_j) = m - 1 + (2m-1)(j-1),$ if j is odd,
$f(u_j) = 3m - 2 + (2m-1)(j-2),$ if j is even,
$f(x^j_{1,i}) = m - 1 + (2m-1)(j-1) - i,$ if j is odd,
$f(x^j_{1,i}) = 3m - 2 + (2m-1)(j-2) - i,$ if j is even,
$f(x^j_{2,i}) = m - 1 + (2m-1)(j-1) + i,$ if j is odd,
$f(x^j_{2,i}) = 3m - 2 + (2m-1)(j-2) + i,$ if j is even.

Let (u_iu_j) be an edge of T, $1 \le i < j \le n$. Let the *ept* P_1 delete the edge (u_iu_j) and add the edge $(u_{i+t}u_{j-t})$ where t is the distance from u_i to u_{i+t} and also the distance from u_j to u_{j-t}. Let the parallel transformation P contain one of the constituent $ept^{'}s$ P_1. Since $(u_{i+t}u_{j-t})$ is an edge of $P(T)$, it follows that $i + t + 1 = j - t$, implies $j = i + 2t + 1$. Therefore i and j are of opposite equivalence, that is, i is even and j is odd or vice-versa.

The induced edge label of (u_iu_j) is given by
$f^*(u_iu_j) = f^*(u_iu_{i+2t+1}) = f(u_i) + f(u_{i+2t+1}) = 4mi + 4mt - 2i - 2t - 1,$
$f^*(u_{i+t}u_{j-t}) = f^*(u_{i+t}u_{i+t+1}) = f(u_{i+t}) + f(u_{i+t+1}) = 4mi + 4mt - 2i - 2t - 1,$
$f^*(u_iu_j) = f^*(u_{i+t}u_{j-t}).$

The induced edge labels are

$f^*(u_j x_{1,1}^j) = 2(m-1) + 2(2m-1)(j-1) - 1,$ if j is odd,

$f^*(u_j x_{1,1}^j) = 2(3m-2) + 2(2m-1)(j-2) - 1,$ if j is even,

$f^*(u_j x_{2,1}^j) = 2(m-1) + 2(2m-1)(j-1) + 1,$ if j is odd,

$f^*(u_j x_{2,1}^j) = 2(3m-2) + 2(2m-1)(j-2) + 1,$ if j is even,

$f^*(u_j u_{j+1}) = 2(2m-1)(j-1) + 4m + 3,$ $1 \leq j \leq n-1,$

For $1 \leq i \leq m-2$,

$f^*(x_{1,i}^j x_{1,i+1}^j) = 2(2m-1)(j-1) + 2(m-1) - 2i - 1,$ if j is odd,

$f^*(x_{1,i}^j x_{1,i+1}^j) = 2(2m-1)(j-2) + 2(3m-2) - 2i - 1,$ if j is even,

$f^*(x_{2,i}^j x_{2,i+1}^j) = 2(2m-1)(j-1) + 2(m-1) + 2i + 1,$ if j is odd,

$f^*(x_{2,i}^j x_{2,i+1}^j) = 2(2m-1)(j-2) + 2(3m-2) + 2i + 1,$ if j is even.

Hence $T\hat{o}2P_m$ is strongly odd harmonious. □

The strongly odd harmonious labeling of $T\hat{o}2P_4$ where T is a T_p-tree with 13 vertices is shown in Figure 4.

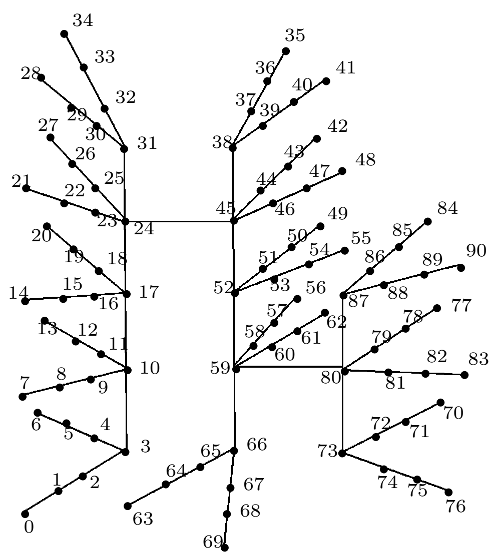

Figure 4: strongly odd harmonious labeling of $T\hat{o}2P_4$

Theorem 2.4. *Every regular bamboo tree is odd harmonious.*

Proof. Let $v_0^j, v_1^j, v_2^j, \cdots, v_{n-1}^j$ be the vertices of the j^{th} path P_n, $1 \leq j \leq m$ where v_0^j is identified with the apex vertex v_0 and v_{n-1}^j is identified with u_0^j which is the centre of the j^{th} star. Let $u_1^j, u_2^j, \cdots, u_t^j$ be the pendant vertices of the j^{th} star. The regular bamboo tree has $m(t+n-1)+1$ vertices and $m(t+n-1)$ edges.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2m(n+t-1)-1\}$ as follows:

Case(i): m is odd

$f(v_0) = 0,$
For $1 \le j \le m,$
$f(v_i^j) = 2j - 1 + m(i-1),$ if i is odd,
$f(v_i^j) = 2 + 4(m-j) + m(i-2),$ if i is even,
$f(u_i^j) = m(n-1) + 2m - 1 + 2m(t-i) - 2(m-j),$ $1 \le i \le t.$

The induced edge labels are
For $1 \le j \le m,$
$f^*(v_0 v_1^j) = 2j - 1,$
$f^*(v_i^j v_{i+1}^j) = 2j + 2m(i-1) + 4(m-j) + 1,$ if i is odd,
$f^*(v_i^j v_{i+1}^j) = 4(m-j) + 2m(i-1) + 2j + 1,$ if i is even,
$f^*(v_{n-1}^j u_i^j) = 2m(n+t-i) - 2j + 1,$ $1 \le i \le t.$

Case (ii): m is even

$f(v_0) = n - 1,$
$f(v_i^1) = n - 1 - i,$ $1 \le i \le n-1,$
For $2 \le j \le m$ and $1 \le i \le t,$
$f(v_i^j) = n + 2(j-2) + (m-1)(i-1),$ if i is odd,
$f(v_i^j) = n + 1 + 4(m-j) + (m-1)(i-2),$ if i is even,
If n is odd, $f(u_i^1) = 2mn + 2(m-1)(t-1) - 2m + 2(t-i) + 7,$
If n is even, $f(u_i^1) = 2mn + 2(m-1)(t-1) + 2(t-i) - 1,$
If n is odd, $f(u_i^j) = m(n-1) + 5 + 2(m-1)(t-i) - 2(m-j),$
If n is even, $f(u_i^j) = m(n-2) + 3 + 2(m-1)(t-i) + 4(m-j).$

The induced edge labels are
$f^*(v_0 v_1^1) = 2n - 3,$
$f^*(v_0 v_1^j) = 2n - 1 + 2(j-2),$ $2 \le j \le m,$
$f^*(v_i^j v_{i+1}^j) = 2n + 2(j-2) + 2(m-1)(i-1) + 4(m-j) + 1,$ $2 \le j \le m.$

For $2 \le j \le m$ and $1 \le i \le t,$
If n is even, $f^*(v_{n-1}^j u_i^j) = 2mn - 2j + 1 + 2(m-1)(t-i).$
If n is odd, $f^*(v_{n-1}^j u_i^j) = 2m(n-1) - 2j + 2(m-1)(t-i) + 9.$

$$f^*(v_{n-1}^1 u_i^1) = \begin{cases} 2mn + 2(m-1)(t-1) - 2m + 2(t-i) + 7 & \text{if } n \text{ is odd} \\ \\ 2mn + 2(m-1)(t-1) + 2(t-i) - 1 & \text{if } n \text{ is even.} \end{cases}$$

Thus every regular bamboo tree is odd harmonious. □

An odd harmonious labeling of a regular bamboo tree with $m = 5, n = 6, t = 2$ is shown in Figure 5.

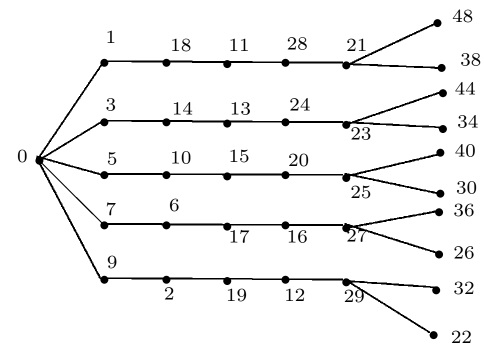

Figure 5: A regular bamboo tree with $m = 5, n = 6, t = 2$

An odd harmonious labeling of a regular bamboo tree with $m = 4, n = 5, t = 2$ is shown in Figure 6.

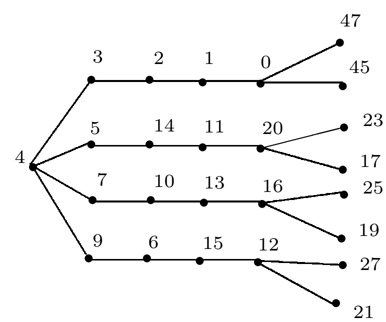

Figure 6: A regular bamboo tree with $m = 4, n = 5, t = 2$

Theorem 2.5. *The graph* $C_n \hat{\circ} P_m$, $n \equiv 0(mod\ 4)$ *is odd harmonious.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of cycle C_n. Let $u_i^0, u_i^1, \cdots, u_i^{m-1}$ be the vertices of path P_m. We identify u_i^0 with u_i, $1 \leq i \leq n$ to obtain $C_n \hat{\circ} P_m$. Then the graph $C_n \hat{\circ} P_m$ has mn edges and vertices.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2nm - 1\}$ as follows:

Case (i): m is odd

$f(u_i) = mi,$ $\quad i = 1, 3, \cdots, n-1,$

$f(u_i) = mi - m - 1,$ $\quad i = 2, 4, \cdots, \frac{n}{2},$

$f(u_i) = mi - m + 1,$ $\quad i = \frac{n}{2} + 2, \cdots, n;$

$f(u_i^j) = mi - j,$ $\quad$ if i is odd and j is even,

$$f(u_i^j) = \begin{cases} mi - j - 2 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ mi - j & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if both } i \text{ and } j \text{ are odd,}$$

$$f(u_i^j) = \begin{cases} mi - m + j - 1 & \text{if } 2 \le i \le \frac{n}{2} \\ mi - m + j + 1 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if both } i \text{ and } j \text{ are even,}$$

$$f(u_i^j) = mi - m + j + 1, \qquad \text{if } i \text{ is even and } j \text{ is odd.}$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = 2im - 1, \qquad 1 \le i \le \frac{n}{2},$$

$$f^*(u_i u_{i+1}) = 2im + 1, \qquad \frac{n}{2} + 1 \le i \le n - 1,$$

$$f^*(u_1 u_n) = mn + 1,$$

$$f(u_i u_i^1) = \begin{cases} 2mi - 3 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ 2mi - 1 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases}$$

$$f(u_i u_i^1) = \begin{cases} 2mi - 2m + 1 & \text{if } 2 \le i \le \frac{n}{2} \\ 2mi - 2m + 3 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$$f(u_i^j u_i^{j+1}) = \begin{cases} 2mi - 2j - 3 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ 2mi - 2j - 1 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if } i \text{ is odd,}$$

$$f(u_i^j u_i^{j+1}) = \begin{cases} 2mi - 2m + 2j - 1 & \text{if } 2 \le i \le \frac{n}{2} \\ 2mi - 2m + 2j + 1 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even.}$$

Case (ii): m is even

$$f(u_i) = mi - 2, \qquad i = 1, 3, \cdots, n - 1,$$

$$f(u_i) = \begin{cases} mi - m + 1 & \text{if } 2 \le i \le \frac{n}{2} \\ mi - m + 3 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even,}$$

$$f(u_i^j) = \begin{cases} mi - j & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ mi - j + 2 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if both } i \text{ and } j \text{ are odd,}$$

$$f(u_i^j) = mi - j - 2, \; 1 \le i \le n - 1, \qquad \text{if } i \text{ is odd and } j \text{ is even,}$$

$$f(u_i^j) = mi - m + j - 1, \; 2 \le i \le n - 2, \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$$f(u_i^j) = \begin{cases} mi - m + j + 1 & \text{if } 2 \le i \le \frac{n}{2} \\ mi - m + j - 1 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if both } i \text{ and } j \text{ are even.}$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = \begin{cases} 2mi - 1 & \text{if } 1 \le i \le \frac{n}{2} \\ 2mi + 1 & \text{if } \frac{n}{2} + 1 \le i \le n \end{cases} \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$$f^*(u_1 u_n) = mn + 1;$$

$$f(u_i u_i^1) = \begin{cases} 2mi - 3 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ 2mi - 1 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if } i \text{ is odd,}$$

$$f^*(u_iu_i^1) = \begin{cases} 2mi-2m+1 & \text{if } 2 \le i \le \dfrac{n}{2} \\ 2mi-2m+3 & \text{if } \dfrac{n}{2}+2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even,}$$

$$f^*(u_i^ju_i^{j+1}) = \begin{cases} 2mi-2j-3 & \text{if } 1 \le i \le \dfrac{n}{2}-1 \\ 2mi-2j-1 & \text{if } \dfrac{n}{2}+1 \le i \le n-1 \end{cases} \qquad \text{if } i \text{ is odd,}$$

$$f^*(u_i^ju_i^{j+1}) = \begin{cases} 2mi-2m+2j+1 & \text{if } 2 \le i \le \dfrac{n}{2} \\ 2mi-2m+2j-1 & \text{if } \dfrac{n}{2}+2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even.}$$

Therefore $C_n\hat{\circ}P_m$ is odd harmonious. □

An odd harmonious labeling of $C_4\hat{\circ}P_5$ and $C_{12}\hat{\circ}P_4$ are shown in Figure 7.

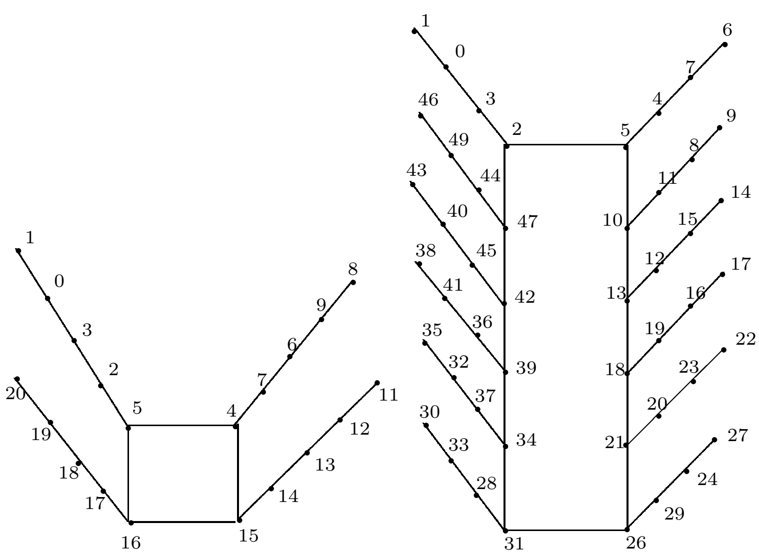

Figure 7: An odd harmonious labeling of $C_4\hat{\circ}P_5$ and $C_{12}\hat{\circ}P_4$

Theorem 2.6. *The graph $C_n\hat{\circ}2P_m$, $n \equiv 0(mod\ 4)$ is odd harmonious.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of C_n. Let $x_{1,0}^j, x_{1,1}^j, x_{1,2}^j, \cdots, x_{1,m-1}^j$ and $x_{2,0}^j, x_{2,1}^j, x_{2,2}^j, \cdots, x_{2,m-1}^j$, $1 \le j \le n$ be the vertices of two disjoint paths P_m. Identify $x_{1,0}^j$ and $x_{2,0}^j$ with u_j, $1 \le j \le n$ to obtain $C_n\hat{\circ}2P_m$. Then the graph $C_n\hat{\circ}2P_m$ has $n(2m-1)$ edges and vertices.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2n(2m-1)-1\}$ as follows:

$$f(u_j) = m-1+(2m-1)(j-1), \qquad j = 1, 3, \cdots, n-1,$$

$$f(u_j) = \begin{cases} 3m-2+(2m-1)(j-2) & \text{if } 2 \le j \le \dfrac{n}{2} \\ 7m-2+(2m-1)(j-4) & \text{if } \dfrac{n}{2}+2 \le j \le n \end{cases} \qquad \text{if } j \text{ is even,}$$

$$f(x_{1,i}^j) = (m-1)+(2m-1)(j-1)-i, \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$f(x_{1,i}^j) = \begin{cases} m+(2m-1)(j-1)-i-1 & \text{if } 1 \leq j \leq \frac{n}{2}-1 \\ m+(2m-1)(j-1)-i+1 & \text{if } \frac{n}{2}+1 \leq j \leq n-1 \end{cases}$ if both i and j are odd,

$f(x_{2,i}^j) = m+(2m-1)(j-1)+i-1$, $1 \leq j \leq n-1$, if i is even and j is odd,

$f(x_{2,i}^j) = \begin{cases} m+(2m-1)(j-1)+i-1 & \text{if } 1 \leq j \leq \frac{n}{2}-1 \\ m+(2m-1)(j-1)+i+1 & \text{if } \frac{n}{2}+1 \leq j \leq n-1 \end{cases}$ if both i and j are odd,

$f(x_{1,i}^j) = \begin{cases} 3m+(2m-1)(j-2)-i-2 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 7m+(2m-1)(j-4)-i-2 & \text{if } \frac{n}{2}+2 \leq j \leq n \end{cases}$ if both i and j are even,

$f(x_{1,i}^j) = \begin{cases} 3m+(2m-1)(j-2)-i-2 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 7m+(2m-1)(j-4)-i-4 & \text{if } \frac{n}{2}+2 \leq j \leq n \end{cases}$ if i is odd and j is even,

$f(x_{2,i}^j) = \begin{cases} 3m+(2m-1)(j-2)+i-2 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 7m+(2m-1)(j-4)+i-2 & \text{if } \frac{n}{2}+2 \leq j \leq n \end{cases}$ if both i and j are even,

$f(x_{2,i}^j) = \begin{cases} 3m+(2m-1)(j-2)+i-2 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 7m+(2m-1)(j-4)+i-4 & \text{if } \frac{n}{2}+2 \leq j \leq n \end{cases}$ if i is odd and j is even.

The induced edge labels are

$f^*(u_ju_{j+1}) = 4m+2(2m-1)(j-1)-3, \qquad 1 \leq j \leq \frac{n}{2}-1,$

$f^*(u_ju_{j+1}) = 8m+2(2m-1)(j-2)-3, \qquad \frac{n}{2}+1 \leq j \leq n-1,$

$f^*(u_jx_{1,1}^j) = \begin{cases} 2m+2(2m-1)(j-1)-3 & \text{if } 1 \leq j \leq \frac{n}{2}-1 \\ 2m+2(2m-1)(j-1)-1 & \text{if } \frac{n}{2}+1 \leq j \leq n-1 \end{cases}$ if j is odd,

$f^*(u_jx_{1,1}^j) = \begin{cases} 6m+2(2m-1)(j-2)-5 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 14m+2(2m-1)(j-4)-7 & \text{if } \frac{n}{2}+2 \leq j \leq n \end{cases}$ if j is even,

$f^*(u_jx_{2,1}^j) = \begin{cases} 2m+2(2m-1)(j-1)-1 & \text{if } 1 \leq j \leq \frac{n}{2}-1 \\ 2m+2(2m-1)(j-1)+1 & \text{if } \frac{n}{2}+1 \leq j \leq n-1 \end{cases}$ if j is odd,

$f^*(u_jx_{2,1}^j) = \begin{cases} 6m+2(2m-1)(j-2)-3 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 14m+2(2m-1)(j-4)-5 & \text{if } \frac{n}{2}+2 \leq j \leq n \end{cases}$ if j is even,

$f^*(x_{1,i}^jx_{1,i+1}^j) = \begin{cases} 4mj-2m-2j-2i-1 & \text{if } 1 \leq j \leq \frac{n}{2}-1 \\ 4mj-2m-2j-2i+1 & \text{if } \frac{n}{2}+1 \leq j \leq n-1 \end{cases}$ if j is odd,

$f^*(x_{1,i}^jx_{1,i+1}^j) = \begin{cases} 6m+2(2m-1)(j-2)-2i-5 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 14m+2(2m-1)(j-4)-2i-7 & \text{if } \frac{n}{2}+2 \leq j \leq n \end{cases}$ if j is even,

$f^*(x_{2,i}^jx_{2,i+1}^j) = \begin{cases} 2m+2(2m-1)(j-1)+2i-1 & \text{if } 1 \leq j \leq \frac{n}{2}-1 \\ 2m+2(2m-1)(j-1)+2i+1 & \text{if } \frac{n}{2}+1 \leq j \leq n-1 \end{cases}$ if j is odd,

$$f^*(x^j_{2,i}x^j_{2,i+1}) = \begin{cases} 6m + 2(2m-1)(j-2) + 2i - 3 & \text{if } 2 \le j \le \dfrac{n}{2} \\ 14m + 2(2m-1)(j-4) + 2i - 5 & \text{if } \dfrac{n}{2} + 2 \le j \le n \end{cases} \quad \text{if } j \text{ is even,}$$

$f^*(u_1u_n) = 8m + (2m-1)(n-4) - 3.$

Thus $C_n\hat{o}2P_m$ is odd harmonious. □

An odd harmonious labeling of $C_8\hat{o}2P_3$ is shown in Figure 8.

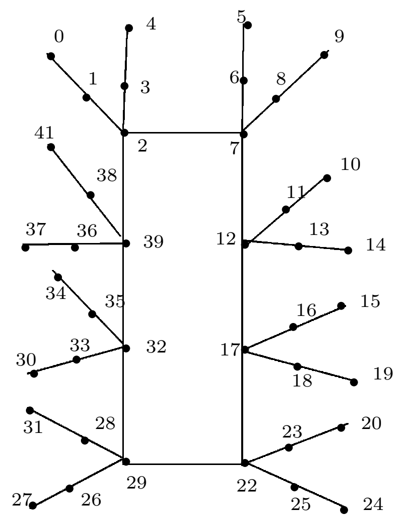

Figure 8: An odd harmonious labeling of $C_8\hat{o}2P_3$

Theorem 2.7. *Every subdivided grid $P_m \times P_m$, $m \ge 2$ is strongly odd harmonious.*

Proof. Let $v_{i,1}, v_{i,2}, \cdots, v_{i,m}$, $1 \le i \le m$ be the vertices of the i^{th} row of $P_m \times P_m$. Let $u_{1,i}$, $u_{2,i}, \cdots, u_{m-1,i}$, $1 \le i \le m$ be the vertices of the subdivided of i^{th} column and $w_{1,i}, w_{2,i}, \cdots, w_{m-1,i}$, $1 \le i \le m$ be the vertices of the subdivided of i^{th} row. Then the subdivided grid graph has $m(3m-2)$ and $4m(m-1)$ vertices and edges respectively.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q = 4m(m-1)\}$ as follows:

$f(v_{i,j}) = 2(j-1) + 2(2m-1)(i-1)$, $\quad 1 \le j \le m$ and i is odd,

$f(v_{i,j}) = 2(3m-j-1) + 2(2m-1)(i-2)$, $\quad 1 \le j \le m$ and i is even,

$f(u_{i,j}) = 2m - 1 + 4(m-j) + 2(2m-1)(i-1)$, $\quad 1 \le j \le m$ and i is odd,

$f(u_{i,j}) = 6m - 3 + 4(j-1) + 2(2m-1)(i-2)$, $\quad 1 \le j \le m$ and i is even,

$f(w_{i,1}) = 2i - 1$, $\quad 1 \le i \le m-1$,

$f(w_{i,j}) = (m-1)(4j-2) - 4i + 2j - 1$, $\quad 1 \le i \le m-1$ and j is even,

$f(w_{i,j}) = (m-1)(4j-6) + 4i + 2j - 5$, $\quad 1 \le i \le m-1$ and j is odd.

The induced edge labels are

$f^*(v_{i,j}u_{i,j}) = 2(j-1)+4(2m-1)(i-1)+2m-1+4(m-j),\ \ 1 \le j \le m$ and i is odd,

$f^*(u_{i,j}v_{i+1,j}) = 2m-1+2(5m-3j-1)+4(2m-1)(i-1),\quad 1 \le j \le m$ and i is odd,

$f^*(u_{i,j}v_{i+1,j}) = 6m-3+6(j-1)+4(2m-1)(i-1),\qquad 1 \le j \le m$ and i is even,

$f^*(v_{i,j}w_{t,k}) = 2(j-1)+2(2m-1)(i-1)+(m-1)(4k-2)-4t-2k-1$, i is odd and k is even,

$f^*(v_{i,j}w_{t,k}) = 2(j-1)+2(2m-1)(i-1)+(m-1)(4k-6)+4t+2k-5$, i is odd and k is odd,

$f^*(v_{i,j}w_{t,k}) = 2(3m-j-1)+2(2m-1)(i-2)+2(m-1)(2k-1)-4t-2k-1$, i is even and k is even,

$f^*(v_{i,j}w_{t,k}) = 2(3m-j-1)+2(2m-1)(i-2)+2(m-1)(2k-3)+4t+2k-5$, i is even and k is odd,

$f^*(v_{1,j}w_{i,1}) = 2(j-1)+2i-1,\qquad 1 \le i$ and $j \le m-1$,

$f^*(w_{i,1}v_{1,j}) = 2(j-1)+2i-1,\qquad 1 \le i \le m-1$ and $2 \le j \le m$.

Therefore every subdivided grid graph is strongly odd harmonious. □

A strongly odd harmonious labeling of subdivided grid $P_4 \times P_4$ is shown in Figure 9.

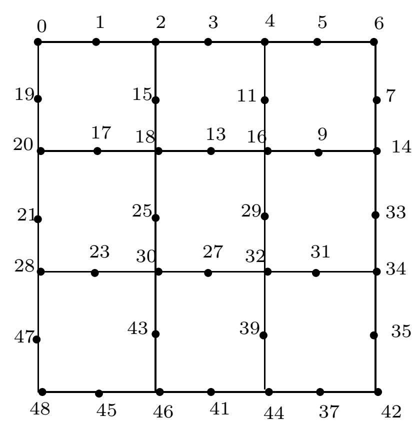

Figure 9: Strongly odd harmonious labeling of subdivided grid $P_4 \times P_4$

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Characterization of Upper Detour Monophonic Domination Number

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ABSTRACT

This paper introduces the concept of *upper detour monophonic domination number* of a graph. For a connected graph G with vertex set $V(G)$, a set $M \subseteq V(G)$ is called minimal detour monophonic dominating set, if no proper subset of M is a detour monophonic dominating set. The maximum cardinality among all minimal monophonic dominating sets is called *upper detour monophonic domination number* and is denoted by $\gamma^+_{dm}(G)$. For any two positive integers p and q with $2 \leq p \leq q$ there is a connected graph G with $\gamma_m(G) = \gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$. For any three positive integers p, q, r with $2 < p < q < r$, there is a connected graph G with $m(G) = p$, $\gamma_{dm}(G) = q$ and $\gamma^+_{dm}(G) = r$. Let p and q be two positive integers with $2 < p < q$ such that $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$. Then there is a minimal DMD set whose cardinality lies between p and q. Let p, q and r be any three positive integers with $2 \leq p \leq q \leq r$. Then, there exist a connected graph G such that $\gamma_{dm}(G) = p, \gamma^+_{dm}(G) = q$ and $|V(G)| = r$.

RESUMEN

Este artículo introduce el concepto de *número de dominación de desvío monofónico superior* de un grafo. Para un grafo conexo G con conjunto de vértices $V(G)$, un conjunto $M \subseteq V(G)$ se llama conjunto dominante de desvío monofónico minimal, si ningún subconjunto propio de M es un conjunto dominante de desvío monofónico. La cardinalidad máxima entre todos los conjuntos dominantes de desvío monofónico minimales se llama *número de dominación de desvío monofónico superior* y se denota por $\gamma^+_{dm}(G)$. Para cualquier par de enteros positivos p y q con $2 \leq p \leq q$ existe un grafo conexo G con $\gamma_m(G) = \gamma_{dm}(G) = p$ y $\gamma^+_{dm}(G) = q$. Para cualquiera tres enteros positivos p, q, r con $2 < p < q < r$, existe un grafo conexo G con $m(G) = p$, $\gamma_{dm}(G) = q$ y $\gamma^+_{dm}(G) = r$. Sean p y q dos enteros positivos con $2 < p < q$ tales que $\gamma_{dm}(G) = p$ y

$\gamma^+_{dm}(G) = q$. Entonces existe un conjunto DMD mínimo cuya cardinalidad se encuentra entre p y q. Sean p, q y r tres enteros positivos cualquiera con $2 \leq p \leq q \leq r$. Entonces existe un grafo conexo G tal que $\gamma_{dm}(G) = p, \gamma^+_{dm}(G) = q$ y $|V(G)| = r$.

Keywords and Phrases: Monophonic number, Domination Number, Detour monophonic number, Detour monophonic domination number, Upper detour monophonic domination number.

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1 Introduction

Consider an undirected connected graph $G(V, E)$ without loops or multiple edges. Let P : $u_1, u_2, ...u_n$ be a path of G. An edge e is said to be a *chord* of P if it is the join of two non adjacent vertices of P. A path is said to be *monophonic path* if there is no chord. If S is a set of vertices of G such that each vertex of G lies on an $u - v$ monophonic path in G for some $u, v \in S$, then S is called *monophonic set. Monophonic number* is the minimum cardinality among all the monophonic sets of G. It is denoted by $m(G)$ [1,2].

A vertex v in a graph G dominates itself and all its neighbours. A set T of vertices in a graph G is a *dominating set* if $N[T] = V(G)$. The minimum cardinality among all the dominating sets of G is called *domination number* and is dented by $\gamma(G)$[4]. A set $T \subset V(G)$ is a *monophonic dominating set* of G if T is both monophonic set and dominating set. The *monophonic domination number* is the minimum cardinality among all the monophonic dominating sets of G and is denoted by $\gamma_m(G)$[5,6]. A monophonic set M in a connected graph G is *minimal monophonic set* if no proper subset of M is a monophonic set. The *upper monophonic number* is the maximum cardinality among all minimal monophonic sets and is denoted by $m^+(G)$[9].

The shortest $x - y$ path is called *geodetic path* and longest $x - y$ monophonic path is called *detour monophonic path*. If every vertex of G lies on a $x - y$ detour monophonic path in G for some $x, y \in M \subseteq V(G)$, M could be identified as a *detour monophonic set*. The minimum cardinality among all the detour monophonic set is the *detour monophonic number* and is denoted by $dm(G)$. A *minimal detour monophonic set* D of a connected graph G is a subset of $V(G)$ whose any proper subset is not a detour monophonic set of G. The maximum cardinality among all minimal detour monophonic sets is called *upper detour monophonic set*, denoted by $dm^+(G)$ [10].

If D is both a detour monophonic set and a dominating set, it could be a *detour monophonic dominating set*. The minimum cardinality among all detour monophonic dominating sets of G is the *detour monophonic dominating number*(DMD number) and is denoted by $\gamma_{dm}(G)$[7,8]. A vertex v is an *extreme vertex* if the sub graph induced by its neighbourhood is complete. A vertex u in a connected graph G is a *cut-vertex* of G, if $G - u$ is disconnected. In this article, we consider

G as a connected graph of order $n \geq 2$ if otherwise not stated. For basic notations and terminology refer [3].

Theorem 1.1 (8)**.** *Each extreme vertex of a connected graph G belongs to every detour monophonic dominating set of G.*

Example 1.1. *Consider the graph G given in Figure 1. Here $M_1 = \{v_1, v_4\}$ is a monophonic set. Therefore $m(G) = 2$. M_1 also dominate G. Hence $\gamma(G) = 2$. The set $M_2 = \{v_1, v_2, v_3\}$ is a minimum detour monophonic set. Thus $dm(G) = 3$. M_2 does not dominate G. $M_2 \cup \{v_4\}$ is a minimum DMD set. Therefore $\gamma_{dm}(G) = 4$.*

2 UDMD Number of a Graph

Definition 2.1. *A detour monophonic dominating set M in a connected graph G is called minimal detour monophonic dominating set if no proper subset of M is a detour monophonic dominating set. The maximum cardinality among all minimal detour monophonic dominating sets is called upper detour monophonic domination number and is denoted by $\gamma^+_{dm}(G)$.*

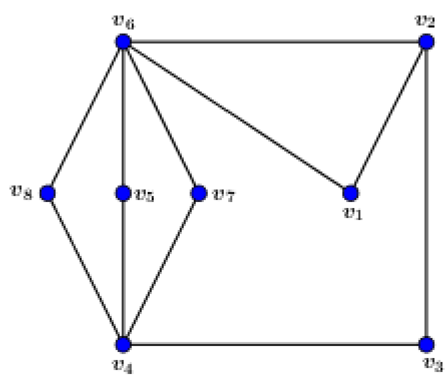

Figure 1: Graph G with UDMD number 5

Example 2.1. *Consider the graph G given in Figure 1. The set $M = \{v_1, v_5, v_6, v_7, v_8\}$ is a minimal DMD set with maximum cardinality. Therefore $\gamma^+_{dm}(G) = 5$.*

Theorem 2.1. *Let G be a connected graph and v an extreme vertex of G. Then v belongs to every minimal detour monophonic dominating set of G.*

Proof. Every minimal detour monophonic dominating set is a minimum detour monophonic set. Since each extreme vertex belongs to every minimum detour monophonic dominating set, the result follows. ∎

Theorem 2.2. *Let v be a cut- vertex of a connected graph G. If M is a minimal DMD set of G, then each component of $G - v$ have an element of M.*

Proof. Suppose let A is a component of $G - v$ having no vertices of M. Let u be any one of the vertex in A. Since M is a minimal DMD set, there exist two vertices p, q in M such that u lies on a $p - q$ detour monophonic path $P : p, u_0, u_1, ..., u, ..., u_m = q$ in G. Consider two sub-paths $P_1 : p - u$ and $P_2 : u - q$ of P. Given v is a cut-vertex of G. Therefore both P_1 and P_2 contain v. Hence P is not a path. This is a contradiction. That is, each component of $G - v$ have an element of every minimal DMD set. ∎

Theorem 2.3. *For a connected graph G of order n, $\gamma_{dm}(G) = n$ if and only if $\gamma^+_{dm}(G) = n$.*

Proof. First, suppose $\gamma^+_{dm}(G) = n$. That is $M = V(G)$ is the unique minimal DMD set of G, so that no proper subset of M is a DMD set. Hence M is the unique DMD set. Therefore $\gamma_{dm}(G) = n$. Conversely, let $\gamma_{dm}(G) = n$. Since every DMD set is a minimal DMD set, $\gamma_{dm}(G) \leq \gamma^+_{dm}(G)$. Therefore $\gamma^+_{dm}(G) \geq n$. Since $V(G)$ is the maximum DMD set, $\gamma^+_{dm}(G) = n$. ∎

3 UDMD Number of Some Standard Graphs

Example 3.1. ***Complete bipartite graph*** $K_{m,n}$

For complete bipartite graph $G = K_{m,n}$,

$$\gamma^+_{dm}(G) = \begin{cases} 2, & if \quad m = n = 1; \\ n, & if \quad n \geq 2, m = 1; \\ 4, & if \quad m = n = 3 \\ max\{m, n\} & if \quad m, n \geq 2, m, n \neq 3 \end{cases}$$

Proof. Case (i): Let $m = n = 1$. Then $K_{m,n} = K_2$. Therefore $\gamma^+_{dm}(G) = 2$.
Case (ii): Let $n \geq 2, m = 1$. This graph is a rooted tree. There are n end vertices. All these are extreme vertices. Therefore they belong to every DMD set and consequently every minimal DMD set.
Case (iii): If $m = n = 3$, then exactly two vertices from both the particians form a minimal DMD set.
Case (iv): Let $m, n \geq 2, m, n \neq 3$. Assume that $m \leq n$. Let $A = \{a_1, a_2, ...a_m\}$ and $B = \{b_1, b_2, ...b_n\}$ be the partitions of G. First, prove $M = B$ is a minimal DMD set. Take a vertex $a_j, 1 \leq j \leq m$, which lies in a detour monophonic path $b_i a_j b_k$ for $k \neq j$ so that M is a detour monophonic set. They also dominate G. Hence M is a DMD set.

Next, let S be any minimal DMD set such that $|S| > n$. Then S contains vertices from both the sets A and B. Since A and B are themselves minimal DMD sets, they do not completely belongs to S. Note that if S contains exactly two vertices from A and B, then it is a minimum DMD set. Thus $\gamma^+_{dm}(G) = n = max\{m, n\}$. ∎

Example 3.2. ***Complete graph*** K_n

For complete graph $G = K_n$, $\gamma^+_{dm}(G) = n$.

Proof. For a complete graph G, every vertex in G is an extreme vertex. By theorem 2.1 they belong to every minimal DMD set. ∎

Example 3.3. ***Cycle graph*** C_n

For Cycle graph $G = C_n$ with n vertices ,

$$\gamma^+_{dm}(G) = \begin{cases} 3, & if \quad n \leq 7, n \neq 4 \\ 2, & if \quad n = 4 \\ 4 + \dfrac{n-7-r}{3}, & if \quad n \geq 8, \quad n-7 \equiv r \; mod(3) \end{cases}$$

Proof. For $n \leq 7$ the results are trivial. For $n \geq 8$, let $C_n : v_1, v_2, v_3, ..., v_n, v_1$ be the cycle with n vertices. Then the set of vertices $\{v_1, v_3, v_{n-1}\}$ is a minimal detour monophonic set but not dominating. This set dominates only seven vertices. There are $n-7$ remaining vertices. If r is the reminder when $n-7$ is divided by 3, then $\dfrac{n-7-r}{3} + 1$ vertices dominate the remaining vertices. Therefore every minimal DMD set contains $4 + \dfrac{n-7-r}{3}$ vertices. ∎

4 Characterization of $\gamma^+_{dm}(G)$

Theorem 4.1. *For any two positive integers p and q with $2 \leq p \leq q$ there is a connected graph G with $\gamma_m(G) = \gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$.*

Proof. Construct a graph G as follows. Let $C_6 : u_1, u_2, u_3, u_4, u_5, u_6, u_1$ be the cycle of order 6. Join $p-1$ disjoint vertices $M_1 = \{x_1, x_2, ..., x_{p-1}\}$ with the vertex u_1. Let $M_2 = \{y_1, y_2, ..., y_{q-p-1}\}$ be a set of $q-p-1$ disjoint vertices. Add each vertex in M_2 with u_4 and u_6. Let x_{p-1} be adjacent with u_2 and u_6. This is the graph G given in Figure 2.

Since all vertices except x_{p-1} in M_1 are extreme, they belong to every minimum monophonic dominating set and DMD set. The set $M = M_1 \cup \{u_4\}$ is a minimum monophonic dominating set. Therefore $\gamma_m(G) = p$. Moreover, the set of all vertices in M form a DMD set and is minimum. That is $\gamma_{dm}(G) = p$.

Next, we prove that $\gamma^+_{dm}(G) = q$. Clearly $N = M_1 \cup M_2 \cup \{u_5, u_6\}$ is a DMD set. N is also a minimal DMD set of G. For the proof, let N' be any proper subset of N. Then there exists at least one vertex $u \in N$ and $u \notin N'$. If $u = y_i$, for $1 \leq i \leq q - p - 1$, then y_i does not lie on any $x - y$ detour monophonic path for some $x, y \in N'$. Similarly if $u \in \{u_5, u_6, x_{p-1}\}$, then that vertex does not lie on any detour monophonic path in N'. Thus N is a minimal DMD set. Therefore $\gamma^+_{dm}(G) \geq q$. ∎

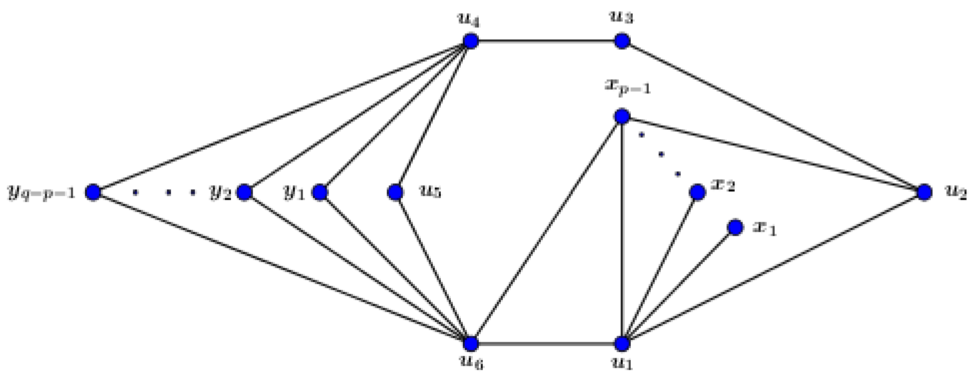

Figure 2: $\gamma_m(G) = \gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$.

Note that N is a minimal DMD set with maximum cardinality. On the contrary, suppose there exists a minimal DMD set, say T, whose cardinality is strictly greater than q. Then there is a vertex $u \in T, u \notin N$. Therefore $u \in \{u_2, u_3, u_4\}$. If $u = u_4$, then $M_1 \cup \{u_4\}$ is a DMD set properly contained in T which is a contradiction. If $u = u_3$, then the set $M_1 \cup \{u_3, u_5\}$ is a DMD set which is a proper subset of T and is a contradiction. If $u = u_2$, then the set $(N - \{u_6\}) \cup \{u_2\}$ is a DMD set properly contained in T and is a contradiction. Thus $\gamma^+_{dm}(G) = q$.

Theorem 4.2. *For any three positive integers p, q, r with $2 < p < q < r$, there is a connected graph G with $m(G) = p$, $\gamma_{dm}(G) = q$ and $\gamma^+_{dm}(G) = r$.*

Proof. Let G be the graph constructed as follows. Take $q - p$ copies of a cycle of order 5 with each cycle C_i has a vertex set $\{d_i, e_i, f_i, g_i, h_i\}$, for $1 \leq i \leq q - p$. Join each e_i with all other vertices in C_i. Also join the vertex f_{i-1} of C_{i-1} with the vertex d_i of C_i. Let $\{u, v\}$ and $\{b_1, b_2, ..., b_{r-q+1}\}$ be two sets of mutually non adjacent vertices. Join each b_i with u and v, for $1 \leq i \leq r - q + 1$. Join another $p - 2$ pendent vertices with u and one pendent vertex with d_1. This is the graph G given in Figure 3.

The set $M_1 = \{a_0, a_1, a_2 ..., a_{p-2}\}$ is the set of all extreme vertices and belongs to every monophonic dominating set and DMD set (Theorem 1.1). Clearly M_1 is not monophonic. But $M_1 \cup \{v\}$ is a monophonic set and is minimum. Therefore $m(G) = p$. Take $M_2 = \{e_1, e_2, ..., e_{q-p}\}$. Then $M_1 \cup M_2 \cup \{v\}$ is a DMD set and is minimum. Therefore $\gamma_{dm}(G) = p - 1 + q - p + 1 = q$. ∎

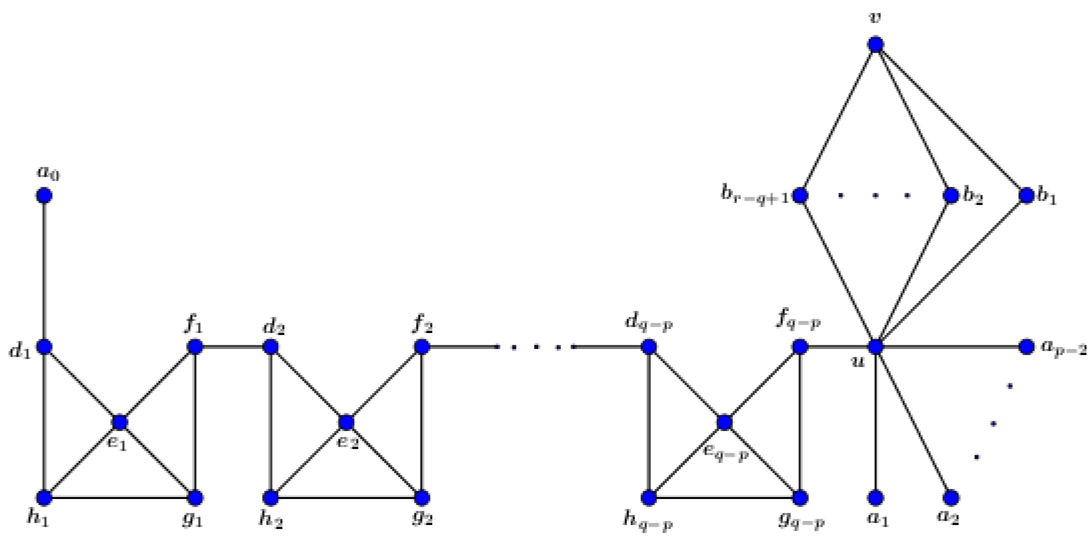

Figure 3: Graph G with $m(G) = p$, $\gamma_{dm}(G) = q$ and $\gamma^+_{dm}(G) = r$.

Let $M_3 = \{b_1, b_2, ..., b_{r-q+1}\}$. Then $M = M_1 \cup M_2 \cup M_3$ is a DMD set. Now M is a minimal DMD set. On the contrary, suppose N is any proper DMD subset of M so that there exists at least one vertex in M which does not belong to N. Let $u \in M$ and $u \notin N$. Clearly $u \notin M_1$ since M_1 is the set of all extreme vertices. If $u = e_i$ for some i, then the vertex e_i does not belong to any detour monophonic path induced by N. Therefore $u \notin M_2$. Similarly $u \notin M_3$. This is a contradiction. Hence M is a minimal DMD set with maximum cardinality. Therefore $\gamma^+_{dm}(G) = |M_1| + |M_2| + |M_3| = (p-1) + (q-p) + (r-q+1) = r$.

Theorem 4.3. *Let p and q be two positive integers with $2 < p < q$ such that $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$. Then there is a minimal DMD set whose cardinality lies between p and q.*

Proof. Consider three sets of mutually disjoint vertices $M_1 = \{a_1, a_2, ..., a_{q-n+1}\}$, $M_2 = \{b_1, b_2, ..., b_{n-p+1}\}$ and $M_3 = \{x, y, z\}$. Join each vertex a_i with x and z and each vertex b_j with y and z. Add $p-2$ pendent vertices $M_4 = \{c_1, c_2, ..., c_{p-2}\}$ with the vertex y. This is the graph G given in Figure 4. Since M_4 is the set of all extreme vertices, it belongs to every DMD set. But M_4 is not a DMD set. The set $M = M_4 \cup \{x, z\}$ is a minimum DMD set. Therefore $\gamma_{dm}(G) = p$.

Consider the set $N = M_1 \cup M_2 \cup M_4$. We claim N is a minimal DMD set with maximum cardinality. On the contrary, suppose there is a set $N' \subset N$ which is a DMD set of G. Then there exists at least one vertex, say u in N which does not belong to N'. Clearly $u \notin M_4$ since it is the set of all extreme vertices. If $u \in M_1$, then $u = a_i$ for some i. Then the vertex a_i does not lie on any detour monophonic path, which is a contradiction. Similarly, if $u \in M_2$, we get a contradiction. Thus N is a minimal DMD set. Therefore $\gamma^+_{dm}(G) \geq q$. ∎

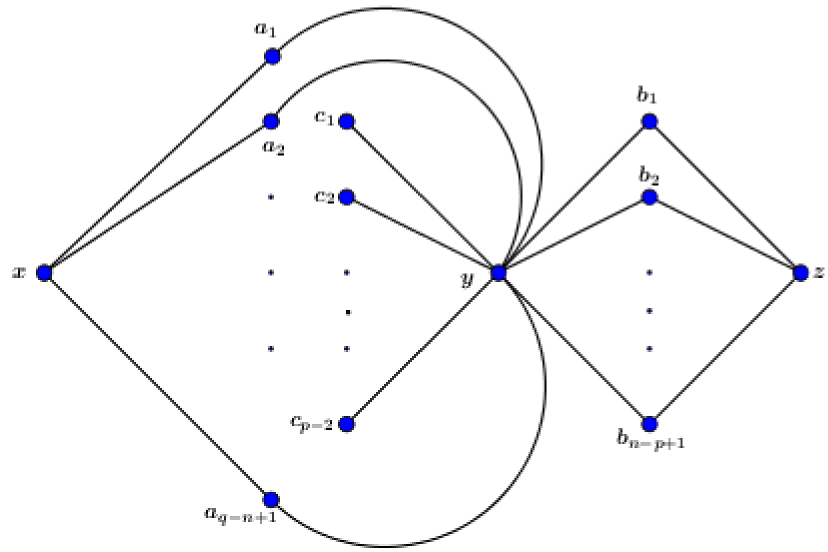

Figure 4: Graph G with $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$

Next, we claim that N has the maximum cardinality of any minimal DMD set. If $\gamma^+_{dm}(G) > q$, there is at least one vertex $v \in V(G), v \notin N$ and belongs to a minimal DMD set. Therefore $v \in M_3$. If $v = x$, then the set $M_2 \cup M_4 \cup \{v\}$ is a minimal DMD set having less than q vertices. Similarly if $v = z$, then the set $M_1 \cup M_4 \cup \{v\}$ is a minimal DMD set. For $v = y$, the set $N \cup \{y\}$ is not a minimal DMD set. Therefore $\gamma^+_{dm}(G) \leq q$.

Let n be any number which lies between p and q. Then there is a minimal DMD set of cardinality n. For the proof, consider the set $T = M_2 \cup M_4 \cup \{x\}$. T is a minimal DMD set. If T is not a minimal DMD set, there is a proper subset T' of T such that T' is a minimal DMD set. Let $u \in T$ and $u \notin T'$. Since each vertex in M_4 is an extreme vertex, $v \notin M_4$. If $u = x$, then the vertex u is not an internal vertex of any detour monophonic path in T'. A similar argument may be made if $u \in M_2$. This leads to a contradiction. Therefore T is a minimal DMD set with cardinality $(n - p + 1) + (p - 2) + 1 = n$.

Theorem 4.4. *Let p , q and r be any three positive integers with $2 \leq p \leq q \leq r$. Then, there exists a connected graph G such that $\gamma_{dm}(G) = p, \gamma^+_{dm}(G) = q$ and $|V(G)| = r$.*

Proof. Let $K_{1,p}$ is a star graph with leaves set $M_1 = \{u_1, u_2, ..., u_p\}$ and let u be the support vertex of $K_{1,p}$. Insert $r - q - 1$ vertices $M_2 = \{v_1, v_2, ..., v_{r-q-1}\}$ in the edges uu_i respectively for $1 \leq i \leq r - q - 1$. Add $q - p$ vertices $M_3 = \{x_1, x_2, ..., x_{q-p}\}$ with this graph and join each x_i with u and u_1. This is the graph G as shown in Figure 5. Here $|V(G)| = (q-p)+p+(r-q-1)+1 = r$. The length of a detour monophonic path is 4.

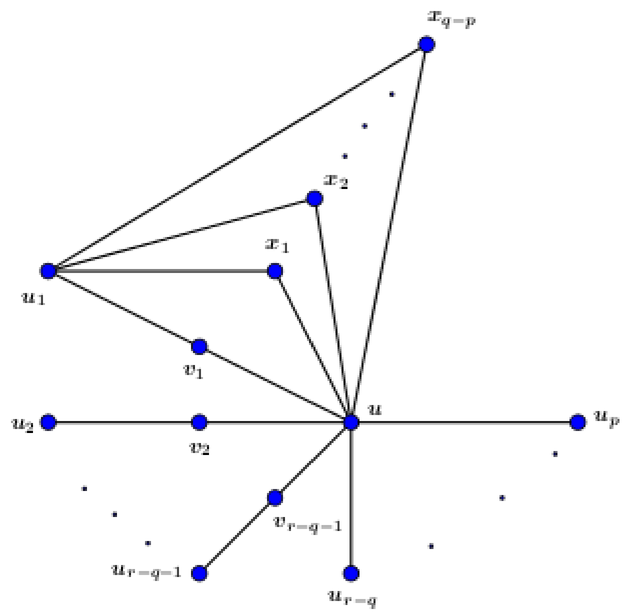

Figure 5: Graph G with $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$

Let $T = M_1 - \{u_1\}$. All the vertices in T are extreme vertices and belong to all DMD sets and minimal DMD sets. Clearly M_1 is a DMD set with minimum cardinality. Therefore $\gamma_{dm}(G) = p$. Let $N = T \cup M_3 \cup \{v_1\}$. Then $|N| = (p-1) + (q-p) + 1 = q$. We claim that N is a minimal DMD set with maximum cardinality.

On the contrary, suppose there is a proper subset N' of N which is a minimal DMD set of G. Then there exists at least one vertex $x \in N, x \notin N'$. Clearly $x \notin T$. If $x \in M_3$, then $x = x_i$ for some i, $1 \leq i \leq q-p$. Then the vertex x_i does not lie on any $u - v$ detour monophonic path for $u, v \in N'$. If $x = v_1$ then v_1 does not lies on any detour monophonic path in N'. Thus no such vertex x exists. This is a contradiction. Therefore $\gamma^+_{dm}(G) \geq q$.

To prove maximum cardinality of N, suppose there exists a minimal DMD set S with $|S| > q$. Since S contains T, the set of all extreme vertices, the vertex x lies on some $u-v$ detour monophonic path for all $x \in \{u, v_2, v_3, .., v_{r-q-1}\}$. Now S is a minimal DMD set having more than q vertices and $u, v_2, v_3, ..., v_{r-q-1} \notin S$. Therefore $S = \{v_1\} \cup M_3 \cup \{u_1\} \cup T$. Then N is properly contained in S. This is a contradiction. Therefore $\gamma^+_{dm}(G) = q$. Hence the proof. ∎

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Existence and Attractivity Theorems for Nonlinear Hybrid Fractional Integrodifferential Equations with Anticipation and Retardation

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ABSTRACT

In this paper, we establish the existence and a global attractivity results for a nonlinear mixed quadratic and linearly perturbed hybrid fractional integrodifferential equation of second type involving the Caputo fractional derivative on unbounded intervals of real line with the mixed arguments of anticipations and retardation. The hybrid fixed point theorem of Dhage is used in the analysis of our nonlinear fractional integrodifferential problem. A positivity result is also obtained under certain usual natural conditions. Our hypotheses and claims have also been explained with the help of a natural realization.

RESUMEN

En este artículo, se establecen resultados de existencia y de atractividad global para una ecuación no lineal cuadrática mixta e híbrida fraccionaria integrodiferencial linealmente perturbada de segundo tipo involucrando la derivada fraccional de Caputo en intervalos no acotados de la recta real con argumentos mixtos de anticipación y retardo. El teorema de punto fijo híbrido de Dhage es usado en el análisis de nuestro problema no lineal fraccionario integrodiferencial. También se obtiene un resultado de positividad bajo ciertas condiciones naturales usuales. Nuestras hipótesis y afirmaciones también se explican con la ayuda de una realización natural.

Keywords and Phrases: Hybrid fractional integrodifferential equation, Dhage fixed point theorem, Existence theorem, Attractivity of solutions, Asymptotic stability.

2020 AMS Mathematics Subject Classification: 34K10, 47H10.

1 Introduction

Let $t_0 \in \mathbb{R}$ be a fixed real number and let $J_\infty = [t_0, \infty)$ be a closed but unbounded interval in $\mathbb{R}$. Let $\mathcal{CRB}(J_\infty)$ denote the class of **pulling functions** $a : J_\infty \to (0, \infty)$ satisfying the following properties:

(i) a is continuous, and

(ii) $\lim\limits_{t\to\infty} a(t) = \infty$.

The notion of the pulling function is introduced in Dhage [15, 17] and Dhage *et al.* [21]. There do exist functions $a : J_\infty \to (0, \infty)$ satisfying the above two conditions. In fact, if $a_1(t) = |t| + 1$, $a_2(t) = e^{|t|}$, then $a_1, a_2 \in \mathcal{CRB}(J_\infty)$. Again, the class of continuous and strictly monotone functions $a : J_\infty \to (0, \infty)$ going to ∞ satisfy the above criteria. Note that if $a \in \mathcal{CRB}(J_\infty)$, then the reciprocal function $\overline{a} : J_\infty \to \mathbb{R}_+$ defined by $\overline{a}(t) = \frac{1}{a(t)}$ is continuous and $\lim\limits_{t\to\infty} \overline{a}(t) = 0$. It has been shown in Dhage [16, 18, 19, 20] and Dhage *et. al* [21] that the pulling functions are useful in proving different asymptotic characterizations of the solutions of nonlinear differential and integral equations. In this paper we employ the pulling functions for characterizing the solutions of a nonlinear hybrid fractional differential equation when the value of independent variable is large.

It is now well-known that several anomalous real world problems in sciences and engineering are adequately modelled on fractional differential equations (see Hilfer [25] and Kilbas *et. al* [27]). Sometimes one may be interested in the behaviour of the anomalous dynamic system in the long duration of time which depend upon both past history as well as the future data of the process in question. In such cases, we take help of fractional differential equations with retardatory and anticipatory arguments on the unbounded intervals of real line. Motivated by this reason, in this paper we discuss asymptotic behaviour of a nonlinear hybrid fractional integrodifferential equation with retardation and anticipation on the unbounded intervals via hybrid fixed point theory of Dhage [8, 9, 16].

We need the following fundamental definitions from fractional calculus (see Podlubny [28], Kilbas *et al.* [27] and references therein) in what follows.

Definition 1.1. *If $J_\infty = [t_0, \infty)$ be an interval of the real line $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, then for any $x \in L^1(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional integral of fractional order $q > 0$ is defined as*

$$I^q_{t_0} x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{1-q}}\, ds, \quad t \in J_\infty,$$

provided the right hand side is pointwise defined on (t_0, ∞), where Γ is the Euler's gamma function defined by $\Gamma(q) = \int_0^\infty e^{-t} t^{q-1}\, dt$.

Definition 1.2. *If $x \in C^n(J_\infty, \mathbb{R})$, then the Caputo fractional derivative ${}^C D_{t_0}^q x$ of x of fractional order q is defined as*

$$ {}^C D_{t_0}^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s)\, ds, \quad t \in J_\infty, $$

where $n-1 < q \leq n,\ n = [q]+1,\ [q]$ denotes the integer part of the real number q, and Γ is the Euler's gamma function. Here $C^n(J_\infty, \mathbb{R})$ denotes the space of real valued functions $x(t)$ which are n times continuously differentiable on J_∞.

Given a pulling function $a \in \mathcal{CRB}(J_\infty) \bigcap C^1(J_\infty, \mathbb{R})$, we consider the following nonlinear hybrid fractional integrofractional differential equation (in short HFRIGDE) involving the Caputo fractional derivative,

$$ \left. \begin{aligned} {}^C D_{t_0}^q \left[\frac{a(t)x(t) - \displaystyle\sum_{j=1}^{m} I^{\alpha_j} h_j(t, x(t), x(\eta(t)))}{f(t, x(t), x(\theta(t)))} \right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned} \right\} \tag{1.1} $$

where ${}^C D_{t_0}^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$, I^{α_j} are the Riemann-Louville fractional integration of fractional order $\alpha_j \geq 0$ for $j = 1, \ldots, m$, $f : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$, $h_j : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, $g : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and $\eta, \theta, \gamma : J_\infty \to J_\infty$ are the continuous functions such that η and θ are anticipatory and γ is retardatory, that is, $\eta(t) \geq t$, $\theta(t) \geq t$ and $\gamma(t) \leq t$ for all $t \in J_\infty$ with $\eta(t_0) = t_0 = \theta(t_0)$.

Definition 1.3. *By a solution for the hybrid fractional differential equation (1.1) we mean a function $x \in C^1(J_\infty, \mathbb{R})$ such that*

(i) the map $(x, y, z) \mapsto \dfrac{a(t)x - \Sigma_{j=1}^{m} I^{\alpha_j} h_j(t, x, z)}{f(t, x, y)}$ is well defined for each $t \in J_\infty$,

(ii) the map $t \mapsto \dfrac{a(t)x(t) - \Sigma_{j=1}^{m} I^{\alpha_j} h_j(t, x(t), x(\theta(t)))}{f(t, x(t), x(\theta(t)))} = z(t)$ is differentiable on J_∞ and $z' \in C(J_\infty, \mathbb{R})$, and

(iii) x satisfies the equations in (1.1) on J_∞,

where $C^1(J_\infty, \mathbb{R})$ is the space of continuous real-valued functions defined on J_∞ whose first derivative x' exists and $x' \in C(J_\infty, \mathbb{R})$.

As the functions θ and γ in the HFRIGDE (1.1) are respectively anticipatory and retardatory, the arguments in the problem (1.1) are deviating over the unbounded interval J_∞. Therefore, the behaviour of the dynamic system modelled on the HFRIGDE (1.1) depends upon both back history as well as future data. As a result the existence analysis of the HFRIGDE (1.1) involves both anticipation and retardation information of the state variable. In a nutshell, the HFRIGDE (1.1) is a nonlinear problem with anticipation and retardation.

The HFRIGDE (1.1) is a mixed linear and quadratic perturbation of second type obtained by multiplying the unknown function under Caputo derivative with a scalar function a together with a subtraction of the term containing unknown function and dividing by a nonlinearity f. The classification of the different types of perturbations of a differential equation is given in Dhage [6]. When $h_j \equiv h$ on $J_\infty \times \mathbb{R} \times \mathbb{R}$, the HFRIGDE (1.1) reduces to the nonlinear ordinary quadratic Caputo fractional differential equation,

$$\left.\begin{aligned} {}^C D^q_{t_0}\left[\frac{a(t)x(t) - h(t, x(t), x(\eta(t)))}{f(t, x(t), x(\theta(t)))}\right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.2}$$

which again, when $h_j \equiv 0$, includes the class of the nonlinear quadratic Caputo fractional differential equations

$$\left.\begin{aligned} {}^C D^q_{t_0}\left[\frac{a(t)x(t)}{f(t, x(t), x(\theta(t)))}\right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.3}$$

as a special case. The HFRIGDE (1.2) is new to the literature whereas the HFRIGDE (1.3) is studied in Dhage [18] for existence and attractivity of the solutions on unbounded interval J_∞. When $f(t, x, y) = 1$ and $g(t, x, y) = g(t, x)$ for all $(t, x, y) \in J_\infty \times \mathbb{R} \times \mathbb{R}$, we obtain the following Caputo fractional differential equation,

$$\left.\begin{aligned} {}^C D^q_{t_0}[a(t)x(t)] &= g(t, x(t)), \quad t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.4}$$

The equation (1.4) is studied in Dhage [17] for existence, uniqueness and asymptotic attractivity and stability of solutions via classical fixed point theory.

We note that when $q = 1$, the hybrid fractional differential equations (in short HFRDEs) (1.2), (1.3) and (1.4) reduce to the ordinary nonlinear hybrid differential equations,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t) - h(t, x(t), x(\eta(t)))}{f(t, x(t), x(\theta(t)))}\right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned}\right\}, \tag{1.5}$$

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t)}{f(t, x(t), x(\theta(t)))}\right] &= g(t, x(t), x(\gamma(t))), \quad t \in J_\infty, \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.6}$$

and

$$\left.\begin{aligned} \frac{d}{dt}\left[a(t)x(t)\right] = g(t,x(t)),\ \ t \in J_\infty, \\ x(t_0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.7}$$

which are discussed in Dhage [17], Dhage [18] and [15] respectively. The hybrid differential equation (1.7) also includes the nonlinear differential equation treated in Burton and Furumochi [4] as the special case. Therefore the existence and attractivity results of this paper include the similar results for the ordinary nonlinear hybrid classical and fractional differential equations (1.2) through (1.7) as special cases.

Now we state a couple of well-known results fractional calculus which are helpful in transforming the Caputo fractional differential equations into Riemann-Louville fractional integral equations and vice versa.

Lemma 1.1 (Kilbas et al. [27])**.** *Suppose that $x \in C^n(J,\mathbb{R})$ and $q \in (n-1,n)$, $n \in \mathbb{N}$. Then, the general solution of the fractional differential equation*

$${}^cD^q_{t_0}x(t) = 0$$

is given by

$$x(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}$$

for all $t \in J$, where c_i, $i = 0,1,\ldots,n-1$ are constants and $C^n(J,\mathbb{R})$ is the space of n times continuously differentiable real-valued functions defined on $J = [a,b]$.

Lemma 1.2. (Kilbas *et al.* [27, page 96]) *Let $x \in C^n(J,\mathbb{R})$ and $q > 0$. Then, we have*

$$I^q_{t_0}\left({}^CD^q_{t_0}x(t)\right) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(t_0)}{k\,!}(t-t_0)^k = x(t) + \sum_{k=0}^{n-1}c_k(t-t_0)^k$$

for all $t \in J = [a,b]$, where $n-1 < q \le n$, $n = [q]+1$ and $c_0,\ldots,c_{n-1}$ are constants.

The converse of the above lemma is not true. It is mentioned in Kilbas *et al.* [27, page 95] that if $q > 0$ and $x \in C(J,\mathbb{R})$, then ${}^CD^q_{t_0}\left(I^q_{t_0}x(t)\right) = x(t)$ for all $t \in J = [a,b]$, however it has been proved recently in Cohen and Salem [1, 2] that it is not true for any continuous function on J.

Remark 1.1. *The conclusion of the above Lemmas 1.1 and 1.2 also remains true if we replace the function spaces $C^n([a,b],\mathbb{R})$ and $C([a,b],\mathbb{R})$ with the function spaces $BC^n(J_\infty,\mathbb{R})$ and $BC(J_\infty,\mathbb{R})$ respectively.*

2 Auxiliary Results

Let X be a non-empty set and let $\mathcal{T} : X \to X$. An invariant point under $\mathcal{T}$ in X is called a fixed point of $\mathcal{T}$, that is, the fixed points are the solutions of the functional equation $\mathcal{T}x = x$. Any statement asserting the existence of fixed point of the mapping $\mathcal{T}$ is called a fixed point theorem for the mapping $\mathcal{T}$ in X. The fixed point theorems are obtained by imposing the conditions on T or on X or on both $\mathcal{T}$ and X. By experience, better the mapping $\mathcal{T}$ or X, we have better fixed point principles. As we go on adding richer structure to the non-empty set X, we derive richer fixed point theorems useful for applications to different areas of mathematics and particularly to nonlinear differential and integral equations. Below we give some fixed point theorems useful in establishing the attractivity and ultimate positivity of the solutions for HFRIGDE (1.1) on unbounded intervals. Before stating these results we give some preliminaries.

Definition 2.1 (Dhage [8, 9, 10])**.** *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\psi(0) = 0$ is called a $\mathcal{D}$-**function** on $\mathbb{R}_+$. Let X be an infinite dimensional Banach space with the norm $\|\cdot\|$. A mapping $\mathcal{T} : X \to X$ is called* $\mathcal{D}$**-Lipschitz** *if there is a $\mathcal{D}$-function $\psi_{\mathcal{T}} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi_{\mathcal{T}}(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$.

If $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is called **Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then $\mathcal{T}$ is called a **contraction** on X with the contraction constant k. Further, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called **nonlinear $\mathcal{D}$-contraction** and the function $\psi_{\mathcal{T}}$ is called $\mathcal{D}$-function of $\mathcal{T}$ on X. There do exist $\mathcal{D}$-functions and the commonly used $\mathcal{D}$-functions are $\psi_{\mathcal{T}}(r) = k\,r$ and $\phi(r) = \dfrac{r}{1+r}$, etc. (see Banas and Dhage [3] and the references therein).

Definition 2.2. *An operator $\mathcal{T}$ on a Banach space X into itself is called totally bounded if for any bounded subset S of X, $\mathcal{T}(S)$ is a relatively compact subset of X. If $\mathcal{T}$ is continuous and totally bounded, then it is called completely continuous on X.*

The operator theoretic technique is a powerful method often times used in the analysis of different types of nonlinear equations. Our essential tool used in the chapter is the following hybrid fixed point theorem of Dhage [9, 16] for a quadratic operator equation involving three operators in a Banach algebra X which uses arguments from analysis and topology. See also Dhage [6, 7, 9, 16] and Dhage and O'Regan [22] for some related results and applications.

Theorem 2.1 (Dhage fixed point theorem [9, 16])**.** *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $\mathcal{A}, \mathcal{C} : X \to X$ and $\mathcal{B} : S \to X$ be three operators such that*

(a) $\mathcal{A}$ and $\mathcal{C}$ are $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively,

(b) $\mathcal{B}$ is completely continuous,

(c) $M_{\mathcal{B}}\,\psi_{\mathcal{A}}(r)+\psi_{\mathcal{C}}(r)<r$, *where* $M_{\mathcal{B}}=\|\mathcal{B}(S)\|=\sup\{\|\mathcal{B}x\|:x\in S\}$, *and*

(d) $x=\mathcal{A}x\,\mathcal{B}y+\mathcal{C}x\Longrightarrow x\in S$ *for all* $y\in S$.

Then the operator equation $\mathcal{A}x\,\mathcal{B}x+\mathcal{C}x=x$ *has a solution in* S.

The above hybrid fixed point theorem of Dhage is a fifth important operator theoretic technique or tool that used in the subject of nonlinear analysis in line with Banach, Schauder, Krasnoselskii and Dhage (see [23],[5]). The nonlinear alternatives related to Dhage fixed point theorem, Theorem 2.1 on the lines of Leray-Schauder and Schafer are also available in the literature (see Dhage [7, 8, 9, 10] and references therein), however the present version is more convenient to apply in the theory of nonlinear hybrid differential equations. A collection of a good number of applicable fixed point theorems may be found in the monographs of Granas and Dugundji [23], Deimling [5], Dhage [16] and the references therein. In the following section we give different types of characterizations of the solutions for nonlinear fractional integrodifferential equations on unbounded intervals of the real line.

3 Characterizations of Solutions

We seek solutions of the HFRIGDE (1.1) in the function space $BC(J_\infty,\mathbb{R})$ of continuous and bounded real-valued functions defined on J_∞. Define a standard supremum norm $\|\cdot\|$ and a multiplication "$\cdot$" in $BC(J_\infty,\mathbb{R})$ by

$$\|x\|=\sup_{t\in J_\infty}|x(t)|$$

and

$$(x\cdot y)(t)=(xy)(t)=x(t)y(t),\ t\in J_\infty.$$

Clearly, $BC(J_\infty,\mathbb{R})$ becomes a Banach algebra w.r.t. the above norm and the multiplication. Let $\mathcal{A},\mathcal{B},\mathcal{C}:BC(J_\infty,\mathbb{R})\to BC(J_\infty,\mathbb{R})$ be three continuous operators and consider the following operator equation in the Banach algebra $BC(J_\infty,\mathbb{R})$,

$$\mathcal{A}x(t)\,\mathcal{B}x(t)+\mathcal{C}x(t)=x(t) \tag{3.1}$$

for all $t\in J_\infty$. Below we give different characterizations of the solutions for the operator equation (3.1) in the function space $BC(J_\infty,\mathbb{R})$.

Definition 3.1. *We say that solutions of the operator equation (3.1) are* **locally attractive** *if there exists a closed ball* $\overline{B}_r(x_0)$ *in the space* $BC(J_\infty,\mathbb{R})$ *for some* $x_0\in BC(J_\infty,\mathbb{R})$ *such that for arbitrary solutions* $x=x(t)$ *and* $y=y(t)$ *of equation (3.1) belonging to* $\overline{B}_r(x_0)$ *we have that*

$$\lim_{t\to\infty}(x(t)-y(t))=0. \tag{3.2}$$

In the case when the limit (3.2) is uniform with respect to the set $\overline{B}_r(x_0)$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leq \epsilon \tag{3.3}$$

for all $x, y \in \overline{B}_r(x_0)$ being solutions of ((3.1) and for $t \geq T$, we will say that solutions of equation (3.1) are **uniformly locally attractive** *on J_∞.*

Definition 3.2. *A solution $x = x(t)$ of equation (3.1) is said to be* **globally attractive** *if (3.2) holds for each solution $y = y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$. In other words, we may say that solutions of the equation (3.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$, the condition (3.2) is satisfied. In the case when the condition (3.2) is satisfied uniformly with respect to the space $BC(J_\infty, \mathbb{R})$, i.e., if for every $\epsilon > 0$ there exists $T > 0$ such that the inequality (3.2) is satisfied for all $x, y \in BC(J_\infty, \mathbb{R})$ being the solutions of (3.1) and for $t \geq T$, we will say that solutions of the equation (3.1) are* **uniformly globally attractive** *on J_∞.*

Remark 3.1. *Let us mention that the details of the global attractivity of solutions may be found in a recent paper of Hu and Yan [26] while the concepts of uniform local and global attractivity (in the above sense) may be found in Banas and Dhage [3], Dhage [10, 12, 13] and references therein.*

Now we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (3.1) in the space $BC(J_\infty, \mathbb{R})$.

Definition 3.3 (Dhage [11]). *A solution x of the equation (3.1) is called* **locally ultimately positive** *if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J_\infty, \mathbb{R})$ for some $x_0 \in BC(J_\infty, \mathbb{R})$ such that $x \in \overline{B}_r(0)$ and*

$$\lim_{t \to \infty} \big[|x(t)| - x(t)\big] = 0. \tag{3.4}$$

In the case when the limit (3.4) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\big|\,|x(t)| - x(t)\big| \leq \epsilon \tag{3.5}$$

for all x being solutions of (3.1) in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we will say that solutions of equation (3.1) are **uniformly locally ultimately positive** *on J_∞.*

Definition 3.4 (Dhage [13]). *A solution $x \in BC(J_\infty, \mathbb{R})$ of the equation (3.1) is called* **globally ultimately positive** *if (3.4) is satisfied. In the case when the limit (3.5) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (3.5) is satisfied for all x being solutions of (3.1) in in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we will say that solutions of equation (3.1) are* **uniformly globally ultimately positive** *on J_∞.*

Finally, we have the the following characterization of the asymptotic stability of the solution of the equation (3.1) on J_∞.

Definition 3.5. *A solution of the equation (3.1) is called* ***asymptotically stable*** *to t-axis or zero if* $\lim_{t\to} x(t) = 0$. *Again, x is called* ***uniformly asymptotically stable*** *to zero if for $\epsilon > 0$ there exists a real number $T \geq t_0$ such that $|x(t)| \leq \epsilon$ for all $t \geq T$.*

Remark 3.2. We note that global attractivity implies the local attractivity and uniform global attractivity implies the uniform local attractivity of the solutions for the operator equation (3.1) on J_∞. Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation (3.1) on an unbounded interval J_∞. However, the converse of the above two statements may not be true.

4 Attractivity and Positivity Results

Now, in this section, we discuss the attractivity results for the ordinary hybrid functional fractional integrodifferential equation (1.1) on J_∞. We need the following definition in the sequel.

Definition 4.1. *A function $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) the map $t \mapsto \beta(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$, and

(ii) the map $(x, y) \mapsto \beta(t, x, y)$ is jointly continuous for each $t \in J_\infty$.

The following lemma is often used in the study of nonlinear differential equations (see Granas *et al.* [24] and references therein).

Lemma 4.1 (Carathéodory). *Let $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the map $(t, x, y) \mapsto \beta(t, x, y)$ is jointly measurable. In particular the map $t \mapsto \beta(t, x(t), y(t))$ is measurable on J_∞ for all $x, y \in C(J_\infty, \mathbb{R})$.*

We need the following hypotheses in the sequel.

(A_1) The function f is continuous and there exists a function $\ell \in BC(J_\infty, \mathbb{R}_+)$ and a constant $K > 0$ such that

$$\left|f(t, x_1, x_2) - f(t, x_1, x_2)\right| \leq \frac{\ell(t)\,\max\{|x_1 - x_2|, |x_2 - y_2|\}}{K + \max\{|x_1 - x_2|, |x_2 - y_2|\}}$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, $\sup\limits_{t \in J_\infty} \ell(t) = L$.

(A_2) The function $t \mapsto |f(t, 0, 0)|$ is bounded with bound F.

(B$_1$) The function g is Carathéodory and bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound M_g.

(C$_1$) The functions h_j are continuous and there exist a functions $\ell_j \in BC(J_\infty, \mathbb{R}_+)$ and a constants $K_j > 0$ such that

$$\big|h_j(t, x_1, x_2) - h_j(t, x_1, x_2)\big| \leq \frac{\ell_j(t) \max\{|x_1 - x_2|, |x_2 - y_2|\}}{K_j + \max\{|x_1 - x_2|, |x_2 - y_2|\}}$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $j = 1, \dots, m$. Moreover, $\sup\limits_{t \in J_\infty} \ell_j(t) = L_j$.

(C$_2$) The function $t \mapsto |h_j(t, 0, 0)|$ is bounded with bound H_j.

(D$_1$) The pulling function a satisfies $\lim\limits_{t\to\infty} \overline{a}(t)\, t^q = 0 = \lim\limits_{t\to\infty} \overline{a}(t)\, t^{\alpha_j}$ for each $j = 1, \dots, m$.

Remark 4.1. *If $a \in \mathcal{CRB}(J_\infty)$, then $\overline{a} \in BC(J_\infty, \mathbb{R}_+)$ and so the number $\|\overline{a}\| = \sup_{t\in J_\infty} \overline{a}(t)$ exists. Again, since the hypothesis (D$_1$) holds, the function $w : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the expression $w(t) = \overline{a}(t)\, t^q$ is continuous on J_∞ and satisfies the relation $\lim\limits_{t\to\infty} w(t) = 0$. So the number $W = \sup_{t \geq t_0} w(t)$ exists. Similarly, the function $w_j : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the expression $w_j(t) = \overline{a}(t)\, t^{\alpha_j}$ is continuous on J_∞ and satisfies the relation $\lim\limits_{t\to\infty} w_j(t) = 0$ for each $j = 1, \dots, m$. Hence, the number $W_j = \sup_{t\geq t_0} w_j(t)$ exists for each $j = 1, \dots, m$.*

The following lemma is useful in the sequel.

Lemma 4.2. *If for any function $h \in L^1(J_\infty, \mathbb{R})$, the function $x \in BC(J_\infty, \mathbb{R})$ is a solution of the HFRIGDE*

$$ {}^C D^q_{t_0}\left[\frac{a(t)x(t) - \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t)))}{f(t, x(t), x(\theta(t)))}\right] = h(t), \ \ t \in J_\infty, \tag{4.1}$$

and

$$x(0) = x_0, \tag{4.2}$$

then x satisfies the hybrid fractional integral equation (in short HFRIE)

$$\begin{aligned} x(t) &= \big[f(t, x(t), x(\theta(t)))\big]\left(c_0\, \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^t (t-s)^{q-1} h(s)\, ds\right) \\ &\quad + \overline{a}(t) \sum_{j=1}^m I^{\alpha_j} h_j(t, x(t), x(\eta(t))) \end{aligned} \tag{4.3}$$

for all $t \in J_\infty$, where $c_0 = \dfrac{a(t_0)x_0}{f(t_0, x_0, x_0)}$ and $x_0 \neq 0$.

Proof. Let $h \in L^1(J_\infty, \mathbb{R})$. Assume first that x is a solution of the HFRIGDE (4.1)-(4.2) defined on J_∞ and $x_0 \neq 0$. We apply the Riemann-Liouville fractional integration $I^q_{t_0}$ of fractional order q from t_0 to t on both sides of the HFRIGDE (4.1). Then, by an application Lemma 1.2, the HFRIGDE (4.1)-(4.2) is transformed into the HFRIE (4.3) on J_∞. ∎

Definition 4.2. *A solution $x \in BC(J_\infty, \mathbb{R})$ of the FRIE (4.3) is called a* ***mild solution*** *of the HFRIGDE (4.1)-(4.2) defined on J_∞.*

In the following we shall deal with the mild solution of the HFRIGDE (1.1) on unbounded interval J_∞ of the real line $\mathbb{R}$. Our main existence and global attractivity result is as follows.

Theorem 4.1. *Assume that the hypotheses (A_1) - (A_2), (B_1), (C_1) - (C_2) and (D_1) hold. Further, assume that*

$$(m+1)\cdot \max\left\{L\Big(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\| + \frac{M_g W}{\Gamma q}\Big), \frac{L_1\,W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m\,W_m}{\Gamma(\alpha_m)}\right\} \\ \leq \min\{K, K_1, \dots, K_m\}. \tag{4.4}$$

Then the HFRIGDE (1.1) has a mild solution and mild solutions are uniformly globally attractive defined on J_∞.

Proof. Now, using Lemma 4.2, it can be shown that the mild solution x of the HFRIGDE (1.1) is equivalent to the nonlinear hybrid fractional integral equation (in short HFRIE)

$$x(t) = \big[f(t,x(t),x(\theta(t)))\big]\left(c_0\,\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right) \\ + \overline{a}(t)\sum_{j=1}^{m} I^{\alpha_j}h_j(t,x(t),x(\eta(t))) \tag{4.5}$$

for all $t \in J_\infty$, where $c_0 = \dfrac{a(t_0)x_0}{f(t_0,x_0,x_0)}$. Set $X = BC(J_\infty, \mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in X centered at origin of radius r given by

$$r = \big(L+F\big)\left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) + \sum_{j=1}^{m}\frac{L_j + H_j}{\Gamma(\alpha_j)}\,W_j.$$

Define three operators $\mathcal{A}$ and $\mathcal{C}$ on X and $\mathcal{B}$ on $\overline{B}_r(0)$ by

$$\mathcal{A}x(t) = f(t,x(t),x(\theta(t))),\ \ t \in J_\infty, \tag{4.6}$$

$$\mathcal{B}x(t) = c_0\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds,\ \ t \in J_\infty \tag{4.7}$$

and

$$\mathcal{C}x(t) = \overline{a}(t)\sum_{j=1}^{m} I^{\alpha_j}h_j(t,x(t),x(\eta(t))),\ \ t \in J_\infty. \tag{4.8}$$

Then the HFRIE (4.5) is transformed into the operator equation as

$$\mathcal{A}x(t)\,\mathcal{B}x(t) + \mathcal{C}x(t) = x(t),\ \ t \in J_\infty. \tag{4.9}$$

We show that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.1 on $BC(J_\infty, \mathbb{R})$. First we we show that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ define the mappings $\mathcal{A}, \mathcal{C} : X \to X$ and $\mathcal{B} : \overline{B}_r(0) \to X$. Let $x \in X$ be arbitrary. Obviously, $\mathcal{A}x$ is a continuous function on J_∞. We show that $\mathcal{A}x$ is bounded on J_∞. Thus, if $t \in J_\infty$, then we obtain:

$$\begin{aligned}
|\mathcal{A}x(t)| &= |f(t, x(t), x(\theta(t)))| \\
&\leq |f(t, x(t), x(\theta(t))) - f(t, 0, 0)| + |f(t, 0, 0)| \\
&\leq \ell(t) \frac{\max\{|x(t)|, |x(\theta(t))|\}}{K + \max\{|x(t)|, |x(\theta(t))|\}} + F \\
&\leq L + F.
\end{aligned}$$

Therefore, taking the supremum over t,

$$\|\mathcal{A}x\| \leq L + F = N.$$

Thus $\mathcal{A}x$ is continuous and bounded on J_∞. As a result $\mathcal{A}x \in X$. Again, we have

$$\begin{aligned}
\big|\mathcal{C}x(t)\big| &\leq \left| \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} h_j(t, x(t), x(\eta(t))) - \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} h_j(t, 0, 0) \right| \\
&\quad + \left| \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} h_j(t, 0, 0) \right| \\
&\leq \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \big| h(t, x(t), x(\eta(t))) - h(t, 0, 0) \big| + \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \big| h_j(t, 0, 0) \big| \\
&\leq \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \frac{\ell_j(t) \, \max\{|x(t)|\,, \, |x(\eta(t))|\}}{K_j + \max\{|x(t)|\,, \, |x(\eta(t))|\}} + \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} H_j \\
&\leq \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \frac{\ell_j(t) \, \|x\|}{K_j + \|x\|} + \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} H_j \\
&\leq \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} L_j + \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} H_j \\
&\leq \sum_{j=1}^{m} \frac{L_j}{\Gamma(\alpha_j)} W_j + \sum_{j=1}^{m} \frac{H_j}{\Gamma(\alpha_j)} W_j \\
&\leq \sum_{j=1}^{m} \frac{L_j + H_j}{\Gamma(\alpha_j)} \, W_j
\end{aligned}$$

for all $t \in t_\infty$. Taking the supremum over t as $t \to \infty$, we obtain

$$\|\mathcal{C}x\| \leq \sum_{j=1}^{m} \frac{L_j + H_j}{\Gamma(\alpha_j)} \, W_j.$$

As a result $(\mathcal{C}x)$ is continuous and bounded on J_∞. Hence, $\mathcal{C}x \in X$. Similarly, it can be shown that $\mathcal{B}x \in X$ and in particular, $\mathcal{A}, \mathcal{C} : X \to X$ and $\mathcal{B} : \overline{B}_r(0) \to X$. We show that $\mathcal{A}$ is a Lipschitz on X. Let $x, y \in X$ be arbitrary. Then, by hypothesis (A$_1$),

$$\begin{aligned}
\|\mathcal{A}x - \mathcal{A}y\| &= \sup_{t\in J_\infty} |\mathcal{A}x(t) - \mathcal{A}y(t)| \\
&\leq \sup_{t\in J_\infty} \ell(t) \frac{\max\{|x(t)-y(t)|,\, |x(\theta(t)) - y(\theta(t))|\}}{K + \max\{|x(t)-y(t)|,\, |x(\theta(t)) - y(\theta(t))|\}} \\
&\leq \frac{L\|x-y\|}{K + \|x-y\|} \\
&= \psi_{\mathcal{A}}(\|x-y\|)
\end{aligned}$$

for all $x, y \in X$. This shows that $\mathcal{A}$ is a $\mathcal{D}$-Lipschitz on X with $\mathcal{D}$-function $\psi_{\mathcal{A}}(r) = \dfrac{Lr}{K+r}$.

Similarly, by hypothesis (C$_1$). we have

$$\begin{aligned}
\|\mathcal{C}x - \mathcal{C}y\| &= \sup_{t\in J_\infty} |\mathcal{C}x(t) - \mathcal{C}y(t)| \\
&\leq \sup_{t\in J_\infty} \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \big| h_j(t, x(t), x(\eta(t))) - h_j(t, y(t), y(\eta(t))) \big| \\
&\leq \sup_{t\in J_\infty} \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \frac{\ell_j(t)\, \max\{|x(t)-y(t)|,\, |x(\theta(t)) - y(\theta(t))|\}}{K_j + \max\{|x(t)-y(t)|,\, |x(\theta(t)) - y(\theta(t))|\}} \\
&\leq \sup_{t\in J_\infty} \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \frac{L_j\, \|x-y\|}{K_j + \|x-y\|} \\
&\leq \sum_{j=1}^{m} \frac{\frac{L_j\, W_j}{\Gamma(\alpha_j)} \|x-y\|}{K_j + \|x-y\|} \\
&\leq \sum_{j=1}^{m} \frac{W_j}{\Gamma(\alpha_j)} \cdot \frac{L_j\, \|x-y\|}{K_j + \|x-y\|} \\
&\leq m \cdot \frac{\max\left\{\frac{L_1\, W_1}{\Gamma(\alpha_1)}, \ldots, \frac{L_m\, W_m}{\Gamma(\alpha_m)}\right\} \|x-y\|}{\min\{K_1, \ldots, K_m\} + \|x-y\|}
\end{aligned}$$

This shows that $\mathcal{C}$ is a $\mathcal{D}$-Lipschitz on X with $\mathcal{D}$-function $\psi_{\mathcal{C}}(r)$ given by

$$\psi_{\mathcal{C}}(r) = m \cdot \frac{\max\left\{\frac{L_1\, W_1}{\Gamma(\alpha_1)}, \ldots, \frac{L_m\, W_m}{\Gamma(\alpha_m)}\right\} r}{\min\{K_1, \ldots, K_m\} + r}.$$

Next, we shows that $\mathcal{B}$ is a completely continuous operator on $\overline{B}_r(0)$. First, we show that $\mathcal{B}$ is continuous on $\overline{B}_r(0)$. To do this, let us fix arbitrarily $\epsilon > 0$ and let $\{x_n\}$ be a sequence of points in $\overline{B}_r(0)$ converging to a point $x \in \overline{B}_r(0)$. Then we get:

$$\begin{aligned}
&|(\mathcal{B}x_n)(t) - (\mathcal{B}x)(t)| \\
&\leq \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} |g(s, x_n(s), x_n(\gamma(s))) - g(s, x(s), x(\gamma(s)))| ds \\
&\leq \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} [|g(s, x_n(s), x_n(\gamma(s)))| + |g(s, x(s), x(\gamma(s)))|] ds \\
&\leq 2M_g \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1}\, ds \\
&= \frac{2M_g}{\Gamma q} \cdot w(t),
\end{aligned}$$

where, $w(t) = \overline{a}(t)\, t^q$.

Hence, by virtue of hypothesis (D_1), we infer that there exists a $T > 0$ such that $w(t) \leq \epsilon$ for $t \geq T$. Thus, for $t \geq T$, from the estimate (3.3) we derive that

$$|(\mathcal{B}x_n)(t) - (\mathcal{B}x)(t)| \leq \frac{2M_g}{\Gamma q}\, \epsilon \quad \text{as} \quad n \to \infty.$$

Furthermore, let us assume that $t \in [t_0, T]$. Then, by dominated convergence theorem, we obtain the estimate:

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[c_0\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} g(s, x_n(s), x_n(\gamma(s)))\, ds \right] \\
&= c_0\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \left[\lim_{n\to\infty} g(s, x_n(s), x_n(\gamma(s))) \right] ds \\
&= c_0\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s), x(\gamma(s)))\, ds \\
&= \mathcal{B}x(t)
\end{aligned}$$

for all $t \in [t_0, T]$. Moreover, it can be shown as below that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas *et al.* [23], it is proved that $\mathcal{B}$ is a a continuous operator on $\overline{B}_r(0)$.

Next, we show that $\mathcal{B}$ is a compact operator on $\overline{B}_r(0)$. To finish, it is enough to show that every sequence $\{\mathcal{B}x_n\}$ in $\mathcal{B}(\overline{B}_r(0))$ has a Cauchy subsequence. Now, proceeding with the earlier arguments it is proved that

$$\|\mathcal{B}x_n\| \leq |c_0|\, \|\overline{a}\| + \frac{M_f W}{\Gamma q} = r$$

for all $n \in \mathbb{N}$. This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in $\mathcal{B}(\overline{B}_r(0))$.

Next, we show that $\{\mathcal{B}x_n\}$ is also a equicontinuous sequence in $\mathcal{B}(\overline{B}_r(0))$. Let $\epsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T_1 > t_0 \geq 0$ such that $|w(t)| < \dfrac{\epsilon}{8M_f/\Gamma(q)}$ for all

$t \geq T_1$. Similarly, since $\lim_{t\to\infty} \overline{a}(t) = 0$, for above $\epsilon > 0$, there is a real number $T_2 > t_0 \geq 0$ such that $|\overline{a}(t)| < \dfrac{\epsilon}{8|c_0|}$ for all $t \geq T_2$. Thus, if $T = \max\{T_1, T_2\}$, then

$$|w(t)| < \frac{\epsilon}{8M_f/\Gamma(q)} \quad \text{and} \quad |\overline{a}(t)| < \frac{\epsilon}{8|c_0|} \tag{4.10}$$

for all $t \geq T$. Let $t, \tau \in J_\infty$ be arbitrary. If $t, \tau \in [t_0, T]$, then we have

$$\begin{aligned}
&|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \left| \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds - \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{\tau} (\tau-s)^{q-1} f(s, x(s))\, ds \right| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \left| \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds - \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{t} (\tau-s)^{q-1} f(s, x(s))\, ds \right| \\
&\quad + \left| \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{t} (\tau-s)^{q-1} f(s, x(s))\, ds - \frac{\overline{a}(\tau)}{\Gamma q} \int_{t_0}^{\tau} (\tau-s)^{q-1} f(s, x(s))\, ds \right| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \frac{M_f}{\Gamma q} \int_{t_0}^{t} \left| \overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1} \right| ds \\
&\quad + \frac{M_f}{\Gamma q} \left| \int_{\tau}^{t} \left| \overline{a}(\tau)(\tau-s)^{q-1} \right| ds \right| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \frac{M_f}{\Gamma q} \int_{t_0}^{T} \left| \overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1} \right| ds \\
&\quad + \frac{M_f\,\|\overline{a}\|}{\Gamma q} \left| (\tau - t)^q \right|.
\end{aligned}$$

Since the functions $t \mapsto \overline{a}(t)$ and $t \mapsto \overline{a}(t)(t-s)^{q-1}$ are continuous on compact $[t_0, T]$, they are uniformly continuous there. Therefore, by the uniform continuity, for above ϵ we have the real numbers $\delta_1 > 0$ and $\delta_2 > 0$ depending only on ϵ such that

$$|t - \tau| < \delta_1 \Longrightarrow |\overline{a}(t) - \overline{a}(\tau)| < \frac{\epsilon}{9|c_0|}$$

and

$$|t - \tau| < \delta_2 \Longrightarrow \left| \overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1} \right| < \frac{\epsilon}{9M_f T/\Gamma q}.$$

Similarly, choose the real number $\delta_3 = \left(\frac{\epsilon}{9M_f\|\overline{a}\|/\Gamma(q)} \right)^{1/q} > 0$ so that

$$|t - \tau| < \delta_3 \Longrightarrow |(t - \tau)^q| < \frac{\epsilon}{9M_f\|\overline{a}\|/\Gamma(q)}.$$

Let $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$|t-\tau| < \delta_4 \Longrightarrow |\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| < \frac{\epsilon}{3}$$

for all $n \in \mathbb{N}$. Again, if $t, \tau > T$, then we have a $\delta_5 > 0$ depending only on ϵ such that

$$\begin{aligned}
&|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \\
&\quad \leq |c_0|\ |a(t) - a(\tau)| + \frac{\overline{a}(t)}{\Gamma q}\left| \int_{t_0}^{t} (t-s)^{q-1} f(s, x_n(s))\, ds \right| \\
&\qquad + \frac{\overline{a}(\tau)}{\Gamma q}\left| \int_{t_0}^{\tau} (\tau-s)^{q-1} f(s, x_n(s))\, ds \right| \\
&\quad \leq \big|c_0\big[\, |\overline{a}(t)| + |\overline{a}(\tau)|\big] + \frac{M_f}{\Gamma(q)}\big[w(t) + w(\tau)\big] \\
&\quad < \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta_5$. Similarly, if $t, \tau \in \mathbb{R}_+$ with $t < T < \tau$, then we have

$$|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \leq |\mathcal{B}x_n(t) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(\tau)|.$$

Take $\delta = \min\{\delta_4, \delta_5\} > 0$ depending only on ϵ. Therefore, from the above obtained estimates, it follows that

$$|\mathcal{B}x_n(t) - \mathcal{B}x_n(T)| < \frac{\epsilon}{2} \text{ and } |\mathcal{B}x_n(T) - \mathcal{B}x_n(\tau)| < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. As a result, $|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| < \epsilon$ for all $t, \tau \in J_\infty$ and for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. This shows that $\{\mathcal{B}x_n\}$ is a equicontinuous sequence in X. Now an application of Arzelà-Ascoli theorem yields that $\{\mathcal{B}x_n\}$ has a uniformly convergent subsequence on the compact subset $[t_0, T]$ of J_∞. Without loss of generality, call the subsequence to be the sequence itself. We show that $\{\mathcal{B}x_n\}$ is Cauchy in X. Now $|\mathcal{B}x_n(t) - \mathcal{B}x(t)| \to 0$ as $n \to \infty$ for all $t \in [t_0, T]$. Then for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{t_0 \leq t \leq T} \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \big| f(s, x_m(s)) - f(s, x_n(s)) \big|\, ds < \frac{\epsilon}{2}$$

for all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$, then we have

$$\begin{aligned}
&\|\mathcal{B}x_m - \mathcal{B}x_n\| \\
&= \sup_{t_0 \leq t < \infty} \left| \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \big| f(s, x_m(s)) - f(s, x_n(s)) \big|\, ds \right| \\
&\leq \sup_{t_0 \leq t \leq T} \left| \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \big| f(s, x_m(s)) - f(s, x_n(s)) \big|\, ds \right| \\
&\quad + \sup_{t \geq T} \left| \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \Big[\big| f(s, x_m(s)) \big| + \big| f(s, x_n(s)) \big| \Big]\, ds \right| \\
&< \epsilon.
\end{aligned}$$

This shows that $\{\mathcal{B}x_n\} \subset \mathcal{B}(\overline{B}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{\mathcal{B}x_n\}$ converges to a point in X. As $\mathcal{B}(\overline{B}_r(0))$ is closed, we have that $\{\mathcal{B}x_n\}$ converges to a point in $\mathcal{B}(\overline{B}_r(0))$. Hence $\mathcal{B}(\overline{B}_r(0))$ is relatively compact and consequently $\mathcal{B}$ is a continuous and compact operator on $B_r(0)$ into itself.

Next, we estimate the value of the constant $M_{\mathcal{B}}$ of the hypothesis (c) of the Theorem 2.1. By definition of $M_{\mathcal{B}}$, one has

$$\begin{aligned}
\|\mathcal{B}(B_r(0))\| &= \sup\{\|\mathcal{B}x\| \ : \ x \in \overline{B}_r(0)\} \\
&= \sup\left\{\sup_{t\in J_\infty} |\mathcal{B}x(t)| \ : \ x \in \overline{B}_r(0)\right\} \\
&\leq \sup_{x\in\overline{B}_r(0)} \left\{ \sup_{t\in J_\infty} \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| |\overline{a}(t)| \right. \\
&\qquad \left. + \frac{1}{\Gamma q} \cdot \sup_{t\in J_\infty} |\overline{a}(t)| \int_{t_0}^{t} (t-s)^{q-1} |g(s,x(s),x(\gamma(s)))|\, ds \right\} \\
&\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g}{\Gamma q} \cdot \sup_{t\in J_\infty} \overline{a}(t) \int_{t_0}^{t} (t-s)^{q-1}\, ds \\
&\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g}{\Gamma q} \cdot \sup_{t\in J_\infty} \overline{a}(t)\, t^q \\
&\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} = M_{\mathcal{B}}.
\end{aligned}$$

Thus,

$$\|\mathcal{B}x\| \leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q} = M_{\mathcal{B}}$$

for all $x \in \overline{B}_r(0)$. Hence, we have

$$\begin{aligned}
&M_{\mathcal{B}}\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) \\
&\quad \leq \frac{L\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) r}{K+r} \\
&\qquad + m \cdot \frac{\max\left\{\frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)}\right\} r}{\min\{K_1,\dots,K_m\} + r} \\
&\quad \leq (m+1) \cdot \frac{\max\left\{L\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right), \frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)}\right\} r}{\min\{K, K_1,\dots,K_m\} + r} \\
&\quad < r
\end{aligned}$$

for $r > 0$, because

$$\begin{aligned}
(m+1)\cdot \max&\left\{L\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right), \frac{L_1 W_1}{\Gamma(\alpha_1)}, \dots, \frac{L_m W_m}{\Gamma(\alpha_m)}\right\} \\
&\leq \min\{K, K_1, \dots, K_m\}
\end{aligned}$$

Therefore, hypothesis (c) of Theorem 2.1 is satisfied.

Next, let $y \in \overline{B}_r(0)$ be arbitrary and let $x = \mathcal{A}x\,\mathcal{B}y + \mathcal{C}x$. Then,

$$\begin{aligned}
|x(t)| &\leq |\mathcal{A}x(t)|\,|\mathcal{B}y(t)| + |\mathcal{C}x(t)| \\
&\leq \|\mathcal{A}x\|\,\|\mathcal{B}y\| + \|\mathcal{C}x\| \\
&\leq \|\mathcal{A}(X)\|\,\|\mathcal{B}(\overline{B}_r(0))\| + \|\mathcal{C}(X)\| \\
&\leq \big(L + F\big)\, M_{\mathcal{B}} + \sum_{j=1}^{m} \frac{L_j + H_j}{\Gamma(\alpha_j)}\, W_j \\
&\leq \big(L + F\big)\, \left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) + \sum_{j=1}^{m} \frac{L_j + H_j}{\Gamma(\alpha_j)}\, W_j
\end{aligned}$$

for all $t \in J_\infty$. Therefore, we have:

$$\|x\| \leq \big(L + F\big)\, \left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\| + \frac{M_g W}{\Gamma q}\right) + \sum_{j=1}^{m} \frac{L_j + H_j}{\Gamma(\alpha_j)}\, W_j = r.$$

This shows that $x \in \overline{B}_r(0)$ and hypothesis (c) of Theorem 2.1 is satisfied. Now we apply Theorem 2.1 to the operator equation $\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x$ to yield that the HFRIGDE (1.1) has a mild solution on J_∞. Moreover, the mild solutions of the HFRIGDE (1.1) are in $\overline{B}_r(0)$. Hence, mild solutions are global in nature.

Finally, let $x, y \in \overline{B}_r(0)$ be any two mild solutions of the HFRIGDE (1.1) on J_∞. Then, from

(4.5) we obtain

$$
\begin{aligned}
|x(t)-y(t)| \leq & \Bigg| \left[f(t,x(t),x(\theta(t))) \right] \times \\
& \times \left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds \right) \\
& - \left[f(t,y(t),y(\theta(t))) \right] \times \\
& \times \left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,y(s),y(\gamma(s)))\,ds \right) \Bigg| \\
& + \sup_{t\in J_\infty} \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \big| h_j(t,x(t),x(\eta(t))) - h_j(t,y(t),y(\eta(t)) \big| \\
\leq & \Bigg| \left[f(t,x(t),x(\theta(t))) - f(t,y(t),y(\theta(t))) \right] \\
& \quad \times \left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds \right) \Bigg| \\
& + \left| f(t,y(t),y(\theta(t))) \right| \times \\
& \times \left| \left(\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds - g(s,y(s),y(\gamma(s)))\,ds \right) \right| \\
& + \sup_{t\in J_\infty} \overline{a}(t) \sum_{j=1}^{m} I^{\alpha_j} \frac{L_j\,\|x-y\|}{K_j+\|x-y\|} \\
\leq & \big| f(t,x(t),x(\theta(t))) - f(t,y(t),y(\theta(t))) \big| \times \\
& \times \left| \left(\left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t)| + \frac{M_gW}{\Gamma q}w(t) \right) \right| \\
& + 2\big[|f(t,x(t),x(\theta(t))) - f(t,0,0)| + |f(t,0,0)| \big] \frac{M_gW}{\Gamma q}w(t) \\
& + \sum_{j=1}^{m} \frac{w_j(t)}{\Gamma(\alpha_j)} \cdot \frac{L_j\,\|x-y\|}{K_j+\|x-y\|} \\
\leq & \ell(t)\frac{|x(t)-y(t)|}{K+|x(t)-y(t)|} \left(\left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + \frac{M_gW}{\Gamma q} \right) \\
& + \frac{2\,M_gW}{\Gamma q} \left[\frac{\ell(t)\max\{|x(t)|\,,\,|x(\theta(t))|\}}{K+\max\{|x(t)|\,,\,|x(\theta(t))|\}} + F \right] w(t) \\
& + \sum_{j=1}^{m} \frac{L_j\,w_j(t)}{\Gamma(\alpha_j)} \\
\leq & \frac{L\left(\left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + \frac{M_gW}{\Gamma q} \right)|x(t)-y(t)|}{K+|x(t)-y(t)|} \\
& + \frac{2\,M_gW}{\Gamma q}(L+F)w(t) + \sum_{j=1}^{m} \frac{L_j\,w_j(t)}{\Gamma(\alpha_j)}. \qquad (4.11)
\end{aligned}
$$

Taking the limit superior as $t \to \infty$ in the above inequality (4.11) yields, $\lim_{t\to\infty} |x(t)-y(t)| = 0$. Therefore, there is a real number $T > 0$ such that $|x(t) - y(t)| < \epsilon$ for all $t \geq T$. Consequently, the mild solutions of HFRIGDE (1.1) are uniformly globally attractive on J_∞. This completes the proof. ∎

Remark 4.2. The conclusion of Theorem 4.1 also remains true under if we replace the hypotheses (A_1), (A_2), (C_1) and (C_2) with the following modified conditions:

(A'_1) The function f is continuous and there exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$\big|f(t,x_1,x_2) - f(t,y_1,y_2)\big| \leq \psi_f\big(\max\{|x_1-y_1|, |x_2-y_2|\}\big)$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(A'_2) The function f is bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound M_f.

(C'_1) The functions $h'_j s$ are continuous and there exist $\mathcal{D}$-functions $\psi_{h_j} \in \mathfrak{D}$ such that

$$\big|h_j(t,x_1,x_2) - h_j(t,y_1,y_2)\big| \leq \psi_{h_j}\big(\max\{|x_1-y_1|, |x_2-y_2|\}\big)$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $j = 1, \ldots, m$.

(C'_2) The functions h_j are bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound M_{h_j}.

Theorem 4.2. *Assume that the hypotheses (A'_1) - (A'_2), (B_1), (C'_1) - (C'_2) and (D_1) hold. Furthermore, assume that*

$$\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + \frac{M_g W}{\Gamma q}\right)\psi_f(r) + \sum_{j=1}^{m} \frac{W_j}{\Gamma(\alpha_j)}\psi_{h_j}(r) < r,\ r > 0. \tag{4.12}$$

Then the HFRIGDE (1.1) has a mild solution and mild solutions are uniformly globally attractive defined on J_∞.

Proof. The proof is similar to Theorem 4.1 and hence we omit the details. ∎

Theorem 4.3. *Assume that the hypotheses (A_1) - (A_2), (B_1), (C_1) - (C_2) and (D_1) hold. Then the HFRIGDE (1.1) has a mild solution and mild solutions are uniformly globally attractive and ultimately positive defined on J_∞.*

Proof. By Theorem 4.1, the HFRIGDE (1.1) has a global mild solution in the closed ball $\overline{B}_r(0)$, where the radius r is given as in the proof of Theorem 4.1, and the mild solutions are uniformly globally attractive on J_∞. We know that for any $x, y \in \mathbb{R}$, one has the inequality,

$$|x|\,|y| = |xy| \geq xy,$$

and therefore,

$$\big||xy| - (xy)\big| \le |x|\,\big||y| - y\big| + \big||x| - x\big|\,|y| \tag{4.13}$$

for all $x, y \in \mathbb{R}$. Now, for any mild solution x of the HFRIGDE (1.1) in $\overline{B}_r(0)$, one has

$$\begin{aligned}
&\big||x(t)| - x(t)\big| \\
&= \Bigg|\Big|f(t,x(t),x(\theta(t)))\Big| \times \\
&\quad \times \Big|\left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right)\Big| \\
&\quad - \big[f(t,x(t),x(\theta(t)))\big] \times \\
&\quad \times \left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\right)\Bigg| \\
&\quad + \Bigg|\sum_{j=1}^{m} I^{\alpha_j}\big|h_j(t,x(t),x(\eta(t)))\big| - \sum_{j=1}^{m} I^{\alpha_j}h_j(t,x(t),x(\eta(t)))\Bigg| \\
&\le \Bigg|\Big|f(t,x(t),x(\theta(t)))\Big|\,\left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big| - \frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right)\overline{a}(t)\Bigg| \\
&\quad + \Big|f(t,x(t),x(\theta(t)))\Big| \times \\
&\qquad \times \Bigg|\Big|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\Big| \\
&\qquad\quad - \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\Bigg| \\
&\quad + \Big||f(t,x(t),x(\theta(t)))| - f(t,x(t),x(\theta(t)))\Big| \\
&\qquad \times \Bigg|\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s),x(\gamma(s)))\,ds\Bigg| \\
&\quad + \sum_{j=1}^{m} I^{\alpha_j}\Big|\,\big|h_j(t,x(t),x(\eta(t)))\big| - h_j(t,x(t),x(\eta(t)))\Big| \\
&\le \left[4(L+F)\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\right]\overline{a}(t) + \left[4(L+F)\,\frac{M_g}{\Gamma q}\right]w(t) \\
&\quad + 2\sum_{j=1}^{m}\frac{H_j}{\Gamma(\alpha_j)}\,w_j(t)
\end{aligned} \tag{4.14}$$

for all $t \in J_\infty$.

Taking the limit superior as $t \to \infty$ in the above inequality (4.14), we obtain the estimate that $\lim\limits_{t\to\infty}\big||x(t)| - x(t)\big| = 0$. Therefore, there is a real number $T > 0$ such that $\big|\,|x(t)| - x(t)\,\big| \le \epsilon$ for all $t \ge T$. Hence, mild solutions of the HFRIGDE (1.1) are uniformly globally attractive as well as ultimately positive defined on J_∞. This completes the proof. ∎

Theorem 4.4. *Assume that the hypotheses (A_1) - (A_2) and (B_1) hold. Then the HFRDE (1.1) has a mild solution and mild solutions are uniformly globally attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on J_∞.*

Proof. By Theorems 4.1 and 4.2, the HFRIGDE (1.1) has a global mild solution in the closed ball $\overline{B}_r(0)$, where the radius r is given as in the proof of Theorem 4.1, and the mild solutions are uniformly globally attractive and uniformly ultimately positive on J_∞. Now, for any mild solution $x \in \overline{B}_r(0)$, we have from (4.10),

$$|x(t)| \leq (L+F)\left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\overline{a}(t) + \frac{M_g}{\Gamma q}w(t)\right) + \sum_{j=1}^{m}\frac{L_j+H_j}{\Gamma(\alpha_j)}\,w_j(t).$$

Taking the limit superior as $t \to \infty$ in the above inequality yields that $\lim_{t\to\infty}|x(t)| = 0$. Therefore, for $\epsilon > 0$ there exists a real number $T \geq t_0$ such that $|x(t)| < \epsilon$ whenever $t \geq T$. Consequently, the mild solution x is a uniformly asymptotically stable to zero defined on J_∞. This completes the proof. ∎

Example 4.1 Let $J_\infty = \mathbb{R}_+ = [0,\infty) \subset \mathbb{R}$. Given a pulling function $a(t) = e^t \in \mathcal{CRB}(\mathbb{R}_+)$, consider the following nonlinear hybrid fractional Caputo differential equation with the mixed arguments of anticipation and retardation,

$$\left.\begin{aligned} {}^{C}D_0^q\left[\frac{e^t x(t) - \dfrac{t}{t^2+1}\,I^{3/2}\left(\dfrac{|x(t)|+|x(3t)|}{4+|x(t)|+|x(3t)|}\right)}{1+\dfrac{1}{t^2+1}\left(\dfrac{|x(t)|+|x(2t)|}{2+|x(t)|+|x(2t)|}\right)}\right] &= \frac{e^{-t}\log\big(1+|x(t)|+|x(t/2)|\big)}{2+|x(t)|+|x(t/2)|},\ t\in\mathbb{R}_+,\\ x(0) &= 0, \end{aligned}\right\} \tag{4.15}$$

for all $t \in \mathbb{R}_+$, where ${}^{C}D_0^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$.

Here, $a(t) = e^t$, $\theta(t) = 2t$, $\eta(t) = 3t$, $\gamma(t) = \dfrac{t}{2}$ for $t \in \mathbb{R}_+$ and hence $\theta(0) = 0 = \eta(0)$. Next, $\alpha = 3/2$ and the functions $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ and $g, h : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined by

$$f(t,x,y) = 1 + \frac{1}{t^2+1}\left[\frac{|x|+|y|}{2+|x|+|y|}\right],$$

$$h(t,x,y) = \frac{t}{t^2+1}\left[\frac{|x|+|x|}{4+|x|+|x|}\right]$$

and

$$g(t,x,y) = \frac{e^{-t}\log(|x|+|y|)}{1+|x|+|y|}.$$

Clearly, the function f is continuous and bounded real function on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$ and in particular, $F = 1$. Now, it can be shown as in Banas and Dhage [3] that the function f satisfies the hypothesis (A_1) with $\ell(t) = \dfrac{1}{t^2+1}$ and $K = 1$. So we have $L = 1$. Furthermore, the function h is also continuous and bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ with bound $M_h = 1$. Next, the function h satisfies the hypothesis (C_1) with the function $\ell_h(t) = \dfrac{t}{t^2+1}$ so that we have $L_h = \dfrac{1}{2}$ and $K_h = 4$. Again, the function g is continuous and bounded on $J_\infty \times \mathbb{R} \times \mathbb{R}$ and therefore, satisfies the hypotheses (B_1) with $M_g = 1$. Next, we have

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} e^{-t} t^q = 0 = \lim_{t\to\infty} e^{-t} t^{3/2} = \lim_{t\to\infty} w_h(t)$$

and so the hypothesis (D_1) is satisfied. Now, $\|\overline{a}\| = \sup_{t\in\mathbb{R}_+} e^{-t} = 1, W = \sup_{t\in\mathbb{R}_+} e^{-t}\, t^q = 1$ and $W_h = 1$. Finally, it is verified that the the functions a, f, g and h satisfy the condition (4.4) of Theorem 4.1. Consequently, the HFRIGDE (4.15) has a mild solution and mild solutions are globally uniformly attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on $\mathbb{R}_+$. In particular, the HFRIGDE

$$\left.\begin{aligned} {}^C D_0^{2/3}\left[\frac{e^t x(t) - \dfrac{t}{t^2+1}\, I^{3/2}\!\left(\dfrac{|x(t)|+|x(3t)|}{4+|x(t)|+|x(3t)|}\right)}{1+\dfrac{1}{t^2+1}\left(\dfrac{|x(t)|+|x(2t)|}{2+|x(t)|+|x(2t)|}\right)}\right] &= \frac{e^{-t}\log\big(1+|x(t)|+|x(t/2)|\big)}{2+|x(t)|+|x(t/2)|},\ t\in\mathbb{R}_+, \\ x(0) &= 0, \end{aligned}\right\}$$

has a mild solution and mild solutions are globally uniformly attractive, uniformly ultimately positive and uniformly asymptotically stable to zero defined on $\mathbb{R}_+$.

Remark 4.3. Finally, we remark that the ideas of this paper may be extended with appropriate modifications to a more general hybrid fractional integrdifferential equation with Caputo fractional derivative,

$$\left.\begin{aligned} {}^C D_{t_0}^q\left[\frac{a(t)x(t) - \sum_{j=1}^{m} I^{\alpha_j} h_j(t, x(t), x(\eta_1(t)), \dots, x(\eta_n))}{f(t, x(\theta_1(t)), \cdots, x(\theta_n(t)))}\right] & \\ = g(t, x(\gamma_1(t)), \cdots, x(\gamma_n(t))),\ t\in J_\infty, & \\ x(t_0) = x_0 \in \mathbb{R}, & \end{aligned}\right\} \tag{4.16}$$

where ${}^C D_{t_0}^q$ is the Caputo fractional derivative of fractional order $0 < q \le 1$, Γ is a Euler's gamma function, $f : J_\infty \times \mathbb{R} \times ...(n \text{ times}) \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g, h_j : J_\infty \times \mathbb{R} \times ...(n \text{ times}) \times \mathbb{R} \to \mathbb{R}$ are continuous and $\theta_i, \gamma_i : J_\infty \to J_\infty$ are continuous functions which are respectively anticipatory and retardatory, that is, $\theta_i(t) \ge t$ and $\gamma_i(t) \le t$ for all $t \in J_\infty$ with $\theta_i(t_0) = t_0 = \eta_i(t_0)$ for $i = 1, \dots, n$.

Remark 4.4. If g is assumed to be continuous function on $J_\infty \times \mathbb{R} \times \mathbb{R}$, then the attractivity and existence results for the HFRIGDE (1.1) may be obtained via another approach of using measure of noncompactness. In that case we need to construct a handy tool for the measure of noncompactness which is not the case with the present approach in the qualitative study of such nonlinear fractional integrodifferential equations. See the details of this procedure that appears in Banas and Dhage [3], Hu and Yan [26], Dhage [11, 14] and the references therein.

5 The Conclusion

From the foregoing discussion, it is clear that the pulling functions and the hybrid fixed point theorems are very much useful for proving the existence theorems as well as characterizing the mild solutions of different types of nonlinear fractional integrodifferential equations on unbounded intervals of the real line when the nonlinearity is not necessarily continuous. The choices of the pulling function and the fixed point theorem depends upon the situations and the circumstances of the nonlinearities involved in the nonlinear problem. The clever selection of the fixed point theorems yields very powerful existence results as well as different characterizations of the nonlinear fractional differential equations. In this article, we have been able to prove in Theorems 4.1, 4.2, 4.3 and 4.4 the existence as well as global attractivity, ultimate positivity and asymptotic stability of the mild solutions for a quadratic type of nonlinear hybrid fractional differential equation (1.1) on the unbounded interval $J_\infty = [t_0, \infty)$ of right half of the real line $\mathbb{R}_+$, however, other nonlinear fractional integrodifferential equations can be treated in the similar way for these and some other characterizations such as monotonic global attractivity, monotonic asymptotic attractivity and monotonic ultimate positivity etc. of the mild solutions on unbounded intervals of the real line. It is known that several real world phenomena in physics and chemistry such as growth and decay of the radioactive elements continue for a very long period of time and the existence results of the type proved in this paper may be applicable for the situation to understand the behavior of the process after a sufficient lapse of time. In a forthcoming paper, it is proposed to discuss the global asymptotic and monotonic attractivity of the mild solutions for nonlinear hybrid fractional integrodifferential equations involving three nonlinearities via classical and applicable hybrid fixed point theory.

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The Multivariable Aleph-function involving the Generalized Mellin-Barnes Contour Integrals

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ABSTRACT

In this paper, we have evaluated three definite integrals involving the product of two hypergeometric functions and multivariable Aleph-function. Certain special cases of the main results are also pointed out.

RESUMEN

En este artículo, hemos evaluado tres integrales definidas que involucran el producto de dos funciones hipergeométricas y la función Aleph multivariada. También se señalan ciertos casos especiales del resultado principal.

Keywords and Phrases: Hypergeometric function, Multivariable Aleph function.

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1 Introduction

The Aleph-function is among very significant special functions and its closely related ones are widely used in physics and engineering. Therefore they are of high interest to physicists and engineers as well as mathematicians. In recent years, many integral formulas involving a diversity of special functions have been presented by many authors (see e.g., [3, 9, 12, 13, 14, 15, 16]). Motivated by these recent papers, three generalized integral formulae involving product of two hypergeometric functions and multivariable Aleph-function are established in the form of three theorems:

For our study, we recall the following three integral formulas (see [5], p. 77, Equations (3.1), (3.2) and (3.3)):

$$\int_0^\infty \left[\left(\alpha x + \frac{\beta}{x}\right)^2 + \gamma\right]^{-\rho-1} dx = \frac{\sqrt{\pi}\,\Gamma\left(\rho+\frac{1}{2}\right)}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}\Gamma\left(\rho+1\right)} \tag{1.1}$$

$$\left(\alpha > 0;\ \beta \geq 0;\ \ \gamma + 4\alpha\beta > 0;\ \Re(\rho) + \frac{1}{2} > 0\right).$$

$$\int_0^\infty \frac{1}{x^2}\left[\left(\alpha x + \frac{\beta}{x}\right)^2 + \gamma\right]^{-\rho-1} dx = \frac{\sqrt{\pi}\,\Gamma\left(\rho+\frac{1}{2}\right)}{2\beta\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}\Gamma\left(\rho+1\right)} \tag{1.2}$$

$$\left(\alpha \geq 0;\ \beta > 0;\ \ \gamma + 4\alpha\beta > 0;\ \Re(\rho) + \frac{1}{2} > 0\right).$$

$$\int_0^\infty \left[\left(\alpha + \frac{\beta}{x^2}\right)\left(\alpha x + \frac{\beta}{x}\right)^2 + \gamma\right]^{-\rho-1} dx = \frac{\sqrt{\pi}\,\Gamma\left(\rho+\frac{1}{2}\right)}{\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}\Gamma\left(\rho+1\right)} \tag{1.3}$$

$$\left(\alpha > 0;\ \beta \geq 0;\ \ \gamma + 4\alpha\beta > 0;\ \Re(\rho) + \frac{1}{2} > 0\right).$$

We also recall the following identity involving the hypergeometric series ${}_2F_1(.)$ ([8] p. 75, Theorem 1): If

$$(1-y)^{\alpha+\beta-\gamma}\,{}_2F_1\left(2\alpha, 2\beta; 2\gamma; y\right) = \sum_{k=1}^{\infty} a_k y^k, \tag{1.4}$$

then

$${}_2F_1\left(a,\ b;\ c+\frac{1}{2};\ y\right){}_2F_1\left(c-a,\ c-b;\ c+\frac{1}{2};\ y\right) = \sum_{k=0}^{\infty}\frac{(c)_k}{\left(c+\frac{1}{2}\right)_k} a_k y^k. \tag{1.5}$$

The multivariable Aleph-function defined by Sharma and Ahmad [6] as:

$$\aleph\left(z_1,\ z_2,\ ...,\ z_r\right)$$

$$= \aleph^{0,\,n:\,m_1 n_1;\ m_2 n_2;\,....;m_r n_r}_{p_i,\,q_i,\,\tau_i;\,R;\,p_{i^{(1)}},\,q_{i^{(1)}},\,\tau_{i^{(1)}};\,R^{(1)},\,...,\,p_{i^{(r)}},\,q_{i^{(r)}},\,\tau_{i^{(r)}};\,R^{(r)}}\left\{\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} B_1 : B_2 \\ B_3 : B_4 \end{matrix}\right\}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\varsigma_1, \cdots, \varsigma_r) \prod_{i=1}^{r} (\phi_i(\varsigma_i)(z_i)^{\varsigma_i})\, d\varsigma_1 \cdots d\varsigma_r \tag{1.6}$$

where, $\omega = \sqrt{-1}$,

$$B_1 = \left(\left(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}\right)_{1,n}\right), \left(\tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)}\right)_{n+1, p_i}\right)$$

$$B_2 = \left(\left(c_j^{(1)}, \gamma_j^{(r)}\right)_{1,n_1}\right), \left(\tau_{i^{(1)}} \left(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1, p_{i^{(1)}}}\right)$$

$$; \cdots ; \left(\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,n_r}\right), \left(\tau_{i^{(r)}} \left(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1, p_{i^{(r)}}}\right)$$

$$B_3 = \left(\tau_i \left(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)}\right)_{m+1, q_i}\right)$$

$$B_4 = \left(\left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,m_1}\right), \left(\tau_{i^{(1)}} \left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1, q_{i^{(1)}}}\right)$$

$$; \cdots ; \left(\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}\right), \left(\tau_{i^{(r)}} \left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)_{m_r+1, q_{i^{(r)}}}\right)$$

and

$$\psi(\varsigma_1, \cdots, \varsigma_r) = \frac{\prod_{j=1}^{n} \Gamma\left(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} \varsigma_k\right)}{\sum_{i=1}^{R} \left[\tau_i \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} \varsigma_k\right) \prod_{j=1}^{q_i} \Gamma\left(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} \varsigma_k\right)\right]}, \tag{1.7}$$

$$\phi_k(\varsigma_k) = \frac{\prod_{j=1}^{m_k} \Gamma\left(d_j^{(k)} - \delta_j^{(k)} \varsigma_k\right) \prod_{j=1}^{n_k} \Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)} \varsigma_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}} \left[\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma\left(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} \varsigma_k\right) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma\left(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} \varsigma_k\right)\right]}, \tag{1.8}$$

The parameters $d_{ji^{(k)}}^{(k)}$ $(j = m_k + 1, \cdots, q_{i^{(k)}})$, $(k = 1, \cdots, r;\ i = 1, \cdots, R\ \&\ i^{(k)} = 1, \cdots, R^{(k)}$ are complex numbers. Also positive real numbers α's, β's, γ' sand δ's for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$- \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leq 0 \tag{1.9}$$

The real numbers $\tau_i > 0$ $(i = 1, ..., R)$ and $\tau_{i^{(k)}} > 0$ $(i = 1, \cdots, R)$. The contour is in the s_k-plane and run from $\sigma - \omega\,\infty \text{to} \sigma + \omega\,\infty$, where σ is real number with loop, if necessary, ensure that the poles of $\Gamma\left(d_j^{(k)} - \delta_j^{(k)}\varsigma_k\right)$ with $j = 1, \ldots, m_k$ are separated from those of $\Gamma\left(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)}\varsigma_k\right)$ with $j = 1, \ldots, n$ and $\Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)}\varsigma_k\right)$ with $j = 1, \ldots, n_k$ to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contours (1.6) can be obtained by extension of corresponding conditions for multi variable H-function as: $|\arg\, z_k| < \frac{1}{2}A_i^{(k)}\pi$ where.

$$A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0 \tag{1.10}$$

with $k = 1, \cdots, r;\;\; i = 1, \cdots, R\;$ and $i^{(k)} = 1, \cdots, R^{(k)}$.

Remark 1: By setting $\tau_i = \tau_{i^{(k)}} = 1$, the multivariable Aleph function reduces to multivariable I-function (see [4, 7]).

Remark 2: By setting $\tau_i = \tau_{i^{(k)}} = 1\,(k = 1, ..., r)$ and $R = R^{(1)} =, ..., R^{(r)} = 1$, the multivariable Aleph-function reduces to multivariable H-function defined by Srivastava and Panda [10].

Remark 3: When we set $r = 1$, the multivariable Aleph function reduces to Aleph-function of one variable defined by Sudland [11].

2 Main Results

Theorem 2.1. *Let* $\alpha > 0$, $\beta \geq 0$, $\gamma + 4\alpha\beta > 0$, $\mu_i > 0$, $\eta \geq 0$, $\Re(\rho) + \frac{1}{2} > 0$; $-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$; $\Re\left(\lambda + \mu_i \min_{1 \leq j \leq m_i} \left\{\frac{Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right\}\right) > 0$ $(i = 1, \cdots, r)$, *and* $\sigma = \left[\left(\alpha\, x + \frac{\beta}{x}\right)^2 + \gamma\right]$ *then the following formula holds:*

$$\int_0^\infty \sigma^{-\rho-1}\,{}_2F_1\left(\alpha,\; \beta;\; \gamma + \frac{1}{2};\, \sigma\right)\; {}_2F_1\left(\gamma - \alpha,\; \gamma - \beta;\; \gamma + \frac{1}{2};\, \sigma\right)$$

$$\times \aleph\left(z_1\sigma^{-\eta_1},\, ...,\, z_r\sigma^{-\eta_r}\right) dx$$

$$= \frac{\sqrt{\pi}}{2\alpha\,(4\alpha\beta + \gamma)^{\rho + \frac{1}{2}}} \sum_{h=0}^{\infty} \frac{1}{(4\alpha\beta + \gamma)^{-h}} \frac{(\gamma)_h}{\left(\gamma + \frac{1}{2}\right)_h} a_h$$

$$\times\, \aleph^{0,\, n+1:\, m_1 n_1;\, m_2 n_2;\, ...; m_r n_r}_{p_i+1,\, q_i+1,\, \tau_i;\, R;\, p_{i^{(1)}},\, q_{i^{(1)}},\, \tau_{i^{(1)}};\, R^{(1)},\, ...,\, p_{i^{(r)}},\, q_{i^{(r)}},\, \tau_{i^{(r)}};\, R^{(r)}}$$

$$\left\{ \begin{array}{c} \frac{z_1}{(4\alpha\beta+\gamma)^{\eta_1}} \\ \vdots \\ \frac{z_r}{(4\alpha\beta+\gamma)^{\eta_r}} \end{array} \left| \begin{array}{c} \left(-\frac{1}{2}-\rho+h;\eta_1,\cdots,\eta_r\right), B_1 : B_2 \\ \left(-\rho+h;\eta_1,\cdots,\eta_r\right), B_3 : B_4 \end{array} \right. \right\}. \tag{2.1}$$

Proof. Assume that Ω in L.H.S. of (2.1), then by virtue of equation (1.5) and (1.6), we have the following

$$\Omega = \int_0^\infty \sigma^{-\rho-1} \sum_{h=0}^\infty \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \sigma^h \aleph^{0,\,n:\,m_1 n_1,\; m_2 n_2,\; \ldots,\, m_r n_r}_{p_i,\,q_i,\,\tau_i;\,R;\,p_{i^{(1)}},\,q_{i^{(1)}},\,\tau_{i^{(1)}};\,R^{(1)},\,\ldots,\,p_{i^{(r)}},\,q_{i^{(r)}},\,\tau_{i^{(r)}};\,R^{(r)}}$$

$$\left\{ \begin{array}{c} z_1\sigma^{-\eta_1} \\ \vdots \\ z_r\sigma^{-\eta_r} \end{array} \left| \begin{array}{c} B_1 : B_2 \\ B_3 : B_4 \end{array} \right. \right\} dx$$

$$= \int_0^\infty \sigma^{-\rho-1} \sum_{h=0}^\infty \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \sigma^h$$

$$\times \left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^r \left(\phi_i\left(\varsigma_i\right)\left(z_i\sigma^{-\eta_i}\right)^{\varsigma_i}\right) d\varsigma_1\cdots d\varsigma_r \right\} dx$$

$$= \sum_{h=0}^\infty \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^r \left(\phi_i\left(\varsigma_i\right)\left(z_i\right)^{\varsigma_i}\right) d\varsigma_1\cdots d\varsigma_r \right\}$$

$$\times \int_0^\infty \sigma^{-\rho-1+h-\sum_{k=1}^s \eta_k\varsigma_k} dx$$

By using equation (1.1), we can obtain the following equation

$$\Omega = \sum_{h=0}^\infty \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^r \left(\varphi_i\left(\varsigma_i\right)\left(z_i\right)^{\varsigma_i}\right) d\varsigma_1\cdots d\varsigma_r \right\}$$

$$\times \frac{\sqrt{\pi}\,\Gamma\left(\rho-h+\sum_{k=1}^s \eta_k\varsigma_k+\frac{1}{2}\right)}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}-h+\sum_{k=1}^s \eta_k\varsigma_k}\,\Gamma\left(\rho-h+\sum_{k=1}^s \eta_k\varsigma_k+1\right)}$$

$$= \sum_{h=0}^\infty \frac{(\gamma)_h\; a_h}{\left(\gamma+\frac{1}{2}\right)_h} \frac{\sqrt{\pi}\,\Gamma\left(\rho-h+\sum_{k=1}^s \eta_k\varsigma_k+\frac{1}{2}\right)}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}-h+\sum_{k=1}^s \eta_k\varsigma_k}\,\Gamma\left(\rho-h+\sum_{k=1}^s \eta_k\varsigma_k+1\right)}$$

$$\times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^r \left(\varphi_i\left(\varsigma_i\right)\left(z_i\right)^{\varsigma_i}\right) d\varsigma_1\cdots d\varsigma_r$$

$$= \frac{\sqrt{\pi}}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}} \sum_{h=0}^\infty \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} \frac{a_h}{\left(4\alpha\beta+\gamma\right)^{-h}}$$

$$\times \frac{\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+\frac{1}{2}\right)}{(4\alpha\beta+\gamma)^{\sum_{k=1}^{s}\eta_k\varsigma_k}\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+1\right)}$$

$$\times \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\varsigma_1,\cdots,\varsigma_r)\prod_{i=1}^{r}\left(\varphi_i(\varsigma_i)(z_i)^{\varsigma_i}\right)d\varsigma_1\cdots d\varsigma_r$$

$$= \frac{\sqrt{\pi}}{2\alpha(4\alpha\beta+\gamma)^{\rho+\frac{1}{2}}}\sum_{h=0}^{\infty}\frac{1}{(4\alpha\beta+\gamma)^{-h}}\frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h}a_h x\left\{\frac{\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+\frac{1}{2}\right)}{\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+1\right)}\right\}$$

$$\times \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\varsigma_1,\cdots,\varsigma_r)\prod_{i=1}^{r}\left(\phi_i(\varsigma_i)\left[\frac{z_i}{(4\alpha\beta+\gamma)^{\eta_i}}\right]^{\varsigma_i}\right)d\varsigma_1\cdots d\varsigma_r$$

we readily arrive at the right hand side of (2.1) in view of the presentation of Aleph function in Mellin Barnes contour integral. □

Theorem 2.2. *Let* $\alpha \geq 0,\ \beta > 0,\ \gamma+4\alpha\beta > 0,\ \mu_i > 0,\ \eta \geq 0,\ \Re(\rho)+\frac{1}{2} > 0;\quad -\frac{1}{2} < \alpha-\beta-\gamma < \frac{1}{2}$ $\Re\left(\lambda+\mu_i \min_{1\leq j\leq m_i}\left\{\frac{Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right\}\right) > 0\ (i=1,\cdots,r)$, *and* $\sigma = \left[\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]$ *then the following formula holds:*

$$\int_0^\infty \frac{1}{x^2}\sigma^{-\rho-1}{}_2F_1\left(\alpha,\ \beta;\ \gamma+\frac{1}{2};\ \sigma\right){}_2F_1\left(\gamma-\alpha,\ \gamma-\beta;\ \gamma+\frac{1}{2};\ \sigma\right)$$
$$\times\aleph\left(z_1\sigma^{-\eta_1},\ ...,\ z_r\sigma^{-\eta_r}\right)dx$$

$$= \frac{\sqrt{\pi}}{2\beta(4\alpha\beta+\gamma)^{\rho+\frac{1}{2}}}\sum_{h=0}^{\infty}\frac{1}{(4\alpha\beta+\gamma)^{-h}}\frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h}a_h$$
$$\times\aleph^{0,\,n+1:\,m_1n_1;\ m_2n_2;\ \dots;m_rn_r}_{p_i+1,\,q_i+1,\,\tau_i;\,R;\,p_{i^{(1)}},\,q_{i^{(1)}},\,\tau_{i^{(1)}};\,R^{(1)},\,\dots,\,p_{i^{(r)}},\,q_{i^{(r)}},\,\tau_{i^{(r)}};\,R^{(r)}}$$
$$\times\left\{\begin{array}{c|c}\frac{z_1}{(4\alpha\beta+\gamma)^{\eta_1}} & \left(-\frac{1}{2}-\rho+h;\eta_1,\cdots,\eta_r\right),B_1:B_2\\ \vdots & \left(-\rho+h;\eta_1,\cdots,\eta_r\right),B_3:B_4\\ \frac{z_r}{(4\alpha\beta+\gamma)^{\eta_r}} & \end{array}\right\}. \tag{2.2}$$

Proof. In the similar manner of Theorem 2.1 and using (1.2) we easily arrive at the result (2.2). □

Theorem 2.3. *Let* $\alpha > 0,\ \beta > 0,\ \gamma+4\alpha\beta > 0,\ \mu_i > 0,\ \eta \geq 0,\ \Re(\rho)+\frac{1}{2} > 0;\quad -\frac{1}{2} < \alpha-\beta-\gamma < \frac{1}{2};$ $\Re\left(\lambda+\mu_i \min_{1\leq j\leq m_i}\left\{\frac{Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right\}\right) > 0\ (i=1,\cdots,r)$, *and* $\sigma = \left[\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]$ *then the following formula holds:*

$$\int_0^\infty \left(\alpha+\frac{\beta}{x^2}\right)\sigma^{-\rho-1}{}_2F_1\left(\alpha,\ \beta;\ \gamma+\frac{1}{2};\ \sigma\right){}_2F_1\left(\gamma-\alpha,\ \gamma-\beta;\ \gamma+\frac{1}{2};\ \sigma\right)$$
$$\times\aleph\left(z_1\sigma^{-\eta_1},\ ...,\ z_r\sigma^{-\eta_r}\right)dx$$

$$= \frac{\sqrt{\pi}}{(4\alpha\beta+\gamma)^{\rho+\frac{1}{2}}} \sum_{h=0}^{\infty} \frac{1}{(4\alpha\beta+\gamma)^{-h}} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h$$

$$\times \aleph^{0,\, n+1:\, m_1 n_1;\; m_2 n_2;\; \ldots;m_r n_r}_{p_i+1,\, q_i+1,\, \tau_i;\, R;\, p_{i(1)},\, q_{i(1)},\, \tau_{i(1)};\, R^{(1)},\, \ldots,\, p_{i(r)},\, q_{i(r)},\, \tau_{i(r)};\, R^{(r)}}$$

$$\left\{ \begin{array}{c} \frac{z_1}{(4\alpha\beta+\gamma)^{\eta_1}} \\ \vdots \\ \frac{z_r}{(4\alpha\beta+\gamma)^{\eta_r}} \end{array} \left| \begin{array}{c} \left(-\frac{1}{2}-\rho+h;\eta_1,\cdots,\eta_r\right), B_1 : B_2 \\ \left(-\rho+h;\eta_1,\cdots,\eta_r\right), B_3 : B_4 \end{array} \right. \right\}. \tag{2.3}$$

Proof. In the similar way of Theorem 2.1 and using (1.3) we easily arrive at the result (2.3). □

3 Special Cases

(1) If we put $\tau_i = 1$, in (2.1), (2.2) and (2.3), we get the results in terms of multivariable I-function [4, 7].

(2) Some suitable parametric changes in (1.1), we obtain single variable I-function, then we arrive at the results due to Chand [1].

(3) Also, multivariable Aleph function reduces to multivariable H-function with some suitable parameters;we get the known result due to Daiya et al. [2].

4 Conclusion

In this article, we analyze the generalized fractional calculus involving definite integrals of Gradshteyn-Ryzhik of the Multivariable Aleph-function. As the special cases of our main results, which are related to I-function, H-function and G-function, we can also get the number of special functions.

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Mild solutions of a class of semilinear fractional integro-differential equations subjected to noncompact nonlocal initial conditions

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ABSTRACT

In this paper, we prove the existence of mild solutions of a class of fractional semilinear integro-differential equations of order $\beta \in (1, 2]$ subjected to noncompact initial nonlocal conditions. We assume that the linear part generates an arbitrarily strongly continuous β-order fractional cosine family, while the nonlinear forcing term is of Carathéodory type and satisfies some fairly general growth conditions. Our approach combines the Monch fixed point theorem with some recent results regarding the measure of noncompactness of integral operators. Our conclusions improve and generalize many earlier related works. An example is provided to illustrate the main results.

RESUMEN

En este artículo, probamos la existencia de soluciones leves de una clase de ecuaciones integro-diferenciales fraccionales semilineales de orden $\beta \in (1, 2]$ con condiciones no-compactas iniciales no-locales. Asumimos que la parte lineal genera una familia coseno de orden fraccional β arbitrariamente fuertemente continua , mientras que el término no-lineal de forzamiento es de tipo Carathéodory y satisface algunas condiciones de crecimiento bastante generales. Nuestro enfoque combina el teorema de punto fijo de Monch con algunos resultados recientes sobre la medida de no-compacidad de operadores integrales. Nuestras conclusiones mejoran y generalizan muchos trabajos anteriores relacionados . Se provee un ejemplo para ilustrar los resultados principales.

Keywords and Phrases: Cosine operator, fractional integro-differential operator, abstract differential equation, noncompact nonlocal condition.

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1 Introduction

In recent years, the investigation of fractional differential equations in Banach spaces has attracted many research works due to its applications in various areas of engineering, physics, bio-engineering, and other applied sciences. Notable contributions have been made to both theory and applications of fractional differential equations; we refer, e.g., to [1, 6, 13, 14, 15, 16, 18, 19, 25] and the references therein. Actually, it has been found that differential equations involving fractional derivatives in time are more realistic to describe many phenomena in practical situations than those of integer order. The most significant advantage of fractional derivatives compared with integer derivatives is that it can be used to describe the property of memory and heredity of various materials and processes [5, 8, 22]. For more details about fractional calculus and fractional differential equations, we refer the reader to [2, 4, 10].

In this paper, we are concerned with the existence of mild solutions of the following class of fractional semilinear integro-differential equations:

$$\begin{cases} {}^cD_t^\beta u(t) &=& Au(t) + f(t, u(t), Gu(t), Su(t)), \quad t \in [0, a], \\ u(0) &=& u_0 + q(u), \\ u'(0) &=& v_0 + p(u), \end{cases} \tag{1.1}$$

where $\beta \in (1, 2]$ and ${}^cD_t^\beta$ is the standard Caputo fractional derivative of order β. The operator A is the infinitesimal generator of a strongly continuous β-order fractional cosine family $\{C_\beta(t) : t \geq 0\}$

in a Banach space E, f, q, p are suitably defined functions satisfying certain conditions to be specified later, x_0, y_0 are given elements of E and G, S are two linear operators defined by

$$Gu(t) = \int_0^t K(t,s)u(s)ds \text{ and } Su(t) = \int_0^a H(t,s)u(s)ds, \quad t \in [0,a], \tag{1.2}$$

where $H \in C\left[[0,a] \times [0,a], \mathbb{R}^+\right]$, $K \in C\left[U, \mathbb{R}^+\right]$, and

$$U = \left\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le a\right\}.$$

Here $\mathbb{R}^+$ refers to the set of nonnegative real numbers. The problem of the existence of mild solutions to (1.1) has been addressed by many investigators in the case where $\beta \in (0,1]$. We quote for instance the contributions by Shu and Wang [21], Qin *et al.* [20], and the pioneering works of Travis and Webb [23, 24]. However, only a few papers have been up to now devoted to the case $\beta \in (1,2]$. We quote the paper [25], where the authors proved the existence of mild solutions to (1.1) with $\beta \in (1,2]$ when p and q are compact. In many applications, nonlocal conditions are not compact. Specifically, periodic $p(u) = u(a)$, anti-periodic $p(u) = -u(a)$, or multipoint discrete nonlocal conditions $p(u) = \sum_{i=1}^m c_i u(t_i)$, $0 < t_1 < \cdots < t_m$ are not compact.

As a matter of fact, the first and major aim of this paper is to address the problem of existence of mild solutions to (1.1) in the case where p and q are not necessarily compact. Moreover, we merely assume that the operator A generates an arbitrarily strongly continuous β-order fractional cosine family, which is an extra interesting feature. Our approach combines the Monch fixed point theorem with some recent results concerning the measure of noncompactness of integral operators.

The outline of the paper is as follows: In Section 2, we present the main technical tools which will be used in this work. In Section 3, we investigate the existence of mild solution to problem (1.1) by means of a fixed point method. Finally, in Section 4, we include an example to illustrate our results.

2 Preliminaries and auxiliary results

In this section, we recall some background and collect several useful results which are crucial for our further work. To do this, let $(E, \|\cdot\|)$ be a Banach space and $C([0,a], E)$ be the space of all continuous functions defined on $[0,a]$ with values in E, equipped with the standard sup-norm. Let $\mathcal{L}(E)$ denote the space of all bounded linear operators on E endowed with the classical operator norm. We first list some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. *[4] For $0 < \gamma < 1$, consider the function of Wright type defined by*

$$\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)} = \frac{1}{2\pi i}\int_\Gamma \mu^{\gamma-1} \exp\left(\mu - z\mu^\gamma\right) d\mu, \tag{2.1}$$

where Γ *is a contour which starts and ends at* $-\infty$ *and encircles the origin once counterclockwise.* $\Phi_\gamma(t)$ *is a probability density function:*

$$\Phi_\gamma(t) \geq 0 \ \ for \ \ t > 0 \ \ and \ \ \int_0^\infty \Phi_\gamma(t)dt = 1. \tag{2.2}$$

Definition 2.2. *[4] The Riemann-Liouville fractional integral of order* $\beta > 0$ *of a function* $f \in L^1([0,a];E)$ *is defined by*

$$I_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds, \quad t > 0, \tag{2.3}$$

where $\Gamma(\cdot)$ *stands for the Gamma function.*

Definition 2.3. *[4] The Riemann-Liouville fractional derivative of order* $1 < \beta \leq 2$ *is defined by*

$$D_t^\beta f(t) = \frac{d^2}{dt^2} I_t^{2-\beta} f(t), \tag{2.4}$$

where $f \in L^1([0,a];E)$ *and* $D_t^\beta f \in L^1([0,a];E)$.

Definition 2.4. *[4] The Caputo fractional derivative of order* $\beta \in (1,2]$ *is defined by*

$${}^cD_t^\beta f(t) = D_t^\beta \left(f(t) - f(0) - f'(0)t\right), \tag{2.5}$$

where $f \in L^1([0,a];E) \cap C^1([0,a];E)$ *and* $D_t^\beta f \in L^1([0,a];E)$.

Consider the following problem

$${}^cD_t^\beta x(t) = Ax(t), \qquad x(0) = \eta, \qquad x'(0) = 0, \tag{2.6}$$

where $\beta \in (1,2]$, $A : D(A) \subset E \to E$ is a closed densely defined linear operator in Banach space E.

Definition 2.5. *[4] Let* $\beta \in (1,2]$. *A family* $\{C_\beta\}_{\beta \geq 0} \subset \mathcal{L}(E)$ *is called a solution operator (or a strongly continuous* β*-order fractional cosine family) for the problem* (2.6) *if the following conditions are satisfied:*

(a) $C_\beta(t)$ *is strongly continuous for* $t \geq 0$ *and* $C_\beta(0) = I$,

(b) $C_\beta(t)D(A) \subset D(A)$ *and* $AC_\beta(t)\eta = C_\beta(t)A\eta$ *for all* $\eta \in D(A)$, $t \geq 0$,

(c) $C_\beta(t)\eta$ *is a solution of* $x(t) = \eta + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Ax(s)ds$ *for all* $\eta \in D(A)$, $t \geq 0$.

In this case, A *is called the infinitesimal generator of* $C_\beta(t)$.

Definition 2.6. *[15] The fractional sine family* $S_\beta : \mathbb{R}^+ \to \mathcal{L}(E)$ *associated with* C_β *is defined by*

$$S_\beta(t) = \int_0^t C_\beta(s)ds, \qquad t \geq 0. \tag{2.7}$$

Definition 2.7. *[15] The fractional Riemann-Liouville family $P_\beta : \mathbb{R}^+ \to \mathcal{L}(E)$ associated with C_β is defined by*

$$P_\beta(t) = I_t^{\beta-1} C_\beta(t) = \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} C_\beta(s) ds, \qquad t \geq 0. \tag{2.8}$$

Definition 2.8. *[4] The strongly continuous β-order fractional cosine family $C_\beta(t)$ is called exponentially bounded if there are constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|C_\beta(t)\| \leq M e^{\omega t}, \qquad t \geq 0. \tag{2.9}$$

An operator A is said to belong to $C^\beta(M,\omega)$, if the problem (2.6) has a strongly continuous β-order fractional cosine family $C_\beta(t)$ satisfying (2.9). Denote $C^\beta(\omega) = \bigcup\{C^\beta(M,\omega); M \geq 1\}$.

Theorem 2.1. *[4, Theorem 3.1] Let $0 < \beta' < \beta \leq 2$, $\gamma = \frac{\beta'}{\beta}$, $\omega \geq 0$. If $A \in C^\beta(\omega)$ then $A \in C^{\beta'}(\omega^{\frac{1}{\gamma}})$ and the following representation holds*

$$C_{\beta'}(t) = \int_0^\infty \varphi_{t,\gamma}(s) C_\beta(s) ds, \quad t > 0, \tag{2.10}$$

where $\varphi_{t,\gamma}(s) := t^{-\gamma}\Phi_\gamma(st^{-\gamma})$ and $\Phi_\gamma(z)$ is defined by (2.1).

For more details regarding β-order fractional cosine families, we refer the reader to [4].

Definition 2.9. *A function ψ defined on the set of all bounded subsets of a Banach space E with values in $\mathbb{R}^+$ is called a measure of noncompactness (MNC in short) on E if for any bounded subset M of E we have $\psi(\overline{co}M) = \psi(M)$, where $\overline{co}M$ stands for the closed convex hull of M. An MNC is said to be*

(i) Full: $\psi(M) = 0$ if and only if M is a relatively compact set.

(ii) Monotone: for all bounded subsets M_1 and M_2 of E, we have

$$M_1 \subset M_2 \Longrightarrow \psi(M_1) \leq \psi(M_2).$$

(iii) Nonsingular: $\psi(M \cup \{x\}) = \psi(M)$, for every bounded subset M of E and for all $x \in E$.

One of most important measures of noncompactness is the Hausdorff measure of noncompactness defined by

$$\chi(M) = \inf\{r > 0; M \text{ can be covered by finitely many balls with radii } \leq r\},$$

for each bounded subset M of E. The Hausdorff measure of noncompactness is *full, monotone* and *nonsingular*. Moreover, it enjoys the following additional properties.

Lemma 2.1. *[3]*

(i) $\chi(M_1 + M_2) \leq \chi(M_1) + \chi(M_2)$.

(ii) $\chi(\lambda M) = |\lambda|\chi(M)$, *for all* $\lambda \in \mathbb{R}$.

(iii) $\chi(\overline{co}(M)) = \chi(M)$.

(iv) $\chi(A + x) = \chi(A), \ \forall x \in E$.

(v) *if* $B\colon E \longrightarrow E$ *is a Lipschitz continuous map with constant* k, *then* $\chi(B(M)) \leq k\chi(M)$ *for all bounded subset* M *of* E.

Lemma 2.2. *[17, 9] If* $\{u_n\}_{n\in\mathbb{N}} \subset L^1([0,a];E)$ *is uniformly integrable, then the function* $t \mapsto \chi(\{u_n(t)\}_{n\in\mathbb{N}})$ *for* $t \in [0,a]$ *is measurable and*

$$\chi\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leq \int_0^t \chi\big(\{u_n(s)\}_{n=1}^{\infty}\big)ds.$$

In the sequel, we use a measure of noncompactness in the space $C(I;E)$ which was investigated in [11, 12]. In order to define this measure, let us fix a nonempty bounded subset Ω of the space $C(I;E)$. Let

$$\operatorname{mod}_C(\Omega) = \sup\left\{\operatorname{mod}_C(\Omega(t)) : t \in I\right\},$$

where

$$\operatorname{mod}_C(\Omega(t)) = \lim_{\delta\to 0}\sup_{x\in\Omega}\left\{\sup\left\{|x(t_2) - x(t_1)| : t_1, t_2 \in (t-\delta, t+\delta)\right\}\right\},$$

and

$$\chi_\infty(\Omega) = \sup\left\{\chi(\Omega(t)) : t \in I\right\},$$

where χ denotes the Hausdorff measure of noncompactness in E. It is worth noticing that χ_∞ and mod_C are monotone nonsingular MNCs on $C(I;E)$ (see [3, 12]). From an application view point, one of the main disadvantages of these MNCs is the lack of fullness. To overcome this problem, we can define the function ψ_C on the family of bounded subsets in $C(I;E)$ by taking

$$\psi_C(\Omega) = \chi_\infty(\Omega) + \operatorname{mod}_C(\Omega)$$

Lemma 2.3. *[11, Lemma 3.1]* ψ_C *is a full monotone and nonsingular MNC on the space* $C(I;E)$.

Finally, we will make use of Monch's fixed point theorem.

Theorem 2.2. *[17] Let* C *be a closed, convex subset of a Banach space* E *with* $x_0 \in C$. *Suppose there is a continuous map* $T : C \to C$ *with the following property:*

$$\begin{cases} D \subseteq C \ \textit{countable and}\ D \subseteq \operatorname{co}\left(\{x_0\} \cup T(D)\right) \\ \quad \textit{imply that}\ D\ \textit{is relatively compact.} \end{cases}$$

Then, T *has at least one fixed point in* C.

Let $\mathcal{F}$ be a function from $[0,+\infty)$ into $\mathcal{L}(E)$. Suppose that $\mathcal{F}$ is continuous for the strong operator topology, namely

$$\text{The mapping } [0,+\infty) \ni t \to \mathcal{F}(t)x \in E \text{ is continuous for every } x \in E. \tag{2.11}$$

Notice that from the uniform boundedness principle, we know that $\mathcal{F}$ is uniformly bounded on any interval $[0,a]$, i.e., $M_a := \sup_{t\in[0,a]} \|\mathcal{F}(t)\|_{\mathcal{L}(E)} < +\infty$. For later use, let us define the quantity

$$\omega(\mathcal{F}(t)) = \lim_{\delta\to 0} \sup_{\|x\|\leq 1} \left\{ \left\|\mathcal{F}(t_2)x - \mathcal{F}(t_1)x\right\|_E : t_1, t_2 \in (t-\delta, t+\delta)\right\}.$$

Recall that a family $(\mathcal{F}(t))_{t\geq 0}$ is said to be equicontinuous if $\{\mathcal{F}(\cdot)x : x \in \Omega\}$ is equicontinuous at any $t > 0$ for any bounded subset $\Omega \subset X$. It is easily seen that a family $(\mathcal{F}(t))_{t\geq 0}$ is equicontinuous if and only if $\omega(\mathcal{F}(t)) = 0$ for any $t > 0$.

Theorem 2.3. *[7] Let $\mathcal{F}$ be a function from $[0,+\infty)$ into $\mathcal{L}(E)$. Suppose that $\mathcal{F}$ is continuous for the strong operator topology. Then, for any bounded set $\Omega \subset E$ and for any $t \geq 0$, we have*

$$\operatorname{mod}_C(\mathcal{F}(t)\Omega) \leq \omega(\mathcal{F}(t))\chi(\Omega).$$

In particular, for any $t \in [0,a]$ we have

$$\operatorname{mod}_C(\mathcal{F}(t)\Omega) \leq 2M_a\chi(\Omega).$$

Now, we present two crucial results concerning the integral operator:

$$(\mathcal{S}_0 f)(t) = \int_0^t \mathcal{F}(t-s)f(s)\mathrm{d}s \quad \text{for } t \in [0,a]$$

where $f \in L^1([0,a];E)$ and $\mathcal{F} : [0,+\infty) \to \mathcal{L}(E)$ verifies (2.11).

Theorem 2.4. *[7] Let $\{f_n\}_{n=1}^{\infty} \subset L^1([0,a];E)$ be integrably bounded, that is,*

$$\|f_n(t)\| \leq \nu(t) \text{ for all } n = 1,2,\cdots \text{ and a.e. } t \in [0,a], \tag{2.12}$$

where $\nu \in L^1([0,a])$. Assume that

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \leq q(t) \tag{2.13}$$

for a.e. $t \in [0,a]$ where $q \in L^1([0,a])$. Then, for every $t \in [0,a]$ we have:

$$\operatorname{mod}_C\left(\{\mathcal{S}_0 f_n(t)\}_{n=1}^{\infty}\right) \leq 4M_a \int_0^t q(s)\mathrm{d}s. \tag{2.14}$$

Theorem 2.5. *[7] Let $\{f_n\}_{n=1}^{\infty} \subset L^1([0,a];E)$ be as in (2.12) Assume that (2.13) holds. Then*

$$\chi\left(\{\mathcal{S}_0 f_n(t)\}_{n=1}^{\infty}\right) \leq 2M_a \int_0^t q(s)\mathrm{d}s, \quad \text{for all } t \in [0,a]$$

3 Existence results

In this section, we discuss the existence of mild solutions to the semilinear fractional integro-differential equation (1.1). Before doing so, it is appropriate to clarify the definition of solution we will consider.

Definition 3.1. *Assume $A \in C^{\beta}(M,\omega)$ and $C_\beta(t)$ is the solution operator. We say that $u \in C[I,E]$ is a mild solution of (1.1) if u satisfies*

$$\begin{aligned} u(t) &= C_\beta(t)\,(u_0+q(u)) + S_\beta(t)\,(v_0+p(u)) \\ &\quad + \int_0^t P_\beta(t-s) f(s,u(s),Gu(s),Su(s))ds, \quad t\in I. \end{aligned} \tag{3.1}$$

To allow the abstract formulation of our problem, we define the operator $T : C([0,a];E) \to C([0,a];E)$ by

$$\begin{aligned} Tu(t) &= C_\beta(t)\,(u_0+q(u)) + S_\beta(t)\,(v_0+p(u)) \\ &\quad + \int_0^t P_\beta(t-s) f(s,u(s),Gu(s),Su(s))ds, \quad t\in [0,a] \end{aligned} \tag{3.2}$$

for all $t \in [0,a]$. It is clear that u is a mild solution of (1.1) if and only if it is a fixed point of T.

Our problem will be investigated under the following assumptions:

(C_1) $p,q : C([0,a];E) \to E$ are continuous functions and there exist nonnegative constants k_p and k_q, such that for all bounded subset $D \subset C([0,a];E)$, we have

$$M_a\chi(q(D)) + aM_a\chi(p(D)) \le (M_a k_q + aM_a k_p)\chi_\infty(D),$$

where $M_a = \sup_{t\in[0,a]} \|C_\beta(t)\|_{\mathcal{L}(E)}$.

(C_2) There exist nondecreasing continuous functions $\sigma_1,\sigma_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{cases} \|q(u)\|_E \le \sigma_1(\|u\|_\infty), & \text{for all } u\in C([0,a];E), \\ \|p(u)\|_E \le \sigma_2(\|u\|_\infty), & \text{for all } u\in C([0,a];E). \end{cases}$$

(C_3)

$$\begin{cases} f : [0,a]\times E\times E\times E \to E \ \text{ is a Carathéodory function, i.e.,} \\ (i) \text{ the map } t\mapsto f(t,u_1,u_2,u_3) \ \text{ is measurable for all} \\ (u_1,u_2,u_3)\in E\times E\times E, \\ (ii) \text{ the functions } u_1\mapsto f(t,u_1,u_2,u_3),\ u_2\mapsto f(t,u_1,u_2,u_3) \ \text{ and} \\ u_3\mapsto f(t,u_1,u_2,u_3) \ \text{ are continuous for almost } t\in[0,a], \end{cases}$$

(C_4) There exist functions $\rho_1,\rho_2,\rho_3 \in L^1((0,a);\mathbb{R}^+)$ and nondecreasing continuous functions $\Omega_1,\Omega_2,\Omega_3 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,u_1,u_2,u_3)\|_E \le \sum_{i=1}^{3}\rho_i(t)\Omega_i(\|u_i\|_E), \quad \text{for all } t\in[0,a] \text{ and } u_i\in E.$$

(C_5) There exist functions $m_1, m_2, m_3 \in L^1([0,a];\mathbb{R}^+)$ such that for all bounded subset $D_1, D_2, D_3 \subset E$

$$\chi(f(t, D_1, D_2, D_3)) \leq \sum_{i=1}^{3} m_i(t)\chi(D_i), \quad \text{for almost every } t \in [0,a].$$

(C_6) $M_a k_q + aM_a k_p + 2\dfrac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1 < 1,$

where

$$m(s) = m_1(s) + ak_0 m_2(s) + ah_0 m_3(s),\ k_0 = \sup\{K(t,s);\ (t,s) \in U\},$$
$$h_0 = \sup\{H(t,s);\ (t,s) \in U\}, \text{ and } U = \left\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq a\right\}.$$

Remark 3.1. *It is easy to prove that for every $t \geq 0$, we have*

$$\sup_{t\in[0,a]} \|S_\beta(t)\|_{\mathcal{L}(E)} \leq aM_a \quad \text{and} \quad \sup_{t\in[0,a]} \|P_\beta(t)\|_{\mathcal{L}(E)} \leq \frac{M_a a^{\beta-1}}{\Gamma(\beta)}. \tag{3.3}$$

In light of this, we shall show that operator T fulfills all conditions of Theorem 2.2. This will be done in a series of lemmas.

Lemma 3.1. *$T : C([0,a];E) \to C([0,a];E)$ is continuous.*

Proof. Let $(u_n) \subset C([0,a];E)$ be a sequence which converges to $u \in C([0,a];E)$. Then

$$\begin{aligned}\|Tu_n - Tu\|_\infty \quad \leq \quad & M_a\|q(u_n) - q(u)\|_E + aM_a\|p(u_n) - p(u)\|_E \\ & + \tfrac{M_a a^{\beta-1}}{\Gamma(\beta)} \int_0^a \|f(s, u_n(s), Gu_n(s), Su_n(s)) \\ & - f(s, u(s), Gu(s), Su(s))\|_E ds.\end{aligned}$$

With assumptions (C_1) and (C_3) in mind, the continuity of G and S entails

$$\lim_{n\to\infty} f(s, u_n(s), Gu_n(s), Su_n(s)) = f(s, u(s), Gu(s), Su(s)).$$

Since (u_n) is convergent then there exists $r > 0$ such that $\|u_n\|_\infty \leq r$, for all $n \in \mathbb{N}$ and $\|u\|_\infty \leq r$. So by (C_4) we have

$$\begin{aligned}&\|f(s, u_n(s), Gu_n(s), Su_n(s)) - f(s, u(s), Gu(s), Su(s))\|_\infty \\ &\qquad \leq 2\left(\rho_1(s)\Omega_1(r) + \rho_2(s)\Omega_2(ak_0 r) + \rho_3(s)\Omega_3(ah_0 r)\right).\end{aligned}$$

Using the dominated convergence theorem, we deduce that T is continuous. □

Lemma 3.2. *Assume that*

$$M_a \liminf_{r\to\infty} \left(\frac{\sigma(r)}{r} + \frac{a^{\beta-1}}{\Gamma(\beta)}\frac{\Omega(r)}{r}\right) < 1, \tag{3.4}$$

where $\sigma(r) = \sigma_1(r) + a\sigma_2(r)$ and

$$\Omega(r) = \Omega_1(r)\|\rho_1\|_{L^1} + \Omega_2(ak_0 r)\|\rho_2\|_{L^1} + \Omega_3(ah_0 r)\|\rho_3\|_{L^1}.$$

Then, there is a $r_0 > 0$ such that T selfmaps the closed ball

$$B_{r_0} = \{u \in C([0,a];E) : \|u\|_\infty \leq r_0\}.$$

Proof. For $u \in B_r$ and $t \in [0,a]$, we have

$$\begin{array}{rcl} \|(Tu)(t)\|_E & \leq & \|C_\beta(t)\,(u_0+q(u))\|_E + \|S_\beta(t)\,(v_0+p(u))\|_E \\ & & + \left\| \int_0^t P_\beta(t-s) f(s,u(s),Gu(s),Su(s))ds \right\|_E \\ & \leq & M_a\left(\|u_0\|_E + \sigma_1(r)\right) + aM_a\left(\|v_0\|_E + \sigma_2(r)\right) \\ & & + \frac{M_a a^{\beta-1}}{\Gamma(\beta)} \int_0^a \Omega_1(r)\rho_1(s) \\ & & + \Omega_2(ak_0 r)\rho_2(s) + \Omega_3(ah_0 r)\rho_3(s)\mathrm{d}s. \end{array}$$

We claim that there exists $r_0 > 0$ such that $Tu \in B_{r_0}$ whenever $u \in B_{r_0}$. If is not the case, then for each $r > 0$ there exists $u \in B_r$ such that $Tu \notin B_r$, that is

$$r < \|Tu\|_\infty \leq M_a\left(\|u_0\|_E + \sigma_1(r)\right) + aM_a\left(\|v_0\|_E + \sigma_2(r)\right) + \frac{M_a a^{\beta-1}}{\Gamma(\beta)}\Omega(r),$$

which implies when dividing by r that

$$1 < \frac{M_a\,\|u_0\|_E + aM_a\|v_0\|_E}{r} + M_a\frac{\sigma(r)}{r} + \frac{M_a a^{\beta-1}}{\Gamma(\beta)}\frac{\Omega(r)}{r}.$$

Taking the lim inf as $r \to \infty$, we obtain

$$1 \leq M_a \liminf_{r\to\infty}\left(\frac{\sigma(r)}{r} + \frac{a^{\beta-1}}{\Gamma(\beta)}\frac{\Omega(r)}{r}\right),$$

which contradicts the assumption (3.4) Therefore, there exists $r_0 > 0$ such that

$$\|Tu\|_\infty \leq r_0, \text{ for all } \|u\| \leq r_0.$$

Thus, $Tu \in B_{r_0}$ for all $u \in B_{r_0}$. □

Lemma 3.3. *Let r_0 be as in Lemma 3.2 and let $x_0 \in B_{r_0}$. Let D be a countable subset of B_{r_0}. Then $D \subseteq \overline{\mathrm{co}}\left(\{x_0\} \cup T(D)\right)$ implies that D is relatively compact.*

Proof. Let $D = \{u_n\}_{n=1}^\infty$ be any countable subset of B_{r_0} such that

$$D \subseteq \overline{\mathrm{co}}\left(\{x_0\} \cup T(D)\right). \tag{3.5}$$

We show that D is relatively compact. Notice first that for each $t \in [0,a]$, we have

$$\begin{aligned} \chi(T(D)(t)) \leq{}& \chi(C_\beta(t)(u_0+q(D))) + \chi(S_\beta(t)(v_0+p(D))) \\ & + \chi\left(\left\{\int_0^t P_\beta(t-s)f(s,u_n(s),Gu_n(s),Su_n(s))ds\right\}_{n=1}^\infty\right). \end{aligned}$$

Since

$$\|f(s,u_n(s),Gu_n(s),Su_n(s))\|_E \leq \Omega_1(r_0)\rho_1(s) + \Omega_2(ak_0r_0)\rho_2(s) + \Omega_3(ah_0r_0)\rho_3(s)$$

and $\rho_1, \rho_2, \rho_3 \in L^1([0,a]; R+)$, then, in view of Theorem 2.5 and Lemma 2.2, we obtain the following estimates:

$$\begin{array}{ll} & \chi(T(D)(t)) \\ \leq & M_a\chi(q(D)) + aM_a\chi(p(D)) \\ & +2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\int_0^t m_1(s)\chi(D(s)) + m_2(s)\chi(G(D(s))) + m_3(s)\chi(S(D(s)))ds \\ \leq & (M_a k_q + aM_a k_p)\chi_\infty(D) \\ & +2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\int_0^t m_1(s)\chi(D(s)) + ak_0 m_2(s)\chi(D(s)) + ah_0 m_3(s)\chi(D(s))ds \\ \leq & \left(M_a k_q + aM_a k_p + 2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\right)\chi_\infty(D). \end{array}$$

Thus,

$$\chi_\infty(T(D)) \leq \left[M_a k_q + aM_a k_p + 2\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\right]\chi_\infty(D). \tag{3.6}$$

On the other hand, referring to Theorem 2.3, Theorem 2.4, and Lemma 2.2, we can see that

$$\begin{array}{lll} \mathrm{mod}_C(T(D)(t)) & \leq & \mathrm{mod}_C(C_\beta(t)q(D)) + \mathrm{mod}_C(S_\beta(t)p(D)) \\ & & +4\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\int_0^t m(s)\chi(D(s))\mathrm{d}s \\ & \leq & 2(M_a k_q + aM_a k_p)\chi_\infty(D) + 4\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\chi_\infty(D). \end{array}$$

Thus,

$$\mathrm{mod}\,_C(T(D)) \leq \left[2(M_a k_q + aM_a k_p) + 4\frac{M_a a^{\beta-1}}{\Gamma(\beta)}\|m\|_1\right]\chi_\infty(D). \tag{3.7}$$

Combining (3.5) and (3.6), we arrive at $\chi_\infty(TD) = \chi_\infty(D) = 0$. By (3.7) we get $\mathrm{mod}\,_C(T(D)) = 0$ and therefore $T(D)$ is equicontinuous. Going back to (3.5) we deduce that D is equicontinuous and so relatively compact in $C([0,a]; E)$. This achieves the proof. □

Theorem 3.1. *Assume that* $(C_1) - -(C_6)$ *hold. Then, the nonlocal problem* (1.1) *has at least one mild solution in* $C([0,a]; E)$, *provided that* (3.4) *holds.*

Proof. Invoking Theorem 2.2 together with Lemmas 3.1, 3.2, and 3.3, we infer that T has at least one fixed point in B_{r_0} which is, in turn, a mild solution of (1.1). □

4 Application

To illustrate the application of the theoretical results of this work, we consider the following integro-differential equation:

$$\begin{cases} {}^cD_t^\beta w(t,x) = \dfrac{\partial^2 w(t,x)}{\partial^2 x} + \rho_1(t) f_1(w(t,x)) + \rho_2(t) f_2\left(\int_0^t \frac{ts}{2} w(s,x)ds\right) \\ +\rho_3(t) f_3\left(\int_0^1 \frac{t^2s^2}{2} w(s,x)ds\right), \quad t\in I=[0,1],\ x\in[0,\pi], \\ w(t,0)=w(t,\pi)=0,\ t\in I, \\ w(0,x)=w_0(x)+\sum\limits_{i=1}^m c_i w(s_i,x), \quad x\in[0,\pi], \\ s_1<s_2<\ldots<s_m,\ t_i\in I,\ c_i\in\mathbb{R}, \\ \dfrac{\partial w(t,x)}{\partial t}\Big|_{t=0} = y_0(x)+\sum\limits_{i=1}^n d_i w(t_i,x),\ x\in[0,\pi], \\ t_1<t_2<\ldots<t_n,\ t_i\in I,\ d_i\in\mathbb{R}, \end{cases} \tag{4.1}$$

where $\beta \in (1,2]$, the functions $\rho_i : I \to \mathbb{R}$ and $f_i : E \to E$ for $i \in \{1,2,3\}$ satisfy appropriate conditions which are specified later.

To allow the abstract formulation of (4.1), let $E = L^2([0,\pi];\mathbb{R})$ be the Banach space of square integrable functions from $[0,\pi]$ into $\mathbb{R}$. Define the operator $A : D(A) \subset E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous }, w'' \in E, w(0) = w(\pi) = 0\}.$$

It is well known that A is the generator of strongly continuous cosine functions $\{C(t) : t \in \mathbb{R}\}$ on E. Moreover A has a discrete spectrum whose eigenvalues are $-n^2$, $n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) = \sqrt{\frac{2}{\pi}}\sin(n\tau),$$

and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $z \in E$, then $Az = -\sum_{n=1}^\infty n^2 < z,\ z_n > z_n$.

(c) For $z \in E$, $C(t)z = \sum_{n=1}^\infty \cos(nt) < z, \ z_n > z_n$, and the associated sine family is $S(t)z = \sum_{n=1}^\infty \frac{\sin(nt)}{n} < z, z_n > z_n$. $S(t)$ is compact for every $t \in I$ and $\|C(t)\|_{\mathcal{L}(E)} = \|S(t)\|_{\mathcal{L}(E)} \le 1$, for every $t \in \mathbb{R}$.

For $\beta \in (1,2]$, since A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, from the subordinate principle (Theorem 2.1), it follows that A is the infinitesimal generator of a strongly continuous exponentially bounded fractional cosine family $C_\beta(t)$.

With $u(t) = w(t, \cdot)$, Equation (4.1) may be written in the abstract form:

$$\begin{cases} {}^cD_t^{\beta} u(t) = Au(t) + f(t, u(t), Gu(t), Su(t)), \ t \in I, \\ u(0) = u_0 + q(u), \\ u'(0) = v_0 + p(u), \end{cases} \tag{4.2}$$

where the function $f : I \times E \times E \times E \to E$ is given by

$$f(t, x, y, z) = \rho_1(t)f_1(x) + \rho_2(t)f_2(y) + \rho_3(t)f_3(z).$$

Here $\rho_i : I \to \mathbb{R}$ is integrable on I, $f_i : E \to E$ is a Lipschitz continuous function with a Lipschitz constant L_i, the functions $p, q : C(I, E) \to E$ are given by

$$q(u) = \sum_{i=1}^{m} c_i u(s_i), \quad 0 < s_1 < s_2 < \cdots < s_m \leq 1,$$

and

$$p(u) = \sum_{i=1}^{n} d_i u(t_i), \quad 0 < t_1 < t_2 < \cdots < t_n \leq 1,$$

and the functions $G, S : C(I, E) \to C(I, E)$ are defined by

$$Gu(t) = \int_0^t \frac{ts}{2} u(s)ds, \quad Su(t) = \int_0^1 \frac{t^2 s^2}{2} u(s)ds,$$

where $h_0 = k_0 = \dfrac{1}{2}$.

In order to obtain a mild solution, our strategy is to apply Theorem 3.1. First, by (c) we have $\|C(t)\|_{\mathcal{L}(E)} \leq 1$, for every $t \in \mathbb{R}^+$. In view of Theorem 2.1 and (2.2) we see that there exists a real number $M_a = 1 > 0$ such that $\|C_\beta(t)\|_{\mathcal{L}(E)} \leq M_a$ for $t \geq 0$. Observe further that the function $f : I \times E \times E \times E \to E$ is given by

$$f(t, x, y, z) = \rho_1(t)f_1(x) + \rho_2(t)f_2(y) + \rho_3(t)f_3(z),$$

where $\rho_i : I \to \mathbb{R}$ is integrable on I and $f_i : E \to E$ is a Lipschitz continuous function with a Lipschitz constant L_i $(i = 1, 2, 3)$. This shows that (C_3) is satisfied. On one hand,

$$\|q(u)\|_E \leq \left(\sum_{i=1}^{m} |c_i| \right) \|u\|_\infty = \sigma_1(\|u\|_\infty) \tag{4.3}$$

and

$$\|p(u)\|_E \leq \left(\sum_{i=1}^{n} |d_i| \right) \|u\|_\infty = \sigma_2(\|u\|_\infty), \tag{4.4}$$

where $\sigma_1(r) = \left(\sum_{i=1}^{m} |c_i|\right) r$ and $\sigma_2(r) = \left(\sum_{i=1}^{n} |d_i|\right) r$. In addition, it is easily seen that for any bounded subset D of $C([0, 1], E)$ we have

$$\chi(q(D)) \leq \sum_{i=1}^{m} |c_i| \, \chi\left(D\left(s_i\right)\right) \leq \left(\sum_{i=1}^{m} |c_i| \right) \chi_\infty(D) = k_q \chi_\infty(D) \tag{4.5}$$

and

$$\chi(p(D)) \leq \sum_{i=1}^{n} |d_i| \, \chi\left(D\left(t_i\right)\right) \leq \left(\sum_{i=1}^{n} |d_i|\right) \chi_\infty(D) = k_p \chi_\infty(D). \tag{4.6}$$

Thus

$$M_a \chi(q(D)) + a M_a \chi(p(D)) \leq (M_a k_q + a M_a k_p) \chi_\infty(D), \tag{4.7}$$

for any bounded subset D of $C([0,1];E)$. This shows that (C_1) and (C_2) are satisfied. Moreover the function f satisfies

$$\begin{aligned}
\|f(t,u_1,u_2,u_3)\|_E &\leq |\rho_1(t)| \|f_1(u_1)\|_E + |\rho_2(t)| \|f_2(u_2)\|_E + |\rho_3(t)| \|f_3(u_3)\|_E \\
&\leq |\rho_1(t)| (\|f_1(0)\|_E + L_1 \|u_1\|_E) + |\rho_2(t)| (\|f_2(0)\|_E + L_2 \|u_2\|_E) \\
&\qquad + |\rho_3(t)| (\|f_3(0)\|_E + L_3 \|u_3\|_E) \\
&\leq |\rho_1(t)| \Omega_1(\|u_1\|_E) + |\rho_2(t)| \Omega_2(\|u_2\|_E) + |\rho_3(t)| \Omega_3(\|u_3\|_E) \\
&\leq \sum_{i=1}^{3} |\rho_i(t)| \Omega_i(\|u_i\|_E),
\end{aligned}$$

where $\Omega_i(\|u_i\|_E) = \|f_i(0)\|_E + L_i \|u_i\|_E$. By virtue of Lemma 2.1, (v) we have

$$\begin{aligned}
\chi(f(t,D_1,D_2,D_3)) &\leq |\rho_1(t)| \chi(f_1(D_1)) + |\rho_2(t)| \chi(f_2(D_2)) + |\rho_3(t)| \chi(f_3(D_3)) \\
&\leq |\rho_1(t)| L_1 \chi(D_1) + |\rho_2(t)| L_2 \chi(D_2) + |\rho_3(t)| L_3 \chi(D_3) \\
&\leq \sum_{i=1}^{3} m_i(t) \chi(D_i),
\end{aligned}$$

for any $t \in [0,a]$ and for any bounded subsets D_1, D_2, D_3 of E. Thus, (C_4) and (C_5) are satisfied. Now the condition (C_6) is given by taking

$$2\frac{a^{\beta-1} M_a}{\Gamma(\beta)} \left(L_1 \|\rho_1\|_{L^1} + \frac{1}{2} L_2 \|\rho_2\|_{L^1} + \frac{1}{2} L_3 \|\rho_3\|_{L^1} \right) + (M_a k_q + a M_a k_p) < 1,$$

because, we have

$$\begin{aligned}
m(s) &= m_1(s) + a k_0 m_2(s) + a h_0 m_3(s) \\
&= L_1 |\rho_1(s)| + \frac{1}{2} L_2 |\rho_2(s)| + \frac{1}{2} L_3 |\rho_3(s)|.
\end{aligned}$$

Then

$$\|m\|_1 = L_1 \|\rho_1\|_{L^1} + \frac{1}{2} L_2 \|\rho_2\|_{L^1} + \frac{1}{2} L_3 \|\rho_3\|_{L^1}.$$

Finally, for

$$\begin{aligned}
\Omega(r) &= \Omega_1(r) \|\rho_1\|_{L^1} + \Omega_2(a k_0 r) \|\rho_2\|_{L^1} + \Omega_3(a h_0 r) \|\rho_3\|_{L^1} \\
&= \Omega_1(r) \|\rho_1\|_{L^1} + \Omega_2(\tfrac{1}{2} r) \|\rho_2\|_{L^1} + \Omega_3(\tfrac{1}{2} r) \|\rho_3\|_{L^1},
\end{aligned}$$

we have

$$\lim_{r\to\infty} \frac{\Omega(r)}{r} = L_1 \|\rho_1\|_{L^1} + \frac{1}{2} L_2 \|\rho_2\|_{L^1} + \frac{1}{2} L_3 \|\rho_3\|_{L^1},$$

and for $\sigma(r) = \sigma_1(r) + a\sigma_2(r) = (k_q + ak_p)r$, notice that

$$\lim_{r\to\infty} \frac{\sigma(r)}{r} = k_q + ak_p.$$

Then

$$\begin{array}{rl} & M_a \liminf_{r\to\infty} \left(\frac{\sigma(r)}{r} + \frac{a^{\beta-1}}{\Gamma(\beta)} \frac{\Omega(r)}{r} \right) \\ = & \frac{a^{\beta-1}M_a}{\Gamma(\beta)} \left(L_1\|\rho_1\|_{L^1} + \frac{1}{2}L_2\|\rho_2\|_{L^1} + \frac{1}{2}L_3\|\rho_3\|_{L^1} \right) + (M_a k_q + aM_a k_p) \\ \leq & 2\frac{a^{\beta-1}M_a}{\Gamma(\beta)} \left(L_1\|\rho_1\|_{L^1} + \frac{1}{2}L_2\|\rho_2\|_{L^1} + \frac{1}{2}L_3\|\rho_3\|_{L^1} \right) + (M_a k_q + aM_a k_p) \\ < & 1. \end{array}$$

Thus, all conditions of Theorem 3.1 are fulfilled. Therefore Equation (4.1) has a mild solution.

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Curves in low dimensional projective spaces with the lowest ranks

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ABSTRACT

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. For each $q \in \mathbb{P}^r$ the X-rank $r_X(q)$ of q is the minimal number of points of X spanning q. A general point of $\mathbb{P}^r$ has X-rank $\lceil (r+1)/2 \rceil$. For $r = 3$ (resp. $r = 4$) we construct many smooth curves such that $r_X(q) \le 2$ (resp. $r_X(q) \le 3$) for all $q \in \mathbb{P}^r$ (the best possible upper bound). We also construct nodal curves with the same properties and almost all geometric genera allowed by Castelnuovo's upper bound for the arithmetic genus.

RESUMEN

Sea $X \subset \mathbb{P}^r$ una curva integral y no-degenerada. Para cada $q \in \mathbb{P}^r$ el X-rango $r_X(q)$ de q es el mínimo número de puntos de X que generan q. Un punto general de $\mathbb{P}^r$ tiene X-rango $\lceil (r+1)/2 \rceil$. Para $r = 3$ (resp. $r = 4$) construimos muchas curvas suaves tales que $r_X(q) \le 2$ (resp. $r_X(q) \le 3$) para todo $q \in \mathbb{P}^r$ (la mejor cota superior posible). También construimos curvas nodales con las mismas propiedades y casi todos los géneros geométricos permitidos por la cota superior de Castelnuovo para el género aritmético.

Keywords and Phrases: X-rank, projective curve, space curve, curve in projective spaces.

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1 Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field with characteristic 0. For each $q \in \mathbb{P}^r$ the *X-rank* $r_X(q)$ of q is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S\rangle$, where $\langle\ \ \rangle$ denotes the linear span. An interesting problem is the maximum of all integers $r_X(q)$, $q \in \mathbb{P}^r$ ([2, 8]). An obvious lower bound for this integer is the *generic X-rank* $r_{\mathrm{gen}}(X)$, i.e. the only integer such there is a non-empty Zariski open subset $U \subset \mathbb{P}^r$ such that $r_X(q) = r_{\mathrm{gen}}(X)$ for all $q \in U$. For each positive integer t set $W^0_t(X) := \{q \in \mathbb{P}^r \mid r_X(q) = t\}$. Let $W_t(X)$ denote the closure of $W^0_t(X)$ in $\mathbb{P}^r$. If $t \le r_{\mathrm{gen}}(X)$ the algebraic set $W_t(X)$ is the t-secant variety $\sigma_t(X)$ of X. Hence if $1 \le t \le r_{\mathrm{gen}}(X)$ the algebraic set $W_t(X)$ is non-empty, irreducible and $\dim W_t(X) \le \min\{r, t(\dim X + 1) - 1\}$ with equality if $\dim X = 1$ ([1, Remark 1.6]). Thus $r_{\mathrm{gen}}(X) = \lceil (r + 1)/2\rceil$ if $\dim X = 1$. For $t > r_{\mathrm{gen}}(X)$ the geometry of $W_t(X)$ is described in [3, Theorem 3.1], assuming of course $W_t(X) \neq \emptyset$, i.e. $W^0_t(X) \neq \emptyset$.

We prove the following results.

Theorem 1.1. *Fix integers $b \ge a > 0$ such that $a + b \ge 5$. Set $d := a + b$ and $\gamma := ab - a - b + 1$. Then there exists an integral and non-degenerate nodal curve $X \subset \mathbb{P}^3$ with geometric genus g, $\deg(X) = d$, exactly $\gamma - g$ ordinary nodes and $W^0_3(X) = \emptyset$.*

Theorem 1.2. *Fix integers a, b such that $a \ge 2$ and $b \ge 2a + 3$. Set $d := a + b$ and $\gamma := 1 + ab - a(a + 1)/2 - b$. Fix an integer g such that $0 \le g \le \gamma$. Then there is an integral nodal curve $X \subset \mathbb{P}^4$ with degree d, geometric genus g, exactly $\gamma - g$ ordinary nodes and with $W^0_4(X) = \emptyset$.*

Question 1.1. *Is there an integral and non-degenerate curve $X \subset \mathbb{P}^5$ with $W^0_4(X) = \emptyset$? Take an odd integer $r > 5$. Is there an integral and non-degenerate curve $X \subset \mathbb{P}^r$ with $W^0_{(r+3)/2}(X) = \emptyset$?*

By [9, Theorem 1] $W^0_3(X) \neq \emptyset$ for X as in Theorem 1.1, but with $(a, b) \in \{(1, 2), (1, 3), (2, 2)\}$. The case $(a, b) = (3, 3)$ of Theorem 1.1 is [9, Theorem 2]. When $a \le b \le a + 1$ the integer γ appearing in Theorem 1.1 is the maximal arithmetic genus of all non-degenerate space curves ([6, Ch. III]).

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2 Preliminaries

Notation 2.1. *For any $q \in \mathbb{P}^r$ let $\ell_q : \mathbb{P}^r \setminus \{q\} \longrightarrow \mathbb{P}^{r-1}$ denote the linear projection from q.*

Let M be a projective scheme. Let $D \subset M$ be an effective Cartier divisor of M. For any zero-dimensional scheme $Z \subset M$ the *residual scheme* $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of M with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and hence $\mathrm{Res}_D(Z)$ is a zero-dimensional scheme. We have $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$ and for any line bundle $\mathcal{L}$ on M we have an exact sequence of coherent sheaves on M:

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \longrightarrow \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D} \longrightarrow 0 \tag{2.1}$$

We will call (2.1) the *residual exact sequence of D* or the *residual exact sequence of D in M*.

Remark 2.1. *Let M be a smooth, projective and rational surface. Thus $h^1(\mathcal{O}_M) = 0$. Assume that $\omega_M^\vee$ is ample. This will be true in the cases in which we apply this remark, i.e. the case in which M is the smooth quadric surface and the case in which M is the Hirzebruch surface F_1. Fix an integer $e \geq 2$, a very ample line bundle $\mathcal{L}$ on M and a nodal curve $D = D_1 \cup \cdots \cup D_e \in |\mathcal{L}|$ with each D_i a smooth and connected curve. Note that $p_a(D) = \sum_{i=1}^e p_a(D_i) + \sharp(\mathrm{Sing}(D)) + 1 - e$. Since $\mathcal{L}$ is very ample, D is connected. Since $\omega_M^\vee$ is ample, we have $D_i \cdot \omega_M < 0$ (intersection number) for all i. A subset $A \subseteq \mathrm{Sing}(D)$ is said to be a* disconnecting *set of nodes if $D \setminus A$ is not connected. Fix a set $A \subset \mathrm{Sing}(D)$ which is not disconnecting and set $g := p_a(D) - \sharp(A)$. With the terminology of [10] we will say that A is the set of* assigned nodes, *while the set $\mathrm{Sing}(D) \setminus A$ is the set of* unassigned nodes. *By [10, Corollary 2.14] there are an affine smooth and connected curve Δ, $o \in \Delta$, and a flat family $\{Y_t\}_{t\in\Delta}$ of elements of $|\mathcal{L}|$ such that $Y_o = D$ and Y_t is integral, nodal and with geometric genus g for all $t \in \Delta \setminus \{o\}$. Moreover, the sets $\{\mathrm{Sing}(Y_t)\}_{t\in\Delta\setminus\{o\}}$ have A as a limit. Thus $p_a(D) = \sharp(\mathrm{Sing}(D)) + 1 - e$. We do not impose (or claim) that all Y_t are singular at the points of A, because it would require very strong restrictions on the integer $\sharp(A)$, only that the nodes of the curves Y_t near D are near A and that Y_t has only $\sharp(A)$ nodes. The quoted result [10, Corollary 2.14] with movable assigned nodes is optimal, as shown by following particular case, the only one we will use. Assume that each D_i is rational. In this case for each integer g with $0 \leq g \leq p_a(D)$ there is a set of assigned nodes $A \subset \mathrm{Sing}(D)$ such that the corresponding family of nodal curves has as a general member an integral nodal curve with geometric genus g.*

Remark 2.2. *Let X be a smooth projective curve, $\mathcal{L}$ a line bundle on X and $V \subseteq H^0(\mathcal{L})$ a linear subspace. Set $g := p_a(X)$, $d := \deg(\mathcal{L})$ and $n := \dim V - 1$. Assume $n \geq 1$. For each $p \in X$ and each integer $t > 0$ set $V(-tp) := V \cap H^0(\mathcal{I}_{tp} \otimes \mathcal{L})$. We get $n+1$ integers $\dim V(-tp)$, $1 \leq t \leq n+1$ ([5, pp. 264–277]). This is also done in details in [9]. The point p is said to be an osculating point of the pair $(\mathcal{L}, V)$ (or of the linear system $\mathbb{P}V$) if $\dim(V(-(n+1)p)) > 0$. Since we are in characteristic zero, there are only finitely many osculating points of $(\mathcal{L}, V)$, say $p_1, \dots, p_s$, and at each point p_i one can associate a positive integer $w(p_i)$ (the weight of p_i), only depending on the $n+1$ integers $\dim V(-tp)$, $1 \leq t \leq n+1$. Moreover, there is an integer δ only depending on g, d and n such that $w(p_1) + \cdots + w(p_s) = \delta$. We have $w(p_i) = 1$ if and only if $\dim V(-np_i) = \dim V(-(n+1)p_i) = 1$. Suppose for instance that $\mathbb{P}V$ induces an embedding of X into $\mathbb{P}^n$ and see X has a curve of $\mathbb{P}^n$. Since $V \subseteq H^0(\mathcal{L})$, X is non-degenerate. The point $p \in X$ is an osculating point if and only if there is a hyperplane $H \subset \mathbb{P}^n$ such that the connected component Z of the scheme $H \cap X$ with p has its reduction has degree $\geq n+1$, i.e. H contains the divisor*

$(n+1)p$. The integer $\deg(Z)$ *is the order of contact of the osculating hyperplane H with X at p. The integer* $\deg(Z)-n$ *is a lower bound for the weight of p. All non-osculating points have weight* 0.

3 Proof of Theorem 1.1

In this section we fix a smooth quadric surface $Q \subset \mathbb{P}^3$. For any irreducible curve $Y \subset \mathbb{P}^3$, Y not a line, let $\tau(Y)$ denote the tangential surface of Y, i.e. the closure in $\mathbb{P}^3$ of the union of all tangent lines of Y at its smooth points. $\tau(Y)$ is a plane if and only if $\langle Y \rangle$ is a plane.

Notation 3.1. *For any reduced curve $X \subset \mathbb{P}^3$ with no irreducible component contained in a plane let $T(X)$ be the set of all pairs (H,p), where $H \subset \mathbb{P}^3$ is a plane, $p \in H \cap X$ and the connected component of the scheme $H \cap X$ with p as its reduction has degree at least* 5.

Remark 3.1. *Let Δ a quasi-projective variety and $\mathcal{X} \subset \mathbb{P}^3 \times \Delta$ a closed algebraic set such that the restriction $u : \mathcal{X} \longrightarrow \Delta$ to $\mathcal{X}$ of the projection $\mathbb{P}^3 \times \Delta \longrightarrow \Delta$ is proper and flat. Assume that all fibers of u are reduced curves with no irreducible component contained in a plane. Let $T(\mathcal{X})$ or $T(u)$ denote the set of all triples (s,H,p), where $s \in \Delta$ and $(H,p) \in T(u^{-1}(s))$. The map $u_{T(\mathcal{X})} : T(\mathcal{X}) \longrightarrow \Delta$ is proper. Thus if Δ is irreducible and if $T(u^{-1}(s_0)) = \emptyset$ for some $s_0 \in \Delta$, then $T(u^{-1}(s)) = \emptyset$ for a general $s \in \Delta$.*

Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Fix $p \in X_{\mathrm{reg}}$. We say that p is a *flex point* of X or a *flex* of X or that the tangent line T_pX is a *flex tangent* of X if the connected component of the zero-dimensional scheme $T_pX \cap X$ with p as its reduction has degree at least 3. We say that p is a *stall point* of X or that T_pX is a *stall* of X if T_pX is not a flex tangent, but the osculating plane $O_p(X)$ of X at p has order of contact at least 4 with X at p. Thus a stall point is an osculating point which is not a flex point.

Remark 3.2. *Fix a smooth element Y either of $|\mathcal{O}_Q(1,1)|$ or of $|\mathcal{O}_Q(2,1)|$ or of $|\mathcal{O}_Q(1,2)|$. Since Y is a rational normal curve in its linear span, it is easy to check that $T(Y) = \emptyset$ and that each $q \in \tau(Y) \setminus Y$ is contained in at most 2 tangent lines of Y.*

We collect in the next remark some standard tools and ideas which are used in the proofs of Lemmas 3.1, 3.2 and 3.3 and which may be used in several other cases. In section 4 we will use this set-up for the Hirzebruch surface F_1 and the line bundle $\mathcal{O}_{F_1}(ah+bf)$.

Remark 3.3. *Fix positive integers a, b and an integral quasi-projective family $\mathcal{F}$ of zero-dimensional subschemes of the smooth quadric Q. Suppose you want to compute the dimension of the family Ψ of all $C \in |\mathcal{O}_Q(a,b)|$ containing at least one $Z \in \mathcal{F}$ or of the family Φ of all smooth $C \in |\mathcal{O}_Q(a,b)|$ containing at least one $Z \in \mathcal{F}$. In most lemmas we will need to check that*

$\dim \Phi < \dim |\mathcal{O}_Q(a,b)|$, *i.e. that a general* $C \in |\mathcal{O}_Q(a,b)|$ *contains no* $Z \in \mathcal{F}$. *Consider the incidence variety* $\mathbb{I} := \{(Z,C) \in \mathcal{F} \times |\mathcal{O}_Q(a,b)| : Z \in C\}$. *Let* $\pi_1 : \mathbb{I} \longrightarrow \mathcal{F}$ *and* $\pi_2 : \mathbb{I} \longrightarrow |\mathcal{O}_Q(a,b)|$ *denote the restriction to* $\mathbb{I}$ *of the projections of* $\mathcal{F} \times |\mathcal{O}_Q(a,b)|$ *onto its factors. Note that* $\Psi = \pi_2(\mathbb{I})$. *The algebraic set* $\mathbb{I}$ *is a closed subset of* $\mathcal{F} \times |\mathcal{O}_Q(a,b)|$. *Thus by Chevalley's theorem* Ψ *is a constructible set ([7, ex. II.3.18 and II.3.19]). If* $\mathbb{I}$ *is irreducible, then* Ψ *is irreducible. Obviously* $\Phi = \emptyset$, *unless at least some* $Z \in \mathcal{F}$ *is curvilinear. Call* $\mathcal{U}$ *the set of all smooth* $C \in |\mathcal{O}_Q(a,b)|$. *Assume that at least some* $Z \in \mathcal{F}$ *is curvilinear and let* $\mathcal{G}$ *denote the set of all curvilinear* $Z \in \mathcal{F}$. *The set* $\mathcal{G}$ *is an open subset of* $\mathcal{F}$. *Since* $\mathcal{F}$ *is assumed to be irreducible,* $\mathcal{G}$ *is irreducible. Set* $\mathbb{J} := \mathbb{I} \cap \mathcal{G} \times \mathcal{U}$. *Usually, if we are only interested in smooth curves* $C \in |\mathcal{O}_Q(a,b)|$ *it is better to start with* $\mathcal{G}$, *i.e. take an irreducible family of curvilinear schemes. Thus from now on we assume* $\mathcal{F} = \mathcal{G}$, *but we use* $\mathbb{I}$, *i.e. we also consider singular curves, to quote below [7, III.9.3, III.9.6, III.9.7]. Suppose there is an integer* $z > 0$ *such that* $h^0(Q, \mathcal{I}_Z(a,b)) = z$ *for all* $Z \in \mathcal{G}$. *With this assumption all fibers of* π_1 *are projective spaces of dimension* $z-1$. *Hence* π_1 *is a proper flat map. Since* $\mathcal{G}$ *is assumed to be irreducible,* $\mathbb{I}$ *is irreducible and* $\dim \mathbb{I} = \dim \mathcal{G} + z$ *([7, III.9.3, III.9.6, III.9.7]). Since* $\mathbb{J}$ *is a non-empty open subset of* $\mathbb{I}$, $\mathbb{J}$ *is irreducible and* $\dim \mathbb{J} = \dim \mathbb{I} = \dim \mathcal{G} + z$. *Thus* Φ *is irreducible and* $\dim \Phi \le \dim \mathcal{G} + z$. *If this inequality is not sufficient to conclude, one should look at a general* $C \in \Phi$ *and try to compute* $\dim(\mathbb{J} \cap \pi_2^{-1}(C))$. *Suppose* $\dim(\mathbb{J} \cap \pi_2^{-1}(C)) = x$ *for a general* $C \in |\mathcal{O}_Q(a,b)|$. *Then* $\dim \Phi = \dim \mathcal{G} + z - x$. *Since* C *is smooth and* $\dim C = 1$, $\dim(\mathbb{J} \cap \pi_2^{-1}(C)) \le x$ *if* $\sharp(Z_{\mathrm{red}}) \le x$ *for all* $Z \in \mathbb{J} \cap \pi_2^{-1}(C)$. *Moreover,* $\dim(\mathbb{J} \cap \pi_2^{-1}(C)) = x$ *if varying* $Z \in \mathbb{J} \cap \pi_2^{-1}(C)$ *the sets* Z_{red} *form an* x*-dimensional family of* x *distinct points of* C. *This set-up is classically summarized by the words " A dimensional count shows that* Φ *has dimension* $\dim \mathcal{G} + z - x$ *". If our family* $\mathcal{G}$ *is not irreducible, we try to study separately each of its irreducible components. Now we drop the assumption that all integers* $h^0(Q, \mathcal{I}_Z(a,b))$ *are the same. There are a non-empty open subset* $\mathcal{G}'$ *of* $\mathcal{G}$ *and an integer* z *such that* $h^0(Q, \mathcal{I}_Z(a,b)) = z$ *for all* $Z \in \mathcal{G}'$. *Moreover, there are a positive integer* s *and integers* $z_i \ge z$, $1 \le i \le s$, *such that* $\mathcal{G} \setminus \mathcal{G}'$ *is the union of finitely many irreducible quasi-projective varieties, say* $\mathcal{G} \setminus \mathcal{G}' = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_s$, *such that* $h^0(Q, \mathcal{I}_Z(a,b)) = z_i$ *for all* $Z \in \mathcal{G}_i$. *Then we use the irreducible families* $\mathcal{G}', \mathcal{G}_1, \ldots, \mathcal{G}_s$ *of curvilinear schemes.*

We will need only the case $a = 1$ of the next lemma, but its proof when $a \ge 2$ requires no modification.

Lemma 3.1. *Fix integers* $a > 0$, $b > 0$ *such that* $a + b \ge 4$. *Let* D *be a general element of* $|\mathcal{O}_Q(a,b)|$. *Then* D *has no flex and* $T(D) = \emptyset$.

Proof. We follow the classical approach outlined in Remark 3.3. The key step in the proof of the lemma is the computation of the integer $h^0(Q, \mathcal{I}_Z(a,b))$ for two types of zero-dimensional schemes Z.

With no loss of generality we may assume $b \geq a$ and hence $b \geq 2$. By Bertini's theorem D is smooth. Since $D \subset Q$, Bezout theorem implies that each flex tangent line of D is contained in Q and hence it is either an element of $|\mathcal{O}_Q(1,0)|$ or an element of $|\mathcal{O}_Q(0,1)|$.

(a) Take $L \in |\mathcal{O}_Q(1,0)|$ and any connected zero-dimensional scheme $F \subset L$ such that $\deg(F) = 3$. Since $\deg(\mathcal{O}_L(a,b)) = b \geq 2$, we have $h^1(L, \mathcal{I}_{F,L}(a,b)) = 0$. Since $h^1(\mathcal{O}_Q(0,b)) = 0$, the residual exact sequence of L gives $h^1(\mathcal{I}_F(a,b)) = 0$, i.e. $h^0(\mathcal{I}_F(a,b)) = h^0(\mathcal{O}_Q(a,b)) - 3$. Since $\dim |\mathcal{O}_Q(1,0)| = 1$ and each $L \in |\mathcal{O}_Q(1,0)|$ contains ∞^1 connected degree 3 subschemes, a general $D \in |\mathcal{O}_Q(a,b)|$ contains no F (for any L), i.e. no $L \in |\mathcal{O}_Q(1,0)|$ is a flex tangent of D.

(b) If $a \geq 2$ step (a) shows that no $R \in |\mathcal{O}_Q(0,1)|$ is a flex tangent of D. Now assume $a = 1$. Since $D \in |\mathcal{O}_Q(a,b)|$, we have $\deg(R \cap D) = 1$ for all $R \in |\mathcal{O}_Q(0,1)|$. Thus no element of $|\mathcal{O}_Q(0,1)|$ is a flex tangent line of D.

By steps (a) and (b) D has no flex. Thus it is sufficient to prove that each osculating plane of D has order of contact 4 with D at the osculating point. Fix a smooth element $A \in |\mathcal{O}_Q(1,1)|$ and $p \in A$. Let E be the connected zero-dimensional subscheme of A such that $E_{\mathrm{red}} = \{p\}$ and $\deg(E) = 5$.

Claim 1: We have $h^1(Q, \mathcal{I}_E(a,b)) = 0$.

Proof of Claim 1: We have $h^1(A, \mathcal{I}_{E,A}(a,b)) = 0$, because $A \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_A(1,b)) = b+1 \geq 4$. Since $E \subset A$, $\mathrm{Res}_A(E) = \emptyset$. Thus it is sufficient to use the residual exact sequence of A in Q and that $h^1(\mathcal{O}_Q(0,b-1)) = 0$.

By Claim 1 we have $h^0(\mathcal{I}_E(a,b)) = (a+1)(b+1) - 5$ for all E. Since $\dim |\mathcal{O}_Q(1,1)| = 3$ and each smooth $A \in |\mathcal{O}_Q(1,1)|$ has ∞^1 points and hence ∞^1 schemes E's. Use Claim 1. $\square$

Notation 3.2. *Let $D \subset Q$ be a reduced curve with no irreducible component of D being an element of $|\mathcal{O}_Q(1,0)|$ or $|\mathcal{O}_Q(1,0)|$ or $|\mathcal{O}_Q(1,1)|$. Let $F(D)$ be denote the set of all $C \in |\mathcal{O}_Q(1,1)|$ such that the scheme $C \cap D$ contains at least two connected components, both of them of degree at least 4.*

Lemma 3.2. *Fix integers $a > 0$ and $b > 0$ such that $a + b \geq 3$. Take a general $D \in |\mathcal{O}_Q(a,b)|$. Then $F(D) = \emptyset$.*

Proof. The curve D is smooth and for each line $L \subset Q$ every connected component of the scheme $L \cap D$ has connected components of degree 1 or 2, with at most one having degree 2. Thus it is sufficient to test the smooth $C \in |\mathcal{O}_Q(1,1)|$. Since $\deg(D \cap C) = a + b$, we may assume $a + b \geq 8$. Call $\mathcal{G}$ the set of all zero-dimensional schemes Z with 2 connected components, both of degree 4 and with Z contained in some smooth $C \in |\mathcal{O}_Q(1,1)|$. Each C contains ∞^2 elements of $\mathcal{G}$. Fix $Z \in \mathcal{G}$ and take C containing it. As in the proof of Lemma 3.1 it is sufficient to observe that $h^1(\mathcal{I}_Z(a,b)) = 0$, because $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(a,b)) = a + b \geq \deg(Z) - 1$ and hence

$h^1(C, \mathcal{O}_C(a,b)(-Z)) = 0$. Since $\mathrm{Res}_C(Z) = \emptyset$ and $h^1(Q, \mathcal{O}_Q(a-1,b-1)) = 0$, the residual exact sequence of C gives $h^1(Q, \mathcal{I}_{Z,Q}(a,b)) = 0$. □

Lemma 3.3. *Fix positive integers a, b, a', b' such that $(a,b) \neq (1,1)$ and $(a',b') \neq (1,1)$. Take a general $(D, D') \in |\mathcal{O}_Q(a,b)| \times |\mathcal{O}_Q(a',b')|$. Set $Y := D \cup D'$. Then there is no $C \in |\mathcal{O}_Q(1,1)|$ such that the scheme $C \cap Y$ has two connected components of degree at least 4.*

Proof. By Bertini's theorem D and D' are smooth and Y is nodal. For a general pair (D, D') for each line $L \subset Q$ the scheme $L \cap D$ has connected components of degree 1 or 2, with at most one being of degree 2. Thus it is sufficient to test all smooth $C \in |\mathcal{O}_Q(1,1)|$. Since (D, D') is general, each C contains at most 2 points of $D \cap D'$. Thus every smooth $C \in |\mathcal{O}_Q(1,1)|$ containing some $p \in D \cap D'$ satisfies the property that the connected component of $C \cap Y$ with p as its reduction has degree ≤ 3. Thus we only need to consider the schemes $C \cap (Y \setminus D \cap D')$ with C smooth. By Lemma 3.2 it is sufficient to exclude the smooth C such that $C \cap D$ has a connected component Z_1 of degree at least 4 and $C \cap (D' \setminus D \cap D')$ has a connected component Z_2 of degree at least 4. We may assume $a + b \geq 4$ and $a' + b' \geq 4$. As in the proof of Lemma 3.1 we find only finitely many smooth $C_i \in |\mathcal{O}_Q(1,1)|$, say C_i, $1 \leq i \leq t$, such that $C \cap D$ has a connected component of degree at least 4. For a general D', the curve D' is transversal to all C_i, $1 \leq i \leq t$. □

Lemma 3.4. *Fix positive integers $e \geq 2$, a_i, b_i, $1 \leq i \leq e$, such that for each $i \in \{1, \dots, e\}$ exactly one among a_i and b_i is 1. Let $D = D_1 \cup \cdots \cup D_s \subset Q$ be a general union with each D_i general in $|\mathcal{O}_Q(a_i, b_i)|$. Then D is nodal, no two of the nodes of D are contained in the same line of Q, each line of Q passing through a singular point of D is transversal to each D_i, $T(D) = \emptyset$ and there is no line $J \subset Q$ such that $J \cap D$ has a connected component of degree at least 3.*

Proof. D is nodal by Bertini's theorem. Lemma 3.1 gives $T(D) \subseteq \mathrm{Sing}(D)$. Fix $p \in \mathrm{Sing}(D)$. Call D_i and D_j the irreducible components of D containing p. Since D is general, neither D_i nor D_j have a osculating plane at p with weight ≥ 2 and the tangent plane to one component, does not contain the tangent line to the other component. Thus $p \notin T(D)$.

For a general $(D_1, \dots, D_e)$ no two of the nodes of D are on the same line of Q, because $a_i b_i \neq 0$ for all i. We also see by induction on e that each line of Q passing through a singular point of D is transversal to each D_i.

Fix any line $J \subset \mathbb{P}^3$. Since $D \subset Q$, we have $\deg(D \cap J) \leq 2$ if $J \nsubseteq Q$. Now assume $L \in |\mathcal{O}_Q(1,0)|$ (resp. $R \in |\mathcal{O}_Q(1,0)|$). We have $\deg(L \cap D) = b$ (resp. $\deg(R \cap D) = a$). By Lemma 3.1 each connected component of the zero-dimensional schemes $L \cap D$ and $R \cap D$ has degree ≤ 2. □

Lemma 3.5. *Fix positive integers a, b and $q \in \mathbb{P}^3 \setminus Q$. Then $q \notin \tau(Y)$ for a general $Y \in |\mathcal{O}_Q(a,b)|$.*

Proof. The polar surface of Q with respect to Q is a plane, H, intersecting transversally Q and $q \in T_pQ$ if and only if $p \in H \cap Q$. Take Y intersecting transversally $H \cap Q$ and not containing the degree 2 subscheme of $\langle \{p, q\}\rangle$ with p as its reduction at all $p \in H \cap Q \cap Y$. $\square$

Lemma 3.6. *Fix positive integers $s \ge 4$, a_i, b_i, $1 \le i \le s$. Take a general $(D_1, \dots, D_s) \in \prod_{i=1}^s |\mathcal{O}_Q(a_i, b_i)|$. Then for every $q \in \mathbb{P}^3 \setminus Q$ there is $S_q \subset \{1, \dots, s\}$ such that $\sharp(S_q) \le 3$ and $q \notin \tau(D_i)$ for all $i \in \{1, \dots, s\} \setminus S_q$.*

Proof. By Lemma 3.5 and the generality of $(D_1, \dots, D_s)$ we have $\dim((\mathbb{P}^3 \setminus Q) \cap \tau(D_1)) = 2$, $\dim((\mathbb{P}^3 \setminus Q) \cap \tau(D_1) \cap \tau(D_2)) \le 1$, $\dim((\mathbb{P}^3 \setminus Q) \cap \tau(D_1) \cap \tau(D_2) \cap \tau(D_3)) \le 0$ and $(\mathbb{P}^3 \setminus Q) \cap \tau(D_1) \cap \tau(D_2) \cap \tau(D_3) \cap \tau(D_4) = \emptyset$. Using all subsets of $\{1, \dots, s\}$ with cardinality 4 we get the lemma. $\square$

Lemma 3.7. *Fix positive integers a, b. Take a general $Y \in |\mathcal{O}_Q(a, b)|$. Then for every $q \in \mathbb{P}^3 \setminus Q$ there are at most 3 points $p \in Y$ such that $q \in T_pY$.*

Proof. With no loss of generality we may assume $b \ge a$. Y is smooth. If $a + b \le 3$, then Y is a rational normal curve in its linear span and the lemma is trivial in this case. Thus we may assume $a + b \ge 4$. The lemma is also easy to check using the linear projection ℓ_q and the genus formula for plane curves if $(a, b) \in \{(2, 2), (1, 3), (2, 3)\}$ (all these cases are discussed in [9]).

For any $q \in \mathbb{P}^3 \setminus Q$ the polar plane H_q of Q with respect to q has the following properties. The curve $C_q := H_q \cap Q$ is a smooth conic and $q \in TpQ$, $p \in Q$, if and only if $p \in C_q$. For any $p \in C_q$ let z_p denote the degree 2 connected zero-dimensional subscheme of the line $\langle\{p, q\}\rangle$ with p as its reduction. For any curve $E \subset Q$ such that $p \in E_{reg}$ we have $q \in T_pE$ if and only if $z_p \subset E$. Let $\mathcal{U}$ denote the set of quadruples (Z_1, Z_2, Z_3, Z_4) with each Z_i a connected degree 2 zero-dimensional subscheme of Q such that there is $q \in \mathbb{P}^3 \setminus Q$ and $(p_1, p_2, p_3, p_4) \in C_q^4$ such that $p_i \ne p_j$ for all $i \ne j$ and $Z_i = z_{p_i}$. The lemma is equivalent to proving that a general Y contains no scheme $Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ with $(Z_1, Z_2, Z_3, Z_4) \in \mathcal{U}$. For each smooth $C \in |\mathcal{O}_Q(1, 1)|$ there is a unique $q \in \mathbb{P}^3 \setminus Q$ such that $C = C_q$. Each smooth $C \in |\mathcal{O}_Q(1, 1)|$ has ∞^4 quadruples of distinct points. Since $\dim |\mathcal{O}_Q(1, 1)| = 3$, we get $\dim \mathcal{U} = 7$. Thus to prove the lemma it is sufficient to prove that $\dim |\mathcal{I}_{Z_1 \cup Z_2 \cup Z_3 \cup Z_4}(a, b)| = \dim |\mathcal{O}_Q(a, b)| - 8$. Fix $(Z_1, Z_2, Z_3, Z_4) \in \mathcal{U}$, say $Z_i = z_{p_i}$ with p_1, p_2, p_3, p_4 distinct points of a smooth $C \in |\mathcal{O}_Q(1, 1)|$. Set $Z := Z_1 \cup Z_2 \cup Z_3 \cup Z_4$. Since $\deg(Z) = 8$, it is sufficient to prove that $h^1(\mathcal{I}_Z(a, b)) = 0$. We have $C \cap Z = \{p_1, p_2, p_3, p_4\}$ (scheme-theoretically), because each tangent line of C is contained in the plane $\langle C\rangle$ and if $C = C_q$, then $q \notin \langle C\rangle$. Hence $\mathrm{Res}_C(Z) = \{p_1, p_2, p_3, p_4\}$. We have $h^1(C, \mathcal{I}_{Z \cap C}(a, b)) = 0$, because $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(a, b)) = a + b$. We have $h^1(C, \mathcal{I}_{\mathrm{Res}_C(Z)}(a - 1, b - 1)) = 0$, because $\deg(\mathcal{O}_C(a - 1, b - 1)) = a + b - 2 \ge 3$. We have $h^1(\mathcal{O}_Q(a - 2, b - 2)) = 0$. Use twice the residual exact sequence of C, first with $\mathcal{I}_Z(a, b)$ as its middle term and then with $\mathcal{I}_{\mathrm{Res}_C(Z)}(a - 1, b - 1)$ as its middle term. $\square$

Lemma 3.8. *Fix positive integers a, b such that $(a, b) \neq (1, 1)$. Take a general $Y \in |\mathcal{O}_Q(a, b)|$. The set of all $q \in \mathbb{P}^3 \setminus Q$ such that there are 2 (resp. 3) points $p \in Y$ with $q \in T_pY$ has dimension ≤ 1 (resp. ≤ 0).*

Proof. Adapt the proof of Lemma 3.7 using $Z_1 \cup Z_2 \cup Z_3$ (resp. $Z_1 \cup Z_2$) instead of $Z_1 \cup Z_2 \cup Z_3 \cup Z_4$. □

Lemma 3.9. *Fix positive integers a_1, b_1, a_2, b_2. Take a general pair $(D_1, D_2) \in |\mathcal{O}_Q(a_1, b_1)| \times |\mathcal{O}_Q(a_2, b_2)|$. For each $q \in \mathbb{P}^3 \setminus Q$ the following properties are true:.*

(a) there is no $(p_1, p_2, p_3, p_4) \in D_1 \times D_1 \times D_2 \times D_2$ such that $p_1 \neq p_2$, $p_3 \neq p_4$ and $q \in T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_2 \cap T_{p_4}D_2$;

(b) there is no $(p_1, p_2, p_3, p_4) \in D_1 \times D_1 \times D_1 \times D_2$ such that $\sharp(\{p_1, p_2, p_3\}) = 3$ and $q \in T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_1 \cap T_{p_4}D_2$.

Proof. Part (b) follows from Lemmas 3.5 and 3.8.

Now we prove part (a). This is trivial if $(a_2, b_2) \in \{(2, 1), (1, 2)\}$, i.e. if D_2 is a rational normal curve. Thus we may assume $a_2 + b_2 \geq 4$. As in the proof of Lemma 3.7 let H_q be the polar hyperplane of Q with respect to q and $C_q := H_q \cap Q$. For any $p \in C_q$ let z_p denote the degree 2 connected zero-dimensional subscheme of the line $\langle\{p, q\}\rangle$ with p as its reduction. Let $\mathcal{U}$ denote the set of all quadruples Z_1, Z_2, Z_3, Z_4 such that there is a smooth $C \in |\mathcal{O}_Q(1, 1)|$ and 4 distinct points $p_i \in C$, $1 \leq i \leq 4$, such that $z_{p_i} = Z_i$ for all i. For a fixed D_1 Lemma 3.8 shows that we have at most ∞^1 pairs (p_1, p_2) which may be prolonged to be the reduction of some (Z_1, Z_2, Z_3, Z_4). For a fixed p_1, p_2 we have $h^0(Q, \mathcal{I}_{p_1,p_2}(1, 1)) = 2$ and hence there are only ∞^1 $C \in |\mathcal{O}_Q(1, 1)|$ containing $\{p_1, p_2\}$. For a fixed C we have ∞^2 pairs $(p_3, p_4) \in C \times C$. We fix the general D_1. To prove that a general D_2 satisfies part (a) of the lemma it is sufficient to prove that $h^1(\mathcal{I}_{Z_1\cup Z_2\cup Z_3\cup Z_4}(a_2, b_2)) \leq 2$. We prove this inequality in the following way. Recall that $C \cap (Z_\cup Z_2 \cup Z_3 \cup Z_4) = \{p_1, p_2, p_3, p_4\}$ (scheme-theoretically), because each tangent line of C is contained in the plane $\langle C\rangle$ and if $C = C_q$, then $q \notin \langle C\rangle$. Thus $\mathrm{Res}_C(Z_1 \cup Z_2 \cup Z_3 \cup Z_4) = \{p_1, p_2, p_3, p_4\}$. Since $a_2 + b_2 \geq 4$, we have $h^1(C, \mathcal{I}_{\{p_1,p_2,p_3,p_4\}}(a_2, b_2)) = 0$ and $h^1(C, \mathcal{I}_{\{p_1,p_2,p_3,p_4\}}(a_2 - 1, b_2 - 1)) \leq 1$. Use twice the residual exact sequence of C, first with $\mathcal{I}_{Z_1\cup Z_2\cup Z_3\cup Z_4}(a_2, b_2)$ as its middle term and then with $\mathcal{I}_{\{p_1,p_2,p_3,p_4\}}(a_2 - 1, b_2 - 1)$ as its middle term. □

Lemma 3.10. *Fix positive integers $a_1, b_1, a_2, b_2, a_3, b_3$ such that $(a_i, b_i) \neq (1, 1)$, $1 \leq i \leq 3$. Take a general $(D_1, D_2, D_3) \in |\mathcal{O}_Q(a_1, b_1)| \times |\mathcal{O}_Q(a_2, b_2)| \times |\mathcal{O}_Q(a_3, b_3)|$. Take any $q \in \mathbb{P}^3 \setminus Q$. There are no $(p_1, p_2, p_3, p_4) \in D_1 \times D_1 \times D_2 \times D_3$ such that $p_1 \neq p_2$ and $q \in T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_2 \cap T_{p_4}D_3$.*

Proof. The proof of part (a) of Lemma 3.9 shows that there are only finitely many triples $(p_1, p_2, p_3) \in D_1 \times D_1 \times D_2$ such that $p_1 \neq p_2$ and $T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_2$ is a point of $\mathbb{P}^3 \setminus Q$. Apply Lemma

3.5 to D_3. $\square$

Proof of Theorem 1.1: Any $Y \in |\mathcal{O}_Q(a,b)|$ has arithmetic genus γ.

Claim 1: There are integers $e \geq 2$, a_i, b_i, $1 \leq i \leq e$, such that for each $i \in \{1, \dots, e\}$ exactly one among a_i and b_i is 1, $a_1 + \cdots + a_e = a$ and $b_1 + \cdots + b_e = b$.

Proof of Claim 1: If $d \equiv 0 \pmod 6$ we take $e = d/3$, $(a_i, b_i) = (1,2)$ for odd i and $(a_i, b_i) = (2,1)$ for even i. If $d \equiv i \pmod 6$, $1 \leq i \leq 5$, we take $e = (d-i)/3$, $(a_1, b_1) = (1, 2+i)$, $(a_i, b_i) = (1,2)$ for odd $i \geq 3$ and $(a_i, b_i) = (2,1)$ for even i.

Take a nodal curve $D = D_1 \cup \cdots \cup D_e \subset Q$ satisfying the thesis of Lemma 3.4. Since each D_i is smooth and rational and $p_a(D) = \gamma$, we have $\sharp(\mathrm{Sing}(D)) = \gamma + e - 1$. Since $0 \leq g \leq \gamma$ and each D_i is irreducible, there is a set $A \subset \mathrm{Sing}(D)$ such that $\sharp(A) = \gamma - g$ and $D \setminus A$ is connected. We fix one such set A and call it the set of all *assigned nodes*. The set $\mathrm{Sing}(D)$ is called the set of all *unassigned nodes* (we are using the terminology of A. Tannenbaum ([10]) who extended to other rational surfaces the classical theory of nodal plane curves due to Severi). Since $D \setminus A$ is connected, [10, Lemma 2.2 and Theorem 2.13] gives the existence of a flat family $\{D_t\}_{t \in \Delta}$, Δ an integral affine curve, and $o \in \Delta$ such that $D_t \in |\mathcal{O}_Q(a,b)|$ for all $t \in \Delta$, $D_o = D$, each D_t, $t \in \Delta \setminus \{o\}$, is integral, nodal and with geometric genus g, and the nodes of D_t, $t \in \Delta \setminus \{o\}$, go to the set of assigned nodes. By Remark 3.1 we have $T(D_t) = \emptyset$ for a general $t \in \Delta$. Fix $c \in \Delta \setminus \{o\}$ such that $T(D_c) = \emptyset$ and set $X := D_c$. X is an integral and nodal curve with geometric genus g. To conclude the proof of the theorem it is sufficient to prove that $r_X(q) = 2$ for all $q \in \mathbb{P}^3 \setminus X$.

(a) Fix $q \in Q$. Let L be the element of $|\mathcal{O}_Q(1,0)|$ containing q. We have $\deg(L \cap X) = b$. By Lemma 3.1 each connected component of $L \cap X$ has degree ≤ 2. Thus $\sharp((L \cap X)_{\mathrm{red}}) \geq \lceil ba/2 \rceil$. Since $b \geq 3$, we get $r_X(q) = 2$.

(b) Fix $q \in \mathbb{P}^3 \setminus Q$. Assume $r_X(q) > 2$, i.e. assume $\ell_{q|X}$ is injective. Since $\ell_q(X)$ has degree $d = a + b$, it has arithmetic genus $(a+b-1)(a+b-2)/2$, while X has arithmetic genus $\gamma = ab - a - b + 1$. We silently use a small modification of Remark 3.1 to get $F(X) = \emptyset$ (for a general partial smoothing X) knowing that $F(D) = \emptyset$. We use Lemmas 3.1, 3.2, 3.3 to get $T(D) = \emptyset$ and hence (Remark 3.1) we get $T(X) = \emptyset$.

(b1) Assume for the moment that q is not in the tangent space of one of the nodes of X. Call o_i, $1 \leq i \leq s$, the points of X_{reg} such that $q \in T_{o_i}X$.

The following observation summarize lemmas 3.1, 3.2, 3.3, 3.4, 3.6, 3.7, 3.9, 3.10 first on D and then on X.

Observation 1: X has no flex, its osculating planes have weight 1 and each point of $\mathbb{P}^3 \setminus Q$ is contained in at most 3 tangent lines to smooth points of X.

A dimensional count similar to the one needed to prove Lemmas 3.2 and 3.7 gives the following observation.

Observation 2: At each $q \in \mathbb{P}^3 \setminus Q$ such that there are 3 different smooth points p_1, p_2, p_3 of $X_{\rm reg}$ with $q \in T_{p_i}X$, no $T_{p_i}(X)$ is a stall. At each point of X at which there are 2 different smooth points p_1, p_2 of $X_{\rm reg}$ with $q \in T_{p_i}X$ at most one among $T_{p_1}X$ and $T_{p_2}X$ is a stall.

By Observations 1 and 2 we have $p_a(\ell_q(X)) \le p_a(X) + 3$. Since $\ell_q(X)$ is a plane curve of degree $a+b$ and $p_a(X) = ab - a - b + 1$, we get $(a+b-1)(a+b-2)/2 \le ab - a - b + 4$, i.e. $a^2 + b^2 \le a + b + 6$, which is false if $a = 1$ and $b \ge 4$ or $a \ge 2$ and $b \ge 3$.

(b2) Assume $g < \gamma$ and that q is contained in at least one tangent plane at X at one of its points.

First assume that q is contained in the tangent cone at one of the nodes, o, of X. For a general D (and hence a general partial smoothing X) no line in the tangent cones of X at its singular points are stalls and tangent cones at different singular points are disjoints. At most another singular point o' of X has tangent plane containing q.

Now assume that q is not contained in any tangent cone at singular points. It is contained in at most 3 tangent spaces of X at its singular points and if at 3 it is not contained in any tangent line at a smooth point of X. We get a contradiction if $(a+b-1)(a+b-2)/2 \ge \gamma + 4$, i.e. if $a^2 + b^2 \ge a + b + 8$, which is true (for positive a, b) if and only if $a + b \ge 5$. $\square$

4 Curves in $\mathbb{P}^4$

Let $F_1 \subset \mathbb{P}^4$ be a smooth and non-degenerate surface such that $\deg(F_1) = 3$. All such surfaces are projectively equivalent. The smooth or nodal curves we use to prove Theorem 1.2 are contained in F_1. The surface F_1 is an embedding of the Hirzebruch surface with the same name ([7, §V.2]). We have $\mathrm{Pic}(F_1) \cong \mathbb{Z}^2$ and we take as free generators of it the class f, of a fiber of the ruling of F_1 and the section h of its ruling with negative self-intersection. We have $h^2 = -1$, $f^2 = 0$ and $h \cdot f = 1$. We have $\mathcal{O}_{F_1}(1) \cong \mathcal{O}_{F_1}(h+2f)$ and h and the elements of the ruling $|f|$ are the only lines contained in F_1. Each $\mathcal{O}_{F_1}(ah+bf)$, $b \ge a \ge 0$, is globally generated; it is ample (and very ample, too) if and only if $b > a > 0$. Fix $D \in |ah+bf|$, $b \ge a > 0$. Since $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h-3f)$, the adjunction formula gives $\omega_D \cong \mathcal{O}_D((a-2)h + (b-3)f)$. Thus $p_a(D) = 1 + ab - a(a+1)/2 - b$. For all $b \ge a-1$ we have $h^1(\mathcal{O}_{F_1}(ah+bf)) = 0$ and $h^0(\mathcal{O}_{F_1}(ah+bf)) = \sum_{i=0}^{a}(b+1-i) = (2b+2-a)(a+1)/2$.

Remark 4.1. *Take any curve $D \subset F_1$ and any line $L \subset \mathbb{P}^4$ such that* $\deg(D \cap L) \ge 3$. *Since F_1 is scheme-theoretically cut out by quadrics and $D \subset F_1$, Bezout theorem gives $L \subset F_1$.*

Lemma 4.1. *Fix an integer $q \in \mathbb{P}^4 \setminus F_1$. Then there is $C \in |h+f|$ such that $q \in \langle C \rangle$.*

Proof. Since 3 is a prime integer and $q \notin F_1$, $\ell_q(F_1)$ is an irreducible degree 3 ruled surface. This surface has a double line L meeting all lines of the ruling of $\ell_q(F_1)$ ([4, Theorem 9.2.1]). Thus there is a plane conic $C \subset F_1$ (a priori even a double line) mapped by ℓ_q onto L. All conics $C \subset F_1$ are elements of $|h+f|$. $\square$

Up to projective transformations there are exactly two degree 3 surfaces $\ell_q(F_1)$, $q \in \mathbb{P}^4 \setminus F_1$, distinguished by the fact that the unique conic $C \in |h+f|$ given by Lemma 4.1 is smooth or not ([4, Theorem 9.2.1]).

Proposition 4.1. *Let $X \subset F_1 \subset \mathbb{P}^4$ be a reduced and non-degenerate curve whose irreducible component have degrees at least 3. Assume the following conditions:*

(1) $\sharp((L\cap X)_{\mathrm{red}}) \ge 2$ *for all* $L \in |f|$*;*

(2) $\sharp((h\cap X)_{\mathrm{red}}) \ge 2$*;*

(3) $\sharp((C\cap X)_{\mathrm{red}}) \ge 3$ *for all smooth* $C \in |h+f|$.

Then $r_X(q) \le 3$ for all $q \in \mathbb{P}^4$.

Proof. The assumptions on the irreducible components of X is equivalent to assuming that $X \cap C$ contains no curve for all $C \in |h+f|$. First assume $q \in F_1$. Let L be the only element of $|f|$ containing q. Since L is a line and $\sharp((L\cap X)_{\mathrm{red}}) \ge 2$, we have $r_X(q) \le 2$.

Now assume $q \notin F_1$. Take $C \in |h+f|$ such that $q \in \langle C\rangle$. Note that $\langle C\rangle$ is a plane. If C is smooth (and hence it is a smooth conic), we have $r_X(q) \le 3$, because $\sharp((C\cap X)_{\mathrm{red}}) \ge 3$ and hence $(C\cap X)_{\mathrm{red}}$ spans $\langle C\rangle$. Now assume that C is singular, i.e. $C = h+L$ for some $L \in |f|$. Both h and L are lines and $h\cap L$ is a single point. By assumption there are $p_1, p_2 \in (L\cap X)_{\mathrm{red}}$ with $p_1 \ne p_2$ and hence $L = \langle\{p_1,p_2\}\rangle$. Since $\sharp((h\cap X)_{\mathrm{red}}) \ge 2$, there is $p_3 \in (h\cap X)_{\mathrm{red}}$ such that $p_3 \ne h\cap L$. Since $h = \langle\{p_3, h\cap L\}\rangle$, we have $\langle C\rangle = \langle\{p_1,p_2,p_3\}\rangle$ and hence $r_X(q) \le 3$. $\square$

Lemma 4.2. *Let Δ a quasi-projective variety and $\mathcal{X} \subset F_1 \times \Delta$ a closed algebraic set such that the restriction $u : \mathcal{X} \longrightarrow \Delta$ to $\mathcal{X}$ of the projection $F_1 \times \Delta \longrightarrow \Delta$ is proper and flat. For each $t \in \Delta$ set $X_t := u^{-1}(t)$. Assume that all fibers of u are reduced curves with no irreducible component of degree ≤ 2. Fix $o \in \Delta$ and assume $\sharp((C\cap X_o)_{\mathrm{red}}) \ge 3$ for all $C \in |h+f|$. Then for a general $t \in \Delta$ we have $\sharp((C\cap X_t)_{\mathrm{red}}) \ge 3$ for all $C \in |h+f|$.*

Proof. Assume that the lemma is false. Taking a neighborhood Ω of o in Δ and then a branch covering of Ω we may assume that for each $t \in \Omega\setminus\{o\}$ there is $C_t \in |h+f|$ with $\sharp((C_t\cap X_t)_{\mathrm{red}}) \le 2$. Since $|h+f|$ is a projective set, the family $\{C_t\}_{t\in\Omega\setminus\{o\}}$ has at least one limit point, C', and $\sharp((C'\cap X_o)_{\mathrm{red}}) \le 2$. $\square$

Lemma 4.3. *Fix integers a, b such that either $a = 1$ and $b \geq 5$ or $a \geq 2$ and $b \geq \max\{4, a\}$. Let X be a general element of $|ah + bf|$. Then $\sharp((C \cap X)_{\mathrm{red}}) \geq 3$ for all smooth $C \in |h + f|$.*

Proof. For each $C \in |h + f|$ we have $\deg(X \cap C) = b$. For $e \in \{1, 2\}$ let $\mathcal{U}(e)$ denote the set of all degree zero-dimensional schemes $Z \subset F_1$ such that $\deg(Z) = b$, Z has exactly e connected components and there is a smooth $C \in |h+f|$ containing Z. Since each smooth $C \in |h+f|$ has ∞^e elements of $\mathcal{U}(e)$, we have $\dim \mathcal{U}(e) = 2 + e$. Thus (since $e \leq 2$) to prove the lemma it is sufficient to prove that $\dim |\mathcal{I}_Z(ah + bf)| = \dim |ah + bf| - 5$ for all $Z \in \mathcal{U}(e)$, $i = 1, 2$. Fix $Z \in \mathcal{U}(e)$ and take a smooth $C \in |h + f|$ containing it. Since $\deg(Z) = b \geq 5$, it is sufficient to prove that $h^1(\mathcal{I}_Z(ah + bf)) = 0$. Since $h^1(\mathcal{O}_{F_1}((a - 1)h + (b - 1)f)) = 0$, the residual exact sequence of C shows that it is sufficient to prove that $h^1(C, \mathcal{I}_{Z,C}(ah + bf)) = 0$. This is true, because $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(ah + bf)) = b$. □

Lemma 4.4. *Fix $q \in F_1$. There is a smooth $C \in |h + f|$ such that $q \in C$ if and only if $q \in F_1 \setminus h$.*

Proof. Since $h \cdot (h + f) = 0$, no irreducible $C \in |h + f|$ (i.e. no smooth $C \in |h + f|$) meets h. Now assume $q \in |h + f|$. Since $\dim |\mathcal{I}_q(h + f)| = \dim |h + f| - 1 = 1$ and there is a unique singular element of $|h + f|$ containing q, there is a smooth $C \in |h + f|$ such that $q \in C$. □

Proposition 4.2. *Fix integer a, b such that $a \geq 1$ and $b \geq 2a + 3$.*

(1) There is a nodal $D \in |ah+bf|$ with exactly a smooth irreducible components, all of them rational and neither lines nor conics, such that $\sharp((D \cap C)_{\mathrm{red}}) \geq 3$ for all $C \in |h + f|$.

(2) If $a \geq 2$ we have $r_D(q) \leq 3$ for all $q \in \mathbb{P}^4$.

Proof. Set $b_i := 2$ for $2 \leq i \leq a$ and $b_1 := b - 2a + 2$. Take a general $(D_1, \dots, D_a) \in \prod_{i=1}^{a} |h + b_i f|$ and set $D := D_1 \cup \cdots \cup D_a$. By Bertini's theorem each D_i is smooth and connected and D is nodal. Set $S := \mathrm{Sing}(D)$. Each D_i is rational and $p_a(D) = 1 + ab - a(a + 1)/2 - b$. Thus $\sharp(S) = p_a(D) + a - 1 = ab - a(a - 1)/2 - b$. In the case $a = 1$ we have $D = D_1$ with D_1 a general element of $|h + bf|$. For a general $(D_1, \dots, D_a)$ the nodal curve D is transversal to h and hence $\sharp((h \cap D)_{\mathrm{red}}) = b - a \geq 4$. Hence part (1) is true for all singular $C \in |h + f|$.

Now we check part (1) for all smooth $C \in |h + f|$.

If $a = 1$ it is sufficient to quote Lemma 4.3.

Now assume $a \geq 2$. Since $\sharp((D_1 \cap C)_{\mathrm{red}}) \geq 3$ by Lemma 4.3, we get part (1) for all smooth $C \in |h + f|$.

Now we prove part (2). By Lemma 4.4 we have $r_D(q) \leq 3$ for all $q \in F_1 \setminus h$. Since $\sharp((h \cap D)_{\mathrm{red}}) = b - a \geq 2$, we have $r_D(q) \leq 2$ for all $q \in h$.

Take $q \in \mathbb{P}^4 \setminus F_1$. Take $C \in |h+f|$ such that $q \in \langle C \rangle$ (Lemma 4.1). If C is smooth we get $r_D(q) \leq 3$ by Proposition 4.1. Now assume C singular, say $C = h \cup L$ with $L \in |f|$. Since D contains $b-a$ points of D, it is sufficient to prove that L contains a point of $D \setminus D \cap h$. This is true, because $a \geq 2$ and D is transversal to h. □

Proof of Theorem 1.2: Take the curve D given by Proposition 4.2. Use Remark 2.1 to get X as in the proof of Theorem 1.1. Apply part (1) of Proposition 4.2 and Lemma 4.2. □

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Toric, $U(2)$, and LeBrun metrics

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ABSTRACT

The LeBrun ansatz was designed for scalar-flat Kähler metrics with a continuous symmetry; here we show it is generalizable to much broader classes of metrics with a symmetry. We state the conditions for a metric to be (locally) expressible in LeBrun ansatz form, the conditions under which its natural complex structure is integrable, and the conditions that produce a metric that is Kähler, scalar-flat, or extremal Kähler. Second, toric Kähler metrics (such as the generalized Taub-NUTs) and $U(2)$-invariant metrics (such as the Fubini-Study or Page metrics) are certainly expressible in the LeBrun ansatz. We give general formulas for such transitions. We close the paper with examples, and find expressions for two examples—the exceptional half-plane metric and the Page metric—in terms of the LeBrun ansatz.

RESUMEN

El ansatz de LeBrun fue diseñado para métricas Kähler escalares-planas con una simetría continua; acá mostramos que es generalizable a clases mucho más amplias de métricas con una simetría. Establecemos las condiciones para que una métrica sea (localmente) expresable con la forma de ansatz de LeBrun, las condiciones bajo las cuales su estructura compleja natural es integrable, y las condiciones que producen una métrica que es Kähler, escalar-plana, o Kähler extremal. En segundo lugar, métricas tóricas Kähler (tales como las Taub-NUT generalizadas) y métricas $U(2)$-invariantes (tales como la métrica de Fubini-Study o la de Page) son ciertamente expresables en el ansatz de LeBrun. Damos fórmulas generales para tales transiciones. Concluimos el artículo con ejemplos, y encontramos expresiones para dos ejemplos—la métrica excepcional del semiplano y la métrica de Page—en términos del ansatz de LeBrun.

Keywords and Phrases: Differential geometry, Kähler geometry, canonical metrics, ansatz.

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1 Introduction

LeBrun [19] created an ansatz for scalar-flat Kähler metrics with a continuous symmetry. This was an expansion of the Gibbons-Hawking ansatz for Ricci-flat Kähler metrics with a symmetry, itself a version of the Kaluza ansatz [18] [6]. In the original construction Kaluza showed that if a Lorentzian 5-metric is endowed with a spacelike continuous symmetry, the Einstein equations will partially linearize, with the linear part being the Maxwell equations. The Gibbons-Hawking construction utilized this idea except in Euclidean signature and a dimension lower, where the Maxwell equations reduce to just the Laplace equation on a potential, and the "gravity" equations (the Ricci-flat equations) fully linearize.

LeBrun's ansatz, which also works for 4-dimensional Riemannian metrics with a circle symmetry, partially linearizes the scalar-flat Kähler (SFK) equations. These SFK equations, normally exceedingly complicated and nonlinear, were shown to reduce to a pair of second order equations, one linear and the other quasilinear.

We show that LeBrun's ansatz is much more general than this original use, and is suitable for expressing interesting 4-metrics that are not scalar-flat, Kähler, or even have an integrable complex structure. We show the conditions under which a metric is expressible in terms of the LeBrun ansatz, and give the explicit transformations into the LeBrun ansatz from two toric Kähler ansätze, and from the $U(2)$-invariant ansatz. In the last section we use these translations to express several common metrics in the LeBrun ansatz. Finally we indicate how the LeBrun ansatz can be used, at least in principle, to create new metrics of special kinds, a subject we shall take up elsewhere.

2 The LeBrun ansatz

We lay out the basic definitions in the LeBrun ansatz and determine when the ansatz possesses an integrable complex structure and when it possesses a closed Kähler 2-form. We end with some expressions for curvature quantities of such metrics, and state when such a metric is extremal Kähler. The reference for this section is [19].

2.1 The ansatz

The LeBrun ansatz is an $\mathbb{S}^1$-fibration $\pi : M^4 \to N^3$ along with the metric

$$g \;=\; we^u\left(dx^2 \,+\, dy^2\right) \,+\, w\,dz^2 \,+\, w^{-1}\left(d\tau \,+\, \pi^*A\right)^2 \tag{2.1}$$

where (x,y,z) are local coordinates on N^3, $w = w(x,y,z)$ and $u = u(x,y,z)$ are functions, and A is a 1-form $A = A_x(x,y,z)dx + A_y(x,y,z)dy + A_z(x,y,z)dz$ on N^3.[1] The coordinate τ is defined after a choice of a transversal: after setting $\tau = 0$ on this transversal, τ is pushed forward via the $\mathbb{S}^1$-action. The field $\frac{d}{d\tau}$ is invariant under rechoosing the transversal so it is globally defined, and it is Killing.

The exterior derivative of A will be important. Because $d\pi^*A = \pi^*dA$, it is immaterial whether we compute on M^4 or N^3. Letting $B = dA$ we have

$$\begin{aligned} B &= B_x\, dy\wedge dz - B_y\, dx\wedge dz + B_z\, dx\wedge dy, \quad \text{where} \\ B_x &= A_{y,x} - A_{x,y}, \quad B_y = A_{x,z} - A_{z,x}, \quad B_z = A_{z,y} - A_{y,z}. \end{aligned} \tag{2.2}$$

In the spirit of Kaluza's work, we may interpret A as a vector potential over 3-space and $B = dA$ as the corresponding Maxwell field strength. It so closely resembles a magnetostatic field that we will sometimes call it the metric's magnetic field. In all curvature computations A never appears; only its field B appears.

A g-compatible almost-complex structure on (M^4, g) is

$$J(dx) = -dy, \quad J(dz) = -w^{-1}(d\tau + \pi^*A), \tag{2.3}$$

which dualizes to

$$J(\nabla x) = \nabla y, \quad J(\nabla z) = \frac{\partial}{\partial \tau} \tag{2.4}$$

where the duality convention is $J(\eta) \triangleq \eta \circ J$ for $\eta \in \bigwedge^1$. The corresponding antisymmetric form is

$$\omega = g(J\cdot, \cdot) = we^u dx\wedge dy + dz \wedge (d\tau + \pi^*A)\,. \tag{2.5}$$

2.2 The complex and symplectic structures

As usual, the almost complex structure splits $\bigwedge^1_{\mathbb{C}} = \bigwedge^1(M^4)\otimes\mathbb{C}$ into holomorphic and antiholomorphic bundles, where $\bigwedge^1_{\mathbb{C}} = \bigwedge^{1,0}\oplus\bigwedge^{0,1}$ are the respective $\pm\sqrt{-1}$ eigenspaces of J. In bases,

$$\begin{aligned} \textstyle\bigwedge^{1,0} &= span_{\mathbb{C}}\left\{ dx + \sqrt{-1}dy,\ dz + \sqrt{-1}w^{-1}(d\tau + \pi^*A) \right\}, \\ \textstyle\bigwedge^{0,1} &= span_{\mathbb{C}}\left\{ dx - \sqrt{-1}dy,\ dz - \sqrt{-1}w^{-1}(d\tau + \pi^*A) \right\}. \end{aligned} \tag{2.6}$$

Of the many ways to check the integrability of an almost-complex structure, the most convenient will be verifying that $d : \bigwedge^{0,1} \to \bigwedge^1_{\mathbb{C}}\wedge\bigwedge^{0,1}$.

Lemma 2.1. *The complex structure (2.3) is integrable if and only if*

$$w_x = B_x \quad \text{and} \quad w_y = B_y. \tag{2.7}$$

[1] LeBrun denotes $\omega = d\tau + \pi^*A$, and interprets this as a connection. Following a different but very standard convention, we shall prefer using the symbol ω for the 2-form $\omega = g(J\cdot,\cdot)$.

Proof. This comes from out of the proof of Proposition 1 of [19]. We compute on bases. Certainly $d(dx - \sqrt{-1}dy) = 0$. Then

$$\begin{aligned} & d\left(dz - \sqrt{-1}w^{-1}(d\tau + \pi^* A)\right) \\ & \quad = \; w^{-1}\Big(dw \wedge \left(dz - \sqrt{-1}w^{-1}(d\tau + \pi^* A)\right) \;-\; dw \wedge dz \;-\; \sqrt{-1}B\Big). \end{aligned} \tag{2.8}$$

From (2.6), the first term is in $\bigwedge^1_{\mathbb{C}} \wedge \bigwedge^{0,1}$. The second and third terms become

$$\begin{aligned} & - dw \wedge dz - \sqrt{-1}B \\ & \quad = \; -(w_x - \sqrt{-1}B_y)dx \wedge dz - (w_y + \sqrt{-1}B_x)dy \wedge dz - \sqrt{-1}B_z dx \wedge dy \\ & \quad = \; \frac{1}{2}\Big((w_x - B_x) - \sqrt{-1}(w_y - B_y)\Big)dz \wedge (dx + \sqrt{-1}dy) \\ & \quad\quad + \frac{1}{2}\Big((w_x + B_x)dz - \sqrt{-1}(B_y + w_y)dz \\ & \quad\quad\quad\quad - \sqrt{-1}B_z(dx + \sqrt{-1}dy)\Big) \wedge (dx - \sqrt{-1}dy). \end{aligned} \tag{2.9}$$

Because $dx - \sqrt{-1}dy \in \bigwedge^{0,1}$ the second term on the right is in $\bigwedge^1_{\mathbb{C}} \wedge \bigwedge^{0,1}$. But the first term is in $\bigwedge^1_{\mathbb{C}} \wedge \bigwedge^{1,0}$. We conclude J is integrable if and only if this term is zero, which is the same as $(w_x - B_x) - \sqrt{-1}(w_y - B_y) = 0$. ☐

Lemma 2.2. *We have* $d\omega = (-B_z + (we^u)_z)\, dz \wedge dx \wedge dy$. *In particular, the antisymmetric form* ω *of (2.5) is closed if and only if* $B_z = (we^u)_z$.

Proof. Using $\omega = dz \wedge (d\tau + \pi^* A) + we^u dx \wedge dy$ and $d\pi^* A = \pi^* dA = \pi^* B$,

$$\begin{aligned} d\omega \; &= \; -dz \wedge d\pi^* A \;+\; (we^u)_z dz \wedge dx \wedge dy \\ &= \; (-B_z \;+\; (we^u)_z)\, dz \wedge dx \wedge dy, \end{aligned} \tag{2.10}$$

from which the assertion follows. ☐

Theorem 2.1. *The triple* (g, J, ω) *always has* $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. *It is*

i) Hermitian if and only if $B_x = w_x$ *and* $B_y = w_y$,

ii) symplectic if and only if $B_z = (we^u)_z$, *and*

iii) Kähler if and only if $B_x = w_x$, $B_y = w_y$, *and* $B_z = (we^u)_z$.

Condition (iii) implies

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0. \tag{2.11}$$

Proof. After Lemmas 2.2 and 2.1, we must only verify equation (2.11). But with $B = dA$, after assuming the relations in (*iii*) then equation (2.11) is just $dB = 0$. □

Remark. The metric is *almost Kähler* if (*ii*) holds but (*i*) does not.

Remark. The original approach of LeBrun [19] was essentially the reverse of this. LeBrun solves (2.11) for w first, and then finds a 1-form A (which will have Dirac string singularities) whose field B satisfies (*iii*). This contrasts with our method which starts with a metric of the form (2.1), finds conditions on A and w that give it special traits, and from such traits *derives* equation (2.11).

We have the following characterization of the LeBrun ansatz.

Theorem 2.2. *Let g be a metric on M^4. Then g can be expressed locally via the LeBrun ansatz if and only if the following three conditions hold:*

i) M^4 has a vector field v and an almost-complex structure J compatible with g so that, letting $\omega = g(J\cdot, \cdot)$ be the associated antisymmetric form, then ω, g, and J are all v-invariant,

ii) Given any simply connected domain $\Omega \subset M^4$, there is a function $z : \Omega \to \mathbb{R}$ with $i_v\omega = dz$, and

iii) The action of ∇z on J, when restricted to the rank-2 distribution $P \subset \bigwedge^1 M^4$ that is null on span$\{v, Jv\}$, is zero.

Remark. Regarding condition (*iii*), P is specifically the distribution $P = \{\eta \in \bigwedge^1_{M^4}$ such that $\eta(v) = 0$ and $\eta(Jv) = 0\}$.

Remark. Condition (*iii*) is certainly the most technical; it exists so that the first two terms in the ansatz can be written in the form $f(x, y, z)(dx^2 + dy^2)$, instead of $f_1dx^2 + f_2(dxdy + dydx) + f_3dy^2$. Condition (*iii*) could also be written $\mathcal{L}_{\nabla z}(J\big|_P) = 0$ where $\mathcal{L}$ is the Lie derivative.

Proof. Supposing g can be expressed via the LeBrun ansatz, we simply set $v = \frac{\partial}{\partial t}$ and let J be as in (2.3) or equivalently (2.4). The work above shows J and ω are v-invariant and $i_v\omega = dz$. We compute $\mathcal{L}_{\nabla z}J\big|_P$ by

$$(\mathcal{L}_{\nabla z}J)(dx) = \mathcal{L}_{\nabla z}(Jdx) - J\mathcal{L}_{\nabla z}dx = \mathcal{L}_{\nabla z}(dy) - J\mathcal{L}_{\nabla z}dx. \tag{2.12}$$

The Cartan formula gives $\mathcal{L}_{\nabla z}dx = di_{\nabla z}dx = d\,\langle dz, dx\rangle$. But this inner product is zero, as is easily verified after computing the inverse matrix g^{ij}. Similarly $\mathcal{L}_{\nabla z}dy = 0$, so we have shown $\mathcal{L}_{\nabla z}J(dx) = 0$. The same argument works for $\mathcal{L}_{\nabla z}J(dy)$, so we have shown that $\mathcal{L}_{\nabla z}(J\big|_P) = 0$.

For the converse we assume g, J, ω are v-invariant, and that $i_v\omega = dz$ for some function z. This allows us to perform a version of the Kähler reduction. Because z is itself v-invariant (due to the fact that $\mathcal{L}_vz = i_vi_v\omega = 0$), the function z passes to the quotient manifold $N^3 = M^4/v$

where the quotient is by the action of the Killing field v—this works if the orbits of v are closed; if not then a second Killing field must exist, and we can take an appropriate linear combination to find a Killing field with closed orbits. Pick a level-set $\Sigma^2_z = \{z = const\}$ on which to place isothermal coordinates (x, y), and then extend (x, y) along trajectories of ∇z so the functions x, y are now defined on some region of N^3. We show that (x, y) remains isothermal on all other nearby level-sets of z; this is a consequence of $J|_P$ being invariant under trajectories of ∇z. To see this, note that $J|_P$ restricts to the Hodge-star $*_2$ on any level-set of z, and x, y are isothermal if and only if $d *_2 dx = d *_2 dy = 0$ and $dx \wedge *dy = 0$. By construction, $d *_2 dx = d *_2 dy = 0$ and $dx \wedge *dy = 0$ holds on *one* level-set of z; to see it is true on all nearby level-sets we compute

$$\mathcal{L}_{\nabla z} d *_2 dx \;=\; d\mathcal{L}_{\nabla z} J|_P dx \;=\; dJ|_P \mathcal{L}_{\nabla z} dx \;=\; dJ|_P d\mathcal{L}_{\nabla z} x \;=\; 0. \tag{2.13}$$

where we used the facts that d always commutes with $\mathcal{L}_{\nabla z}$, that by hypothesis $\mathcal{L}_{\nabla z} J|_P = 0$, and that by construction $\mathcal{L}_{\nabla z} x = 0$. Therefore $d *_2 dx$ remains zero on all level-sets. Similarly we compute

$$\begin{aligned} \mathcal{L}_{\nabla z}\left(dx \wedge *_2 dy\right) \;&=\; \left(\mathcal{L}_{\nabla z} dx\right) \wedge *_2 dy + dx \wedge \left(\mathcal{L}_{\nabla z} *_2 dy\right) \\ &=\; dx \wedge *_2 \left(\mathcal{L}_{\nabla z} dy\right) \;=\; 0 \end{aligned} \tag{2.14}$$

where again we used $\mathcal{L}_{\nabla z} dx = \mathcal{L}_{\nabla z} dy = 0$ and $\mathcal{L}_{\nabla z} *_2 = \mathcal{L}_{\nabla z} J|_P = 0$.

Now, because the functions x, y remain an isothermal system on any level-set of z, we may express the metric g_3 on the quotient manifold N^3 in the form $g_3 = f_1(x, y, z)dz^2 + f_2(x, y, z)\left(dx^2 + dy^2\right)$. We define the functions w, e^u by

$$\begin{aligned} w \;&\triangleq\; |dz|^{-2}_{g_3} \;=\; f_1 \\ we^u \;&\triangleq\; |dx|^{-2}_{g_3} \;=\; |dy|^{-2}_{g_3} = f_2. \end{aligned} \tag{2.15}$$

The functions x and y pull back from N^3 to M^4, where we now have three coordinate functions x, y, and z. For the fourth coordinate τ, after choosing a transversal to v, we may set $\tau = 0$ along this transversal, and push τ along trajectories of v—incidentally, this establishes $\frac{\partial}{\partial \tau} = v$ and $J\nabla z = \frac{\partial}{\partial \tau}$. We now have coordinates (x, y, z, τ) on M^4.

From (2.15) we have $w^{-1} = |dz|^2 = |\nabla z|^2 = |J\nabla z|^2 = |\partial/\partial\tau|^2$. We define functions C, A_x, A_y, and A_z in terms of the complex structure J by

$$-C\left(d\tau + A_x dx + A_y dy + A_z dz\right) \;=\; Jdz. \tag{2.16}$$

We can compute the value of C. Transvecting both sides of (2.16) with $\frac{\partial}{\partial \tau}$ gives

$$-C \;=\; Jdz\left(\tfrac{\partial}{\partial \tau}\right) \;=\; \left\langle \nabla z,\, J\tfrac{\partial}{\partial \tau}\right\rangle \;=\; -|\nabla z|^2 \;=\; -|dz|^2 = -w^{-1}. \tag{2.17}$$

Therefore $C = w^{-1}$. Finally because the distribution $\{\nabla x, \nabla y\}$ is perpendicular to the distribution $\{\nabla z,\, \partial/\partial\tau\}$, we arrive at the expression

$$g \;=\; we^u\left(dx^2+dy^2\right) \;+\; wdz^2 \;+\; w^{-1}\left(d\tau+A_xdz+A_ydy+A_zdz\right)^2. \tag{2.18}$$

□

2.3 Curvature quantities

Proposition 2.1. *Assume the metric (2.1) is Kähler, meaning (iii) of Theorem 2.1 holds. Then the Ricci curvature of g is*

$$\mathrm{Ric} \;=\; -\frac{1}{2}\Big(\mathrm{Hess}\;u\;(\cdot,\,\cdot)+\mathrm{Hess}\;u\;(J\cdot,\,J\cdot)\Big) \tag{2.19}$$

Proof. The proof of Proposition 1 of [19] gives Ricci form and Ricci curvature

$$\begin{aligned}\rho \;&=\; -\sqrt{-1}\partial\bar{\partial}u, \quad \text{and}\\ \mathrm{Ric} \;&=\; \rho(\cdot,\,J\cdot) \;=\; -\frac{1}{2}\big(\mathrm{Hess}\,u\;(\cdot,\,\cdot)+\mathrm{Hess}\,u\;(J\cdot,\,J\cdot)\big).\end{aligned} \tag{2.20}$$

□

Proposition 2.2. *Assume the metric (2.1) is Kähler, meaning (iii) of Theorem 2.1 holds. Then the scalar curvature s of g is*

$$s \;=\; -\frac{1}{we^u}\left(u_{xx}+u_{yy}+(e^u)_{zz}\right). \tag{2.21}$$

Proof. This is computed in the proof of Proposition 1 of [19]. □

Proposition 2.3 (The extremal condition)**.** *Assume the metric (2.1) is Kähler. Then it is an extremal Kähler metric if constants $m, b\in\mathbb{R}$ exist so*

$$-\frac{1}{we^u}\left(u_{xx}+u_{yy}+(e^u)_{zz}\right) \;=\; mz \;+\; b. \tag{2.22}$$

Proof. If (2.22) holds then $s=mz+b$ and so $\nabla s=m\nabla z$ and $J\nabla s=m\frac{\partial}{\partial\tau}$; thus $J\nabla s$ is a Killing field. The proposition is established after recalling that a Kähler metric is extremal if and only if $J\nabla s$ is Killing [7] [8]. □

Remark. Whether g is Kähler or not, its scalar curvature is

$$\begin{aligned} s \;=\; -\frac{1}{we^u}\Bigg(&\Big(u_{xx}+u_{yy}+(e^u)_{zz}\Big)+\frac{1}{w}\Big(w_{xx}+w_{yy}+(we^u)_{zz}\Big)\\ &+\frac{1}{2w^2}(B_x^2-(w_x)^2)+\frac{1}{2w^2}(B_y^2-(w_y)^2)+\frac{e^{-u}}{2w^2}(B_z^2-\big((we^u)_z\big)^2\Bigg).\end{aligned} \tag{2.23}$$

3 Expressing Toric Kähler metrics using the LeBrun ansatz

The LeBrun ansatz operates on 4-manifolds with one symmetry. On Kähler 4-manifolds with *two* holomorphic symmetries, there are more specialized ansätze. Letting $\mathcal{X}^1$, $\mathcal{X}^2$ be commuting holomorphic Killing fields (recall that "holomorphic" means $\mathcal{L}_{\mathcal{X}^i}J = 0$, just as Killing means $\mathcal{L}_{\mathcal{X}^i}g = 0$), then $(M^4, g, J, \mathcal{X}^1, \mathcal{X}^2)$ can be considered a *toric Kähler* 4-manifold. This situation has been studied in [17] [1] [13] [14] [2] [9] and many other works. Certainly a toric Kähler metric can be translated into the LeBrun ansatz once a distinguished Killing field is chosen. We do this here.

3.1 The two toric ansätze

There are two standard presentations for toric Kähler 4-manifolds. These were originally explored by Guillemin [17], who also discovered that they are equivalent via a Legendre transform. The LeBrun ansatz is a mixture of the two.

The first of the two presentations is the *symplectic ansatz*. If $\{\mathcal{X}^1, \mathcal{X}^2\}$ are independent commuting holomorphic Killing fields, we can use the Arnold-Liouville construction [3] to produce the so-called action-angle coordinates on M^4. To execute this construction, one defines *action variables* (up to a constant) by $\nabla\varphi^i = -J\mathcal{X}^i$ or equivalently by $d\varphi^i = i_{\mathcal{X}^i}\omega$, and defines *angle variables*, denoted θ_1, θ_2, by choosing a transversal and then pushing forward the action of the fields $\mathcal{X}^1$, $\mathcal{X}^2$. In these coordinates, the ansatz demands the metric be expressed

$$g \;=\; U_{ij}d\varphi^i \otimes d\varphi^j \;+\; U^{ij}d\theta_i \otimes d\theta_j \tag{3.1}$$

where $U = U(\varphi^1, \varphi^2)$ is a convex function of the action variables. The matrix (U_{ij}) is defined by $U_{ij} \triangleq \frac{\partial^2 U}{\partial\varphi^i\varphi^j}$, and we define $(U^{ij}) \triangleq (U_{ij})^{-1}$.

The map $M^4 \to \mathbb{R}^2$ given by $p \mapsto (\varphi^1(p), \varphi^2(p))$ sends M^4 to a region $\Sigma^2 \subset \mathbb{R}^2$; this is sometimes called the *Arnold-Liouville reduction* or, by abuse of terminology, the *moment map*. If M^4 is compact then its image Σ^2 is a compact polygon in $\mathbb{R}^2$. This polygon encodes the topology of M^4, via the Delzant gluing rules [11]. If M^4 is non-compact, then Σ^2 need not be a polygon nor even be topologically closed.

The second ansatz, the *holomorphic ansatz*, also begins with the fields $\{\mathcal{X}^1, \mathcal{X}^2\}$. Again we may produce corresponding coordinates θ_1, θ_2 after choosing a transversal. Because $\mathcal{X}^1$, $\mathcal{X}^2$ are not only symplectomorphic but holomorphic, the variables θ^i are actually pluriharmonic, meaning $d(Jd\theta_i) = 0$. The Poincaré lemma then guarantees functions ξ_1, ξ_2 exist (at least locally) so that $d\xi^i = Jd\theta_i$, and we have two holomorphic functions $f_i = \xi_i + \sqrt{-1}\theta_i$ which constitute a holomorphic chart $(f_1, f_2) : \Omega \to \mathbb{C}^2$ on some subdomain $\Omega \subseteq M^4$. The Kähler form on this chart, as usual, can be expressed $\omega = \sqrt{-1}\partial\bar{\partial}V$ for some pseudoconvex function V. Because V is θ_1-θ_2 invariant, it is convex instead of just pseudoconvex. The metric is then

$$g \;=\; V^{ij} d\xi_i \otimes d\xi_i \;+\; V^{ij} d\theta_i \otimes d\theta_j \tag{3.2}$$

where (V^{ij}) is the matrix with components $V^{ij} \triangleq \frac{\partial^2 V}{\partial \xi_i \partial \xi_j}$.

We might consider the map $p \mapsto (\xi_1(p),\, \xi_2(p))$ for $p \in M^4$, just as we considered the moment map $p \mapsto (\varphi^1(p), \varphi^2(p))$. But it is much less interesting than the moment map. If M^4 is compact then its image is all of $\mathbb{R}^2$. In particular there is no way to read off the topology of M^4 from its image.

A duality relationship exists between the symplectic system $(\varphi^1, \theta_1, \varphi^2, \theta_2)$ with its symplectic potential U and the holomorphic system $(\xi_1, \theta_1, \xi_2, \theta_2)$ with its Kähler potential V. As shown in [17], they are Legendre transforms of each other:

$$\begin{aligned} \xi_i \;&=\; \frac{\partial U}{\partial \varphi^i}, \quad \varphi^i \;=\; \frac{\partial V}{\partial \xi_i}, \quad \text{and} \\ U(\varphi^i) \;+\; V(\xi_i) \;&=\; \sum_i \varphi^i \xi_i. \end{aligned} \tag{3.3}$$

3.2 Translation to the LeBrun Ansatz

It is now possible to relate these two systems to the LeBrun ansatz, which is a mixed symplectic-holomorphic system. We define the LeBrun variable τ to be the angle variable θ_1 corresponding to $\mathcal{X}^1$, and y the angle variable θ_2 corresponding to $\mathcal{X}^2$. Let z be the *symplectic* variable corresponding to the angle τ, meaning $z = \varphi^1$, and x the *holomorphic* variable corresponding the angle variable y, meaning $x = \xi_2$. Then we create the LeBrun functions w and u, and determine the 1-form A. We record the change of frame from the symplectic frame $\left\{\frac{\partial}{\partial \varphi^1}, \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \varphi^2}, \frac{\partial}{\partial \theta_2}\right\}$ to the LeBrun frame $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. One easily computes

$$\begin{aligned} \tfrac{\partial}{\partial \varphi^1} \;&=\; \tfrac{\partial}{\partial z} + U_{21}\tfrac{\partial}{\partial x} & \qquad d\varphi^1 \;&=\; dz \\ \tfrac{\partial}{\partial \theta_1} \;&=\; \tfrac{\partial}{\partial \tau} & d\theta^1 \;&=\; d\tau \\ \tfrac{\partial}{\partial \varphi^2} \;&=\; U_{22}\tfrac{\partial}{\partial x} & d\varphi^2 \;&=\; -\tfrac{U_{21}}{U_{22}} dz \;+\; \tfrac{1}{U_{22}} dx \\ \tfrac{\partial}{\partial \theta_1} \;&=\; \tfrac{\partial}{\partial y} & d\theta_2 \;&=\; dy. \end{aligned} \tag{3.4}$$

Upon substituting the symplectic frame components into the LeBrun metric (2.1), we find the functions w, u and the components A_x, A_y, and A_z to be

$$\begin{aligned} w \;&=\; 1/U^{11}, \quad u \;=\; \log\left(U^{11}U^{22} - (U^{12})^2\right) \\ A_x \;&=\; 0, \quad A_y \;=\; \frac{U^{12}}{U^{11}}, \quad A_z \;=\; 0. \end{aligned} \tag{3.5}$$

We express this in the form of a proposition.

Proposition 3.1. *Assume $(M^4, J, g, \mathcal{X}^1, \mathcal{X}^2)$ is a toric Kähler manifold. Let $(\varphi^1, \theta_1, \varphi^2, \theta_2)$ be symplectic coordinates and $(\xi_1, \theta_1, \xi_2, \theta_2)$ holomorphic coordinates on M^4. There exists a convex function $U(\varphi^1, \varphi^2)$ on Σ^2, where Σ^2 is the image of the moment map $(\varphi^1, \varphi^2) : M^4 \to \mathbb{R}^2$, so that*

$$g \;=\; U_{ij} d\varphi^i \otimes d\varphi^j + U^{ij} d\theta_i \otimes d\theta_j \tag{3.6}$$

where $U_{ij} = \frac{\partial^2 U}{\partial\varphi^i \varphi^j}$ and $(U^{ij}) = (U_{ij})^{-1}$. There also exists a convex function $V = V(\xi_1, \xi_2)$ on $\mathbb{R}^2$ so that

$$g \;=\; V^{ij} d\xi_i \otimes d\xi_j + V^{ij} d\theta_i \otimes d\theta_j \tag{3.7}$$

where $V^{ij} = \frac{\partial^2 V}{\partial \xi_i \xi_j}$. These systems are related via the Legendre transform:

$$\begin{aligned} \varphi^i \;&=\; \frac{\partial V}{\partial \xi_i}, \quad \xi_i \;=\; \frac{\partial U}{\partial \varphi^i}, \\ U(\varphi^1, \varphi^2) + V(\xi_1, \xi_2) \;&=\; \varphi^1 \xi_1 + \varphi^2 \xi_2. \end{aligned} \tag{3.8}$$

The metric $(M^4, g, J, \mathcal{X}^1, \mathcal{X}^2)$ can be expressed in the LeBrun ansatz after setting

$$\big(z,\, \tau,\, x,\, y\big) \;=\; \big(\varphi^1,\, \theta_1,\, \xi_2,\, \theta_2\big). \tag{3.9}$$

A LeBrun ansatz expression of g is obtained by setting

$$\begin{aligned} u \;&=\; \log \det U^{ij} \;=\; \log\left(U^{11}U^{22} - (U^{12})^2\right), \\ w \;&=\; \frac{1}{U^{11}}, \quad \text{and} \quad A \;=\; A_y dy \;=\; \frac{U^{12}}{U^{11}} dy \end{aligned} \tag{3.10}$$

(the components A_x and A_z are zero). The components of the magnetic 2-form are $B_x = -A_{y,z}$, $B_y = 0$, and $B_z = A_{y,x}$.

3.3 Variation of LeBrun structures

In our construction of Section 3.2 we began by setting $\tau = \theta_1$, but we could have chosen $\tau = \theta_2$ or indeed any linear combination of the cyclic variables. Up to scale a toric metric automatically has a 1-parameter family of distinct LeBrun structures. If $\alpha \in [0, \pi/2]$ is a constant and $\mathcal{X}^1$, $\mathcal{X}^2$ are symplectomorphic Killing fields, then for each α we may select the field

$$\mathcal{X} \;=\; \cos(\alpha)\mathcal{X}^1 \,+\, \sin(\alpha)\mathcal{X}^2. \tag{3.11}$$

Then, referring to the construction of Section 3.2, the corresponding angle variable is $\tau = \cos(\alpha)\theta_1 + \sin(\alpha)\theta_2$ with conjugate momentum variable $z = \cos(\alpha)\varphi^1 + \sin(\alpha)\varphi^2$. The holomorphic variables are then $x = -\sin(\alpha)\xi_1 + \cos(\alpha)\xi_2$ and $y = -\sin(\alpha)\theta_1 + \cos(\alpha)\theta_2$.

This allows for a "tuning" or selection of a distinguished 1-parameter symmetry field form which the LeBrun ansatz metric can be constructed. The variable y remains cyclic (that is, its field remains a symmetry direction), and u, w will remain functions of x and z. These functions will change with α, so we may write $u = u_\alpha(x,z)$ and $w = w_\alpha(x,z)$. We remark that a third auxiliary function $\dot{u}_\alpha \triangleq \frac{d}{d\alpha}u_\alpha$ exists. If the u_α solve the LeBrun equation $(u_\alpha)_{xx} + (e^{u_\alpha})_{zz} = 0$ then $\dot{u}_\alpha$ will solve the linearized equation $(\dot{u}_\alpha)_{xx} + (\dot{u}_\alpha e^{u_\alpha})_{zz} = 0$. Under some conditions u_α will be positive, and setting $w = \dot{u}_\alpha$ we have an entirely new LeBrun metric.

4 Expressing $U(2)$-invariant metrics in the LeBrun ansatz

The usual ansatz for $U(2)$-invariant metrics is

$$g \;=\; A\,dr^2 \;+\; B\,(\eta_1)^2 \;+\; C\left((\eta_2)^2 + (\eta_3)^2\right) \tag{4.1}$$

where $\{\eta_1, \eta_2, \eta_3\}$ is a standard left-invariant coframe on $\mathbb{S}^3$, and A, B, C are functions of the radial variable r. If (ψ, φ, θ) are Euler coordinates on on $\mathbb{S}^3$, the usual frame transitions are

$$\begin{aligned} \eta_1 \;&=\; \frac{1}{2}\left(d\psi \;+\; \cos(\theta)d\varphi\right) \\ \eta_2 \;&=\; \frac{1}{2}\left(\sin(\theta)\cos(\psi)d\varphi - \sin(\psi)d\theta\right) \\ \eta_3 \;&=\; \frac{1}{2}\left(\sin(\theta)\sin(\psi)d\varphi + \cos(\psi)d\theta\right). \end{aligned} \tag{4.2}$$

From this we deduce $(\eta_2)^2 + (\eta_3)^2 = \frac{1}{4}\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right)$, so in Euler coordinates

$$g \;=\; A\,dr^2 \;+\; \frac{B}{4}\left(d\psi + \cos(\theta)d\varphi\right)^2 \;+\; \frac{C}{4}\left(d\theta^2 \;+\; \sin^2(\theta)d\varphi^2\right) \tag{4.3}$$

This is already close to LeBrun ansatz form. To place it precisely in LeBrun ansatz form we make the change of variables

$$x \;=\; \log\cot\frac{\theta}{2}, \qquad y \;=\; \varphi, \qquad z \;=\; \frac{1}{2}\int\sqrt{AB}\,dr, \qquad \tau \;=\; \psi. \tag{4.4}$$

This gives $d\theta^2 + \sin^2(\theta)d\varphi^2 = \operatorname{sech}^2(x)(dx^2 + dy^2)$, and the metric now reads

$$g \;=\; \frac{4}{B}dz^2 \;+\; \frac{B}{4}\left(d\tau + \tanh(x)\,dy\right)^2 \;+\; \frac{C}{4}\operatorname{sech}^2(x)\left(dx^2 + dy^2\right). \tag{4.5}$$

Reading off the LeBrun ansatz quantities from (2.1), we have

$$\begin{aligned} &w \;=\; \frac{4}{B}, \quad u \;=\; \log\left(\frac{BC}{16}\operatorname{sech}^2(x)\right) \\ &A_x \;=\; 0, \quad A_y \;=\; \tanh(x), \quad A_z \;=\; 0 \end{aligned} \tag{4.6}$$

where B and C are now functions of the new variable z, via the transition from r to z given in (4.4). Because $U(2)$ has a rank 2 toral subgroup, any $U(2)$-invariant metric is also $\mathbb{T}^2$-invariant—if the metric is Kähler then it is toric. One can see directly that the metric (4.5) has no τ- or y-dependency so has $\mathbb{T}^2$ symmetry.

5 Examples

We give two examples of our method. The exceptional half-plane metric from [21] was originally written in a toric ansatz, and the Page metric on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$ was originally written in the $U(2)$ ansatz. We use our methods to express both in the LeBrun ansatz. In the last section we outline methods for creating new metrics that are Einstein, half-conformally flat, or Bach-flat.

5.1 The exceptional half-plane metric on $\mathbb{C}^2$.

This toric SFK metric on $\mathbb{C}^2$ appears in [21]. It has one translational and one rotational field. In rectangular coordinates (x_1, y_1, x_2, y_2) on $\mathbb{C}^2$, these fields are $\mathcal{X}^1 = \frac{\partial}{\partial y_1}$ and $\mathcal{X}^2 = -y_2\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_2}$, which are clearly translational and rotational, respectively. Let $U = U(\varphi^1, \varphi^2)$ be the symplectic potential

$$U \;=\; \frac{1}{2}\left(\frac{(\varphi^2)^2}{1+2M\varphi^1} + \varphi^1\log(\varphi^1) + M(\varphi^1)^2\right) \tag{5.1}$$

where $M \geq 0$ is a constant. The case $M = 0$ produces the flat metric. When $M > 0$, the resulting metric is the exceptional half-plane metric; the fact that (5.1) is the correct symplectic potential for the exceptional half-plane metric can be verified directly from equations (6-1) and (6-3) of [21]. The Kähler potential V is difficult to write explicitly, as it involves inverting a function with transcendental and algebraic parts. However it is possible to find LeBrun coordinates, which in terms of the symplectic coordinates are

$$x \;=\; \frac{\varphi^2}{1+2M\varphi^1}, \quad y \;=\; \theta_2, \quad z \;=\; \varphi^1, \quad \tau = \theta_1. \tag{5.2}$$

The LeBrun functions w and u are

$$w \;=\; M + \frac{1}{2z}, \quad u \;=\; \log\left(2z\right) \tag{5.3}$$

and the vector potential and field strength are

$$\begin{aligned} A \;&=\; 2Mx\,dy, \qquad \text{which is} \quad A_x = 0,\, A_y = 2Mx,\, A_z = 0, \\ B \;&=\; 2M dx\wedge dy, \text{ which is} \quad B_x = 0,\, B_y = 0,\, B_z = 2M. \end{aligned} \tag{5.4}$$

We notice that $u = \log(2z)$ gives what LeBrun called the *hyperbolic ansatz* in section 4 of [19]. If $M = 0$ this is the flat metric, which LeBrun wrote down on p. 233 of [19] (unfortunately LeBrun's equations are mostly unnumbered). The exceptional half-plane metric in LeBrun ansatz form is

$$g = (1+2Mz)(dx^2+dy^2) + \frac{1+2Mz}{2z}dz^2 + \frac{2z}{1+2Mz}(d\tau + 2Mx\,dy)^2. \tag{5.5}$$

5.2 The Page metric

The Page metric was originally developed in [20], and can be found explicitly in (3.25) of [16] (unfortunately its expression in the appendix of [15] has a typo). Methods for building Ricci-flat metrics, including the Page metric, can be found [4]; see also 9.125 of [5]. This metric exists on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$; it is Einstein, Hermitian, and Bach-flat, but not half-conformally flat. It is conformal to an extremal Kähler metric, which Calabi [7] [8] independently wrote down; see [10] for the specific conformal transformation, or [12] for a more general theory of conformal transformations between extremal Kähler and Einstein metrics on 4-manifolds. From [16] the Page metric is

$$\begin{aligned} g \;=\; & \frac{3(1+\nu^2)}{\Lambda}\Bigg[\frac{1-\nu\cos^2(r)}{3-\nu^2-\nu^2(1+\nu^2)\cos^2(r)}\,dr^2+ \\ & +\frac{3-\nu^2-\nu^2(1+\nu^2)\cos^2(r)}{(3+\nu^2)^2(1-\nu\cos^2(r))}\sin^2(r)\eta_1^2+4\frac{1-\nu^2\cos^2(r)}{3+6\nu^2-\nu^4}\left(\eta_2^2+\eta_3^2\right)\Bigg]. \end{aligned} \tag{5.6}$$

The method of Section 4 gives its expression in the LeBrun ansatz:

$$\begin{aligned} & g = we^u\left(dx^2+dy^2\right) \;+\; w\,dz^2 \;+\; \frac{1}{w}\left(d\tau+\tanh(x)dy\right)^2, \quad \text{where} \\ & w = \frac{F(z)}{G(z)} \;\text{ and }\; we^u = \frac{1}{3\Lambda(1+\nu^2)(3+6\nu^2-\nu^4)}H(z)\,\mathrm{sech}^2(x) \end{aligned} \tag{5.7}$$

and F, G, H are the polynomials

$$\begin{aligned} F(z) \;=\; & 27(1+\nu^2-\nu^4-\nu^6)+36(4\nu^2+4\nu^4+\nu^6)\Lambda z \\ & \qquad -12(9\nu^2+6\nu^4+\nu^6)\Lambda^2z^2 \\ G(z) \;=\; & 27(1+\nu^2-\nu^4-\nu^6)+3(-9+9\nu^2+11\nu^4+15\nu^6)\Lambda z \\ & \qquad -24(3\nu^2+3\nu^4-\nu^6)\Lambda^2z^2+4(9\nu^2+6\nu^3+\nu^6)\Lambda^3z^3 \\ H(z) \;=\; & 9(1+\nu^2-\nu^4-\nu^6)+12(3\nu^2+16\nu^4+\nu^6)\Lambda z \\ & \qquad -4(9\nu^2+6\nu^4+\nu^6)\Lambda^2z^2. \end{aligned} \tag{5.8}$$

The domain for (x, z) is $x \in \mathbb{R}$ and $z \in \left[0,\, \frac{3(1+\nu^2)}{\Lambda(3+\nu^2)}\right]$.

5.3 New metrics

Creation of special metrics, namely Einstein, half-conformally flat, or Bach-flat metrics are of considerable importance in differential geometry. One may regard the metric g, if expressed in the LeBrun ansatz, as a dynamic variable with five unknowns w, u, B_x, B_y, B_z which are each functions of the coordinates (x, y, z). These values can be specified independently, subject to the single requirement that $B_{x,x}+B_{y,y}+B_{z,z} = 0$ which is equivalent to the definition of B from (2.2), which is that $B = dA$ for a 1-form A. In a sense, there are four completely independent variables that may be chosen, with the choice of a fifth being partially constrained.

Letting W^+ be the self-dual part of the Weyl tensor, one might consider the condition $W^+ = 0$. Because the operator $W^+ : \bigwedge^+ \to \bigwedge^+$ has three eigenvalues which are subject to the condition that they sum to zero, the condition $W^+ = 0$ imposes two differential identities on our five variables. With the fifth constraint discussed above, we arrive at an underdetermined system, which surely has a large solutions space. There remain many obstacles, both technical and theoretical, to fully understanding this system. Similar comments hold for systems like $\mathring{\mathrm{Ric}} = 0$ and $\mathrm{B} = 0$ where $\mathring{\mathrm{Ric}}$ is the trace-free Ricci tensor and B is the Bach tensor. This subject will be taken up elsewhere.

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