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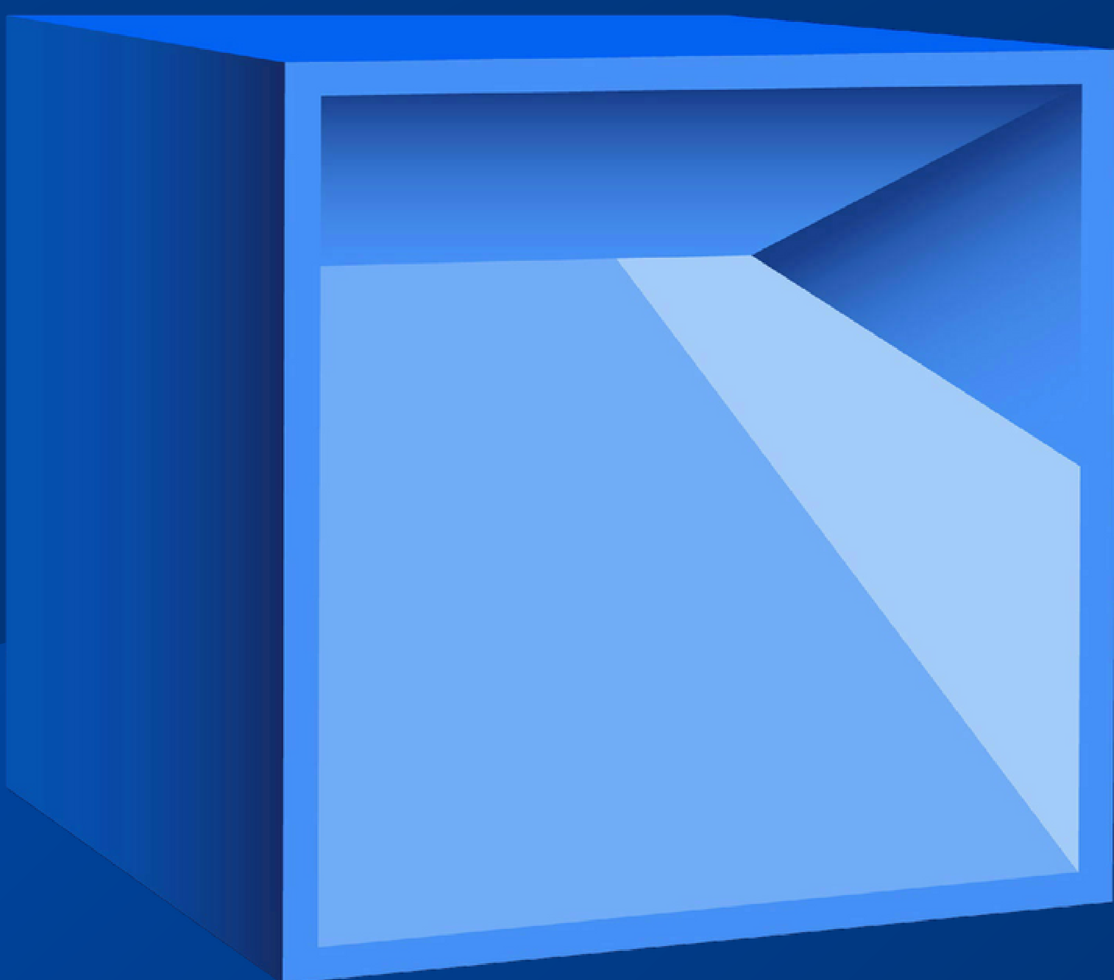

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SUMMARY

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The Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold

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ABSTRACT

The object of the paper is to study a type of canonical linear connection, called the Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold.

RESUMEN

El objetivo de este artículo es estudiar un tipo de conexión lineal canónica, llamada la conexión canónica paracontacto de Zamkovoy en una variedad para-Kenmotsu.

Keywords and Phrases: Para-Kenmotsu manifold; Zamkovoy canonical paracontact connection; local ϕ-symmetry; local ϕ-Ricci symmetry; recurrent; η-Einstein manifold.

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1 Introduction

In recent years, many authors started to the study of paracontact geometry due to its unexpected relation with the most activated contact geometry. As a result of this, in 1985, S. Kaneyuki and F. L. Williams [9] introduced the notion of paracontact metric manifold as a natural counter part of the well known contact metric manifold. Since then, several authors studied these manifolds by focusing on various special cases. A systematic study of paracontact metric manifolds and their subclasses were carried out by S. Zamkovoy [25] by emphasizing similarities and differences with respect to the contact case. Further, the notion of para-Kenmotsu manifold was introduced by J. Welyczko [23] for 3-dimensional normal almost paracontact metric structures. This structure is an analogy of Kenmotsu manifold [10] in paracontact geometry. Again the similar notion called P-Kenmotsu manifold was studied by B. B. Sinha and K. L. Sai Prasad [20] and they obtained many results. At this point, we refer the papers [1, 4, 14, 15, 16, 26] and the references therein to reader for a wide and detailed overview of the results on para-Kenmotsu manifolds.

In the context of para-Kenmotsu geometry, author A. M. Blaga [2] studied certain canonical linear connections (Levi-Civita, Schouten-van Kampen, Golab and Zamkovoy canonical paracontact connections) with a special view towards ϕ-conjugation. Some properties of generalized dual connections of the above said canonical linear connections on a para-Kenmotsu manifold was also studied in [3]. As a continuation of this, we are considering one of such canonical linear connection on a para-Kenmotsu manifold. So we undertake the study of Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold. This connection on a paracontact manifolds was adapted and studied rigorously by S. Zamkovoy [25]. This connection plays the role of the (generalized) Tanaka-Webster connection [22] in paracontact geometry. The main feature of this connection is that, it is metrical but not symmetrical. Throughout the paper, we refer the canonical linear connection as *Zamkovoy canonical paracontact connection*.

On the other hand, the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent. In 1977, T. Takahashi [21] introduced the notion of local ϕ-symmetry on a Sasakian manifold as a weaker version of local symmetry of such a manifold. Since then, several authors studied this notion on various structures and their generalizations or extension in [6, 7, 12, 13, 17, 18, 20]. A para-Kenmotsu manifold is said to be locally ϕ-symmetric if its curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)U) = 0 \tag{1.1}$$

for any vector fields X, Y, U, W orthogonal to ξ on M, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Recently, U. C. De and A. Sarkar [5] introduced the notion of local ϕ-Ricci symmetry on a Sasakian manifold. Further, this notion was studied by S. Ghosh and U. C. De [8] in the context of (κ, μ)-

contact metric manifolds and obtained interesting results. A para-Kenmotsu manifold M is said to be locally ϕ-Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2((\nabla_W Q)X) = 0,$$

for any vector fields X, W orthogonal to ξ on M and $S(X, W) = g(QX, W)$.

The object of the present paper is to study the Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold. This paper is organized as follows: Section 2 is devoted to preliminaries on para-Kenmotsu manifolds. In section 3, we give a brief account of information regarding the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ on a para-Kenmotsu manifold and obtain a relationship between the Levi-Civita connection ∇ and the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$. In section 4, we characterize locally ϕ-symmetric and locally concircular ϕ-symmetric para-Kenmotsu manifolds with respect to the connection $\nabla^{\mathcal{Z}}$. It is prove that the notion of local ϕ-symmetry (also, locally concircular ϕ-symmetry) with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇ are equivalent. Section 5, covers the study of locally ϕ-Ricci symmetric para-Kenmotsu manifold with respect to the connection $\nabla^{\mathcal{Z}}$ and prove that a para-Kenmotsu manifold is locally ϕ-symmetric with respect to the connection $\nabla^{\mathcal{Z}}$, then the manifold is Ricci symmetric and hence it is an Einstein manifold. A para-Kenmotsu manifold whose curvature tensor is covariant constant with respect to the connection $\nabla^{\mathcal{Z}}$ and the manifold is recurrent with respect to the connection ∇ is studied in section 6 and shown that in this situation the manifold is η-Einstein manifold. Finally, we construct an example of a 3-dimensional para-Kenmotsu manifold admitting the connection $\nabla^{\mathcal{Z}}$ to illustrate some results.

2 Preliminaries

Let M be an n-dimensional differentiable manifold, n is odd, with an almost paracontact structure (ϕ, ξ, η), that is, ϕ is a $(1,1)$-tensor field, ξ is a vector field, and η is a 1-form such that

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\phi\xi = 0, \quad \eta \cdot \phi = 0, \quad rank(\phi) = n - 1. \tag{2.2}$$

Let g be a pseudo-Riemannian metric compatible with (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.3}$$

for any vector fields $X, Y \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M, then the manifold is said to be an almost paracontact metric manifold. From (2.3) it can be easily deduce that

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X, \xi) = \eta(X), \tag{2.4}$$

for any vector fields $X, Y \in \chi(M)$. An almost paracontact metric manifold becomes a paracontact metric manifold [25] if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g and is denoted by (M, g). If moreover,

$$(\nabla_X \phi)Y = g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the pseudo-Riemannian connection of g holds, then (M, g) is called an para-Kenmotsu manifold. From (2.5), it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

Moreover, in a para-Kenmotsu manifold (M, g) of dimension n, the curvature tensor R and the Ricci tensor S satisfy [23]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

for any vector fields $X, Y, Z \in \chi(M)$.

A para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X, W) = ag(X, W) + b\eta(X)\eta(W),$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M reduces to an Einstein manifold with some constant a.

3 Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold

In the following, we consider a connection $\nabla^{\mathcal{Z}}$ on an almost paracontact metric manifold using the Levi-Civita connection ∇ of the structure [25]:

$$\nabla^{\mathcal{Z}}_X Y = \nabla_X Y + (\nabla_X \eta)Y.\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{3.1}$$

If we use (2.6) and (2.7) in (3.1), we obtain

$$\nabla^{\mathcal{Z}}_X Y = \nabla_X Y + g(X, Y)\xi - \eta(Y)X + \eta(X)\phi Y, \tag{3.2}$$

for any vector fields $X, Y \in \chi(M)$. We call the connection $\nabla^{\mathcal{Z}}$ defined by (3.2) on a para-Kenmotsu manifold, the *Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold*.
The expression for the curvature tensor $R_{\nabla^{\mathcal{Z}}}$ with respect to the connection $\nabla^{\mathcal{Z}}$ is defined by

$$R_{\nabla^{\mathcal{Z}}}(X,Y)U = \nabla^{\mathcal{Z}}_X \nabla^{\mathcal{Z}}_Y U - \nabla^{\mathcal{Z}}_Y \nabla^{\mathcal{Z}}_X U - \nabla^{\mathcal{Z}}_{[X,Y]} U.$$

Then, in a para-Kenmotsu manifold, we have

$$R_{\nabla^{\mathcal{Z}}}(X,Y)U = R(X,Y)U + g(Y,U)X - g(X,U)Y, \tag{3.3}$$

where $R(X,Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U$, is the curvature tensor of M with respect to the connection ∇. The expression (3.3) is treated as the curvature tensor of a para-Kenmotsu manifold with respect to the connection $\nabla^{\mathcal{Z}}$.

Proposition 3.1. *A para-Kenmotsu manifold is Ricci-flat with respect to the Zamkovoy canonical paracontact connection if and only if it is an Einstein manifold of the form $S(Y,U) = -(n-1)g(Y,U)$.*

Proof. In a para-Kenmotsu manifold M, the Ricci tensor $S_{\nabla^{\mathcal{Z}}}$ and scalar curvature $r_{\nabla^{\mathcal{Z}}}$ of the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ are defined by

$$S_{\nabla^{\mathcal{Z}}}(Y,U) = S(Y,U) + (n-1)g(Y,U), \tag{3.4}$$

$$r_{\nabla^{\mathcal{Z}}} = r + n(n-1), \tag{3.5}$$

where S and r denote the Ricci tensor and scalar curvature of Levi-Civita connection ∇, respectively.

Remark 3.2. *For a para-Kenmotsu manifold M with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$:*
(a) The curvature tensor $R_{\nabla^{\mathcal{Z}}}$ is given by (3.3),
(b) The Ricci tensor $S_{\nabla^{\mathcal{Z}}}$ is given by (3.4),
(c)$R_{\nabla^{\mathcal{Z}}}(X,\xi)U = R_{\nabla^{\mathcal{Z}}}(\xi,Y)U = R_{\nabla^{\mathcal{Z}}}(X,Y)\xi = 0$,
(d) $R'_{\nabla^{\mathcal{Z}}}(X,Y,U,V) + R'_{\nabla^{\mathcal{Z}}}(X,Y,V,U) = 0$,
(e) $R'_{\nabla^{\mathcal{Z}}}(X,Y,U,V) + R'_{\nabla^{\mathcal{Z}}}(Y,X,V,U) = 0$,
(f) $R'_{\nabla^{\mathcal{Z}}}(X,Y,U,V) - R'_{\nabla^{\mathcal{Z}}}(U,V,X,Y) = 0$,
(g) $R_{\nabla^{\mathcal{Z}}}(X,\xi)U = R_{\nabla^{\mathcal{Z}}}(\xi,Y)U = R_{\nabla^{\mathcal{Z}}}(X,Y)\xi = 0$,
(h) $S_{\nabla^{\mathcal{Z}}}(Y,\xi) = 0$,
(i) The Ricci tensor $S_{\nabla^{\mathcal{Z}}}$ is symmetric,
(j) The scalar curvature $r_{\nabla^{\mathcal{Z}}}$ is given by (3.5).

Next, suppose that a para-Kenmotsu manifold is Ricci flat with respect to the Zamkovoy canonical paracontact connection. Then from (3.4) we get

$$S(Y,U) = -(n-1)g(Y,U). \tag{3.6}$$

Conversely, if the manifold is an Einstein manifold of the form $S(Y,U) = -(n-1)g(Y,U)$, then from (3.4) it follows that $S_{\nabla}(Y,U) = 0$: □

Proposition 3.3. *If in a para-Kenmotsu manifold the curvature tensor of the Zamkovoy canonical paracontact connection vanishes, then the sectional curvature of the plane determined by two vectors $X, Y \in \xi^{\perp}$ is -1.*

Proof. Let $\xi^{\perp}$ denote the $(n-1)$-dimensional distribution orthogonal to ξ in a para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection whose curvature tensor vanishes. Then for any $X \in \xi^{\perp}$, $g(X,\xi) = 0$ or, $\eta(X) = 0$. Now we shall determine the sectional curvature $'R$ of the plane determine by the vectors $X, Y \in \xi^{\perp}$. Taking inner product on both sides of (3.3) with X and then for $U = Y$, we have

$$R_{\nabla^{\mathcal{Z}}}(X,Y,Y,X) = R(X,Y,Y,X) + g(Y,Y)g(X,X) - g(X,Y)g(X,Y). \tag{3.7}$$

Putting $R_{\nabla^{\mathcal{Z}}} = 0$ in (3.7) we get

$$'R(X,Y) = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = -1.$$

This proves the require result. □

4 Local ϕ-symmetry and local concircular ϕ-symmetry with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇

Definition 4.1. *A para-Kenmotsu manifold is said to be locally ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if its curvature tensor $R_{\nabla^{\mathcal{Z}}}$ with respect to the connection $\nabla^{\mathcal{Z}}$ satisfies the condition*

$$\phi^2((\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U) = 0, \tag{4.1}$$

for any vector fields X, Y, U, W orthogonal to ξ.

Proposition 4.2. *A para-Kenmotsu manifold is locally ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if and only if it is so with respect to the Levi-Civita connection ∇.*

Proof. Let us suppose that a para-Kenmotsu manifold M is locally ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$. Then, by the help of (3.2), (4.1) simplifies as follow

$$\begin{aligned} (\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U &= (\nabla_W R_{\nabla^{\mathcal{Z}}})(X,Y)U + g(W, R_{\nabla^{\mathcal{Z}}}(X,Y)U)\xi \\ &- \eta(R_{\nabla^{\mathcal{Z}}}(X,Y)U)W + \eta(W)\phi R_{\nabla^{\mathcal{Z}}}(X,Y)U. \end{aligned} \tag{4.2}$$

By virtue of $\eta(R_{\nabla^{\mathcal{Z}}}(X,Y)U)=0$, (4.2) reduces to

$$\begin{aligned} &(\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U \\ = \;& (\nabla_W R_{\nabla^{\mathcal{Z}}})(X,Y)U + g(W, R_{\nabla^{\mathcal{Z}}}(X,Y)U)\xi + \eta(W)\phi R_{\nabla^{\mathcal{Z}}}(X,Y)U. \end{aligned} \tag{4.3}$$

Now covariant differentiation of (3.3) with respect to W, we obtain

$$(\nabla_W R_{\nabla^{\mathcal{Z}}})(X,Y)U = (\nabla_W R)(X,Y)U. \tag{4.4}$$

Using (4.4) in (4.3), we get

$$\begin{aligned} &(\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U \\ = \;& (\nabla_W R)(X,Y)U + \{R'(X,Y,U,W) + g(Y,U)g(X,W) - g(X,U)g(Y,W)\}\xi \\ + \;& \eta(W)\{\phi R(X,Y)U + g(Y,U)\phi X - g(X,U)\phi Y\}. \end{aligned} \tag{4.5}$$

Applying ϕ^2 on both sides of (4.5); then using (2.1) and (2.2), we obtain

$$\begin{aligned} \phi^2(\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U &= \phi^2(\nabla_W R)(X,Y)U \\ &+ \eta(W)\{\phi R(X,Y)U + g(Y,U)\phi X - g(X,U)\phi Y\}. \end{aligned} \tag{4.6}$$

If we consider X, Y, U, W orthogonal to ξ, (4.6) gives to

$$\phi^2((\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U) = \phi^2((\nabla_W R)(X,Y)U).$$

It completes the proof. □

Definition 4.3. *For an n-dimensional $(n>1)$ para-Kenmotsu manifold the concircular curvature tensor $C_{\nabla^{\mathcal{Z}}}$ with respect to the Zamkovoy canonical paracontact connection is defined by*

$$C_{\nabla^{\mathcal{Z}}}(X,Y)U = R_{\nabla^{\mathcal{Z}}}(X,Y)U - \frac{r_{\nabla^{\mathcal{Z}}}}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{4.7}$$

where $R_{\nabla^{\mathcal{Z}}}$ and $r_{\nabla^{\mathcal{Z}}}$ are the Riemannian curvature tensor and scalar curvature with respect to the connection $\nabla^{\mathcal{Z}}$, respectively.

Using (3.3) and (3.5) in (4.7), we get

$$C_{\nabla^{\mathcal{Z}}}(X,Y)U = C(X,Y)U, \tag{4.8}$$

where

$$C(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}[g(Y,U)X - g(X,U)Y] \tag{4.9}$$

is the concircular curvature tensor [24] with respect to the Levi-Civita connection ∇. Thus, the concircular curvature tensor with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇ are equal.

Definition 4.4. *A para-Kenmotsu manifold is said to be locally concircular ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if its concircular curvature tensor $C_{\nabla^{\mathcal{Z}}}$ with respect to the connection $\nabla^{\mathcal{Z}}$ satisfies the condition*

$$\phi^2((\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U) = 0, \tag{4.10}$$

for any vector fields X, Y, U, W orthogonal to ξ.

Proposition 4.5. *A para-Kenmotsu manifold is locally concircular ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if and only if it is so with respect to the Levi-Civita connection ∇.*

Proof. If a para-Kenmotsu manifold M is locally concircular ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$, then using (3.2), (4.10) simplifies to

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U &= (\nabla_W C_{\nabla^{\mathcal{Z}}})(X,Y)U + g(W, C_{\nabla^{\mathcal{Z}}}(X,Y)U)\xi \\ &- \eta(C_{\nabla^{\mathcal{Z}}}(X,Y)U)W + \eta(W)(\phi C_{\nabla^{\mathcal{Z}}})(X,Y)U.\end{aligned} \tag{4.11}$$

Now covariant differentiation of (4.8) with respect to W, yields

$$(\nabla_W C_{\nabla^{\mathcal{Z}}})(X,Y)U = (\nabla_W C)(X,Y)U. \tag{4.12}$$

Making use of (4.8) and (4.12) in (4.11) we obtain

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U = (\nabla_W C)(X,Y)U + g(W, C(X,Y)U)\xi \\ - \eta(C(X,Y)U)W + \eta(W)(\phi C)(X,Y)U.\end{aligned} \tag{4.13}$$

Taking account of (4.9), we write (4.13) as

$$\begin{aligned}&(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U \\ =\ & (\nabla_W C)(X,Y)U + R'(X,Y,U,W)\xi + \eta(W)\phi R(X,Y)U \\ -\ & \frac{r}{n(n-1)}\{g(Y,U)(g(X,W)\xi + \eta(W)\phi X) - g(X,U)(g(Y,W)\xi + \eta(W)\phi Y)\} \\ -\ & \left[\frac{r}{n(n-1)} + 1\right]\{g(X,U)\eta(Y)W - g(Y,U)\eta(X)W\}.\end{aligned} \tag{4.14}$$

Applying ϕ^2 on both sides of above equation; then using (2.1) and (2.2) in (4.14) we have

$$\begin{aligned}&\phi^2(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U \\ =\ & \phi^2(\nabla_W C)(X,Y)U + \eta(W)\phi R(X,Y)U \\ -\ & \frac{r}{n(n-1)}\{g(Y,U)\phi X - g(X,U)\phi Y)\}\eta(W) \\ -\ & \left[\frac{r}{n(n-1)} + 1\right]\{g(X,U)\eta(Y) - g(Y,U)\eta(X)\}(W - \eta(W)\xi).\end{aligned} \tag{4.15}$$

If we consider X, Y, U, W orthogonal to ξ, (4.15) reduces to

$$\phi^2((\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U) = \phi^2((\nabla_W C)(X,Y)U). \tag{4.16}$$

This ends the proof of the required result. □

Proposition 4.6. *Let M be an n-dimensional $(n > 1)$ locally concircular ϕ-symmetric para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$. If the scalar curvature r with respect to the Levi-Civita connection ∇ is constant, then M is locally ϕ-symmetric.*

Proof. Now, from (4.9) we have

$$(\nabla_W C)(X,Y)U = (\nabla_W R_\nabla)(X,Y)U - \frac{(\nabla_W r)}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{4.17}$$

From (4.17) in (4.16) we obtain

$$\phi^2((\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U) = \phi^2((\nabla_W R)(X,Y)U) - \frac{(\nabla_W r)}{n(n-1)}[g(Y,U)\phi^2 X - g(X,U)\phi^2 Y]. \tag{4.18}$$

By virtue of (2.1) in (4.18) and then taking X, Y, U, W orthogonal to ξ, we get

$$\phi^2(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U = \phi^2((\nabla_W R)(X,Y)U) - \frac{(\nabla_W r)}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{4.19}$$

If r is constant, then $\nabla_W r$ is zero. Therefore, (4.19) gives

$$\phi^2(C_{\nabla^{\mathcal{Z}}})(X,Y)U = \phi^2((\nabla_W R)(X,Y)U).$$

Hence, it completes the proof of the required result. □

5 Local ϕ-Ricci symmetry with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇

Definition 5.1. *A para-Kenmotsu manifold M is said to be locally ϕ-Ricci symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if its Ricci operator $Q_{\nabla^{\mathcal{Z}}}$ satisfies*

$$\phi^2((\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X) = 0, \tag{5.1}$$

for any vector fields X, W orthogonal to ξ, and $S_{\nabla^{\mathcal{Z}}}(X,W) = g(Q_{\nabla^{\mathcal{Z}}}X, W)$.

Proposition 5.2. *If a para-Kenmotsu manifold is locally ϕ-Ricci symmetric with respect to the Zamkovoy canonical paracontact connection, then the manifold is Ricci symmetric.*

Proof. Let us consider a para-Kenmotsu manifold, which is locally ϕ-Ricci symmetric with respect to the connection $\nabla^{\mathcal{Z}}$. Then by virtue of (2.1) it follows from (5.1) that

$$(\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X - \eta((\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X)\xi = 0. \tag{5.2}$$

From (3.4) we can write

$$Q_{\nabla^{\mathcal{Z}}}X = QX + (n-1)X. \tag{5.3}$$

Again we have

$$(\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X = \nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}}X - Q_{\nabla^{\mathcal{Z}}}(\nabla^{\mathcal{Z}}_W X), \tag{5.4}$$

Using (5.3) in (5.4) we get

$$(\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X = (\nabla^{\mathcal{Z}}_W Q)X. \tag{5.5}$$

Taking account of (5.5), (5.2) reduces to

$$(\nabla^{\mathcal{Z}}_W Q)X - \eta((\nabla^{\mathcal{Z}}_W Q)X)\xi = 0. \tag{5.6}$$

From (3.2) it follows that,

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W Q)X &= \nabla^{\mathcal{Z}}_W QX - Q(\nabla^{\mathcal{Z}}_W X),\\ &= (\nabla_W Q)X + S(W,X)\xi + (n-1)(\eta(X)W + g(W,X)\xi)\\ &+ \eta(X)QW + \eta(W)(\phi QX - Q\phi X)\end{aligned} \tag{5.7}$$

and

$$\eta((\nabla^{\mathcal{Z}}_W Q)X) = \eta((\nabla_W Q)X) + S(W,X) + (n-1)g(W,X). \tag{5.8}$$

Using (5.7) and (5.8) we get from (5.6) that

$$(\nabla_W Q)X + (n-1)\eta(X)W + \eta(W)(\phi QX - Q\phi X) + \eta(X)QW - \eta((\nabla_W Q)X)\xi = 0. \tag{5.9}$$

Taking inner product with U of (5.9) and considering X, W, U orthogonal to ξ, we get

$$(\nabla_W S)(X,U) = 0, \tag{5.10}$$

which implies that the manifold is Ricci symmetric with respect to the Levi-Civita connection ∇. Hence the proof. □

Proposition 5.3. *A locally ϕ-Ricci symmetric para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection is an Einstein manifold.*

Proof. Putting $X = \xi$ in (5.10) and using (2.11), we get

$$S(W,U) = -(n-1)g(W,U), \tag{5.11}$$

for any vector fields $W, U \in \chi(M)$.
This ends the required proof. □

6 A para-Kenmotsu manifold M whose curvature tensor is covariant constant with respect to the connection $\nabla^{\mathcal{Z}}$ and M is recurrent with respect to the connection ∇

Definition 6.1. *A para-Kenmotsu manifold M with respect to the Levi-Civita connection is said to be recurrent [11] if its curvature tensor R satisfies the condition.*

$$(\nabla_W R)(X,Y)U = A(W)R(X,Y)U, \tag{6.1}$$

where A is a non-zero 1-form and $X,Y,U,W \in \chi(M)$.

Proposition 6.2. *If in a para-Kenmotsu manifold the curvature tensor is covariant constant with respect to the Zamkovoy canonical paracontact connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an η-Einstein manifold.*

Proof. From (3.2), we can write (6.1) as

$$\begin{aligned}
(\nabla^{\mathcal{Z}}_W R)(X,Y)U &= \nabla^{\mathcal{Z}}_W R(X,Y)U - R(\nabla^{\mathcal{Z}}_W X,Y)U - R(X,\nabla^{\mathcal{Z}}_W Y)U - R(X,Y)\nabla^{\mathcal{Z}}_W U,\\
&= (\nabla_W R)(X,Y)U + g(W,R(X,Y)U)\xi - \eta(R(X,Y)U)W\\
&+ \eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W\\
&+ \eta(W)\{\phi R(X,Y)U - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\}\\
&- g(X,W)R(\xi,Y)U - g(Y,W)R(X,\xi)U - g(U,W)R(X,Y)\xi.
\end{aligned} \tag{6.2}$$

Using (2.7)-(2.9) in (6.2), we obtain

$$\begin{aligned}
(\nabla^{\mathcal{Z}}_W R)(X,Y)U &= (\nabla_W R)(X,Y)U + g(W,R(X,Y)U)\xi\\
&+ \eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W\\
&+ \eta(W)\{\phi R(X,Y)U - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\}\\
&- g(X,U)\eta(Y)W + g(Y,U)\eta(X)W - g(X,W)\{\eta(U)Y - g(Y,U)\xi\}\\
&- g(Y,W)\{g(X,U)\xi - \eta(U)X\} - g(W,U)\{\eta(X)Y - \eta(Y)X\}.
\end{aligned} \tag{6.3}$$

Let $(\nabla^{\mathcal{Z}}_W R)(X,Y)U = 0$, then from (6.3), it follows that

$$\begin{aligned}
&(\nabla_W R)(X,Y)U + g(W,R(X,Y)U)\xi\\
+\ &\eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W\\
+\ &\eta(W)\{\phi R(X,Y)U - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\}\\
-\ &g(X,U)\eta(Y)W + g(Y,U)\eta(X)W - g(X,W)\{\eta(U)Y - g(Y,U)\xi\}\\
-\ &g(Y,W)\{g(X,U)\xi - \eta(U)X\} - g(W,U)\{\eta(X)Y - \eta(Y)X\} = 0.
\end{aligned} \tag{6.4}$$

Now using (6.1) in (6.4), we have

$$\begin{aligned} & A(W)R(X,Y)U + g(W,R(X,Y)U)\xi \\ + \quad & \eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W \\ + \quad & \eta(W)\{\phi R(X,Y)U) - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\} \\ - \quad & g(X,U)\eta(Y)W + g(Y,U)\eta(X)W - g(X,W)\{\eta(U)Y - g(Y,U)\xi\} \\ - \quad & g(Y,W)\{g(X,U)\xi - \eta(U)X\} - g(U,W)\{\eta(X)Y - \eta(Y)X\} = 0. \end{aligned} \tag{6.5}$$

Taking the inner product of (6.5) with ξ and using (2.2) and (2.9), it follows that

$$\begin{aligned} & A(W)\{g(X,U)\eta(Y) - g(Y,U)\eta(X)\} + g(W,R(X,Y)U) \\ + \quad & \eta(W)\{g(\phi Y,U)\eta(X) - g(\phi X,U)\eta(Y) - g(X,\phi U)\eta(Y) \\ + \quad & g(Y,\phi U)\eta(X)\} + g(X,W)g(Y,U) - g(Y,W)g(X,U) = 0. \end{aligned} \tag{6.6}$$

Contracrting (6.6) over X and W, we obtain

$$S(Y,U) = \{A(\xi) - (n-1)\}g(Y,U) - A(U)\eta(Y). \tag{6.7}$$

Since the Ricci tensor S with respect to the connection ∇ is symmetric; then from (6.7), we get

$$A(U)\eta(Y) = A(Y)\eta(U). \tag{6.8}$$

Putting $Y = \xi$ in (6.8) and using (2.2) we have

$$A(U) = A(\xi)\eta(U). \tag{6.9}$$

Combining (6.7) and (6.9), it follows that

$$S(Y,U) = \{A(\xi) - (n-1)\}g(Y,U) - A(\xi)\eta(Y)\eta(U). \tag{6.10}$$

This results shows that the manifold is an η-Einstein manifold. Hence the proof. □

7 Example

We consider the 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$ where (x,y,z) are the standard coordinates in R^3. The vector fields (see [27], example of section 7)

$$X = \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y}, \quad \xi = (x+2y)\frac{\partial}{\partial x} + (2x+y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point of M^3.

The 1-form $\eta = dz$ defines an almost paracontact structure on M^3 with characteristic vector field ξ. Let g, ϕ be the semi-Riemannian metric and the $(1,1)$ tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -(x+2y) \\ 0 & -1 & (2x+y) \\ -(x+2y) & (2x+y) & 1-(2x+y)^2+(x+2y)^2 \end{pmatrix}$$

$$\varphi = \begin{pmatrix} 0 & 1 & -(2x+y) \\ 1 & 0 & -(x+2y) \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. Clearly, (ϕ, ξ, η, g) defines an almost paracontact metric structure on M^3. Let ∇ be the Levi-Civita connection with metric g, then we have

$$[X, \phi X] = 0, \quad [X, \xi] = X + 2\phi X, \quad [\phi X, \xi] = 2X + \phi X.$$

Next, by using the well-known Koszul's formula, we obtain

$$\begin{aligned} &\nabla_X X = -\xi, \quad \nabla_{\phi X} X = 0, \quad \nabla_\xi X = -2\phi X, \\ &\nabla_X \phi X = 0, \quad \nabla_{\phi X} \phi X = \xi, \quad \nabla_\xi \phi X = -2X, \\ &\nabla_X \xi = X, \qquad \nabla_{\phi X} \xi = \phi X, \qquad \nabla_\xi \xi = 0. \end{aligned}$$

Hence, from the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a para-Kenmotsu manifold. By the above results, one can easily compute

$$\begin{aligned} &R(X, \phi X)\xi = 0, \quad R(\phi X, \xi)\xi = -\phi X, \quad R(X, \xi)\xi = -X, \\ &R(X, \phi X)\phi X = X, \quad R(\phi X, \xi)\phi X = -\xi, \quad R(X, \xi)\phi X = 0, \\ &R(X, \phi X)X = \phi X, \quad R(\phi X, \xi)X = 0, \quad R(X, \xi)X = \xi. \end{aligned} \tag{7.1}$$

Using (7.1), we have constant scalar curvature as follows:

$$r = S(X, X) - S(\phi X, \phi X) + S(\xi, \xi) = -6.$$

Now consider the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ defined by (3.2) such that

$$\begin{aligned} &\nabla^{\mathcal{Z}}_X X = 0, \quad \nabla^{\mathcal{Z}}_{\phi X} X = 0, \quad \nabla^{\mathcal{Z}}_\xi X = -\phi X, \\ &\nabla^{\mathcal{Z}}_X \phi X = 0, \quad \nabla^{\mathcal{Z}}_{\phi X} \phi X = 0, \quad \nabla^{\mathcal{Z}}_\xi \phi X = -X, \\ &\nabla^{\mathcal{Z}}_X \xi = 0, \quad \nabla^{\mathcal{Z}}_{\phi X} \xi = 0, \quad \nabla_\xi \xi = 0. \end{aligned}$$

Again, by the above results we can compute the components of curvature tensors with respect to the connection $\nabla^{\mathcal{Z}}$ as follows:

$$\begin{aligned} &R_{\nabla^{\mathcal{Z}}}(X, \phi X)\xi = 0, \quad R_{\nabla^{\mathcal{Z}}}(\phi X, \xi)\xi = 0, \quad R_{\nabla^{\mathcal{Z}}}(X, \xi)\xi = 0, \\ &R_{\nabla^{\mathcal{Z}}}(X, \phi X)\phi X = 0, \quad R_{\nabla^{\mathcal{Z}}}(\phi X, \xi)\phi X = 0, \quad R_{\nabla^{\mathcal{Z}}}(X, \xi)\phi X = 0, \\ &R_{\nabla^{\mathcal{Z}}}(X, \phi X)X = 0, \quad R_{\nabla^{\mathcal{Z}}}(\phi X, \xi)X = 0, \quad R_{\nabla^{\mathcal{Z}}}(X, \xi)X = 0. \end{aligned} \tag{7.2}$$

Using (7.2), we have constant scalar curvature r_{∇^z} as follows:

$$r_{\nabla^z} = S_{\nabla^z}(X, X) - S_{\nabla^z}(\phi X, \phi X) + S_{\nabla^z}(\xi, \xi) = 0.$$

The above arguments easily verifies all the properties of Remark 3.2 and Proposition 3.1.

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Coincidence point results of nonlinear contractive mappings in partially ordered metric spaces

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ABSTRACT

In this paper, we proved some coincidence point results for f-nondecreasing self-mapping satisfying certain rational type contractions in the context of a metric space endowed with a partial order. Moreover, some consequences of the main result are given by involving integral type contractions in the space. Some numerical examples are illustrated to support our results. As an application, we have discussed the existence of a unique solution of integral equation.

RESUMEN

En este artículo, probamos algunos resultados sobre puntos de coincidencia para un auto-mapeo no decreciente f satisfaciendo ciertas contracciones de tipo racional en el contexto de un espacio métrico dotado de un orden parcial. Más aún, se entregan algunas consecuencias del resultado principal que involucran contracciones de tipo integral en el espacio. Se ilustran algunos ejemplos numéricos en apoyo a nuestros resultados. Como una aplicación, discutimos la existencia de una única solución de una ecuación integral.

Keywords and Phrases: Ordered metric spaces; rational contractions; compatible mappings; weakly compatible mappings; coupled fixed point; common fixed point.

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1 Introduction

A remarkable fixed point theorem was first introduced by Banach [4] in 1922, which is one of the most influential results in analysis. It is being used widely in many different areas of mathematics and its applications. It needs the structure of complete metric spaces together with a contractive condition on the self map which is easy to test in many circumstances. Basically this principle gives a sequence of approximate solutions and also give a valuable information about the convergence rate of a fixed point. This kind of iteration process has been used both in mathematics and computer science. In particular, fixed point iterations together with monotone iterative techniques are the central methods when solving a large class of problems in theoretical and applied mathematics and play an important role in many algorithms. Many authors have extended this theorem by introducing more generalized contractive conditions, which imply the existence of a fixed point [6, 7, 8, 9, 11, 12, 13, 14, 15, 16].

The existence of fixed point results for self-mappings in partially ordered sets have been considered first by Ran and Reurings [36] and presented some applications to matrix equations therein. These results were again generalized and extended by Nieto et al. [32, 33] in partially ordered sets and applied their results to study the ordinary differential equations. Prominent works on various existence and uniqueness theorems on fixed point and common fixed point for monotone mappings in cone metric spaces, partially ordered metric spaces and others spaces, refer the readers to [5, 10, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45], which generate natural interest to establish usable fixed point theorems by weakening its hypothesis. Various types of contraction conditions have been used to find a fixed point of a single and multivalued mappings on metric spaces by Altun et al. [1], Aslantas et al. [2, 3], and Sahin et al. [37]. It is well known that a powerful technique for proving existence results for nonlinear problems is the method of upper and lower solutions. In many cases it is possible to find a minimal and a maximal solution between the lower and the upper solution by an iterative scheme: the monotone iterative technique. This method provides a constructive procedure for the solutions and it is also useful for the investigation of qualitative properties of solutions. This method has been used to acquire the unique solution of periodic boundary value problems of ordinary and partial differential equations, integro ordinary and partial differential equations by several authors, some of which are in [23, 32, 33].

The aim of this paper is to prove the coincidence point and common fixed point results for f-nondecreasing self-mapping satisfying generalized contractive conditions of rational type in the context of partially ordered metric spaces. These results generalize and extend the result of [7, 12, 14, 25, 26] in partially ordered metric spaces. Some consequences of the main results are given in terms of integral type contractions in the same space. Further, some examples and an application for the existence of the unique solution for an integral equation are presented at the end.

2 Preliminaries

The following definitions are frequently used in our study.

Definition 2.1. *[40] The triple $(X, d, \leq)$ is called a partially ordered metric space, if $(X, \leq)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 2.2. *[40] If (X, d) is a complete metric space, then the triple $(X, d, \leq)$ is called a complete partially ordered metric space.*

Definition 2.3. *[38] Let $(X, \leq)$ be a partially ordered set. A mapping $f : X \to X$ is said to be strictly increasing (strictly decreasing), if $f(x) < f(y)$ $(f(x) > f(y))$ for all $x, y \in X$ with $x < y$.*

Definition 2.4. *[40] A point $x \in A$, where A is a non-empty subset of a partially ordered set $(X, \leq)$ is called a common fixed (coincidence) point of two self-mappings f and T, if $fx = Tx = x$ $(fx = Tx)$.*

Definition 2.5. *[39] The two self-mappings f and T defined over a subset A of a partially ordered metric space $(X, d, \leq)$ are called commuting, if $fTx = Tfx$ for all $x \in A$.*

Definition 2.6. *[39] Two self-mappings f and T defined over $A \subset X$ are compatible, if for any sequence $\{x_n\}$ with $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} Tx_n = \mu$ for some $\mu \in A$, then $\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0$.*

Definition 2.7. *[40] Two self-mappings f and T defined over $A \subset X$ are said to be weakly compatible, if they commute only at their coincidence points (i.e., if $fx = Tx$, then $fTx = Tfx$).*

Definition 2.8. *[40] Let f and T be two self-mappings defined over a partially ordered set $(X, \leq)$. A mapping T is called monotone f-nondecreasing, if*

$$fx \leq fy \text{ implies } Tx \leq Ty, \text{ for all } x, y \in X.$$

Definition 2.9. *[38] Let A be a non-empty subset of a partially ordered set $(X, \leq)$. If every two elements of A are comparable, then it is called a well ordered set.*

Definition 2.10. *[39] A partially ordered metric space $(X, d, \leq)$ is called an ordered complete, if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, one of the following conditions holds:*

- *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \leq x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$ or,*
- *if $\{x_n\}$ is a non-increasing sequence in X such that $x_n \to x$ implies $x \leq x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$.*

3 Main Results

We start this section with the following coincidence point theorem in the context of a partially ordered metric space.

Theorem 3.1. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$\begin{aligned} d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)\,[1 + d(fy, Ty)]}{1 + d(fx, fy)} + \beta \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} \\ + \gamma\,[d(fx, Tx) + d(fy, Ty)] + \delta\,[d(fx, Ty) + d(fy, Tx)] \\ + \lambda d(fx, fy), \end{aligned} \tag{3.1}$$

for all x, y in X for which $fx \neq fy$ are comparable, and for some $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leq \alpha + \beta + 2(\gamma + \delta) + \lambda < 1$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings f and T are compatible, then f and T have a coincidence point in X.

Proof. Suppose for some $x_0 \in X$ such that $fx_0 \leq Tx_0$. From the hypothesis, we have $T(X) \subseteq f(X)$, then choose a point $x_1 \in X$ such that $fx_1 = Tx_0$. But $Tx_1 \in f(X)$, then there exists another point $x_2 \in X$ such that $fx_2 = Tx_1$. As by a similar argument above, we obtain a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$ for all $n \geq 0$.

Since, $fx_0 \leq Tx_0 = fx_1$ and T is monotone f-nondecreasing mapping, then we have that $Tx_0 \leq Tx_1$. Similarly, we get $Tx_1 \leq Tx_2$ as $fx_1 \leq fx_2$. Continuing the same process, we obtain that

$$Tx_0 \leq Tx_1 \leq \ldots \leq Tx_n \leq Tx_{n+1} \leq \ldots\ .$$

Now, we discuss the following two cases.

Case 1: If $d(Tx_{n_0}, Tx_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then $Tx_{n_0+1} = Tx_{n_0}$ and by the above argument, we have $Tx_{n_0+1} = Tx_{n_0} = fx_{n_0+1}$. Therefore, x_{n_0+1} is a coincidence point of T and f, and so we have the result.

Case 2: If $d(Tx_n, Tx_{n+1}) > 0$ for all $n \in \mathbb{N}$, then from contraction condition (3.1), we have

$$\begin{aligned} d(Tx_{n+1}, Tx_n) \leq \alpha \frac{d(fx_{n+1}, Tx_{n+1})\,[1 + d(fx_n, Tx_n)]}{1 + d(fx_{n+1}, fx_n)} + \beta \frac{d(fx_{n+1}, Tx_{n+1})\ d(fx_n, Tx_n)}{d(fx_{n+1}, fx_n)} \\ + \gamma\,[d(fx_{n+1}, Tx_{n+1}) + d(fx_n, Tx_n)] + \delta\,[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})] \\ + \lambda d(fx_{n+1}, fx_n), \end{aligned}$$

which implies that

$$\begin{aligned} d(Tx_{n+1}, Tx_n) \leq \alpha d(Tx_n, Tx_{n+1}) + \beta d(Tx_n, Tx_{n+1}) \\ + \gamma\,[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)] \\ + \delta\,[d(Tx_n, Tx_n) + d(Tx_{n-1}, Tx_{n+1})] + \lambda d(Tx_n, Tx_{n-1}). \end{aligned}$$

Therefore, we arrive at

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\gamma+\delta+\lambda}{1-\alpha-\beta-\gamma-\delta}\right) d(Tx_n, Tx_{n-1}).$$

Continuing the same process up to n times, we obtain that

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\gamma+\delta+\lambda}{1-\alpha-\beta-\gamma-\delta}\right)^n d(Tx_1, Tx_0).$$

Let $k = \frac{\gamma+\delta+\lambda}{1-\alpha-\beta-\gamma-\delta} < 1$. Moreover, from the triangular inequality for $m \geq n$, we have

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + ... + d(Tx_{n+1}, Tx_n) \\ &\leq \left(k^{m-1} + k^{m-2} + ... + k^n\right) d(Tx_1, Tx_0) \\ &\leq \frac{k^n}{1-k}\, d(Tx_1, Tx_0), \end{aligned}$$

as $m, n \to +\infty$, $d(Tx_m, Tx_n) \to 0$, this shows that the sequences $\{Tx_n\}$ is a Cauchy sequence in X. So, by the completeness of X, there exists a point $\mu \in X$ such that $Tx_n \to \mu$ as $n \to +\infty$.

The continuity of T implies that

$$\lim_{n\to+\infty} T(Tx_n) = T\left(\lim_{n\to+\infty} Tx_n\right) = T\mu.$$

Since, $fx_{n+1} = Tx_n$ then $fx_{n+1} \to \mu$ as $n \to +\infty$. Further, the compatibility of T and f, we have

$$\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0.$$

From the triangular inequality of a metric d, we have

$$d(T\mu, f\mu) = d(T\mu, Tfx_n) + d(Tfx_n, fTx_n) + d(fTx_n, f\mu),$$

on taking limit as $n \to +\infty$ in the above inequality and using the fact that T and f are continuous, we obtain that $d(T\mu, f\mu) = 0$. Thus, $T\mu = f\mu$. Hence, μ is a coincidence point of T and f in X. □

We obtain the following consequences from Theorem 3.1 on taking zero value to $\alpha, \beta, \gamma, \delta$ and λ as special cases.

Corollary 3.2. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following contraction conditions*

(a)

$$\begin{aligned} d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)\,[1 + d(fy, Ty)]}{1 + d(fx, fy)} + \gamma\,[d(fx, Tx) + d(fy, Ty)] \\ + \delta\,[d(fx, Ty) + d(fy, Tx)] + \lambda d(fx, fy), \end{aligned} \tag{3.2}$$

for some $\alpha, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leq \alpha + 2(\gamma + \delta) + \lambda < 1$,

(b)

$$d(Tx,Ty) \leq \alpha \frac{d(fx,Tx)\left[1+d(fy,Ty)\right]}{1+d(fx,fy)} + \gamma\left[d(fx,Tx)+d(fy,Ty)\right] + \lambda d(fx,fy), \quad (3.3)$$

where $\alpha, \gamma, \lambda \in [0,1)$ *such that* $0 \leq \alpha + 2\gamma + \lambda < 1$,

(c)

$$d(Tx,Ty) \leq \alpha \frac{d(fx,Tx)\left[1+d(fy,Ty)\right]}{1+d(fx,fy)} + \delta\left[d(fx,Ty)+d(fy,Tx)\right] + \lambda d(fx,fy), \quad (3.4)$$

there exist $\alpha, \delta, \lambda \in [0,1)$ *such that* $0 \leq \alpha + 2\delta + \lambda < 1$,

(d)

$$d(Tx,Ty) \leq \gamma\left[d(fx,Tx)+d(fy,Ty)\right] + \delta\left[d(fx,Ty)+d(fy,Tx)\right] + \lambda d(fx,fy), \quad (3.5)$$

for some $\gamma, \delta, \lambda \in [0,1)$ *with* $0 \leq 2(\gamma+\delta) + \lambda < 1$,

for all x, y *in* X *for which* $fx \neq fy$ *are comparable. If there exists a point* $x_0 \in X$ *such that* $fx_0 \leq Tx_0$ *and the mappings* T *and* f *are compatible, then* T *and* f *have a coincidence point in* X.

Corollary 3.3. *Let* $(X,d,\leq)$ *be a complete partially ordered metric space. Suppose that the mappings* $f,T : X \to X$ *are continuous,* T *is a monotone* f*-nondecreasing,* $T(X) \subseteq f(X)$ *and satisfying the following contraction conditions*

(i)

$$\begin{aligned} d(Tx,Ty) \leq \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} &+ \gamma\left[d(fx,Tx)+d(fy,Ty)\right] \\ &+ \delta\left[d(fx,Ty)+d(fy,Tx)\right] + \lambda d(fx,fy), \end{aligned} \quad (3.6)$$

where $\beta, \gamma, \delta, \lambda \in [0,1)$ *such that* $0 \leq \beta + 2(\gamma+\delta) + \lambda < 1$,

(ii)

$$d(Tx,Ty) \leq \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \gamma\left[d(fx,Tx)+d(fy,Ty)\right] + \lambda d(fx,fy), \quad (3.7)$$

for some $\beta, \gamma, \lambda \in [0,1)$ *with* $0 \leq \beta + 2\gamma + \lambda < 1$,

(iii)

$$d(Tx,Ty) \leq \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \delta\left[d(fx,Ty)+d(fy,Tx)\right] + \lambda d(fx,fy), \quad (3.8)$$

there exist $\beta, \delta, \lambda \in [0,1)$ *such that* $0 \leq \beta + 2\delta + \lambda < 1$,

(iv)

$$d(Tx,Ty) \leq \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \lambda d(fx,fy), \tag{3.9}$$

where $\alpha, \beta, \lambda \in [0,1)$ such that $0 \leq \alpha + \beta + \lambda < 1$,

for all x, y in X for which $fx \neq fy$ are comparable. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Corollary 3.4. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a mapping such that for all comparable $x, y \in X$, the contraction condition(s) in Theorem 3.1 (or Corollaries 3.2 and 3.3) is satisfied. Assume that T satisfies the following hypotheses:*

(i). T is continuous,

(ii). $T(Tx) \leq Tx$ for all $x \in X$.

If there exists a point $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point in X.

Proof. Follow from Theorem 3.1 by taking $f = I_X$ (the identity map). □

We may remove the continuity criteria of T in Theorem 3.1, is still valid by assuming the following hypothesis in X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Theorem 3.5. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that $T, f : X \to X$ are two mappings such that T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying*

$$\begin{aligned} d(Tx,Ty) \leq\ & \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} \\ & + \gamma\,[d(fx,Tx)+d(fy,Ty)] + \delta\,[d(fx,Ty)+d(fy,Tx)] \\ & + \lambda d(fx,fy), \end{aligned} \tag{3.10}$$

for all x, y in X for which $fx \neq fy$ are comparable and where $\alpha, \beta, \gamma, \delta, \lambda \in [0,1)$ such that $0 \leq \alpha+\beta+2(\gamma+\delta)+\lambda < 1$. Assume that there exists $x_0 \in X$ such that $fx_0 \leq Tx_0$ and $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$. If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X.

Further, if T and f are weakly compatible then T and f have a common fixed point in X. Moreover, the set of common fixed points of T and f are well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Suppose $f(X)$ is a complete subset of X. As we know from Theorem 3.1, the sequence $\{Tx_n\}$ is a Cauchy sequence and hence, $\{fx_n\}$ is also a Cauchy sequence in $(f(X), d)$ as $fx_{n+1} = Tx_n$ and $T(X) \subseteq f(X)$. Since $f(X)$ is complete then there exists $fu \in f(X)$ such that

$$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} fx_n = fu. \tag{3.11}$$

Also note that the sequences $\{Tx_n\}$ and $\{fx_n\}$ are nondecreasing and from the hypothesis, we have $Tx_n \leq fu$ and $fx_n \leq fu$ for all $n \in \mathbb{N}$. Since T is a monotone f-nondecreasing, we get $Tx_n \leq Tu$ for all n. Letting $n \to +\infty$, we obtain $fu \leq Tu$.

Suppose that $fu < Tu$, define a sequence $\{u_n\}$ by $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n \in \mathbb{N}$. An argument similar to that in the proof of Theorem 3.1 yields that $\{fu_n\}$ is a nondecreasing sequence and

$$\lim_{n\to+\infty} fu_n = \lim_{n\to+\infty} Tu_n = fv \text{ for some } v \in X. \tag{3.12}$$

So from the hypothesis, we have that $\sup\limits_{n\in N} fu_n \leq fv$ and $\sup\limits_{n\in N} Tu_n \leq fv$.

Notice that

$$fx_n \leq fu \leq fu_1 \leq fu_2 \leq ... \leq fu_n \leq ... \leq fv.$$

Now, we discuss the following two cases:

Case 1: If there exists some $n_0 \geq 1$ with $fx_{n_0} = fu_{n_0}$, then we have

$$fx_{n_0} = fu = fu_{n_0} = fu_1 = Tu,$$

this is a contradiction to $fu < Tu$. Thus, $fu = Tu$, that is, u is a coincidence point of T and f in X.

Case 2: Suppose $fx_n \neq fu_{n+1}$ for all n. Then from condition (3.10), we have

$$\begin{aligned}
d(fx_{n+1}, fu_{n+1}) = d(Tx_n, Tu_n)& \\
&\leq \alpha \frac{d(fx_n, Tx_n)\,[1 + d(fu_n, Tu_n)]}{1 + d(fx_n, fu_n)} + \beta \frac{d(fx_n, Tx_n)\; d(fu_n, Tu_n)}{d(fx_n, fu_n)} \\
&\quad + \gamma\,[d(fx_n, Tx_n) + d(fu_n, Tu_n)] + \delta\,[d(fx_n, Tu_n) + d(fu_n, Tx_n)] \\
&\quad + \lambda d(fx_n, fu_n).
\end{aligned}$$

On taking limit as $n \to +\infty$ in the above inequality and from equations (3.11) and (3.12), we get

$$\begin{aligned}
d(fu, fv) &\leq (2\delta + \lambda)\; d(fu, fv) \\
&< \; d(fu, fv), \text{ since } 2\delta + \lambda < 1.
\end{aligned}$$

So, we have

$$fu = fv = fu_1 = Tu,$$

this is again a contradiction to $fu < Tu$. Hence, we conclude that u is a coincidence point of T and f in X.

Now, we suppose that T and f are weakly compatible. Let w be the coincidence point then

$$Tw = Tfz = fTz = fw, \text{ since } w = Tz = fz, \text{ for some } z \in X.$$

Now from (3.10), we have

$$\begin{aligned} d(Tz, Tw) &\leq \alpha \frac{d(fz, Tz)\,[1 + d(fw, Tw)]}{1 + d(fz, fw)} + \beta \frac{d(fz, Tz)\; d(fw, Tw)}{d(fz, fw)} \\ &\quad + \gamma\,[d(fz, Tz) + d(fw, Tw)] + \delta\,[d(fz, Tw) + d(fw, Tz)] + \lambda d(fz, fw) \\ &\leq (2\gamma + 2\delta + \lambda)\; d(Tz, Tw). \end{aligned}$$

As $2\gamma + 2\delta + \lambda < 1$, then $d(Tz, Tw) = 0$. Therefore, $Tz = Tw = fw = w$. Hence, w is a common fixed point of T and f in X.

Now, suppose that the set of common fixed points of T and f is well ordered, we have to show that the common fixed point of T and f is unique. Let u and v be two common fixed points of T and f such that $u \neq v$, then from condition (3.10), we have

$$\begin{aligned} d(u, v) &\leq \alpha \frac{d(fu, Tu)\,[1 + d(fv, Tv)]}{1 + d(fu, fv)} + \beta \frac{d(fu, Tu)\; d(fv, Tv)}{d(fu, fv)} \\ &\quad + \gamma\,[d(fu, Tu) + d(fv, Tv)] + \delta\,[d(fu, Tv) + d(fv, Tu)] + \lambda d(fu, fv) \\ &\leq (2\gamma + 2\delta + \lambda)\; d(u, v) \\ &< d(u, v), \text{ since } 2\gamma + 2\delta + \lambda < 1, \end{aligned}$$

which is a contradiction and hence, $u = v$. Conversely, suppose T and f have only one common fixed point, then the set of common fixed points of T and f being a singleton is well ordered. □

Besides, in Corollary 3.2 and Corollary 3.3 by relaxing the continuity criteria on T and satisfying the hypotheses given in Theorem 3.5, then T and f have a coincidence point, a common fixed point and its uniqueness in X.

Corollary 3.6. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a mapping such that for all comparable $x, y \in X$, the contraction condition* (3.10) *is satisfied. Suppose that the following hypotheses are satisfied*

- *(i). if $\{x_n\}$ is a non-decreasing sequence in X with respect to $\leq$ such that $x_n \to x \in X$ as $n \to +\infty$, then $x_n \leq x$, for all $n \in \mathbb{N}$ and*
- *(ii). $T(Tx) \leq Tx$ for all $x \in X$.*

If there exists a point $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point in X.

Proof. Follow from Theorem 3.5 by taking $f = I_X$ (the identity map). □

Remark 3.7. *(i). If $\alpha = \gamma = \delta = 0$ in Theorem 3.1 and 3.5, we obtain Theorem 2.1 and 2.3 of Chandok [25].*

(ii). If $f = I$ and $\alpha = \gamma = \delta = 0$ in Theorem 3.1 and 3.5, then we get Theorem 2.1 and 2.3 of Harjani et al. [26].

Some other consequences of the main result for the self mappings involving the integral type contractions are as follows.

Let χ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(a) each φ is Lebesgue integrable function on every compact subset of $[0, +\infty)$ and

(b) for any $\epsilon > 0$, we have $\int_0^\epsilon \varphi(t)dt > 0$, for $t \in [0, +\infty)$.

Corollary 3.8. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the mappings $T, f : X \to X$ are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying*

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt \\ + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt \\ + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.13}$$

for all x, y in X for which $fx \neq fy$ are comparable, $\varphi \in \chi$ and where $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1)$ such that $0 \leq \alpha + \beta + 2(\gamma + \delta) + \lambda < 1$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Similarly, we obtain the following results from Corollaries 3.2 and 3.3 in a complete partially ordered metric space.

Corollary 3.9. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the self-mappings f, T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ satisfying the following contraction conditions*

(a)

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt \\ + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.14}$$

for some $\alpha, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leq \alpha + 2(\gamma + \delta) + \lambda < 1$,

(b)

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.15}$$

where $\alpha, \gamma, \lambda \in [0, 1)$ with $0 \leq \alpha + 2\gamma + \lambda < 1$,

(c)

$$\int_0^{d(Tx,Ty)} \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.16}$$

where $\alpha, \delta, \lambda \in [0, 1)$ such that $0 \leq \alpha + 2\delta + \lambda < 1$,

(d)

$$\int_0^{d(Tx,Ty)} \leq \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.17}$$

there exist $\gamma, \delta, \lambda \in [0, 1)$ such that $0 \leq 2(\gamma + \delta) + \lambda < 1$,

for all x, y in X for which $fx \neq fy$ are comparable, and where $\varphi \in \chi$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Corollary 3.10. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the mappings $f, T : X \to X$ are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following integral type contraction conditions:*

(i)

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.18}$$

for some $\beta, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leq \beta + 2(\gamma + \delta) + \lambda < 1$,

(ii)

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.19}$$

where $\beta, \gamma, \lambda \in [0, 1)$ such that $0 \leq \beta + 2\gamma + \lambda < 1$,

(iii)

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.20}$$

there exist $\beta, \delta, \lambda \in [0,1)$ such that $0 \leq \beta + 2\delta + \lambda < 1$,

(iv)

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.21}$$

where $\alpha, \beta, \lambda \in [0,1)$ with $0 \leq \alpha + \beta + \lambda < 1$,

for all x, y in X for which $fx \neq fy$ are comparable, and where $\varphi \in \chi$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Remark 3.11. *If $\alpha = \gamma = \delta = 0$ in Corollary 3.8, then we obtain the Corollary 2.5 of Chandok [25].*

Now, we give the examples for the main Theorem 3.1.

Example 3.12. *Define a metric $d : X \times X \to [0, +\infty)$ by $d(x,y) = |x-y|$, where $X = [0,1]$ with usual order $\leq$. Let T and f be two self mappings on X such that $Tx = \frac{x^2}{2}$ and $fx = \frac{2x^2}{1+x}$, then T and f have a coincidence point in X.*

Proof. Note that (X,d) is a complete metric space and thus, $(X,d,\leq)$ be a complete partially ordered metric space with respect to usual order $\leq$. Let $x_0 = 0 \in X$ then $fx_0 \leq Tx_0$ and also note that T and f are continuous, T is a monotone f-nondecreasing and $T(X) \subseteq f(X)$.

Now consider the following for any x, y in X with $x < y$,

$$\begin{aligned}
d(Tx,Ty) &= \frac{1}{2}|x^2 - y^2| = \frac{1}{2}(x+y)|x-y| \leq \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y| \\
&\leq \alpha \frac{2x^2|3-x|\left[(1+y) + y^2|3-y|\right]}{4(1+x)(1+y) + 2|x-y|(x+y+xy)} + \frac{\beta}{4}\frac{x^2y^2}{(x+y+xy)}\frac{|x-3||y-3|}{|x-y|} \\
&\quad + \frac{\gamma}{2}\frac{x^2(1+y)|x-3| + y^2(1+x)|y-3|}{(1+x)(1+y)} \\
&\quad + \delta\frac{(1+y)|4x^2 - y^2(1+x)| + (1+x)|4y^2 - x^2(1+y)|}{2(1+x)(1+y)} + \lambda\frac{2(x+y+xy)}{(1+x)(1+y)}|x-y|
\end{aligned}$$

$$\begin{aligned}
&\le \alpha \frac{\frac{x^2|x-3|}{2(1+x)} \cdot \frac{2(1+y)+y^2|3-y|}{2(1+y)}}{1+\frac{2|x-y|(x+y+xy)}{(1+x)(1+y)}} + \beta \frac{\frac{x^2|x-3|}{2(1+x)} \cdot \frac{y^2|y-3|}{2(1+y)}}{2|x-y|\frac{x+y+xy}{(1+x)(1+y)}} + \gamma \left[\frac{x^2|x-3|}{2(1+x)} + \frac{y^2|y-3|}{2(1+y)}\right] \\
&\quad + \delta \left[\left|\frac{x^2}{(1+x)} - \frac{y^2}{2}\right| + \left|\frac{2y^2}{(1+y)} - \frac{x^2}{2}\right|\right] + \lambda \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y| \\
&\le \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta \frac{d(fx,Tx)\; d(fy,Ty)}{d(fx,fy)} + \gamma\,[d(fx,Tx)+d(fy,Ty)] \\
&\quad + \delta\,[d(fx,Ty)+d(fy,Tx)] + \lambda d(fx,fy).
\end{aligned}$$

Then, the contraction condition in Theorem 3.1 holds by selecting proper values of $\alpha, \beta, \gamma, \delta, \lambda$ in $[0,1)$ such that $0 \le \alpha+\beta+2(\gamma+\delta)+\lambda < 1$. Therefore, T and f have a coincidence point $0 \in X$. $\square$

Example 3.13. *Define a distance function $d : X \times X \to [0,+\infty)$ by $d(x,y) = |x-y|$, where $X = [0,1]$ with usual order $\le$. Let T and f be two self mappings on X such that $Tx = x^3$ and $fx = x^4$, then T and f have two coincidence points 0, 1 in X with $x_0 = \frac{1}{4}$.*

4 Applications

Now our aim is to give an existence theorem for a solution of the following integral equation.

$$h(x) = \int_0^M \mu(x,y,h(y))dy + g(x), \quad x \in [0,M], \tag{4.1}$$

where $M > 0$. Let $X = C[0,M]$ be the set of all continuous functions defined on $[0,M]$. Now, define $d : X \times X \to \mathbb{R}^+$ by

$$d(u,v) = \sup_{x\in[0,M]} \{|u(x)-v(x)|\}$$

then, $(X,\le)$ is a partially ordered set. Now, we prove the following result.

Theorem 4.1. *Suppose the following hypotheses holds:*

(i) *$\mu : [0,M]\times[0,M]\times\mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous,*

(ii) *for each $x, y \in [0,M]$, we have*

$$\mu\left(x, y, \int_0^M \mu(x,z,h(z))dz + g(x)\right) \le \mu(x,y,h(y)),$$

(iii) *there exists a continuous function $N : [0,M]\times[0,M] \to [0,+\infty]$ such that*

$$|\mu(x,y,a) - \mu(x,y,b)| \le N(x,y)|a-b| \text{ and}$$

(iv)

$$\sup_{x\in[0,M]}\int_0^M N(x,y)dy \le \gamma$$

for some $\gamma < 1$. *Then, the integral equation* (4.1) *has a solution* $a \in C[0,M]$.

Proof. Define $T : C[0,M] \to C[0,M]$ by

$$Tw(x) = \int_0^M \mu(x,y,w(x))dx + g(x),\ \ x \in [0,M].$$

Now, we have

$$\begin{aligned}
T(Tw(x)) &= \int_0^M \mu(x,y,Tw(x))dx + g(x)\\
&= \int_0^M \mu\left(x,y,\int_0^M \mu(x,z,w(z))dz + g(x)\right)dx + g(x)\\
&\le \int_0^M \mu(x,y,w(z))dz + g(x)\\
&= Tw(x)
\end{aligned}$$

Thus, we have $T(Tx) \le Tx$ for all $x \in C[0,M]$. For any $x^*, y^* \in C[0,M]$ with $x \le y$, we have

$$\begin{aligned}
d(Tx^*,Ty^*) &= \sup_{x\in[0,M]} |Tx^*(x) - Ty^*(x)|\\
&= \sup_{x\in[0,M]} \left|\int_0^M \mu(x,y,x^*(x)) - \mu(x,y,y^*(x))dx\right|\\
&\le \sup_{x\in[0,M]} \int_0^M |\mu(x,y,x^*(x)) - \mu(x,y,y^*(x))|\, dx\\
&\le \sup_{x\in[0,M]} \int_0^M N(x,y)|x^*(x) - y^*(x)|dx\\
&\le \sup_{x\in[0,M]} |x^*(x) - y^*(x)| \sup_{x\in[0,M]} \int_0^M N(x,y)dx\\
&= d(x^*,y^*) \sup_{x\in[0,M]} \int_0^M N(x,y)dx\\
&\le \gamma d(x^*,y^*).
\end{aligned}$$

Moreover, $\{x_n^*\}$ is a nondecreasing sequence in $C[0,M]$ such that $x_n^* \to x^*$, then $x_n^* \le x^*$ for all $n \in \mathbb{N}$. Thus all the required hypotheses of Corollary 3.6 are satisfied. Thus, there exists a solution $a \in C[0,M]$ of the integral equation (4.1). □

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Existence results for a multipoint boundary value problem of nonlinear sequential Hadamard fractional differential equations

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ABSTRACT

In this paper, existence and uniqueness results are established for a nonlinear sequential Hadamard fractional differential equation with multi-point boundary conditions, via Banach and Krasnosel'skiĭ's fixed point theorems and Leray-Schauder nonlinear alternative. An example illustrating the existence of a unique solution is also constructed.

RESUMEN

En este artículo se establecen resultados de existencia y unicidad para una ecuación diferencial fraccional nolineal secuencial de Hadamard con condiciones de borde multi-punto, a través de teoremas de punto fijo de Banach y Krasnosel'skiĭ y la alternativa nolineal de Leray-Schauder. Se construye un ejemplo ilustrando la existencia de una única solución.

Keywords and Phrases: Hadamard fractional integral; Hadamard fractional derivative; multi-point boundary conditions; existence; fixed point theorems.

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1 Introduction

Fractional calculus has been extensively developed during the last few decades as the techniques of this branch of mathematics considerably improved the mathematical modeling of many scientific phenomena, for instance, see [16, 17]. In particular, fractional-order nonlocal boundary value problems are found to be of significant interest for many researchers. Much of the literature on this class of problems is based on Riemann-Liouville or Liouville-Caputo type fractional order differential equations. For details, we refer the reader to some recent works [27] and the references cited therein. In addition to Riemann-Liouville and Caputo type derivatives, there is another kind of derivative, which contains logarithmic function of arbitrary exponent in its definition. This derivative is known as Hadamard derivative [14] and its construction is invariant in relation to dilation and is quite suitable for the problems with semi-infinite domain. For example, Lamb-Bateman integral equation is the one containing Hadamard fractional derivatives of order 1/2 [8]. In [11], a modified Lamb-Bateman equation involving Hadamard derivative and fractional Hyper-Bessel-type operators was studied. One can find application details of Hadamard fractional differential equations in the articles [12, 20]. For some recent results on Hadamard type fractional differential equations, for instance, see [2, 4, 5, 10, 18, 19, 21, 22, 23, 25, 26]. In a recent monograph [3], one can find a detailed description of initial/boundary value problems and inequalities involving Hadamard fractional differential equations and inclusions. New multiple positive solutions for Hadamard-type fractional differential equations with nonlocal conditions on an infinite interval were studied in [28]. In [6], the authors studied a coupled system of Caputo-Hadamard type sequential fractional differential equations supplemented with nonlocal boundary conditions involving Hadamard fractional integrals. A Caputo-Hadamard fractional turbulent flow model was studied in [24]. However, the Hadamard-type fractional boundary value problems are not sufficiently studied in the mainstream literature.

In this paper, motivated by aforementioned work on Hadamard fractional differential equations, we introduce and study a nonlocal multipoint boundary value problem involving a nonlinear sequential Hadamard fractional differential equation to enrich the related literature. Precisely, we investigate the existence criteria for the following problem:

$$\begin{cases} \big({}^{H}D^{\alpha} + \lambda \, {}^{H}D^{\alpha-1} \big) x(t) = f(t, x(t)), \quad 1 < \alpha \leq 2, \quad 1 < t < T, \\ x(1) = 0, \quad x(T) = \sum\limits_{j=1}^{m} \beta_j \, x(t_j), \end{cases} \tag{1.1}$$

where ${}^{H}D^{(\cdot)}$ denotes the Hadamard fractional derivative of order α, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda \in \mathbb{R}^{+}$, t_j, $j = 1, 2, \ldots, m$, are given points with $1 \leq t_1 \leq \ldots \leq t_m < T$, and β_j are appropriate real numbers. An existence and uniqueness result is proved via Banach's fixed point theorem and also two existence results are established by using Krasnosel'skiĭ's fixed point theorem and Leray-Schauder nonlinear alternative.

The remaining part of the paper is structured as follows: In Section 2 we recall the related background material and establish a lemma regarding a linear variant of the problem (1.1), useful to transform the problem (1.1) into an equivalent fixed point problem. Section 3 contains the main results for the problem (1.1). An example illustrating the existence and uniqueness result is also included.

2 Preliminaries

We introduce notations and definitions of fractional calculus.

Definition 2.1. *([3, 17]) The Hadamard fractional integral of order $q \in \mathbb{C}$, $\Re(q) > 0$, for a function $g \in L^p[a,b]$, $0 \le a \le t \le b \le \infty$, is defined as*

$$\begin{aligned} I^q_{a+} g(t) &= \frac{1}{\Gamma(q)} \int_a^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \\ I^q_{b-} g(t) &= \frac{1}{\Gamma(q)} \int_t^b \left(\log \frac{s}{t}\right)^{q-1} \frac{g(s)}{s} ds. \end{aligned}$$

Definition 2.2. *([3, 17]) Let $[a,b] \subset \mathbb{R}$, $\delta = t\frac{d}{dt}$ and $AC^n_\delta[a,b] = \{g : [a,b] \to \mathbb{R} : \delta^{n-1}(g(t)) \in AC[a,b]\}$. The Hadamard derivative of fractional order q for a function $g \in AC^n_\delta[a,b]$ is defined as*

$$\begin{aligned} D^q_{a+} g(t) &= \delta^n (I^{n-q}_{a+})(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds, \\ D^q_{b-} g(t) &= (-\delta)^n (I^{n-q}_{b-})(t) = \frac{1}{\Gamma(n-q)} \left(-t\frac{d}{dt}\right)^n \int_t^b \left(\log \frac{s}{t}\right)^{n-q-1} \frac{g(s)}{s} ds, \end{aligned}$$

where $n-1 < q < n$, $n = [q]+1$ and $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

For more details of the Hadamard fractional integrals and derivatives, we refer the reader to Section 2.7 in the text [17].

Lemma 2.3. *Let $x \in C^2_\delta([1,T],\mathbb{R})$ and $g \in C([1,T],\mathbb{R})$. The (integral) solution of the linear Hadamard fractional boundary value problem:*

$$\begin{cases} \left({}^H D^\alpha + \lambda\, {}^H D^{\alpha-1}\right) x(t) = g(t), \quad 1 < \alpha \le 2, \quad 1 < t < T, \\ x(1) = 0, \quad x(T) = \sum\limits_{j=1}^m \beta_j\, x(t_j), \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned}x(t) &= \frac{1}{\gamma}\Big(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\Big)\Bigg\{\frac{\sum_{j=1}^m \beta_j\, t_j^{-\lambda}}{\Gamma(\alpha-1)}\int_1^{t_j} s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{g(r)}{r}dr\Big)ds \\ &\quad -\frac{T^{-\lambda}}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{g(r)}{r}dr\Big)ds\Bigg\} \\ &\quad +\frac{t^{-\lambda}}{\Gamma(\alpha-1)}\int_1^t s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{g(r)}{r}dr\Big)ds,\end{aligned} \tag{2.2}$$

where it is assumed that

$$\gamma := T^{-\lambda}\int_1^T s^{\lambda-1}(\log s)^{\alpha-2}ds - \sum_{j=1}^m \beta_j\, t_j^{-\lambda}\int_1^{t_j} s^{\lambda-1}(\log s)^{\alpha-2}ds \neq 0. \tag{2.3}$$

Proof. The linear Hadamard fractional differential equation in (2.1) can be rewritten as

$$^{H}D^{\alpha-1}(tD+\lambda)x(t) = g(t), \quad t\in[1,T]. \tag{2.4}$$

Applying the Hadamard fractional operator $I^{\alpha-1}$ on both sides of (2.4), we get

$$\Big(D+\frac{\lambda}{t}\Big)x(t) = t^{-1}\Big(c_1(\log t)^{\alpha-2} + I^{\alpha-1}g(t)\Big),$$

which can be rewritten as

$$D\Big(t^{\lambda}x(t)\Big) = c_1 t^{\lambda-1}(\log t)^{\alpha-2} + t^{\lambda-1}I^{\alpha-1}g(t). \tag{2.5}$$

Integrating (2.5) from 1 to t, we get

$$x(t) = c_0 t^{-\lambda} + c_1 t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds + t^{-\lambda}\int_1^t s^{\lambda-1}\, I^{\alpha-1}\, g(s)ds, \tag{2.6}$$

where $c_i, (i=0,1)$ are unknown arbitrary constants. Using the initial condition $x(1)=0$ in (2.6) implies that $c_0=0$, which leads to

$$x(t) = c_1 t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds + t^{-\lambda}\int_1^t s^{\lambda-1}\, I^{\alpha-1}\, g(s)ds. \tag{2.7}$$

Now using the condition $x(T) = \sum_{j=1}^m \beta_j\; x(t_j)$ in (2.7), we have

$$\begin{aligned}&c_1 T^{-\lambda}\int_1^T s^{\lambda-1}(\log s)^{\alpha-2}ds + T^{-\lambda}\int_1^T s^{\lambda-1}\, I^{\alpha-1}\, g(s)ds \\ &\qquad = c_1\sum_{j=1}^m \beta_j\, t_j^{-\lambda}\int_1^{t_j} s^{\lambda-1}(\log s)^{\alpha-2}ds + \sum_{j=1}^m \beta_j\, t_j^{-\lambda}\int_1^{t_j} s^{\lambda-1}\, I^{\alpha-1}\, g(s)ds,\end{aligned}$$

which, on solving for c_1 together with (2.3), yields

$$c_1 = \frac{1}{\gamma}\Bigg[\sum_{j=1}^m \beta_j\, t_j^{-\lambda}\int_1^{t_j} s^{\lambda-1}\, I^{\alpha-1}\, g(s)ds - T^{-\lambda}\int_1^T s^{\lambda-1}\, I^{\alpha-1}\, g(s)ds\Bigg].$$

Substituting the above value of c_1 in (2.7), we get the desired solution (2.2). The converse of the lemma follows by a direct computation. This completes the proof. ∎

The following lemma contains certain estimates that we need in the sequel.

Lemma 2.4. *For $g \in C([1,T],\mathbb{R})$ with $\|g\| = \sup_{t\in[1,T]} |g(t)|$, we have*

(i) $\left| t^{-\lambda} \int_1^t s^{\lambda-1} \left(\int_1^s \left(\log \frac{s}{r} \right)^{\alpha-2} \frac{g(r)}{r} dr \right) ds \right| \leq \frac{(\log T)^{\alpha}}{\alpha(\alpha-1)} \|g\|.$

(ii) $\left| t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \right| \leq \frac{(\log T)^{\alpha-1}}{(\alpha-1)}.$

Proof. Note that
$$\int_1^s \left(\log \frac{s}{r} \right)^{\alpha-2} \frac{1}{r} dr = \frac{(\log s)^{\alpha-1}}{(\alpha-1)}.$$
Since $s^{\lambda} \leq t^{\lambda}$ for $1 < s < t$, then
$$\begin{aligned}
&\left| t^{-\lambda} \int_1^t s^{\lambda-1} \left(\int_1^s \left(\log \frac{s}{r} \right)^{\alpha-2} \frac{g(r)}{r} dr \right) ds \right| \\
\leq\ & \sup_{t\in[1,T]} \left| t^{-\lambda} \int_1^t s^{\lambda-1} \left(\int_1^s \left(\log \frac{s}{r} \right)^{\alpha-2} \frac{g(r)}{r} dr \right) ds \right| \\
\leq\ & \|g\| \sup_{t\in[1,T]} \left| t^{-\lambda} \int_1^t s^{\lambda-1} \left(\frac{(\log s)^{\alpha-1}}{(\alpha-1)} \right) ds \right| \\
\leq\ & \frac{\|g\|(\log T)^{\alpha}}{\alpha(\alpha-1)}. \quad \square
\end{aligned}$$

3 Existence and uniqueness results

Let $\mathcal{G} = C([1,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[1,T]$ to $\mathbb{R}$ endowed with the usual norm $\|x\| = \sup\{|x(t)| : t \in [1,T]\}$, and $C_{\delta}^{n}([1,T],\mathbb{R})$ denotes the Banach space of all real valued functions g such that $\delta^n g \in \mathcal{G}$.

Using Lemma 2.3, we can transform the problem (1.1) into a fixed point problem as $x = \mathcal{P}x$, where the operator $\mathcal{P} : \mathcal{G} \to \mathcal{G}$ is defined by

$$\begin{aligned}
(\mathcal{P}x)(t) \quad = \quad & \frac{1}{\gamma}\Big(t^{-\lambda} \int_1^t s^{\lambda-1} (\log s)^{\alpha-2} ds \Big) \\
& \times \Bigg\{ \frac{\sum_{j=1}^{m} \beta_j \, t_j^{-\lambda}}{\Gamma(\alpha-1)} \int_1^{t_j} s^{\lambda-1} \Big(\int_1^s \Big(\log \frac{s}{r} \Big)^{\alpha-2} \frac{f(r,x(r))}{r} dr \Big) ds \\
& - \frac{T^{-\lambda}}{\Gamma(\alpha-1)} \int_1^T s^{\lambda-1} \Big(\int_1^s \Big(\log \frac{s}{r} \Big)^{\alpha-2} \frac{f(r,x(r))}{r} dr \Big) ds \Bigg\} \qquad (3.1) \\
& + \frac{t^{-\lambda}}{\Gamma(\alpha-1)} \int_1^t s^{\lambda-1} \Big(\int_1^s \Big(\log \frac{s}{r} \Big)^{\alpha-2} \frac{f(r,x(r))}{r} dr \Big) ds, \quad t \in [1,T].
\end{aligned}$$

For computational convenience, we set

$$\Lambda = \frac{(\log T)^{\alpha-1}}{|\gamma|(\alpha-1)}\left[\frac{\sum_{j=1}^{m}|\beta_j|\,(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.2}$$

In the next theorem, we prove the uniqueness of solutions for problem (1.1) via Banach's fixed point theorem.

Theorem 3.1. *Let $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function and there exists a constant $L>0$ such that:*

(H_1) $|f(t,x)-f(t,y)|\le L|x-y|$, $\forall t\in[1,T]$ *and* $x,y\in\mathbb{R}$.

Then, problem (1.1) has a unique solution on $[1,T]$ if $L\Lambda<1$, where Λ is given by (3.2).

Proof. Let us define M be finite number given by $M=\sup_{t\in[1,T]}|f(t,0)|$, and show that $\mathcal{P}B_r\subset B_r$, where $B_r=\{x\in C[1,T]:\|x\|\le r\}$ with $r\ge\dfrac{M\Lambda}{1-L\Lambda}$.

For $x\in B_r$, $t\in[1,T]$, using (H_1), we get

$$\begin{aligned}
|f(t,x(t))| &= |f(t,x(t))-f(t,0)+f(t,0)| \\
&\le |f(t,x(t))-f(t,0)|+|f(t,0)| \\
&\le L\|x\|+M\le Lr+M.
\end{aligned}$$

Then

$$\begin{aligned}
|\mathcal{P}(x)(t)| \le\ & \sup_{t\in[1,T]}\Bigg\{\frac{1}{|\gamma|}\Big(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\Big) \\
&\times\Bigg[\frac{\sum_{j=1}^{m}|\beta_j\,t_j^{-\lambda}|}{\Gamma(\alpha-1)}\int_1^{t_j}s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))|}{r}dr\Big)ds \\
&+\frac{T^{-\lambda}}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))|}{r}dr\Big)ds\Bigg] \\
&+\frac{t^{-\lambda}}{\Gamma(\alpha-1)}\int_1^t s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))|}{r}dr\Big)ds\Bigg\} \\
\le\ & (Lr+M)\left[\frac{(\log T)^{\alpha-1}}{|\gamma|(\alpha-1)}\left(\frac{\sum_{j=1}^{m}|\beta_j|\,(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
\le\ & \Lambda(Lr+M)\le r.
\end{aligned}$$

In consequence, $\|\mathcal{P}x\|\le r$, for any $x\in B_r$, which shows that $\mathcal{P}B_r\subset B_r$.

Now we prove that the operator $\mathcal{P}$ is a contraction. For $(x,y)\in C([1,T],\mathbb{R})$ and for each $t\in[1,T]$, we obtain

$$
\begin{aligned}
& |(\mathcal{P}x)(t)-(\mathcal{P}y)(t)| \\
\leq\ & \sup_{t\in[1,T]}\Bigg\{\frac{1}{|\gamma|}\Big(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\Big) \\
& \times\Bigg[\frac{\sum_{j=1}^{m}|\beta_j\, t_j^{-\lambda}|}{\Gamma(\alpha-1)}\int_1^{t_j}s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))-f(r,y(r))|}{r}dr\Big)ds \\
& +\frac{T^{-\lambda}}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))-f(r,y(r))|}{r}dr\Big)ds\Bigg] \\
& +t^{-\lambda}\int_1^t s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))-f(r,y(r))|}{r}dr\Big)ds\Bigg\} \\
\leq\ & L\left[\frac{(\log T)^{\alpha-1}}{|\gamma|(\alpha-1)}\left(\frac{\sum_{j=1}^{m}|\beta_j|\,(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
\leq\ & L\Lambda\|x-y\|.
\end{aligned}
$$

By the given condition $L\Lambda<1$, it follows that the operator $\mathcal{P}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (the Banach fixed point theorem). The proof is complete. □

The following existence result is based on the Leray-Schauder nonlinear alternative.

Theorem 3.2 (Nonlinear alternative for single valued maps [13])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u\in\partial U$ (the boundary of U in C) and $\nu\in(0,1)$ with $u=\nu F(u)$.*

Theorem 3.3. *Let $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following conditions hold:*

(H_2) *There exists a function $k\in C([1,T],\mathbb{R}^+)$ and a nondecreasing function $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,x)|\leq k(t)\Psi(\|x\|)$ for all $(t,x)\in[1,T]\times\mathbb{R}$;*

(H_3) *There exists a positive constant $S>0$ such that*

$$\frac{S}{\Psi(S)\|k\|\Lambda}>1,$$

where $\|k\|=\sup_{t\in[1,T]}|k(t)|$ and Λ is defined by (3.2).

Then problem (1.1) has at least one solution on $[1,T]$.

Proof. Firstly, we shall show that the operator $\mathcal{P}$ defined by (3.1) maps bounded sets into bounded sets in $C([1,T],\mathbb{R})$. For a number $r>0$, let $B_r=\{x\in C[1,T]:\|x\|\le r\}$ be a bounded set in $C([1,T],\mathbb{R})$. Then, by assumption (H_2), we obtain

$$\begin{aligned}
|(\mathcal{P}x)(t)| &\le \sup_{t\in[1,T]}\Bigg\{\frac{1}{|\gamma|}\Big(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\Big)\\
&\quad\times\Bigg[\frac{\sum_{j=1}^m|\beta_j\,t_j^{-\lambda}|}{\Gamma(\alpha-1)}\int_1^{t_j}s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,y(r))|}{r}dr\Big)ds\\
&\quad+\frac{T^{-\lambda}}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))|}{r}dr\Big)ds\Bigg]\\
&\quad+\frac{t^{-\lambda}}{\Gamma(\alpha-1)}\int_1^t s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{|f(r,x(r))|}{r}dr\Big)ds\Bigg\}\\
&\le \Psi(\|x\|)\|k\|\Bigg[\frac{(\log T)^{\alpha-1}}{|\gamma|(\alpha-1)}\Bigg(\frac{\sum_{j=1}^m|\beta_j|(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Bigg)+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Bigg],
\end{aligned}$$

and consequently,

$$\|\mathcal{P}x\|\le\Lambda\Psi(r)\|k\|.$$

Next we show that $\mathcal{P}$ maps bounded sets into equicontinuous sets of $C([1,T],\mathbb{R})$. Let $\tau_1,\tau_2\in[1,T]$ with $\tau_1<\tau_2$ and $x\in B_r$. Then, we have

$$\begin{aligned}
|(\mathcal{P}x)(\tau_2)-(\mathcal{P}x)(\tau_1)| &\le \Psi(r)\|k\|\Bigg\{\frac{1}{|\gamma|}\Bigg(|\tau_1^{-\lambda}-\tau_2^{-\lambda}|\int_1^{\tau_1}s^{\lambda-1}(\log s)^{\alpha-2}ds\\
&\quad+\tau_2^{-\lambda}\int_{\tau_1}^{\tau_2}s^{\lambda-1}(\log s)^{\alpha-2}ds\Bigg)\\
&\quad\times\Bigg[\frac{\sum_{j=1}^m|\beta_j||t_j^{-\lambda}|}{\Gamma(\alpha-1)}\int_1^{\tau_1}s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{1}{r}dr\Big)ds\\
&\quad+\frac{T^{-\lambda}}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{1}{r}dr\Big)ds\Bigg]\\
&\quad+\frac{|\tau_1^{-\lambda}-\tau_2^{-\lambda}|}{\Gamma(\alpha-1)}\int_1^{\tau_1}s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{1}{r}dr\Big)ds\\
&\quad+\frac{\tau_2^{-\lambda}}{\Gamma(\alpha-1)}\int_{\tau_1}^{\tau_2}s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{1}{r}dr\Big)ds.
\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x\in B_r$ as $\tau_2-\tau_1\to0$. Therefore, by the Arzelá-Ascoli Theorem, the operator is completely continuous.

The result will follow from Theorem 3.2 once it is established that the set of all solutions to equations $x=\nu\mathcal{P}x$ for $\nu\in(0,1)$ is bounded. Let x be a solution of problem (1.1). Then, for $t\in[1,T]$, as in the first step, we can find that

$$\|x\|=\sup_{t\in[1,T]}\{\nu(\mathcal{P}x)(t)\}\le\Lambda\Psi(\|x\|)\|k\|,$$

which leads to

$$\frac{\|x\|}{\Lambda\Psi(\|x\|)\|k\|} \leq 1.$$

By condition (H_3), there exists $S > 0$ such that $\|x\| \neq S$. Let us set $U = \{x \in C([1,T],\mathbb{R}) : \|x\| < S\}$. Note that the operator $\mathcal{P} : \overline{U} \to C([1,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu\mathcal{P}x$ for some $\nu \in (0,1)$. Consequently, we deduce by Theorem 3.2 that $\mathcal{P}$ has a fixed point $x \in \overline{U}$, which is a solution of problem (1.1). This completes the proof. □

Our final existence result is based on Krasnosel'skiĭ's fixed point theorem.

Theorem 3.4. *(*Krasnosel'skiĭ's fixed point theorem*) Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be the operators such that*

(i) $Ax + By \in M$ whenever $x, y \in M$,

(ii) B is a contraction mapping,

(iii) A is compact and continuous.

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.5. *Let $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition (H_1). In addition, we assume that:*

(H_4) *$|f(t,x)| \leq \mu(t)$ for all $(t,x) \in [1,T] \times \mathbb{R}$, $\mu \in C([1,T],\mathbb{R}^+)$.*

Then, the boundary value problem (1.1) has at least one solution on $[1,T]$, provided that

$$L\Big(\Lambda - \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\Big) < 1, \tag{3.3}$$

where Λ is given by (3.2).

Proof. Consider $B_\rho = \{x \in \mathcal{G} : \|x\| \leq \rho\}$, $\|\mu\| = \sup_{t\in[0,1]} |\mu(t)|$, with $\rho \geq \|\mu\|\Lambda$. Then we define the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ on B_ρ as

$$\begin{aligned}
(\mathcal{P}_1 x)(t) &= \frac{1}{\gamma}\Big(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\Big) \\
&\quad \times \Bigg\{\frac{\sum_{j=1}^m \beta_j\, t_j^{-\lambda}}{\Gamma(\alpha-1)}\int_1^{t_j} s^{\lambda-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{f(r,x(r))}{r}dr\Big)ds \\
&\quad -\frac{T^{-\lambda}}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{f(r,x(r))}{r}dr\Big)ds\Bigg\}, \quad t\in[1,T],
\end{aligned}$$

$$(\mathcal{P}_2 x)(t) = \frac{t^{-\lambda}}{\Gamma(\alpha-1)}\int_1^t s^{\lambda-1}\Big(\int_1^s \Big(\log\frac{s}{r}\Big)^{\alpha-2}\frac{f(r,x(r))}{r}dr\Big)ds, \quad t\in[1,T].$$

As in 3.1 we can prove that $\|\mathcal{P}_1 x + \mathcal{P}_2 y\| \leq \|\mu\|\Lambda < \rho$, and thus, $\mathcal{P}_1 x + \mathcal{P}_2 y \in B_\rho$. By using condition (3.3) it is easy to prove that $\mathcal{P}_1$ is a contraction (see also 3.1). Moreover the continuous operator $\mathcal{P}_2$ is uniformly bounded, as

$$\|\mathcal{P}_2\| \leq \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\|\mu\|,$$

and equicontinuous as

$$\begin{aligned} |(\mathcal{P}_2 x)(\tau_2) - (\mathcal{P}_2 x)(\tau_1)| \quad &\leq \quad \frac{|\tau_1^{-\lambda} - \tau_2^{-\lambda}|}{\Gamma(\alpha-1)} \int_1^{\tau_1} s^{\lambda-1}\Big(\int_1^s \left(\log\frac{s}{r}\right)^{\alpha-2}\frac{1}{r}dr\Big)ds \\ &\quad + \frac{\tau_2^{-\lambda}}{\Gamma(\alpha-1)} \int_{\tau_1}^{\tau_2} s^{\lambda-1}\Big(\int_1^s \left(\log\frac{s}{r}\right)^{\alpha-2}\frac{1}{r}dr\Big)ds. \end{aligned}$$

Hence, by Arzelá-Ascoli Theorem, $\mathcal{P}_2$ is compact on B_ρ. Thus all the assumptions of 3.4 are satisfied and the conclusion of 3.4 implies that the boundary value problem (1.1) has at least one solution on $[1, T]$. The proof is completed. □

Example 3.6. *Consider the boundary value problem for Hadamard fractional differential equations*

$$\begin{cases} \left({}^H D^{7/4} + 2\ {}^H D^{3/4}\right)x(t) = f(t, x(t)), \quad t \in [1, e], \\ x(1) = 0, \quad x(e) = \sum\limits_{j=1}^{3} \beta_j\ x(t_j). \end{cases} \tag{3.4}$$

Here, $\alpha = 7/4,\ \lambda = 2,\ T = e,\ m = 3,\ \beta_1 = 1/3,\ \beta_2 = 1/9, \beta_3 = 1/27,\ t_1 = 5/4,\ t_2 = 3/2,\ t_3 = 7/4$ *and* $f(t, x) = \dfrac{1}{13\sqrt{t^2+24}}\dfrac{|x|}{1+|x|} + \dfrac{1}{t+2} + \log t$.

Clearly, $L = 1/65$ *as* $|f(t,x) - f(t,y)| \leq (1/65)|x-y|$. *Using the given data, we have* $|\gamma| \approx 0.691358$ *and* $\Lambda \approx 1.104500$. *Then* $L\Lambda \approx 0.016992 < 1$. *Thus, by 3.1, the boundary value problem (3.4) has a unique solution on* $[1, e]$.

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Free dihedral actions on abelian varieties

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ABSTRACT

We give a simple construction for hyperelliptic varieties, defined as the quotient of a complex torus by the action of a finite group G that contains no translations and acts freely, with G any dihedral group. This generalizes a construction given by Catanese and Demleitner for D_4 in dimension three.

RESUMEN

Damos una construcción simple de variedades hiperelípticas, definidas como el cociente de un toro complejo por la acción de un grupo finito G que no contiene traslaciones y actúa libremente, con G cualquier grupo diedral. Esto generaliza la construcción de Catanese y Demleitner para D_4 en dimensión tres.

Keywords and Phrases: Abelian varieties, dihedral group, free action.

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1 Introduction

A Generalized Hyperelliptic Manifold X is defined as a quotient $X = T/G$ of a complex torus T by the free action of a finite group G which contains no translations. The manifold X is called a Generalized Hyperelliptic Variety if the torus T is also projective, i.e., it is an Abelian variety. These have Kodaira dimension zero, as Mistretta showed it to be the case for any étale finite quotient of a torus [5]. Furthermore, these type of manifolds are Kähler and their fundamental group is given by the group of complex affine transformations of T which are lifts of transformations of the group G, as stated by Catanese and Corvaja in [2].

Uchida and Yoshihara showed that the only non Abelian group that gives such an action in dimension three is the dihedral group D_4 of order 8 [6]. Later, Catanese and Demleitner gave a simple and explicit construction for that action [4] and completed the characterization of three-dimensional hyperelliptic manifolds [3]. Some authors have worked with these objects in higher dimension as well. For example, Auffarth and Lucchini Arteche showed that they can be constructed using any finite Abelian group, and that there are simple ways to construct some varieties using non Abelian groups [1], all of these formed as products of manifolds of lower dimension. However, not much is known about hyperelliptic manifolds of dimension greater than 3, and a characterization of these manifolds is far from completed.

The purpose of this note is to help with the understanding of Generalized Hyperelliptic Varieties in higher dimension, showing that Catanese and Demleitner's construction is actually generalizable (in quite a natural way) to every odd dimension. Specifically, for every $n \in \mathbb{N}$ we give a Generalized Hyperelliptic Variety of dimension $2n+1$ defined by the action of the dihedral group D_{4n} of order $8n$ acting on a family of Abelian varieties, from which the construction by Catanese and Demleitner remains as the particular case for $n = 1$, and we end with a simple corollary that explains how this allows us to create this type of varieties using any dihedral group.

2 The construction

Let E, E' be any two elliptic curves,

$$E = \mathbb{C}/\left(\mathbb{Z} + \mathbb{Z}\tau\right),\ E' = \mathbb{C}/\left(\mathbb{Z} + \mathbb{Z}\tau'\right).$$

Now, for $n \in \mathbb{N}$ set $A' := E^{2n} \times E'$ and $A := A'/\langle w \rangle$, where $w := (1/2,\ 1/2,\ ...,\ 1/2,\ 0)$.

Theorem 2.1. *The Abelian Variety A admits a free action with no translations of the dihedral group D_{4n} of order $8n$.*

Proof. First, let us recall that for $k \in \mathbb{N}$, the dihedral group of order $2k$ is defined as

$$D_k := \left\langle r, s \,|\, r^k = 1,\, s^2 = 1,\, (rs)^2 = 1 \right\rangle .$$

So, in order to prove the result, we need to find automorphisms of A with the required characteristics, that satisfy the relations described above. And for that, we will use automorphisms of A' that descends to those of A in a useful way. Now, set, for $z := (z_1,\, z_2,\, ...,\, z_{2n},\, z_{2n+1}) \in A'$, the following linear automorphisms:

$$\begin{aligned} R(z) &:= (-z_{2n},\, z_1,\, ...,\, z_{2n-1},\, z_{2n+1}) , \\ S(z) &:= (-z_{2n},\, -z_{2n-1},\, ...,\, -z_2,\, -z_1,\, -z_{2n+1}) , \end{aligned}$$

and with these, we define the following automorphisms of A':

$$\begin{aligned} r(z) &:= R(z) + \left(0,\, ...,\, 0,\, \tfrac{1}{4n}\right) , \\ s(z) &:= S(z) + (b_1,\, b_2,\, ...,\, b_{2n},\, 0), \end{aligned}$$

where, for $i = 1,\, ...,\, n$, $b_{2i-1} := 1/2 + \tau/2$ and $b_{2i} := \tau/2$.

Step 1. It is easy to verify that r and R have order exactly $4n$ on A'. Also, since R is linear and $R(w) = w$, then $r(z + w) = r(z) + w$, and so r descends to an automorphism of A of order exactly $4n$. Moreover, any power r^j, for $0 < j < 4n$, acts freely on A since the $(2n + 1)$-th coordinate of $r^j(z)$ equals $z_{2n+1} + \frac{j}{4n}$ (and so is distinct from z_{2n+1}). Also, clearly none of these powers are translations, due to the fact that their linear part modifies at least one dimension.

Step 2. $s^2(z) = z+w$, since for $i = 1,\, ...,\, 2n$, $b_i - b_{2n+1-i} = 1/2$, and so s does not have order 2 on A'. Neverthless, since the linearity of S and the fact that $S(w) = w$ imply that $s(z+w) = s(z)+w$, s descends to an automorphism of A of order exactly 2.

Step 3. We have

$$rs(z) = zM + b'$$

where

$$M = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ \vdots & & I & & \vdots \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & \dots & 0 & -1 \\ 0 & & -1 & 0 \\ \vdots & -1 & & \vdots \\ -1 & \dots & 0 & 0 \end{bmatrix}$$

and $b' = \left(-b_{2n},\, b_1,\, ...,\, b_{2n-1},\, \frac{1}{4n}\right)$.

Hence, by simple computations, we have that

$$\begin{aligned}(rs)^2(z) &= zM^2 + b'M + b' \\ &= z,\end{aligned}$$

Also, because it was already shown for r and s, it is also true that $rs(z+w) = rs(z) + w$, and so rs descends to an automorphism of A of order 2. Thus, we have an action of D_{4n} on A, since the orders of r, s and rs are precisely $4n$, 2 and 2, respectively.

Step 4. We claim that also the reflections in D_{4n} are not translations, noticing that, since dihedral groups representing even polygons have two conjugacy classes of reflections, those of s and rs, it suffices to observe that these two transformations are not translations.

In the next step we show that they both act freely on A.

Step 5. It is rather immediate that rs acts freely in A, since $rs(z) = z$ in A is equivalent to the difference

$$rs(z) - z = (-b_{2n},\, -z_{2n} - z_2 + b_1,\, ...,\, -z_2 - z_{2n} + b_{2n-1},\, -2z_{2n+1} + \tfrac{1}{4n})$$

being a multiple of w in A', but this is absurd since the only multiples of w are zero and w itself, while $-b_{2n} = \tau/2 \neq 0,\, 1/2$.

On the other hand, s acts freely on A because $s(z) = z$ in A is equivalent to the difference

$$s(z) - z = (-z_{2n} - z_1 + b_1,\, -z_{2n-1} - z_2 + b_2,\, ...,\, -z_1 - z_{2n} + b_{2n},\, -2z_{2n+1})$$

being a multiple of w in A', but the first and $2n$-th coordinate of multiples of w are equal, while here the difference between them is $b_1 - b_{2n} = 1/2 \neq 0$. □

3 Using any dihedral group

Notice that, although the previous construction is somewhat restrictive because it works with very specific dihedral groups, since it is true that $D_n \subseteq D_{kn}$ for all $n, k \in \mathbb{N}$, we can always make a bigger dihedral group act on some variety using the method above, and then restrict ourselves to a smaller one. So we have the following corollary:

Corollary 3.1. *For all $n \in \mathbb{N}$, there exists a free action of the dihedral group D_n of order $2n$ on some Abelian variety of dimension $\frac{\mathbf{lcm}(4,n)}{2} + 1$ that contains no translations.*

It is interesting to observe that, although the relation is far away from being one-to-one, we have shown that for every odd dimension there is an Abelian variety and a dihedral group acting on it, and for every dihedral group there is an Abelian variety of odd dimension on which it acts, in a way that the quotient forms a Generalized Hyperelliptic Variety.

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Approximate solution of Abel integral equation in Daubechies wavelet basis

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ABSTRACT

This paper presents a new computational method for solving Abel integral equation (both first kind and second kind). The numerical scheme is based on approximations in Daubechies wavelet basis. The properties of Daubechies scale functions are employed to reduce an integral equation to the solution of a system of algebraic equations. The error analysis associated with the method is given. The method is illustrated with some examples and the present method works nicely for low resolution.

RESUMEN

Este artículo presenta un nuevo método computacional para resolver la ecuación integral de Abel (tanto de primer como de segundo tipo). El esquema numérico está basado en aproximaciones en la base de ondeletas de Daubechies. Se emplean las propiedades de las funciones de escala de Daubechies para reducir una ecuación integral a la solución de un sistema algebraico de ecuaciones. Se entrega el análisis de error asociado con el método. El método es ilustrado con algunos ejemplos donde el método presentado funciona bien en baja resolución.

Keywords and Phrases: Abel integral equation, Daubechies scale function, Daubechies wavelet, Gauss-Daubechies quadrature rule.

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1 Introduction

The theory of integral equations is a very important tool to deal with problems arising in mathematical physics. Abel integral equation appears in many physical problems of water waves, astrophysics, solid mechanics and in many applied sciences (see [1, 2, 3, 4]). In the year 1823, Abel integral equation was derived directly from the tautochorone problem in physics. In fact this gave birth to the topic known as integral equation.

Before 1930, the branch of mathematics which is related to wavelet began with Joseph Fourier with his theories of frequency analysis, now often referred to Fourier synthesis (see [5]). The concept of wavelet was first mentioned in an appendix of the thesis of A. Haar (see [6]), but the formulation of problems involving wavelets has been developed mostly over last 30 years. Grossman and Morelet [7] developed the continuous wavelet transform and the orthogonal one was developed by Lamarie and Meyer [8]. Daubechies (see [9, 10]) constructed a compactly supported orthogonal wavelet basis that can be generated from a single function with the aim to serve the multiresolution analysis (MRA of $L^2(\mathbb{R})$). Wavelets allow to represent variety of functions and operators very accurately. Furthermore, wavelets setup a connections with fast numerical algorithms [11]. Hence wavelets are used as an efficient tool to solve integral equations.

In this paper we consider the Abel integral equations in the form

$$\text{First kind :} \qquad \int_0^x \frac{y(t)\,\mathrm{d}t}{(x-t)^\mu} = f(x), \tag{1.1}$$

$$\text{Second kind :} \qquad y(x) + \lambda \int_0^x \frac{y(t)\,\mathrm{d}t}{(x-t)^\mu} = f(x). \tag{1.2}$$

Here $0<\mu<1$, $0 \le x \le 1$ and the forcing term $f(x) \in C[0,1]$ in order to confirm the existence and uniqueness of the solution $y(x) \in C[0,1]$, the space of all continuous function defined on $[0,1]$.

The Abel integral equation has been solved earlier analytically and numerically by various methods in the literature. For instance, Yousefi [12] constructed a numerical scheme based on Legendre multiwavelets to solve Abel integral equation. A system of generalized Abel integral equations was solved using Fractional calculus by Mandal et al [13]. Liu and Tao [14] applied mechanical quadrature methods for solving first kind Abel integral equation. Numerical solution of Abel integral equation is obtained using orthogonal functions by Derili and Sohrabi [15]. Alipour and Rostamy [16] used Bernstein polynomials to solve Abel integral equations. Shahsavaram [17] used Haar wavelet as the basis function in the collocation method to solve Volterra integral equation with weakly singular kernel.

In this paper, the unknown function in the integral equation is expanded by employing Daubechies wavelet basis with unknown coefficients. The integral equation is converted into a system algebraic equations utilizing the properties of Daubechies scale functions. After evaluating the unknown coefficients, the values of the unknown function in the integral equations can be determined at any dyadic point in $[0, 1]$.

2 Preliminary concept of Daubechies scale function

Here some important properties of Daubechies scale function with a compact support are presented in a finite interval $[a, b] \subset \mathbb{R}$, where a and $b(> a)$ are integers.

2.1 Two-scale relations

Daubechies constructed a whole new class of orthogonal wavelets that can be generated from a single function $\phi(x)$, known as Daubechies scale function. This scale function has some interesting features like compact support, fractal nature, and unknown structure at all resolutions. Daubechies -K (Dau-K) scale function($K \in \mathbb{N}$) has $2K$ filter coefficients and compact support $[0, 2K-1]$. The two-scale relation of scale function is given by

$$\phi(\cdot) = \sqrt{2}H^T\Phi(\cdot), \tag{2.1}$$

where

$$H = [h_0, h_1, h_2, ..., h_{2K-1}]^T_{2K\times 1} \tag{2.2}$$

and

$$\Phi(\cdot) = [\phi(2\cdot), \phi(2\cdot -1), \phi(2\cdot -2), ..., \phi(2\cdot -2K+1)]^T_{2K\times 1} \tag{2.3}$$

with the normalization condition

$$\int_{\mathbb{R}} \phi(x)\mathrm{d}x = 1. \tag{2.4}$$

The elements h_l $(l = 0, 1, 2, ..., 2K-1)$ are known as filter coefficients or low pass filters. These filter coefficients satisfy the following algebraic relations

$$\sum_{l=0}^{2K-1} h_l = \sqrt{2} \quad ; \quad \sum_{l=0}^{2K-1} h_l h_{l-2m} = \delta_{m0}. \tag{2.5}$$

Here we define two operators, one is the translation operator T and other is the scale transformation operator D as

$$T^k\phi(x) = \phi_k(x) = \phi(x-k) \tag{2.6}$$

and

$$D^j\phi(x) = 2^{\frac{j}{2}}\phi(2^jx). \tag{2.7}$$

For a specific value of resolution j, the translate of scaling functions are orthonormal to each other viz.

$$\int_{\mathbb{R}} \phi_{jk_1}(x)\phi_{jk_2}(x)\mathrm{d}x = \delta_{k_1k_2} \tag{2.8}$$

where

$$\phi_{jk}(x) = 2^{\frac{j}{2}}\phi(2^j x - k). \tag{2.9}$$

It is evident that all the properties of scaling functions are applicable on $\mathbb{R}$. But in the finite interval $[a,b]$ the translation property (2.6) does not hold good for all $k \in \mathbb{Z}$ as well as the orthogonalization condition (2.8) cannot be applied for $\phi_{jk}(x)$. So in order to apply the machinery of Dau-K scale function on a finite interval $[a,b]$, we divide the translate of $\phi(x)$ for a specific resolution j into three classes (cf. Mouley et al. [18] and Panja et al. [19])

$$\begin{array}{rcll} \phi^L_{jk}(\cdot) & = & \phi_{jk}(\cdot)\chi_k(x) & \text{if } k \in \left\{a2^j - 2K + 2, ..., a2^j - 1\right\}, \\ \phi^I_{jk}(\cdot) & = & \phi_{jk}(\cdot)\chi_k(x) & \text{if } k \in \left\{a2^j, ..., b2^j - 2K + 1\right\}, \\ \phi^R_{jk}(\cdot) & = & \phi_{jk}(\cdot)\chi_k(x) & \text{if } k \in \left\{b2^j - 2K + 2, ..., b2^j - 1\right\}. \end{array} \tag{2.10}$$

Here $\chi_k(x)$ is the characteristic function assuming the value 1 or 0 according as $x \in [a,b]$ or $x \notin [a,b]$.

2.2 Scale function at dyadic points

A number of the form $\frac{m}{2^n}$ is known as a dyadic fraction or dyadic rational (m is an integer and n is a natural number). It has extensive application in measurement, the inch is normally subdivided in dyadic rather than decimal fraction. The ancient Egyptians also used dyadic fractions in measurement, with denominators up to 64 [20]. After knowing the value of scale function at integer points within support, it is possible to determine the scale function at any dyadic point with in the support [21] . Using the two-scale relation (2.1) the value of Dau-K scale function $\phi(x)$ at $x = \frac{m}{2^n}$ is calculated as

$$\phi\left(\frac{m}{2^n}\right) = \sum_{l_1=0}^{2K-1} \sqrt{2} h_{l_1} \phi\left(\frac{m - 2^{n-1}l_1}{2^{n-1}}\right). \tag{2.11}$$

Again using the two-scale relation (2.1) we get

$$\phi\left(\frac{m}{2^n}\right) = \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} 2h_{l_1}h_{l_2}\phi\left(\frac{m - 2^{n-1}l_1 - 2^{n-2}l_2}{2^{n-2}}\right). \tag{2.12}$$

Repeating the two-scale relation (2.1) n times, we get

$$\phi\left(\frac{m}{2^n}\right) = \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} ... \sum_{l_n=0}^{2K-1} 2^{\frac{m}{2}} h_{l_1}h_{l_2}...h_{l_n}\phi(m - 2^{n-1}l_1 - 2^{n-2}l_2 ... 2l_{n-1} - l_n). \tag{2.13}$$

3 Multiresolution analysis (MRA) and Daubechies wavelet

Basic concepts of MRA and Daubechies wavelet are discussed in most of the texts on wavelets (see [9, 10, 18, 19, 21]). Why wavelet has started to dominate in different applications such as technology, engineering and applied mathematics, one serious reason behind it is MRA. A MRA on $\mathbb{R}$ is defined as a sequence of nested subspaces V_j of function L^2 on $\mathbb{R}$ with scaling function $\phi(x)$ if the following properties hold,

$$\forall j \in \mathbb{Z}, \quad V_j \subseteq V_{j+1}, \tag{3.1}$$

$$Clos_{L^2}\left(\cup_{j\in\mathbb{Z}} V_j\right) = L^2\left(\mathbb{R}\right), \tag{3.2}$$

$$\cap_{j\in\mathbb{Z}}\, V_j = \{0\}\,, \tag{3.3}$$

$$\phi(x) \in V_j \Leftrightarrow \phi(2x) \in V_{j+1}, \quad \forall j \in \mathbb{Z}. \tag{3.4}$$

Here V_j's are called approximation spaces. The scale function $\phi(x)$ belongs to V_0 and the set $\{\phi(x-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0. The scale function $\phi(x)$ satisfies the two-scale relation (2.1). Also the set $\{\phi_{jk}(x) : k \in \mathbb{Z}\}$ given by (2.9) is a Riesz basis of V_j. From the property (3.1), it is evident that each element of V_{j+1} can be uniquely written as the orthogonal sum of an element in V_j and an element in W_j that contains the complementary details i.e.

$$V_{j+1} = V_j \oplus W_j = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus ... \oplus W_j. \tag{3.5}$$

Let W_j be the span of $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, which is called wavelet function. The wavelet function $\psi(x)$ satisfies the relation

$$\psi(\cdot) = \sqrt{2}G^T\Phi(\cdot) \tag{3.6}$$

where

$$\mathbf{G} = [g_0, g_1, g_2, ..., g_{2K-1}]^T_{2K\times 1}. \tag{3.7}$$

Here $\Phi(\cdot)$ is given by (2.3) and g_l $(l = 0, 1, 2, ..., 2K-1)$ are known as high pass filter coefficients and are given by

$$g_l = (-1)^l h_{2K-1-l}. \tag{3.8}$$

4 Method of approximation

We approximate the unknown function of the integral equations (1.1) and (1.2) in the approximation space V_j as

$$\begin{aligned} y(x) &\approx y_j^{MS}(x) \\ &= \sum_{k=0}^{2^j-1} c_{jk}\phi_{jk}(x) \\ &= \sum_{k=0}^{2^j-2K+1} c^I_{jk}\phi^I_{jk}(x) + \sum_{k=2^j-2K+2}^{2^j-1} c^R_{jk}\phi^R_{jk}(x) \\ &= \mathbf{C}^T\vec{\Phi}(x). \end{aligned} \tag{4.1}$$

As the support of $\phi(x)$ is $[0, 2K-1]$, so $y_j^{MS}(x)$ always vanishes at $x = 0$. The value of $y(x)$ at $x = 0$ for second kind Abel integral equation is obviously $f(0)$ but for the first kind Abel integral equation $y(x)$ cannot be evaluated at $x = 0$ but as $y(x)$ can be evaluated at any dyadic point in $(0, 1]$, it can be evaluated very close to $x = 0$ by making the resolution fairly large. Here $\mathbf{C}$ and $\vec{\Phi}(x)$ both are $2^j \times 1$ vectors, given by

$$\mathbf{C} = \left[c_{j0}^I, c_{j1}^I, ..., c_{j2^j-2K+1}^I, c_{j2^j-2K+2}^R, ..., c_{j2^j-1}^R\right]^T \tag{4.2}$$

and

$$\vec{\Phi}(x) = \left[\phi_{j0}^I(x), \phi_{j1}^I(x), ..., \phi_{j2^j-2K+1}^I(x), \phi_{j2^j-2K+2}^R(x), ..., \phi_{j2^j-1}^R(x)\right]^T. \tag{4.3}$$

Using the approximate form of $y(x)$ in (4.1) in both the first and second kind integral equations (1.1) and (1.2) we get,

$$\mathbf{C}^T \int_0^x \frac{\vec{\Phi}(t)\mathrm{d}t}{(x-t)^\mu} = f(x) \tag{4.4}$$

and

$$\mathbf{C}^T \left[\vec{\Phi}(x) + \lambda \int_0^x \frac{\vec{\Phi}(t)\mathrm{d}t}{(x-t)^\mu}\right] = f(x). \tag{4.5}$$

We choose total 2^j number of points by $x_{jk'} = \frac{k'}{2^j}$ $(k' = 1, 2, 3, ..., 2^j)$ and substituting these points in both the equations (4.4) and (4.5) we get,

$$\mathbf{C}^T \mathbf{B}^{(k')} = f\left(\frac{k'}{2^j}\right) \tag{4.6}$$

and

$$\mathbf{C}^T \left[\mathbf{A}^{(k')} + \lambda \mathbf{B}^{(k')}\right] = f\left(\frac{k'}{2^j}\right) \tag{4.7}$$

where

$$\begin{aligned}\mathbf{A}^{(k')} &= \vec{\Phi}\left(\frac{k'}{2^j}\right)\\ &= \left[\phi_{j0}^I\left(\frac{k'}{2^j}\right), \phi_{j1}^I\left(\frac{k'}{2^j}\right), ..., \phi_{j2^j-2K+1}^I\left(\frac{k'}{2^j}\right), \phi_{j2^j-2K+2}^R\left(\frac{k'}{2^j}\right), ..., \phi_{j2^j-1}^R\left(\frac{k'}{2^j}\right)\right]^T\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}&\mathbf{B}^{(k')} =\\ &\left[\int_0^{\frac{k'}{2^j}} \frac{\phi_{j0}^I(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}, ..., \int_0^{\frac{k'}{2^j}} \frac{\phi_{j2^j-2K+1}^I(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}, \int_0^{\frac{k'}{2^j}} \frac{\phi_{j2^j-2K+2}^R(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}, ..., \int_0^{\frac{k'}{2^j}} \frac{\phi_{j2^j-1}^R(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}\right]^T.\end{aligned} \tag{4.9}$$

As $k = 0, 1, 2, ..., 2^j - 1$ and $k' = 1, 2, 3, ..., 2^j$, each of the equation (4.6) and (4.7) represents a system of 2^j equations in 2^j variables c_{jk}^I and c_{jk}^R. Solving these systems the unknown coefficients c_{jk}^I and c_{jk}^R are obtained.

In the last part of this section, we explain the procedure for calculating the matrix elements of the matrix $\mathbf{B}^{(k^{'})}$. We use the notation

$$\mathbf{I}_{\mu\ j}(k^{'},k)=\int_0^{\frac{k^{'}}{2^j}}\frac{\phi_{jk}\left(t\right)\mathrm{d}t}{(\frac{k^{'}}{2^j}-t)^{\mu}}. \tag{4.10}$$

In the relation (4.10), for $0\leq k\leq 2^j-2K+1$, $\phi_{jk}\left(t\right)$ means $\phi^{I}_{jk}\left(t\right)$ and for $2^j-2K+2\leq k\leq 2^j-1$, $\phi_{jk}\left(t\right)$ means $\phi^{R}_{jk}\left(t\right)$. Using (2.9) we find

$$\mathbf{I}_{\mu\ j}(k^{'},k)=2^{\left(\mu-\frac{1}{2}\right)j}\mathcal{L}_{\mu}(k^{'}-k), \tag{4.11}$$

where

$$\mathcal{L}_{\mu}(k)=\int_0^{k}\frac{\phi\left(t\right)\mathrm{d}t}{(k-t)^{\mu}}. \tag{4.12}$$

As the support of Dau-K scale function $\phi(t)$ is $[0,2K-1]$, so if $k\leq 0$ the range of the integration in (4.12) is completely outside of the support. In this case $\mathcal{L}_{\mu}(k)$ vanishes. Again if $k\geq 2K,\mathcal{L}_{\mu}(k)$ has no singularity within the support $[0,2K-1]$. Using Gauss-Daubechies quadrature rule involving Daubechies scale function [22], $\mathcal{L}_{\mu}(k)$ is evaluated as

$$\mathcal{L}_{\mu}(k)=\sum_{i=1}^{M}\frac{w_i}{(k-t_i)^{\mu}},\quad (k\geq 2K). \tag{4.13}$$

Here w_i , t_i are weights are nodes of Gauss-Daubechies quadrature rule involving Daubechies scale function [22].

For $0<k\leq 2K-1,\ \mathcal{L}_{\mu}(k)$ has integrable singularity at the upper limit so that evaluation of such integrals by using the quadrature rule may not provide their approximate value with desired order of accuracy within less computational time. The two-scale relation (2.1) for $\phi(t)$, may be used to obtain a recurrence relation for $\mathcal{L}_{\mu}(k)$ as

$$\mathcal{L}_{\mu}(k)=2^{\mu-\frac{1}{2}}\sum_{l=0}^{2K-1}h_l\mathcal{L}_{\mu}(2k-l). \tag{4.14}$$

Using the symbols

$$\mathcal{H}_K=\begin{pmatrix} h_1 & h_0 & 0 & 0\cdots & 0 & 0\\ h_3 & h_2 & h_1 & h_0\cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\ldots & \vdots & \vdots\\ 0 & 0 & 0 & 0\cdots & h_{2K-2} & h_{2K-3}\\ 0 & 0 & 0 & 0\cdots & 0 & h_{2K-1}\end{pmatrix} \tag{4.15}$$

and

$$\mathbf{b}_{\mu\ K}=\begin{pmatrix} 0\\ 0\\ \vdots\\ \sum_{l=0}^{2K-4}h_l\mathcal{L}_{\mu}(4K-4-l)\\ \sum_{l=0}^{2K-2}h_l\mathcal{L}_{\mu}(4K-2-l)\end{pmatrix} \tag{4.16}$$

the relation (4.14) can be put in the form

$$\left(I - 2^{\mu-\frac{1}{2}}\mathcal{H}_K\right)\mathcal{L}_\mu = \mathbf{b}_{\mu\ K}. \tag{4.17}$$

So, the singular integrals in $\mathcal{L}_\mu$ are found as

$$\mathcal{L}_\mu = \left(I - 2^{\mu-\frac{1}{2}}\mathcal{H}_K\right)^{-1}\mathbf{b}_{\mu\ K}. \tag{4.18}$$

Thus, evaluation of $\mathcal{L}_\mu(k)$ is summarized as

$$\mathcal{L}(k) = \begin{cases} 0 & k \leq 0, \\ \text{solution obtained by (4.18)} & 1 \leq k \leq 2K-1, \\ \sum_{i=0}^{M} \dfrac{w_i}{(k-t_i)^\mu} & k \geq 2K. \end{cases} \tag{4.19}$$

Table 1: Values of $\mathcal{L}(k)$

k	$\mu = \frac{1}{4}$	$\mu = \frac{1}{3}$	$\mu = \frac{1}{2}$
1	0.925995	1.098666	1.643812
2	1.064183	1.042183	0.954199
3	0.808341	0.759600	0.682604
4	0.748236	0.679445	0.560703
5	0.699178	0.620553	0.488824

In Table 1 the values of $\mathcal{L}(k)$ for $k = 1, 2, ..., 5$ are given taking Dau-3 scale function for $\mu = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. For other values of μ $(0 < \mu < 1)$ these can be easily calculated.

Table 2: Accuracy of $\mathcal{L}(2K)$ for Dau-3 scale function

μ	Detemined by (4.13)	Detemined by (4.18)
1/4	0.662722	0.662722
1/3	0.577792	0.577792
1/2	0.439182	0.439182

In Table 2 the values of $\mathcal{L}(2K)$ for Dau-3 scale function are presented for $\mu = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ using the relations (4.13) and (4.18) separately. For the two methods the values of $\mathcal{L}(2K)$ are found to be same. The values of $\mathcal{L}(2K)$ establish the efficiency of the relation (4.18) in the determination of $\mathcal{L}(k)$ $(k = 0, 1, 2, ..., 2K-1)$.

5 Error estimation

In this section, the error of the proposed method is estimated in detail. For this we need the following definitions and theorems.

Definition 5.1 ([23]). *In a σ-finite measure space $(X, \mathcal{F}, \mu^*)$ (X denotes underlying space, $\mathcal{F}$ is the σ-algebra of measurable sets and μ^* is the measure) the L^p-norm $(1 \leq p < \infty)$ of a function f is defined by*

$$\|f\|_{L^p(X,\ \mathcal{F},\ \mu^*)} = \left(\int_X |f(x)|^p \, d\mu^*(x)\right)^{\dfrac{1}{p}}.$$

The abbreviations $\|f\|_{L^p(X)}$, $\|f\|_{L^p}$, $\|f\|_p$ are also used to mean L^p- norm.

Definition 5.2 ([24]). *The inner product of two functions f and g on a measure space X is defined by*

$$< f, g >= \int_X f\bar{g}\, d\mu.$$

Theorem 5.3 (Minkowski [23]). *If $1 \leq p < \infty$ and $f, g \in L^p$ then $f + g \in L^p$ and $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.*

Theorem 5.4. *Let $\{\phi_{jk}(x) : k \in \mathbb{Z}\}$ and $\{\psi_{jk}(x) : k \in \mathbb{Z}\}$ be the Riesz bases of approximation space V_j and detail space W_j. If $N^B_{j:k,k'} = \int_a^b \phi^B_{jk}(x)\phi^B_{jk'}(x)dx$ and $T^B_{j:k,k'} = \int_a^b \psi^B_{jk}(x)\psi^B_{jk'}(x)dx$ (B stands for L or R) then*

$$T^B_{j:k,k'} = \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} g_{l_1}g_{l_2}N^B_{j+1:2k+l_1,2k'+l_2}.$$

Proof. Here

$$N^B_{j:k,k'} = \int_a^b \phi^B_{jk}(x)\phi^B_{jk'}(x)\mathrm{d}x.$$

Now

$$\begin{aligned}
T^B_{j:k,k'} &= \int_a^b \psi^B_{jk}(x)\psi^B_{jk'}(x)\mathrm{d}x \\
&= 2^j \int_a^b \psi^B(2^jx - k)\psi^B(2^jx - k')\mathrm{d}x \qquad \text{(using expression of } \psi_{j,k}(x)) \\
&= \int_{a2^j}^{b2^j} \psi^B(z - k)\psi^B(z - k')\mathrm{d}z \\
&= \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} g_{l_1}g_{l_2}\int_{a2^{j+1}}^{b2^{j+1}} \phi^B(z - 2k - l_1)\phi^B(z - 2k' - l_2)\mathrm{d}z \qquad \text{(using equation (3.6))} \\
&= \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} g_{l_1}g_{l_2}N^B_{j+1:2k+l_1,2k'+l_2}.
\end{aligned}$$

This completes the proof. □

So to evaluate $T^B_{j:k,k'}$, we need to evaluate $N^B_{j+1:2k+l_1,2k'+l_2}$ $(l_1, l_2 = 0, 1, 2, ..., 2K-1)$. The values of $N^B_{j:k,k'}$ are tabulated in Table 3 and Table 4 in [25].

In section 3 to find the approximate solution, the projection of the unknown function $y^{MS}_j(x)$ is used in the approximation space (the linear span of $\phi_{jk}(x), k = 0, 1, 2, ...2^j - 1$). To estimate the error of the unknown function $y(x) \in L^2([0, 1])$ satisfying both the integral equations (1.1) and (1.2), we employ the fact that the multiscale expansion of $y(x)$ (the projection of $y(x)$ into the approximation space V_j and detail space W_j) is

$$y(x) = \sum_{k=0}^{2^j-1} c_{jk}\phi_{jk}(x) + \sum_{j'=j}^{\infty}\sum_{k=0}^{2^{j'}-1} d_{j'k}\psi_{j'k}(x) \tag{5.1}$$

where

$$c_{jk} \approx \int_0^1 \phi_{jk}(x)y(x)\mathrm{d}x, \tag{5.2}$$

and

$$d_{jk} \approx \int_0^1 \psi_{jk}(x)y(x)\mathrm{d}x. \tag{5.3}$$

Using the two-scale relation (2.1) and the equation (3.6), (5.2) and (5.3) are reduced to

$$c_{jk} = \sum_{l=0}^{2K-1} h_l c_{j+1,2k+l} \ , \tag{5.4}$$

$$d_{jk} = \sum_{l=0}^{2K-1} g_l c_{j+1,2k+l}. \tag{5.5}$$

To evaluate c_{jk} and $d_{jk}, (k = 0, 1, 2, ..., 2^j - 1)$ at level j, we need the values of $c_{j+1,2k+l}$ and $d_{j+1,2k+l}$ at level $j+1$. If $0 \le k \le 2^j - 2K + 1$, c_{jk} and d_{jk} are denoted by c^I_{jk} and d^I_{jk} respectively. Again if $2^j - 2K + 2 \le k \le 2^j - 1$, c_{jk} and d_{jk} are denoted by c^R_{jk} and d^R_{jk} respectively.

Now using the expression for $y^{MS}_j(x)$ given by (4.1), (5.1) is reduced to

$$y(x) = y^{MS}_j(x) + \sum_{j'=j}^{\infty} \delta y_{j'} \tag{5.6}$$

where $\delta y_{j'}$ is given by

$$\begin{aligned} \delta y_{j'} &= \sum_{k=0}^{2^{j'}-1} d_{j'k}\psi_{j'k}(x) \\ &= \sum_{k=0}^{2^{j'}-2K+1} d^I_{j'k}\psi^I_{j'k}(x) + \sum_{k=2^{j'}-2K+2}^{2^{j'}-1} d^R_{j'k}\psi^R_{j'k}(x). \end{aligned} \tag{5.7}$$

The error in the multiscale approximation is given by

$$\begin{aligned} e(x) &= y(x) - y^{MS}_j(x) \\ &= \sum_{j'=j}^{\infty} \delta y_{j'}. \end{aligned} \tag{5.8}$$

Now

$$\begin{aligned}\|e(x)\|^2_{L^2[0,1]} &= \left\| \sum_{j'=j}^{\infty} \delta y_{j'} \right\|^2_{L^2[0,1]} \\ &\leq \sum_{j'=j}^{\infty} \|\delta y_{j'}\|^2_{L^2[0,1]} \\ &= \|\delta y_j\|^2_{L^2[0,1]} \left[1 + \frac{\|\delta y_{j+1}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + \frac{\|\delta y_{j+2}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + ... \right]\end{aligned} \tag{5.9}$$

We choose $\max_{\eta} \frac{\|\delta y_{j+\eta}\|^2_{L^2[0,1]}}{\|\delta y_{j+\eta-1}\|^2_{L^2[0,1]}} = \tau$ for $\eta = 1, 2, 3, ...$ and τ is found to satisfy the condition $0 < \tau < 1$, which is verified by taking a few examples of Abel first kind and second kind integral equations. The values of τ are different for different examples. Then the expression in (5.9) becomes

$$\begin{aligned}\|\delta y_j\|^2_{L^2[0,1]} \left[1 + \frac{\|\delta y_{j+1}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + \frac{\|\delta y_{j+2}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + ... \right] &\leq \|\delta y_j\|^2_{L^2[0,1]} \left[1 + \tau + \tau^2 + \tau^3 + ... \right] \\ &= \|\delta y_j\|^2_{L^2[0,1]} \frac{1}{1-\tau}.\end{aligned} \tag{5.10}$$

The expression for $\|\delta y_j\|^2_{L^2[0,1]}$ is obtained by using orthonormality property of $\psi_{jk}(x)$ within its support and Theorem 5.4 for the partial support of $\psi_{jk}(x)$. This is given by

$$\begin{aligned}\|\delta y_j\|^2_{L^2[0,1]} &= \left\langle \sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(x), \sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(x) \right\rangle \\ &= \sum_{k=0}^{2^j-2K+1} \sum_{k'=0}^{2^j-2K+1} d^I_{jk} d^I_{jk'} \delta_{kk'} + \sum_{k=2^j-2K+2}^{2^j-1} \sum_{k'=2^j-2K+2}^{2^j-1} d^R_{jk} d^R_{jk'} T^R_{j:kk'}.\end{aligned} \tag{5.11}$$

As $\int_0^1 \psi^R_{jk}(x)\psi^I_{jk'}(x)\mathrm{d}x$ and $\int_0^1 \psi^I_{jk}(x)\psi^R_{jk'}(x)\mathrm{d}x$ vanish, so we neglect those terms in the expression (5.11) which contain these specific integrals.

The bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$ can be estimated from the inequality (5.10).

6 Illustrative examples

Example 1

Consider the first kind Abel integral equation

$$\int_0^x \frac{y(t)\mathrm{d}t}{(x-t)^{\mu}} = B\left(1-\mu, 1+\nu\right) x^{1+\nu-\mu}, \quad 0 < \mu < 1, \quad \nu > 0$$

which has the exact solution $y(x) = x^{\nu}$. Here $B(m, n)$ is the beta function and defined by $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x,\ m > 0,\ n > 0.$

Table 3 shows the exact and approximate solutions of the example 1 at the points $x = \frac{i}{8}$ for $i = 1, 2, ..., 7$ taking Dau-3 scale function and $M = 5$. In this table, four sets of values of μ and ν are considered taking both fraction and integer values of ν.

Table 3: Comparison of exact and approximate solutions of Example 1

	x	Exact Solution	Approximate solution $j=4$	$j=6$	$j=8$
$\mu=\frac{1}{4}, \nu=\frac{1}{2}$	1/8	0.353553	0.309319	0.352867	0.353554
	2/8	0.500000	0.486212	0.499995	0.500000
	3/8	0.612372	0.608044	0.612374	0.612373
	4/8	0.707107	0.705733	0.707108	0.707107
	5/8	0.790569	0.790135	0.790570	0.790569
	6/8	0.866025	0.865890	0.866026	0.866025
	7/8	0.935414	0.935375	0.935415	0.935414
$\mu=\frac{1}{4}, \nu=3$	1/8	0.001953	0.001805	0.001951	0.001953
	2/8	0.015625	0.015478	0.015623	0.015625
	3/8	0.052734	0.052588	0.052732	0.052734
	4/8	0.125000	0.124854	0.124998	0.125000
	5/8	0.244141	0.243995	0.244138	0.244141
	6/8	0.421875	0.421730	0.421873	0.421875
	7/8	0.669922	0.669776	0.669920	0.669922
$\mu=\frac{3}{4}, \nu=\frac{1}{2}$	1/8	0.353553	0.358049	0.353775	0.353575
	2/8	0.500000	0.500476	0.500099	0.500009
	3/8	0.612372	0.613000	0.612433	0.612378
	4/8	0.707107	0.707550	0.707149	0.707111
	5/8	0.790569	0.790915	0.790602	0.790572
	6/8	0.866025	0.866305	0.866051	0.866028
	7/8	0.935414	0.935647	0.935436	0.935416
$\mu=\frac{3}{4}, \nu=3$	1/8	0.001953	0.001870	0.001951	0.001953
	2/8	0.015625	0.015525	0.015623	0.015625
	3/8	0.052734	0.052630	0.052732	0.052734
	4/8	0.125000	0.124892	0.124998	0.125000
	5/8	0.244141	0.244030	0.244139	0.244141
	6/8	0.421875	0.421763	0.421873	0.421875
	7/8	0.669922	0.669809	0.669920	0.669922

Table 4: Values of $\|\delta y_j\|^2_{L^2[0,1]}$ for different resolution j

	j	For d^I_{jk}	For both d^I_{jk} and d^R_{jk}
$\mu = \frac{1}{4}, \nu = \frac{1}{2}$	4	5.75007×10^{-8}	1.14422×10^{-3}
	5	1.43777×10^{-8}	5.71189×10^{-4}
	6	3.59444×10^{-9}	2.85378×10^{-4}
	7	8.9861×10^{-10}	1.42637×10^{-4}
	8	2.24653×10^{-10}	7.13055×10^{-5}
	9	5.61631×10^{-11}	3.56496×10^{-5}
$\mu = \frac{1}{4}, \nu = 3$	4	3.92808×10^{-9}	1.15222×10^{-3}
	5	7.16154×10^{-11}	5.74173×10^{-4}
	6	1.19898×10^{-12}	2.86245×10^{-4}
	7	1.93591×10^{-14}	1.42869×10^{-4}
	8	3.07368×10^{-16}	7.13653×10^{-5}
	9	4.84076×10^{-18}	3.56647×10^{-5}
$\mu = \frac{3}{4}, \nu = \frac{1}{2}$	4	1.28416×10^{-7}	1.14442×10^{-3}
	5	3.21065×10^{-8}	5.71226×10^{-4}
	6	2.85385×10^{-9}	2.86245×10^{-4}
	7	2.00666×10^{-9}	1.42638×10^{-4}
	8	5.01665×10^{-10}	7.13058×10^{-5}
	9	1.25416×10^{-10}	3.56496×10^{-5}
$\mu = \frac{3}{4}, \nu = 3$	4	3.92901×10^{-9}	1.15223×10^{-3}
	5	7.16226×10^{-11}	5.74174×10^{-4}
	6	1.9904×10^{-12}	2.86245×10^{-4}
	7	1.93595×10^{-14}	1.42869×10^{-4}
	8	3.07371×10^{-16}	7.13653×10^{-5}
	9	4.84079×10^{-18}	3.56648×10^{-5}

Table 5: Comparison of Sup error and bound of L^2-norm of error $\|e(x)\|_{L^2[0,1]}$

	j	Sup error	Bound of $\|e(x)\|_{L^2[0,1]}$ taking d^I_{jk}	taking d^I_{jk} and d^R_{jk}
$\mu = \frac{1}{4}, \nu = \frac{1}{2}$	4	4.423400×10^{-2}	2.768973×10^{-4}	4.783764×10^{-2}
	6	6.867130×10^{-4}	6.923054×10^{-5}	3.389050×10^{-2}
	8	7.100990×10^{-7}	1.730766×10^{-5}	1.194198×10^{-2}

$\mu = \frac{1}{4}, \nu = 3$	4	1.48274×10^{-4}	6.324620×10^{-5}	4.800458×10^{-2}
	6	2.27344×10^{-6}	1.104969×10^{-6}	1.691878×10^{-2}
	8	3.55446×10^{-8}	1.769186×10^{-8}	1.194201×10^{-2}
$\mu = \frac{3}{4}, \nu = \frac{1}{2}$	4	4.49596×10^{-3}	4.165755×10^{-4}	4.784182×10^{-2}
	6	2.21534×10^{-4}	1.041480×10^{-4}	2.389079×10^{-2}
	8	2.13190×10^{-5}	2.603701×10^{-5}	1.194201×10^{-2}
$\mu = \frac{3}{4}, \nu = 3$	4	8.32861×10^{-5}	6.316013×10^{-5}	4.800479×10^{-2}
	6	1.68850×10^{-6}	1.105110×10^{-6}	2.023927×10^{-2}
	8	2.90795×10^{-8}	1.769375×10^{-8}	1.194699×10^{-2}

Example 2

Consider the second kind Abel integral equation [12]

$$y\left(x\right) = x^2 + \frac{16}{5}x^{\frac{5}{2}} - \int_0^x \frac{y(t)\mathrm{d}t}{\sqrt{x-t}}$$

which has the exact solution $y(x) = x^2$.

Table 6 shows the exact and approximate solutions of the example 2 at the points $x = \frac{i}{8}$ for $i = 0, 1, 2, ..., 7$ taking Dau-3 scale function and $M = 5$.

Table 6: Comparison of exact and approximate solutions of example 2

x	Exact Solution	Approximate solution		
		$j = 4$	$j = 6$	$j = 8$
0	0	0	0	0
1/8	0.015625	0.015508	0.015624	0.015625
2/8	0.062500	0.062463	0.062499	0.062500
3/8	0.140625	0.140603	0.140625	0.140625
4/8	0.250000	0.249984	0.250000	0.250000
5/8	0.390625	0.390613	0.390625	0.390625
6/8	0.562500	0.562490	0.562500	0.562500
7/8	0.765625	0.765617	0.765625	0.765625

Table 7: Values of $\|\delta y_j\|^2_{L^2[0,1]}$ for different resolution j

j	For d^I_{jk}	For both d^I_{jk} and d^R_{jk}
4	1.45784×10^{-12}	1.15094×10^{-3}
5	4.66357×10^{-14}	5.73210×10^{-4}
6	1.46128×10^{-15}	2.85926×10^{-4}
7	4.53812×10^{-17}	1.42779×10^{-4}
8	1.40555×10^{-18}	7.13417×10^{-5}
9	4.35409×10^{-20}	3.56587×10^{-5}

Table 8: Comparison of Sup error and bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$

j	Sup error	Bound of $\|e(x)\|_{L^2[0,1]}$	
		taking d^I_{jk}	taking d^I_{jk} and d^R_{jk}
4	1.16723×10^{-4}	1.22720×10^{-6}	4.79779×10^{-2}
6	9.56852×10^{-7}	3.88534×10^{-8}	2.39134×10^{-2}
8	1.34341×10^{-8}	1.20500×10^{-9}	1.19450×10^{-2}

Example 3

Consider the second kind Abel integral equation [17]

$$y\left(x\right) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{4}\int_0^x \frac{y(t)\mathrm{d}t}{\sqrt{x-t}}$$

which has the exact solution $y(x) = \dfrac{1}{\sqrt{x+1}}$.

Table 9 shows the exact and approximate solutions of the example 3 at the points $x = \frac{i}{8}$ for $i = 0, 1, 2, ..., 7$ taking Dau-3 scale function and $M = 5$.

Table 9: Comparison of exact and approximate solutions of Example 3

x	Exact Solution	Approximate solution		
		$j=4$	$j=6$	$j=8$
0	1	1	1	1
1/8	0.942809	0.964541	0.947179	0.943883
2/8	0.894427	0.905166	0.897201	0.895110
3/8	0.852803	0.861371	0.854894	0.853318
4/8	0.816497	0.823468	0.818192	0.816914
5/8	0.784465	0.790355	0.785898	0.784818
6/8	0.755929	0.761042	0.757173	0.756236
7/8	0.730297	0.734819	0.731398	0.730568

Table 10: Values of $\|\delta y_j\|^2_{L^2[0,1]}$ for different resolution j

j	For d^I_{jk}	For both d^I_{jk} and d^R_{jk}
4	4.60915×10^{-6}	5.80526×10^{-4}
5	2.39761×10^{-6}	2.88967×10^{-4}
6	1.22773×10^{-6}	1.44416×10^{-4}
7	6.23369×10^{-7}	7.20037×10^{-5}
8	3.14886×10^{-7}	3.59832×10^{-5}
9	1.58536×10^{-7}	1.79872×10^{-5}

Table 11: Comparison of Sup error and bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$

j	Sup error	Bound of $\|e(x)\|_{L^2[0,1]}$	
		taking d^I_{jk}	taking d^I_{jk} and d^R_{jk}
4	2.17315×10^{-2}	3.09877×10^{-3}	3.40742×10^{-2}
6	4.37017×10^{-3}	1.59930×10^{-3}	1.69951×10^{-2}
8	1.07361×10^{-3}	8.09946×10^{-4}	8.48330×10^{-3}

We present in Tables 4, 7 and 10 the values of $\|\delta y_j\|^2_{L^2[0,1]}$ ($j = 4, 5, 6, \ldots\ldots, 9$) given by equation (5.11) for the examples 1, 2 and 3 respectively. Second column of all tables present the values $\|\delta y_j\|^2_{L^2[0,1]}$ taking only d^I_{jk} i.e. taking only first term of (5.11), whereas third column presents the values $\|\delta y_j\|^2_{L^2[0,1]}$ taking both d^I_{jk} and d^R_{jk}. From these tables it appears that the values of $\|\delta y_j\|^2_{L^2[0,1]}$ gradually decrease if the resolution j increases. The presence of a few d^R_{jk} in (5.11) makes a lot of difference in calculating $\|\delta y_j\|^2_{L^2[0,1]}$ taking only d^I_{jk} and taking both d^I_{jk} and d^R_{jk}.

In Tables 5, 8 and 11, the Sup errors are compared with the bound of L^2-norm of error $\|e(x)\|_{L^2[0,1]}$ taking d^I_{jk} and taking both d^I_{jk} and d^R_{jk} for examples 1, 2 and 3 respectively. To evaluate bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$, $\tau = 0.250044, \tau = 0.50$; $\tau = 0.250044, \tau = 0.50$; $\tau = 0.250044, \tau = 0.50$ and $\tau = 0.250044, \tau = 0.50$ are used for the four sets of values of μ and ν taking only d^I_{jk} and taking both d^I_{jk} and d^R_{jk} for example 1. Also to evaluate bound of L^2 norm of error $\|e(x)\|_{L^2[0,1]}$, $\tau = 0.032$, $\tau = 0.50$ and $\tau = 0.52$, $\tau = 0.50$ are used for Examples 2 and 3 respectively. Sup errors are calculated taking maximum absolute difference of exact and approximate solutions from Tables 3, 6 and 9.

Figures 1 to 6 display the exact and approximate solutions of examples 1, 2 and 3 for different resolutions ($j = 4, 6, 8$). We observe from these figures that as j increases, an approximate solution becomes closer to exact solution. This demonstrates efficiency of the proposed method.

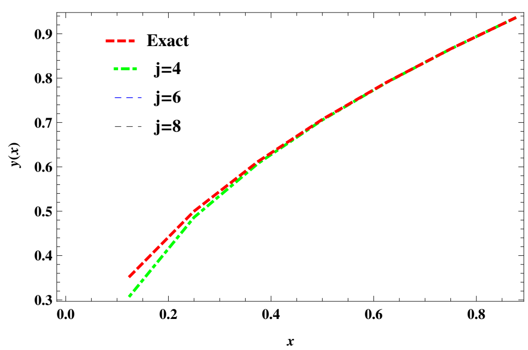

Figure 1: Example 1 ($\mu = \frac{1}{4}, \nu = \frac{1}{2}$)

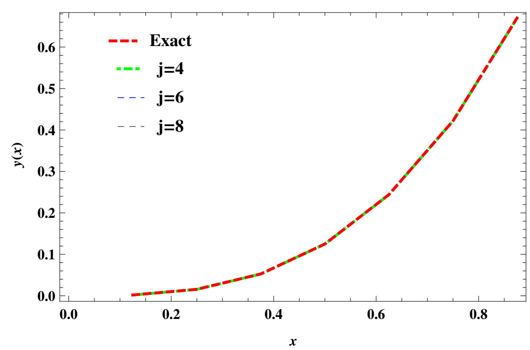

Figure 2: Example 1 ($\mu = \frac{1}{4}, \nu = 3$)

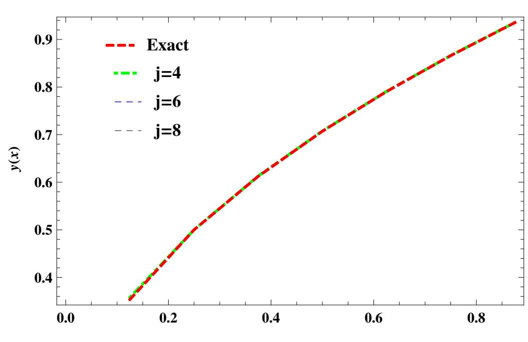

Figure 3: Example 1 ($\mu = \frac{3}{4}, \nu = \frac{1}{2}$)

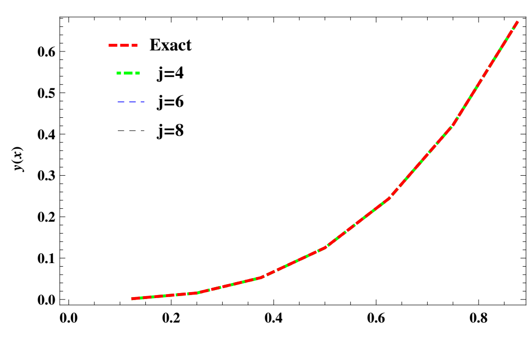

Figure 4: Example 1 ($\mu = \frac{3}{4}, \nu = 3$)

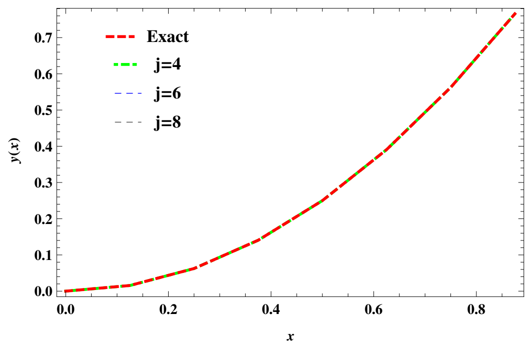

Figure 5: Example 2

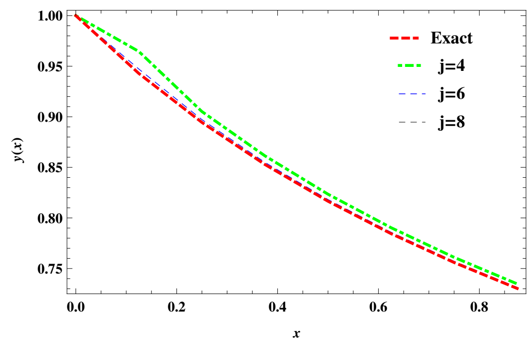

Figure 6: Example 3

7 Conclusion

The purpose of the present work is to develop an efficient and accurate numerical scheme based on Daubechies wavelet basis to solve Abel integral equation. As wavelets are orthogonal systems, they have different resolution capabilities. The detail error estimation shows that the bound of L^2-norm of error $\|e(x)\|_{L^2[0,1]}$ depends on resolution j. From Tables 3, 6 and 9 it appears that the present numerical scheme works nicely for low resolution ($j = 4, 6, 8$). The results can be further improved by taking larger resolution j.

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On Rellich's Lemma, the Poincaré inequality, and Friedrichs extension of an operator on complex spaces

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ABSTRACT

This paper is mainly concerned with: (i) a generalization of the Rellich's Lemma to a Riemann subdomain of a complex space, (ii) the Poincaré inequality, and (iii) Friedrichs extension of a Schrödinger type operator. Applications to the eigenfunction expansion problem associated to the modified Laplacian are also given.

RESUMEN

Este artículo se enfoca principalmente en: (i) una generalización del Lema de Rellich a un subdominio de Riemann de un espacio complejo, (ii) la desigualdad de Poincaré, y (iii) la extensión de Friedrichs de un operador de tipo Schrödinger. Se entregan también aplicaciones al problema de expansión de funciones propias asociado al Laplaciano modificado.

Keywords and Phrases: Weighted Sobolev-Schrödinger product, Friedrichs extension, resolvent mapping.

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1 Introduction

A milestone of pure and applied analysis since the last century is a selection principle of F. Rellich ([15], [4, p. 414]): *Given a family of $\mathscr{C}^1$-functions f in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary such that both the functions f and their first partial derivatives are uniformly bounded in the $L^2(\Omega)$-norm, then $\{f\}$ contains a Cauchy subsequence with respect to the $L^2(\Omega)$-norm.* One consequence of this far reaching result is the *Rellich's Principle*: The Laplacian with zero boundary condition on a bounded domain $\Omega \subset \mathbb{R}^N$ has a compact resolvent. Thus the eigenfunctions of the equation

$$-\Delta\psi = \mu\psi \quad \text{in } \Omega, \quad \psi|\partial\Omega = 0,$$

form a complete orthogonal basis of $L^2(\Omega)$. In view of some recurring interest concerning the Rellich embedding property on non-flat domains ([14], [13, 3.9.3], [11]), it seems of value to consider the question as to on what general domain the Rellich embedding and its consequences will remain valid. A main purpose of this note is to propose a setting of Riemann subdomains (in a complex space) on which an affirmative answer could be sought.

In view of the fact that on a singular space analytic tools are not yet sufficiently developed [1], the basic notions of the Sobolev spaces on a Riemann subdomain (see [20]) need be properly formulated (to be recalled below). Assume throughout this paper that (D, p) is a (relatively compact) *Riemann subdomain* in a complex space Y of dimension m, meaning that D is a relatively compact open subset of Y and (as a subspace of Y) admits a holomorphic map $p : D \to \mathbb{C}^m$ with discrete fibers. Note that the pair (D, p) is a *Riemann domain* in the sense of [3, p. 19] and [8, p. 135]. If in addition p is a local homeomorphism, then (D, p) is said to be *unramified*. Every complex space of pure dimension m admits locally an open neighborhood (of each given point) and a finite, open holomorphic map onto a domain in $\mathbb{C}^m$ ([9, pp. 107-108]), hence is locally realizable as a Riemann subdomain.

Let $\mathfrak{h}$ denote a $(2m+1)$-tuple $(h_0, h_1, \cdots, h_{2m})$ of locally integrable functions h_j on D. Set $V := -h_0$, and denote by $\underline{\mathfrak{h}}$ the $(2m+1)$-tuple obtained by replacing the initial entry by 0 (thus $\underline{\mathfrak{h}} = (0, h_1, \cdots, h_{2m})$). Let μ be a non-negative constant. A $(2m+1)$-tuple $\mathfrak{h}$ (as above) is called an *allowable weight for* (D, μ) if each component h_j, $j = 1, \cdots, m$, is positive a. e. on D, and either $(\mu, V) = (0, 0)$ or $\mathfrak{c}_0 := \mu - \operatorname{ess\,sup}_D V > 0$. (For convenience) call a $2m$-tuple $\mathfrak{h}' = (h_1, \cdots, h_{2m})$ with positive a. e., locally integrable components an *allowable weight on* D, and set $\underset{\rightarrow}{\mathfrak{h}'} := (0, \mathfrak{h}')$. (Unless explicitly specified) in the following let $\mathfrak{h}$ denote an allowable weight for (D, μ). Let D^* be the largest open subset of D on which the map $p = (p_1, \cdots, p_m)$ is locally biholomorphic, and set $\tilde{x}_j := \mathrm{Re}(p_j)$, $\tilde{y}_j := \mathrm{Im}(p_j)$, and

[1] See Ruppenthal [17, p. 7]) "$\cdots$ whereas geometric and algebraic methods $\cdots$ are very well developed on singular spaces, most analytic tools are still missing".

$$\partial_j := \frac{\partial}{\partial \tilde{x}_k}, \;\; j = 2k-1, \;\; \partial_j := \frac{\partial}{\partial \tilde{y}_k}, \;\; j = 2k, \quad 1 \le k \le m,$$

on D^*. The space $\mathscr{C}^1(\overline{D})$ is equipped with the (*weighted*) *Sobolev-Schrödinger product*

$$\langle w, v\rangle_{\mu,\mathfrak{h}} := ((\mu - V)w, \bar{v})_D + [w, v]_{D,(h_1,\cdots,h_{2m})} \quad \forall w, v \in \mathscr{C}^1(\overline{D}), \tag{1.1}$$

where $[\;,\;]_{D,\mathfrak{h}'}$ denotes the (*weighted*) *Dirichlet product*

$$[w, v]_{D,(h_1,\cdots,h_{2m})} := \sum_{k=1}^{m} \int_{D^*} \big(h_{2k-1} \frac{\partial w}{\partial \tilde{x}_k}\frac{\partial \bar{v}}{\partial \tilde{x}_k} + h_{2k}\, \frac{\partial w}{\partial \tilde{y}_k}\frac{\partial \bar{v}}{\partial \tilde{y}_k}\big)\, d\tilde{v}. \tag{1.2}$$

Here $d\tilde{v}$ denotes the pull-back of the Euclidean volume form on $\mathbb{C}^m$ under p.

Clearly for all $g \in \mathscr{C}^1(\overline{D})$ one has

$$\|g\|^2_{\mu,\mathfrak{h}} \ge \int_D (\mu - V)|g|^2\, d\tilde{v} \;\ge\; \mathfrak{c}_0 \|g\|^2_{L^2(D)}.$$

The sesqilinear form "$\langle\;,\;\rangle_{\mu,\mathfrak{h}}$" being elliptic, defines a scalar product on $\mathscr{C}^1(\overline{D})$. With respect to the induced norm $\|\quad\|_{\mu,\mathfrak{h}}$ the completion of $\mathscr{C}^1(\overline{D})$ gives rise to the Sobolev space $H^1_{\mu,\mathfrak{h}}(D)$, with induced scalar product (denoted by the same) and induced norm $\|\quad\|_{\mu,\mathfrak{h}}$. Similarly the completion of the space $\mathscr{C}^{\infty,c}(D)$ of test functions defines the *Sobolev space* $H^1_{\mu,\mathfrak{h},c}(D)$ (see [20] and also [19]). Observe also that the Dirichlet product

$$[w,\, v]_{D,(h_1,\cdots,h_{2m})} = \langle w,\, v\rangle_{1,(0,h_1,\cdots,h_{2m})} - (w,\, \bar{v})_D$$

where $w, v \in \mathscr{C}^{1,c}(\overline{D})$ (given explicitly by the equation (1.2)) extends to a scalar product on $H^1_{0,(0,h_1,\cdots,h_{2m})}(D)$. The norm of $f \in H^1_{1,(0,h_1,\cdots,h_{2m})}(D)$ is given by

$$\|f\|_{1,(0,h_1,\cdots,h_{2m})} = \big(\int_D |f|^2\, d\tilde{v} \,+\, [f,f]_{D,(h_1,\cdots,h_{2m})}\big)^{\frac{1}{2}}.$$

As an application of the inner products (1.1)-(1.2), an explicit representation of the Friedrichs extension of the weighted Schrödinger operator [2] on a Riemann subdomain, allowing possibly singular points, is derived (Theorem 3.1).

It is well-known that the employment of the Poincaré inequality plays a central role in the study of compact Sobolev embeddings. The connection between the (classical) Poincaré inequality and the Rellich embedding theorem (for Euclidean domains) was clarified by Galaz-Fontes [7]. [3] Given an allowable $2m$-weight $\mathfrak{h}'$ on a non-flat subdomain (D,p), questions remain open, however, as to

[2] This operator is motivated by the classical Schrödinger operator (appearing in the Schrödinger equation); the latter has been considered as "one of the most interesting objects in mathematical physics $\cdots$" (see [12, p. 3]).

[3] The Poincaré inequality is sometimes referred to as the *Poincaré-Friedrichs inequality.* An underlying incentive for this paper is provided by a motivational remark in [7] (where by the "Friedrichs inequality" is meant the "Poincaré inequality"), "an explicit connection between the "Friedrichs inequality" and the Rellichs theorem has not been reported" (at least for a Riemann subdomain).

whether (either of) the following properties holds: (1) (D,p) has the *Poincaré property relative to* $\mathfrak{h}'$, in the sense that there exists a constant $C_{D,\mathfrak{h}'}$ such that the following Poincaré inequality (with variable coefficients) holds:

$$\|f\|_{L^2(D)} \le C_{D,\mathfrak{h}'}[f,f]^{\frac{1}{2}}_{D,\mathfrak{h}'} \quad \forall f \in H^1_{1,(0,\mathfrak{h}'),c}(D); \tag{1.3}$$

(2) (D,p) is a *Rellich domain with respect to* $\mathfrak{h}'$, namely the *Rellich property*, "the embedding $H^1_{1,\underset{\rightarrow}{\mathfrak{h}'},c}(D) \hookrightarrow L^2(D)$ is compact".

The Rellich embedding property will be proved in §4 (independently of the Poincaré inequality for constant allowable weight) only for the case where (D,p) satisfies the following conditions: (i) (D,p) is *of finite volume*, namely, $\int_{D^*} d\tilde{v} < \infty$, and (ii) (D,p) is *of Sobolev type*, that is, there exists a constant $\alpha > 2$ such that, for all $f \in H^1_{1,(0,1,\cdots,1),c}(D)$,

$$\|f\|_{L^\alpha(D)} \le \text{Const.}\ \|f\|_{1,(0,1,\cdots,1)}. \tag{1.4}$$

Every Euclidean domain in $\mathbb{C}^m$ is of Sobolev type ([6, Theorem 6, p. 270]). Other examples are provided by finitely quasiregular Riemann subdomains (see §4).

Theorem 1.1. (*Rellich embedding theorem*): *If (D,p) is of finite volume and of Sobolev type, then (D,p) is a Rellich domain with respect to any constant allowable $2m$-weight on D.*

Theorem 1.2. *If (D,p) is an unramified Riemann subdomain in a complex space and if p defines an étale covering*[4]*, then (D,p) has the Poincaré property relative to any allowable $2m$-weight $\mathfrak{h}'$ on D with*

$$\mathfrak{h}_D := \min\{\operatorname{ess\,inf}_D(h_j) \,|\, 1 \le j \le 2m\} > 0.$$

The Rellich embedding property can be extended to sections of a vector bundle over a differentiable manifold (see [10, p. 88 and p. 93], [22, p. 111]). It would be interesting to characterize those Riemann subdomains of an m-dimensional complex space which can be realized as a Rellich domain with respect to some allowable $2m$-weight. A form of the Poincaré inequality (with respect to a continuous allowable $2m$-weight) holds as a consequence of the Rellich embedding property (see Proposition 4.3).

For a Riemann subdomain D with the Poincaré property relative to an allowable $2m$-weight $\mathfrak{h}'$, the defining norms of the Sobolev spaces $H^1_{1,\underset{\rightarrow}{\mathfrak{h}'},c}(D)$ and $H^1_{0,\underset{\rightarrow}{\mathfrak{h}'},c}(D)$ [5] are equivalent. On such a domain the inhomogeneous Dirichlet problem (for the Poisson equation) admits a weak solution (Corollary 4.5). As a further application, consider the following (inhomogeneous) Dirichlet problem (to be referred to as a Poisson problem):

$$-\sum_{j=1}^{2m} \partial_j(h_j\partial_j\psi) + \alpha\psi = g \quad \text{a.e. in } D, \quad \psi|dD = 0, \tag{1.5}$$

[4] Defined in §4.

[5] This equivalence is sometimes called the *Friederichs's Lemma*. The traditional notation for the Sobolev space $H^1_{1,(0,1,\cdots,1),c}(D)$ is $W^{1,2}_0(D)$.

where $g \in L^2(D)$ [6]. A (*weak*) *solution on* D of this problem is taken in the following sense: ψ is an element in $H^1_{1,\underset{\rightarrow}{\mathfrak{h}'},c}(D)$ satisfying the equation

$$-\sum_{j=1}^{2m} \partial_j(h_j \partial_j \psi) + \alpha\psi = g$$

weakly in D, namely, ψ satisfies the functional equation

$$[\psi, v]_{D,\mathfrak{h}'} + \alpha(\psi, \bar{v})_D \;=\; (g, \bar{v})_D, \quad \forall v \in \mathscr{C}^{\infty,c}(D). \tag{1.6}$$

To solve this problem one must also determine the eigenvalues $\lambda := -\alpha$. To this end, an alternative but more expedient formulation of the above equation is available (see Lemma 6.2). The latter requires the use of an operator, to be called a *resolvent map*, which can be introduced as follows: for each allowable $2m$-weight $\mathfrak{h}'$ on D (as above with respect to $\mathfrak{h}'$), there is a linear mapping $\mathscr{R}_{D,\mathfrak{h}'} : L^2(D) \to L^2(D)$ with image in $H^1_{1,\underset{\rightarrow}{\mathfrak{h}'},c}(D)$ defined by the equation

$$[\mathscr{R}_{D,\mathfrak{h}'} f, v]_{D,\mathfrak{h}'} \;=\; (f, \bar{v})_D \tag{1.7}$$

for all $(f, v) \in L^2(D) \times H^1_{1,\underset{\rightarrow}{\mathfrak{h}'},c}(D)$. On a Rellich domain this map (in the special case of $\mathscr{R}_{D,(1,\cdots,1)}$) is in fact a compact mapping:

Theorem 1.3. *For any relatively compact Rellich domain* D *in* Y, *the resolvent map* $\mathscr{R}_{D,\{1,\cdots,1\}} : L^2(D) \to L^2(D)$ *defined by*

$$[\mathscr{R}_{D,\{1,\cdots,1\}} g, v]_{D,\{1,\cdots,1\}} = (g, \bar{v})_D, \quad \forall (g, v) \in L^2(D) \times H^1_{1,(0,1,\cdots,1),c}(D),$$

is a compact, self-adjoint mapping.

If (D, p) is a Rellich domain of type $\mathfrak{h}'$, then the behavior of the solutions of the boundary-eigenvalue problem

$$-\sum_{j=1}^{2m} \partial_j(h_j \partial_j \psi) + \alpha\psi = 0 \quad \text{a.e. in } D, \quad \psi|dD \;=\; 0, \tag{1.8}$$

can be determined, and similarly for the case of the Poisson problem (1.5). For completeness this spectral analysis is carried out in §6. Given mathematical physicists' interest in complexified space-time models [7], it is hoped that the results of this paper may be of use in some applications.

[6] Here (and in the following) dD denotes the (maximal) boundary manifold of D_{reg} in the manifold Y_{reg} of simple points of Y, oriented towards the exterior of D_{reg} ([18, p. 218]).

[7] See, e. g., Hansen, R. O. and Newman, E. T. A complex Minkowski approach to twistors, GRG Vol.6, No.4 (1975), 361-385.

2 Preliminaries

In what follows every complex space is assumed to be reduced and has a countable basis of topology. For the definition and basic properties of differential forms on a complex space, see [18, §4.1]. In particular, the exterior differentiation d, the operators ∂, $\bar{\partial}$ and $d^c := (1/4\pi i)(\partial - \bar{\partial})$ are well-defined ([18, Chap. 4]). For an open subset G of Y, denote by $\mathscr{C}^{\nu}_k(G)$ the set of $\mathbb{C}$-valued k-forms of class $\mathscr{C}^{\nu}$ on G, $\mathscr{C}^{\nu}_{k,c}(G)$ the subspace of compactly supported k-forms (dropping the degree if $k=0$), with $\nu = \beta$ to mean locally bounded, $\nu = \mathfrak{m}$ measurable, and $\nu = \lambda$ locally Lipschitzian ([18, §4]). Similarly for $\mathscr{C}^{\nu}_k(\overline{G})$. A measurable function[8] g on Y is said to be *locally integrable* ($g \in L^1_{\text{loc}}(Y)$) provided the form $g\,\chi$ is locally integrable on Y_{reg} for every $2m$-form $\chi \in \mathscr{C}^0_{2m}(Y)$. Similarly define $L^1(Y)$ and $L^2_{\text{loc}}(Y)$ (for the latter, the above $g\chi$ is replaced by $|g|^2\chi$).

Denote by $\|z\|$ the Euclidean norm of $z = (z_1, \cdots, z_m) \in \mathbb{C}^m$, where $z_j = x_j + i\,y_j$. Let the space $\mathbb{C}^m$ be oriented so that the form $\upsilon^m := (dd^c\|z\|^2)^m$ is positive. Let $\mathbb{B}(r)$ denote the r-ball in $\mathbb{C}^m$ centered at the origin and $\mathbb{B}[1] = \{z \in \mathbb{C}^m \mid \|z\| \le 1\}$. Let $p : Y \to \mathbb{C}^m$ be a holomorphic map. If $S \subseteq Y$, let $S' := p(S)$; and in particular write $a' = p(a)$. Set $p^{[a]} := p - a'$, $\forall a \in Y$. Clearly the form

$$\upsilon_p := dd^c\|p^{[a]}\|^2 = \left(\frac{i}{2\pi}\right)\partial\bar{\partial}\,\|p^{[a]}\|^2 \tag{2.1}$$

is non-negative and independent of a. Denote by dv the Euclidean volume element of $\mathbb{C}^m$ and define $d\tilde{v} := p^*(dv)$ on Y. If (D, p) is a Riemann subdomain, then $d\tilde{v}$ is a semivolume form on D and

$$d\tilde{v} = \frac{c_m\pi^m}{m!}\upsilon_p^m,$$

where $c_m := (-1)^{\frac{m(m-1)}{2}}$. If $f,\, \psi \in L^1_{\text{loc}}(D)$, set

$$(f, \psi)_{_D} := \int_D f\,\psi\,d\tilde{v},$$

provided the integral exists. Each element $f \in L^1_{\text{loc}}(D)$ gives rise naturally to a top-dimensional current, $T = [f] : \chi \mapsto \int f \wedge \chi$, with induced functional (cf. [21]) defined by

$$\langle [f],\, \phi\rangle := (-1)^{\frac{m(m-1)}{2}}\frac{\pi^m}{m!}\int f\phi\,\upsilon_p^m, \quad \forall \phi \in \mathscr{C}^{\infty,c}(D).$$

3 Friedrichs extension of a Schrödinger type operator

Given an allowable $(2m+1)$-weight $\mathfrak{h}$ for (D, μ) and $g \in L^2(D)$, the antilinear functional $v \mapsto (g,\, \bar{v})_{_D}$ is well-defined and continuous on $H^1_{\mu,\mathfrak{h},c}(D)$. By the Riesz representation theorem, there is a unique element $w \in H^1_{\mu,\mathfrak{h},c}(D)$ satisfying the equation

$$\langle w,\, v\rangle_{_{\mu,\mathfrak{h}}} = (g,\, \bar{v})_{_D}, \quad \forall v \in H^1_{\mu,\mathfrak{h},c}(D), \tag{3.1}$$

[8] A function g on Y is said to be *measurable on* Y is so is the restriction $g|Y_{\text{reg}}$.

with $\|w\|_{\mu,\mathfrak{h}} \leq$ const. $\|g\|_{L^2(D)}$. The assignment $g \mapsto G_{\mu,\mathfrak{h}} g := w$ defines a linear, continuous mapping $G_{\mu,\mathfrak{h}}$ of $L^2(D)$ onto its image $\mathfrak{R}_{\mu,\mathfrak{h}}(D) := G_{\mu,\mathfrak{h}}(L^2(D))$. Clearly one has

$$\langle G_{\mu,\mathfrak{h}} g, v\rangle_{\mu,\mathfrak{h}} = (g, \bar{v})_D, \quad \forall v \in H^1_{\mu,\mathfrak{h},c}(D). \tag{3.2}$$

Thus $G_{\mu,\mathfrak{h}}$ may be regarded as a (weak) *Green's map* for the weighted Sobolev-Schrödinger operator. Observe that

$$\langle G_{\mu,\mathfrak{h}} v, v\rangle_{\mu,\mathfrak{h}} = \|v\|^2_{L^2(D)}, \quad \forall v \in H^1_{\mu,\mathfrak{h},c}(D), \tag{3.3}$$

and

$$\langle G_{\mu,\mathfrak{h}} \psi, v\rangle_{\mu,\mathfrak{h}} = \overline{\langle G_{\mu,\mathfrak{h}} v, \psi\rangle_{\mu,\mathfrak{h}}} = \langle \psi, G_{\mu,\mathfrak{h}} v\rangle_{\mu,\mathfrak{h}} \quad \forall (\psi, v) \in H^1_{\mu,\mathfrak{h},c}(D) \times H^1_{\mu,\mathfrak{h},c}(D).$$

By the equation (3.3) the Green's map $G_{\mu,\mathfrak{h}} : L^2(D) \to H^1_{\mu,\mathfrak{h},c}(D)$ is injective, and the inverse map $\mathscr{F}_{\mu,\mathfrak{h}} := G^{-1}_{\mu,\mathfrak{h}} : \mathfrak{R}_{\mu,\mathfrak{h}}(D) \to L^2(D)$ is well-defined. Also, it is positive definite since given $g \in L^2(D)$, the equation (3.2) implies that

$$\begin{aligned}(\mathscr{F}_{\mu,\mathfrak{h}}(G_{\mu,\mathfrak{h}} g), \overline{G_{\mu,\mathfrak{h}} g})_D &= (g, \overline{G_{\mu,\mathfrak{h}} g})_D \\ &= \langle G_{\mu,\mathfrak{h}} g, G_{\mu,\mathfrak{h}} g\rangle_{\mu,\mathfrak{h}} = \|G_{\mu,\mathfrak{h}} g\|^2_{\mu,\mathfrak{h}} \\ &\geq \text{const.}\, \|G_{\mu,\mathfrak{h}} g\|^2_{L^2(D)}.\end{aligned}$$

For an allowable $(2m+1)$-weight $\mathfrak{h}$ for (D, μ) of class $\mathscr{C}^1$ (namely, so is each component h_j of $\mathfrak{h}$ on D), define the *Schrödinger operator* $\mathcal{S}_{\mu,\mathfrak{h}}$ acting on $\mathscr{C}^1(D)$ (in the classical sense) by

$$\mathcal{S}_{\mu,\mathfrak{h}} \psi := (\mu - V)\psi - \sum_{j=1}^{2m} \partial_j (h_j\, \partial_j \psi) \text{ on } D^*. \tag{3.4}$$

Assume (in the following theorem) that $\mathfrak{h}$ is an allowable $(2m+1)$-weight for (D, μ) of class $\mathscr{C}^1$. Define $\mathscr{D}_{\mu,\mathfrak{h}} := \{\psi \in \mathscr{C}^{2,c}(D) | \, \mathcal{S}_{\mu,\mathfrak{h}} \psi \in L^2(D)\}$, and $\mathfrak{H}(D, \mathcal{S}_{\mu,\mathfrak{h}}) := H^1_{\mu,\mathfrak{h},c}(D) \cap \{w \in L^2(D) | \, \mathcal{S}_{\mu,\mathfrak{h}}[w] \in L^2(D)\}$ [9].

Theorem 3.1. *(Friedrichs extension of the operator $\mathcal{S}_{\mu,\mathfrak{h}}$) Let D be a Riemann subdomain in Y. Then the weighted Sobolev-Schrödinger operator $\mathcal{S}_{\mu,\mathfrak{h}} : \mathscr{D}_{\mu,\mathfrak{h}} \to L^2(D)$ admits a positive, self-adjoint extension $\mathscr{F}_{\mu,\mathfrak{h}} : \mathfrak{H}(D, \mathcal{S}_{\mu,\mathfrak{h}}) \to L^2(D)$ with the property that for each $w \in \mathfrak{H}(D, \mathcal{S}_{\mu,\mathfrak{h}})$,*

$$(\mathscr{F}_{\mu,\mathfrak{h}} w, \bar{v})_D = \langle w, v\rangle_{\mu,\mathfrak{h}}, \quad \forall v \in H^1_{\mu,\mathfrak{h},c}(D), \tag{3.5}$$

and

$$(\mathscr{F}_{\mu,\mathfrak{h}} w, \bar{v})_D = (w, \mathcal{S}_{\mu,\mathfrak{h}} \bar{v})_D, \quad \forall v \in \mathscr{C}^{\infty,c}(D). \tag{3.6}$$

Proof. It will be shown that the inverted mapping $\mathscr{F}_{\mu,\mathfrak{h}} := G^{-1}_{\mu,\mathfrak{h}}$ acts as an extension of the operator $\mathcal{S}_{\mu,\mathfrak{h}}$ to $\mathfrak{R}_{\mu,\mathfrak{h}}(D) = G_{\mu,\mathfrak{h}}(L^2(D))$. It follows from the symmetry of $G^{-1}_{\mu,\mathfrak{h}} : L^2(D) \to \mathfrak{R}_{\mu,\mathfrak{h}}(D)$ that $\mathscr{F}_{\mu,\mathfrak{h}} : \mathfrak{R}_{\mu,\mathfrak{h}}(D) \to L^2(D)$ is self-adjoint, and satisfies the equation

$$(\mathscr{F}_{\mu,\mathfrak{h}} w, \bar{v})_D = \langle w, v\rangle_{\mu,\mathfrak{h}}, \quad \forall (w, v) \in \mathfrak{R}_{\mu,\mathfrak{h}}(D) \times H^1_{\mu,\mathfrak{h},c}(D). \tag{3.7}$$

[9] "$\mathcal{S}_{\mu,\mathfrak{h}}[w]$"] denotes the (weak) action of $\mathcal{S}_{\mu,\mathfrak{h}}$ on the functional $[w]$.

If further v is an element of $\mathscr{C}^{2,c}(D)$, then

$$\langle w, v\rangle_{\mu,\mathfrak{h}} = \lim_{j\to\infty} \langle w_j, v\rangle_{\mu,\mathfrak{h}} = \lim_{j\to\infty} (w_j,\, \mathcal{S}_{\mu,\mathfrak{h}}\bar{v})_D.$$

This implies that

$$(\mathscr{F}_{\mu,\mathfrak{h}} w,\, \bar{v})_D = (w,\, \mathcal{S}_{\mu,\mathfrak{h}}\bar{v})_D. \tag{3.8}$$

To see that the mapping $\mathscr{F}_{\mu,\mathfrak{h}}$ is indeed defined on $\mathfrak{H}(D, \mathcal{S}_{\mu,\mathfrak{h}})$ one needs to check that

$$\mathfrak{R}_{\mu,\mathfrak{h}}(D) = \mathfrak{H}(D, \mathcal{S}_{\mu,\mathfrak{h}}).$$

Let $w \in \mathfrak{R}_{\mu,\mathfrak{h}}(D)$. Clearly then $w \in H^1_{\mu,\mathfrak{h},c}(D)$ and $w = G_{\mu,\mathfrak{h}} g$ for some $g \in L^2(D)$. Then by the equation (3.8) one has, for all $v \in \mathscr{C}^{\infty,c}(D)$,

$$(w,\, \mathcal{S}_{\mu,\mathfrak{h}} v)_D = (\psi,\, v)_D$$

with $\psi := \mathscr{F}_{\mu,\mathfrak{h}} w \in L^2(D)$, and thus $\mathcal{S}_{\mu,\mathfrak{h}}[w] = \psi$ as desired. Hence $w \in \mathfrak{H}(D, \mathcal{S}_{\mu,\mathfrak{h}})$. Conversely if $w \in \mathfrak{H}(D, \mathcal{S}_{\mu,\mathfrak{h}})$, then $w \in H^1_{\mu,\mathfrak{h},c}(D)$ and $\mathcal{S}_{\mu,\mathfrak{h}}[w] \in L^2(D)$. Thus there exists an element $g \in L^2(D)$ such that

$$\left(\mathcal{S}_{\mu,\mathfrak{h}}[w],\, v\right)_D = (g,\, v)_D, \quad \forall v \in \mathscr{C}^{\infty,c}(D).$$

For each $v \in H^1_{\mu,\mathfrak{h},c}(D)$ and a sequence $\{v_j\}$ in $\mathscr{C}^{\infty,c}(D)$ tending to v,

$$\langle w, v\rangle_{\mu,\mathfrak{h}} = \lim_{j\to\infty} \langle w, v_j\rangle_{\mu,\mathfrak{h}} = \lim_{j\to\infty} (w,\, \mathcal{S}_{\mu,\mathfrak{h}}\bar{v}_j)_D = \lim_{j\to\infty} (g,\, \bar{v}_j)_D = (g,\, \bar{v})_D,$$

hence w satisfies the equation (3.2), thereby proving that $w \in \mathfrak{R}_{\mu,\mathfrak{h}}(D)$. Therefore the formula (3.5) follows from the equation (3.7). Consequently the mapping $\mathscr{F}_{\mu,\mathfrak{h}}$ gives the Friedrichs extension of $\mathcal{S}_{\mu,\mathfrak{h}} : \mathscr{D}_{\mu,\mathfrak{h}} \to L^2(D)$. □

Remark 3.2. *Especially the preceding theorem ensures that the existence of the Friedrichs extension of the Laplacian*[10] *:* $-\triangle_{\{p\}} : \mathscr{D}_{\mu,(1,\cdots,1)} \to L^2(D)$. *This extension is given by the mapping* $\mathscr{F} : \mathfrak{H}(D, -\triangle_{\{p\}}) \to L^2(D)$, *and admits the representation*

$$(\mathscr{F} w,\, \bar{v})_D = [w,\, v]_{D,(1,\cdots,1)}, \quad \forall (w, v) \in \mathfrak{H}(D, -\triangle_{\{p\}}) \times H^1_{1,(0,1,\cdots,1),c}(D).$$

In particular, for all $(w, v) \in \mathfrak{H}(D, -\triangle_{\{p\}}) \times \mathscr{C}^{\infty,c}(D)$,

$$(\mathscr{F} w,\, \bar{v})_D = (w,\, -\triangle_{\{p\}}\bar{v})_D.$$

4 The Rellich Embedding Theorem and Poincaré inequality

Let $p : Y \to Y'$ be a continuous mapping (between topological spaces). An open subset $W \subseteq Y'$ is called a *base domain evenly covered by* p, if $p^{-1}(W)$ is a disjoint union of open subsets $B_l \subseteq Y$

[10] Here "$\triangle_{\{p\}}$" denotes the (local) pullback to D^* under p of the Laplace operator of the Euclidean metric on $\mathbb{C}^m$.

each of which is homeomorphic to W (under p); as such B_l will also be referred to as a *covering sheet of* p *lying evenly over* W (relative to p).

If $p : D \to \mathbb{C}^m$ is a light mapping [11], then there exists an open connected neighborhood $U_a \Subset D$ of each $a \in D$, called a *pseudoball at* a ([21, §2]) such that, when restricted to the regular part $\hat{U}_a := U_a \backslash p^{-1}(\Delta')$ (Δ being the branch locus), the map $p_{_a} = p : \hat{U}_a \to U' \backslash \Delta'$ is an unramified finite holomorphic covering, where $U' := \mathbb{B}_{[a']}(r)$ is an open ball in $\mathbb{C}^m$ with center a' (see [1, §2]). The sheet number of $p_{_a}$ is equal to $\nu_p(a)$, the multiplicity of p at (the center) a ([ibid., §2]). Thus at each point $z' \in U' \backslash \Delta'$ there is an open ball $W^{z'} \Subset U' \backslash \Delta'$ (of radius < 1) (here z' will be denoted by z'_k and $W^{z'}$ by W^k for reasons to become clear later), which is a base domain evenly covered by $p_{_a}$ with covering sheets $B^\ell = B^{z'}_l \subseteq D$, $l = 1, \cdots, l(k)$, over W^k, each being (necessarily) biholomorphic to W^k under p. Here the number $l(k)$ is equal to $\nu_p(a)$ (for each z'_k) and B^k_l has compact closure in $\hat{U}_a$.

Definition 4.1. *An admissible covering of a compact subset* $\mathcal{K}$ *of* D^* *is an open covering of* $\mathcal{K}$ *consisting of open subsets* $B^\ell \subseteq D^*$ *with* ℓ *varying in a finite range such that each* B^ℓ *is equal to some* B^k_l *(namely some* $B^{z'_k}_l$*) with* k *in a finite range and* $l \in \mathbb{Z}[1, l(k)]$; *moreover, for each fixed* k *the* B^k_l *(with varying* l*) lie evenly (relative to* p*) over a base domain* $W^k := W^{z'_k}$ *contained in some open ball* $\hat{W}^k := \hat{W}^{z'_k} \subset \mathbb{C}^m$ *of volume* < 1.

Note that every bounded domain D in $\mathbb{C}^m$ admits an admissible covering (via a dilatation of the identity map). Also, if U_a is a pseudoball contained in D, then every compact subset $\mathcal{K}$ of $\hat{U}_a$ admits an admissible covering.

The set D^* being σ-compact, one can write D^* as a union of an exhausting (increasing) sequence $\{\mathcal{K}_j\}$ of compact subsets. Choose a $\mathscr{C}^\infty$-partition of unity $\{\beta^{j,\ell}\}_{1\le\ell\le N_j}$ on $\mathcal{K}_j$ subordinate to an admissible covering $\{B^{j,\ell}\}$ for $\mathcal{K}_j$. One has, for each $f \in \mathscr{C}^{\infty,c}(D)$, setting $f^{\{j\}}_{_\ell} := (\beta^{j,\ell})^{\frac{1}{2}} f$ (the ℓ-th partitioned function of f relative to $\{B^{j,\ell}\}$),

$$\int_{K_j} |f|^2 \, d\tilde{v} \le \sum_{\ell=1}^{N_j} \int_{D^*} |f^{\{j\}}_{_\ell}|^2 \, d\tilde{v}. \tag{4.1}$$

A Riemann subdomain (D, p) in Y is said to be *finitely quasiregular* if (i) each compact subset $\mathcal{K}$ of D^* admits a (finite) admissible covering $\{B^k_l\}$ with (corresponding) base domains W^k with $\mathscr{C}^1$-boundary contained in some open ball of finite radius in $\mathbb{C}^m$; and (ii) the family $\{W^k\}$ has finite cardinality $\mathfrak{K}$ which depends only on D. If further D is unramified, then (D, p) is said to be *finitely regular.*

The above definition of "finitely quasiregular domain" is equivalent to the following: "(D, p) is *finitely quasiregular* if the regular part D^* is a finite union of local covering sheets $B^{\{\ell\}}$, $\ell =$

[11] A holommorphic mapping $f : Y \to Y'$ (between complex spaces) is light on $D \subseteq Y$ if for each $a \in D$, $\dim_a f^{-1}(f(a)) = 0$.

$1, \cdots, \mathcal{N}$". For, the above definition implies that, given any (increasing) sequence $\{\mathcal{K}_j\}$ of compact exhaustion of D^*, the index sequence $\{N_j\}$ (with N_j arising from a corresponding $\mathcal{K}_j$) can be replaced by a finite sequence $\{1, \cdots, \mathcal{N}\}$ with $\mathcal{N} = l(1) + \cdots + l(\mathfrak{K})$. Thus D^* is contained in (at most) $\mathcal{N}$ local covering sheets $B^{\{\ell\}}$. The converse assertion is trivially true.

Note that every bounded domain in $\mathbb{C}^m$ is finitely regular. Also, each relatively compact subset D of the projective space $\mathbb{P}^m(\mathbb{C})$ contained in the open set $\mathbb{Q}^{\{s\}}$, $s \in \mathbb{Z}[1, m]$, is finitely regular relative to the Riemann covering $p = p^{[s]}$. For a further example, consider the complex space $Y = \{(z, w) \in \mathbb{C}^2 \,|\, w^2 = z^3\}$. Relative to the projection $(z, w) \mapsto z$ the open set $D := \{(z, w) \in Y | \|z\| \le r_0\}$ is a finitely quasiregular Riemann subdomain in Y (with two covering sheets).

While for a general domain the Rellich Embedding Theorem may not be valid, the requirement that the domain be of Sobolev type is indeed somewhat restrictive. A characterization of the latter remains an open question. The following lemma is of some use:

Lemma 4.2. *Every finitely quasiregular Riemann subdomain (D, p) in Y is of finite volume and of Sobolev type.*

Proof. The regular part D^* of a finitely quasiregular Riemann subdomain (D, p) admits a finite covering by local covering sheets B^ℓ with ℓ varying in a finite range, say $\{1, \cdots, \mathcal{N}\}$. It is easy to show that D has finite volume with respect to p. One can write D^* as a union of an exhausting (increasing) sequence $\{\mathcal{K}_j\}$ of compact subsets. Each $\mathcal{K}_j$ admits a finite admissible covering $\{B^{j,\ell}\}_{1 \le \ell \le N_j}$. The Sobolev Embedding Inequality on a bounded Euclidean domain ([6, Theorem 6, p. 270] (applied to each $B^{j,\ell}$) implies that, for all $g \in \mathscr{C}^{\infty,c}(D)$ [12],

$$\begin{aligned}
(\|g\|_{L^\alpha(D)})^\alpha &\le \lim_{j\to\infty} \sum_{\ell=1}^{N_j} \int_{B^{j,\ell}} |g|^\alpha \, d\tilde{v} = \lim_{j\to\infty} \sum_{\ell=1}^{N_j} (\|g\|_{L^\alpha(B^{j,\ell})})^\alpha \\
&\le (C_{B_m})^\alpha \lim_{j\to\infty} \sum_{\ell=1}^{N_j} (\|g\|_{H^1_{1,(0,1,\cdots,1)}(B^{j,\ell})})^\alpha \\
&\le (C_{B_m})^\alpha \lim_{j\to\infty} \sum_{\ell=1}^{N_j} (\|g\|_{H^1_{1,(0,1,\cdots,1)}(D)})^\alpha
\end{aligned}$$

for some constant C_{B_m} (independent of D). Since the local covering sheets $B^{j,\ell}$ can be selected from the finite set $\{B^\ell\}_{\ell=1,\cdots,\mathcal{N}}$, the above index N_j can be replaced by $\mathcal{N}$. Thus

$$(\|g\|_{L^\alpha(D)})^\alpha \le \text{Const.} \, (\|g\|_{H^1_{1,(0,1,\cdots,1)}(D)})^\alpha.$$

The space $\mathscr{C}^{\infty,c}(D)$ being dense in $H^1_{1,(0,1,\cdots,1),c}(D)$, each element f therein can be approximated in the L^2-norm by a sequence $\{g_n\} \subset \mathscr{C}^{\infty,c}(D)$. Thus the Sobolev inequality (1.4) follows. □

[12] For clarity denote the Sobolev norm of $g \in H^1_{1,(0,1,\cdots,1)}(B^\ell)$ by $\|g\|_{H^1_{1,(0,1,\cdots,1)}\cdot(B^\ell)}$ and similarly for $g \in H^1_{1,(0,1,\cdots,1),c}(D)$.

Proof of Theorem. 1.1. For later reference, assume (unless otherwise mentioned) that $\mathfrak{h}$ denotes a general allowable weight as indicated in the beginning of §1. Let $\{f_n\}$ be a bounded sequence in $H^1_{\mu,\mathfrak{h},c}(D)$. It contains a subsequence $\{f_{n_k}\}$ which converges weakly to some element $f \in H^1_{\mu,\mathfrak{h},c}(D)$. By considering $\{f_{n_k} - f\}$ it may be assumed that $\{f_{n_k}\} \subset \{f_n\}$ is a subsequence converging weakly to 0. In the following for simplicity denote this subsequence by the same notation $\{f_n\}$. Since $\mathscr{C}^{\infty,c}(D)$ is dense in $H^1_{\mu,\mathfrak{h},c}(D)$, each f_n is the limit of a sequence $\{\phi^n_l\} \subset \mathscr{C}^{\infty,c}(D)$ with respect to the $\|\ \ \|_{\mu,\mathfrak{h}}$-norm on $H^1_{\mu,\mathfrak{h},c}(D)$. Thus, for each $n \in \mathbb{N}$ there exists an element $\phi^{\{n\}} \in \mathscr{C}^{\infty,c}(D)$ such that

$$\|f_n - \phi^{\{n\}}\|_{\mu,\mathfrak{h}} < \frac{1}{n}.$$

Then for any given $\rho \in H^1_{\mu,\mathfrak{h},c}(D)$,

$$(\phi^{\{n\}}, \rho)_{\mu,\mathfrak{h}} = (\phi^{\{n\}} - f_n, \rho)_{\mu,\mathfrak{h}} + (f_n, \rho)_{\mu,\mathfrak{h}} \to 0, \quad n \to \infty,$$

namely $\phi^{\{n\}}$ tends to 0 weakly in $H^1_{\mu,\mathfrak{h},c}(D)$. Since

$$\|f_n - \phi^{\{n\}}\|_{L^2(D)} \leq \text{Const.} \|f_n - \phi^{\{n\}}\|_{\mu,\mathfrak{h}},$$

to prove the Rellich Embedding Theorem it suffices to consider the case where $\{f_n\}$ lies in $\mathscr{C}^{\infty,c}(D)$ and converges weakly to 0 in $H^1_{\mu,\mathfrak{h},c}(D)$ (in which case $\{f_n\}$ is uniformly bounded in $H^1_{\mu,\mathfrak{h},c}(D)$). In the rest of the proof, assume that $(\mu, V) = (1, 0)$. The goal is to show (possibly by passing to a subsequence) that $\lim_{n\to\infty} \|f_n\|^2_{L^2(D)}$ exists and equals zero. For this purpose it will be necessary to swap the order of the limits in the following relation

$$\lim_{n\to\infty} \lim_{j\to\infty} \int_{K_j} |f_n|^2 \, d\tilde{v} = \lim_{j\to\infty} \lim_{n\to\infty} \int_{K_j} |f_n|^2 \, d\tilde{v} \tag{4.2}$$

(where $\{K_j\}$ is an increasing exhausting sequence of compact subsets of D^*). This can be justified by verifying two conditions: (a) the sequence of functions

$$\Phi_j(n) := \int_{K_j} |f_n|^2 \, d\tilde{v} \quad \text{converges to} \quad \Phi(n) := \lim_{j\to\infty} \int_{K_j} |f_n|^2 \, d\tilde{v}$$

(as $j \to \infty$) uniformly in n, and (b) for each fixed j the limit "$\lim_{n\to\infty} \int_{K_j} |f_n|^2 \, d\tilde{v}$" exists (in fact, equals 0). More explicitly, the assertion (a) amounts to showing that, for each $\epsilon > 0$, $\exists\, J_\epsilon$ and $N_\epsilon \in \mathbb{N}$ such that

$$\int_{(K_{J_\epsilon})^c} |f_n|^2 \, d\tilde{v} \leq \epsilon, \quad \forall n \geq N_\epsilon. \tag{4.3}$$

To prove this condition, set $\frac{1}{q'} := \frac{2}{\alpha}$ and $\frac{1}{p'} := 1 - \frac{1}{q'}$. Then by Hölder's inequality (with $f = 1$ and $g = |f_n|^2$) the left-hand side of the above inequality is dominated by

$$\int_{(K_{J_\epsilon})^c} |1 \cdot |f_n|^2| \, d\tilde{v} \leq \left(\int_{(K_{J_\epsilon})^c} 1^{p'} d\tilde{v} \right)^{\frac{1}{p'}} \left(\int_{(K_{J_\epsilon})^c} (|f_n|^2)^{q'} \, d\tilde{v} \right)^{\frac{1}{q'}}. \tag{4.4}$$

Since $(|f_n|^2)^{q'} = |f_n|^\alpha,$ and

$$(\||f_n|^2\|)_{L^{q'}((K_{J_\epsilon})^c)} = \left(\int_{(K_{J_\epsilon})^c} |f_n|^\alpha \, d\tilde{v}\right)^{\frac{2}{\alpha}} = (\|f_n\|_{L^\alpha((K_{J_\epsilon})^c)})^2,$$

the inequality (4.4) becomes

$$\int_{(K_{J_\epsilon})^c} |f_n|^2 \, d\tilde{v} \leq |\mathrm{vol}((K_{J_\epsilon})^c)|^{1-\frac{2}{\alpha}} \|f_n\|^2_{L^\alpha((K_{J_\epsilon})^c)}.$$

Therefore (if α is chosen to satisfy the Sobolev Inequality (1.4)) the second factor on the right-hand side is uniformly bounded, hence one has

$$\int_{(K_{J_\epsilon})^c} |f_n|^2 \, d\tilde{v} \leq \text{Const. } |\mathrm{vol}((K_{J_\epsilon})^c)|^{1-\frac{2}{\alpha}}.$$

Finally, since (D, p) has finite volume, by letting $J_\epsilon \to \infty$ one has $\mathrm{vol}((K_{J_\epsilon})^c) \to 0$. Thus the condition for uniform convergence (4.3) follows.

By choice of an open covering of K_j by local covering sheets B^ℓ (with notations as in Definition 4.1) each of which is contained in a fixed open neighborhood $D_j \Subset D^*$ (the sheets B^ℓ and D_j being dependent only on j), all the terms $f^{\{j\}}_{n,\ell}$ have support contained in $\overline{D_j}$. Denoting by $\hat{f}^{\{j\}}_{n,\ell}$ the direct image of $f^{\{j\}}_{n,\ell}|B^\ell$ (under the biholomorphic map $p : B^\ell \to \mathcal{B}^\ell = p(B^\ell)$, the resulting family $\{\hat{f}^{\{j\}}_{n,\ell}\}$ is uniformly bounded in $H^1_{\mu,\mathfrak{h},c}(\mathcal{B}^\ell)$ with respect to n, ℓ. Inscribe $\mathcal{B}^\ell$ in a cube C (independent of ℓ, and it may be assumed that $C \subset \mathbb{B}(1)$ by rescaling and translation). Every element $\hat{\rho} \in \mathscr{C}^{\infty,c}(\mathcal{B}^\ell)$ can be extended to C trivially (by setting $\hat{\rho}$ to be zero off $\mathcal{B}^\ell$).

The weak convergence of $\{f_n\}$ (to 0) in $H^1_{\mu,\underline{\mathfrak{h}},c}(B^\ell)$ implies that of $f^{\{j\}}_{n,\ell}$, hence $\hat{f}^{\{j\}}_{n,\ell}$ converges weakly to 0 in $H^1_{1,\underline{\mathfrak{h}},c}(\mathcal{B}^\ell)$. The preceding discussion shows that the compactness of the embedding "$H^1_{1,\mathfrak{h},c}(D) \hookrightarrow L^2(D)$" is a consequence of the following "claim":

If $\{\hat{f}^{\{j\}}_{n,\ell}\}$ is a sequence (indexed by n) in $H^1_{1,(0,\mathfrak{h}'),c}(\mathcal{B}^\ell)$ which converges weakly to 0 in $H^1_{1,(0,\mathfrak{h}'),c}(\mathcal{B}^\ell)$, the $\mathfrak{h}'$ being an allowable constant $2m$-tuple, then

$$\lim_{n\to\infty} \int_{D^*} |f^{\{j\}}_{n,\ell}|^2 \, d\tilde{v} = 0.$$

Upon taking $\mathfrak{h} = (0, \underline{\underline{h}})$ with $\underline{\underline{h}} = (1, \cdots, 1)$, this is an immediate consequence of the Rellich Embedding Theorem for a bounded Euclidean domain (here the open ball $\mathcal{B}^\ell$). This proves the assertion (b). By interchanging the "n-" and the "j-limit" and making use of the relations (4.2) and (4.1), one has

$$\begin{aligned}\lim_{n\to\infty}\lim_{j\to\infty} \int_{K_j} |f_n|^2 \, d\tilde{v} &= \lim_{j\to\infty}\lim_{n\to\infty} \int_{K_j} |f_n|^2 \, d\tilde{v} \\ &\leq \lim_{j\to\infty}\lim_{n\to\infty} \sum_{\ell=1}^{N_j} \int_{D^*} |f^{\{j\}}_{n,\ell}|^2 \, d\tilde{v} = 0.\end{aligned}$$

Consequently Theorem 1.1 is proved in this case. Let $\mathfrak{h}' = (c_1, \cdots, c_{2m})$ be a $2m$-tuple of positive constants. Define $\hat{p} = \hat{p}_{\{\mathfrak{h}'\}} = (\hat{p}_1, \cdots, \hat{p}_m) : D \to \mathbb{C}^m$ by setting

$$\hat{p}_k(z) := \hat{x}_k + i\hat{y}_k = (c_{2k-1})^{-1/2}\tilde{x}_k + i(c_{2k})^{-1/2}\tilde{y}_k, \quad 1 \le k \le m,$$

Then the volume element

$$d\hat{v}(\mathfrak{h}') := \hat{p}^*(dv) = c_1^{-1/2} \cdots c_{2m}^{-1/2}\, d\tilde{v}.$$

It is easy to show that the norm of an element f of $\hat{H}^1_{1,\{0,1,\cdots,1\},c}(D)$, that is, the space $H^1_{1,\{0,1,\cdots,1\},c}(D)$ defined w. r. t. $(D, \hat{p}_{\{\mathfrak{h}'\}})$ with $\mathfrak{h}' = (1, \cdots, 1)$, is equivalent to the norm of f in $H^1_{1,\{0,c_1,\cdots,c_{2m}\},c}(D)$ w. r. t. (D, p). Hence the Sobolev space $\hat{H}^1_{1,\{0,1,\cdots,1\},c}(D)$ is the same as $H^1_{1,\{0,c_1,\cdots,c_{2m}\},c}(D)$. Since the embedding "$\hat{H}^1_{1,\{0,1,\cdots,1\},c}(D) \hookrightarrow L^2(D)$" is compact, the same is true for the embedding "$H^1_{1,\{0,c_1,\cdots,c_{2m}\},c}(D) \hookrightarrow L^2(D)$". This completes the proof Theorem 1.1. □

A Riemann subdomain (D, p) in Y is said to define a (*distinguished*) *étale covering*[13] (of $p(D^*)$) if (i) each compact subset $\mathcal{K}$ of D^* admits a (finite) admissible covering $\{B^k_l\}$ (with corresponding base domains W^k); and (ii) the family $\{W^k\}$ is pair-wise disjoint. Note that every bounded domain D in $\mathbb{C}^m$ defines a distinguished étale covering (via a rescaled identity map). As another example, consider the m-dimensional projective space $\mathbb{P}^m(\mathbb{C})$ and let $\mathbb{Q}^{\{s\}} := \{a = [a_0, \cdots, a_m] | a_s \neq 0\}$, where $s \in \mathbb{Z}[1, m]$. Then $\mathbb{Q}^{\{s\}}$ can be regarded as an open Riemann subdomain in $\mathbb{P}^m(\mathbb{C})$ relative to the s-th canonical coordinate map $p^{[s]} : \mathbb{Q}^{\{s\}} \to \mathbb{C}^m$ given by

$$p^{[s]} : a \mapsto \left(\frac{a_0}{a_s}, \cdots, \widehat{\frac{a_s}{a_s}}, \cdots, \frac{a_m}{a_s}\right)$$

(here "$\widehat{\ }$" denotes omission). Clearly every relatively compact subset $D \Subset \mathbb{Q}^{\{s\}}$ defines a distinguished étale covering via the Riemann covering $p = p^{[s]}$.

Proof of Theorem 1.2. Let $\mathfrak{h}' = (h_1, \cdots, h_{2m})$ be any allowable weight on D. To prove the Poincaré inequality (1.3), it suffices to verify it for all elements $g \in \mathscr{C}^{\infty,c}(D)$. For, given $f \in H^1_{1,\underrightarrow{\mathfrak{h}'},c}(D)$, there exists a sequence $\{g_n\}$ in $\mathscr{C}^{\infty,c}(D)$ converging to f in $H^1_{1,\underrightarrow{\mathfrak{h}'},c}(D)$, that is,

$$(\|f - g_n\|^2_{L^2(D)} + [f - g_n, f - g_n]_{D,\mathfrak{h}'})^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty. \tag{4.5}$$

If

$$\|g_n\|_{L^2(D)} \le C_{D,\mathfrak{h}'} [g_n, g_n]^{\frac{1}{2}}_{D,\mathfrak{h}'}, \quad \forall n \ge 1,$$

then the limit relation (4.5) implies that inequality (1.3) holds. Thus the above claim holds true. Assume that (D, p) defines an étale covering $\{B^k_l\}$ of $p(D^*)$. Let $g \in \mathscr{C}^{1,c}(D)$ and $K := \mathrm{Spt}(g)$. The regular part D^* being σ-compact, the set $K \cap D^*$ admits an exhausting sequence of increasing compact subsets $\{\mathcal{K}_j\}$ of $K \cap D^*$, each of which is contained in a finite union of covering sheets

[13] See Barlet [2, p. 110] for a closely related notion of "revêtement analytique étale".

B^ℓ (namely some B^k_l as above). Choose a C^∞-partition of unity $\{\beta^{j,\ell}\}_{1\le\ell\le N^j}$ on $\mathcal{K}_j$ subordinate to such a covering $\{B^\ell\}_{\ell=1,\cdots,N^j}$ (with monotonically increasing index N^j). Assume (without loss of generality) that $\sum_{1\le\ell\le N^j}\beta^{j,\ell}=1$ on $D_j:=\cup\{B^\ell|\,1\le\ell\le N^j\}$. Let $\hat{g}^{\{j,\ell\}}:\mathcal{B}^\ell:=p(B^\ell)\to\mathbb{C}$ be the direct image of $g^{\{j,\ell\}}:=\beta^{j,\ell}g$ (under the map p). Assume now that D is unramified. Then $D=D^*$, hence the $\{\mathcal{K}_j\}$ can be chosen to be a finite sequence with $K_j=K$ for large enough j. Then one has

$$\begin{aligned}\int\limits_D|g|^2\,d\tilde{v}&=\int\limits_{D_j}|g|^2\,d\tilde{v}=\int\limits_{D_j}\sum_\ell(\beta^{j,\ell})^2\,|g|^2\,d\tilde{v}\\&=\int\limits_{D_j}\sum_\ell(\beta^{j,\ell})^2\,|g|^2\,d\tilde{v}+2\int\limits_{D_j}\left(\sum_{\ell\neq\ell'}\beta^{j,\ell}\beta^{j,\ell'}\right)|g|^2\,d\tilde{v}\\&=\sum_\ell\int\limits_{\mathcal{B}^\ell}|\hat{g}^{\{j,\ell\}}|^2\,dv\end{aligned}$$

where the added integral (in the second equality) vanishes since the base domains of the covering sheets B^ℓ are pair-wise disjoint. Note that $g=\sum_{1\le\mathfrak{l}\le N^j}\beta^{j,\{\mathfrak{l}\}}g$ on D_j, By resorting to the Poincaré inequality for the Euclidean unit ball, one has [14], for a given $g\in\mathscr{C}^{\infty,c}(D)$,

$$\begin{aligned}\frac{\mathfrak{h}_D}{(\mathcal{P}_{\mathbb{B}})^2}\sum_\ell\int\limits_{\mathcal{B}^\ell}|\hat{g}^{\{j,\ell\}}|^2\,dv&\le\mathfrak{h}_D\sum_\ell\int\limits_{\mathcal{B}^\ell}|\nabla\hat{g}^{\{j,\ell\}}|^2\,dv\le\int_{D_j}\sum_\ell\sum_{\lambda=1}^{2m}h_\lambda|\partial_\lambda(g^{\{j,\ell\}})|^2\,d\tilde{v}\\&=\int_{D_j}\sum_{\lambda=1}^{2m}h_\lambda\sum_{\ell,\,\ell'}\partial_\lambda(g^{\{j,\ell\}})\partial_\lambda(\bar{g}^{\{j,\ell'\}})\,d\tilde{v}\\&=\int_{D_j}\sum_{\lambda=1}^{2m}h_\lambda\left(\sum_\ell\partial_\lambda(g^{\{j,\ell\}})\right)\left(\sum_{\ell'}\partial_\lambda(\bar{g}^{\{j,\ell'\}})\right)d\tilde{v}\\&=\int_{D_j}\sum_{\lambda=1}^{2m}h_\lambda\partial_\lambda\left(\sum_\ell g^{\{j,\ell\}}\right)\partial_\lambda\left(\sum_{\ell'}\bar{g}^{\{j,\ell'\}}\right)d\tilde{v}\\&=[g,g]_{D_j,(h_1,\cdots,h_{2m})}=[g,g]_{D,(h_1,\cdots,h_{2m})},\end{aligned}$$

where $\mathfrak{h}_D=\min\{\operatorname{ess\,inf}_D(h_j)\,|\,1\le j\le 2m\}>0$, and $\mathcal{P}_{\mathbb{B}}$ denotes the Poincaré's constant for the unit ball. Consequently the Poincaré inequality (1.3) follows. □

Proposition 4.3 (Generalized Poincaré-Wirtinger Inequality[15]). *Assume that (D,p) is a Riemann subdomain such that, with respect to a continuous allowable weight $\mathfrak{h}'$ on D, the Rellich-Kondrachov embedding property holds: $H^1_{1,\underset{\to}{\mathfrak{h}'}}(D)\hookrightarrow L^2(D)$ is compact. Then there exists a constant C_D such that*

$$\left\|f-\frac{1}{\operatorname{vol}(D)}\int_D f\,d\tilde{v}\right\|_{L^2(D)}\le C_D[f,f]^{\frac12}_{D,\mathfrak{h}'},\quad\forall f\in H^1_{1,\underset{\to}{\mathfrak{h}'}}(D).\tag{4.6}$$

[14] On the right-side of the following, in the integral of the second inequality a sum of terms "const $\int_{D_j}\sum_\lambda h_\lambda\sum_{\ell\neq\ell'}\partial_\lambda(g^{\{j,\ell\}})\partial_\lambda(\bar{g}^{\{j,\ell'\}})d\tilde{v}$" may be added, since it vanishes owing to the fact that the base domains W^k are pair-wise disjoint.

[15] A version of this inequality is presented in Deny-Lions [5, (5.5), p.329].

The proof (by way of reductio ad absurdum) is omitted.

Definition 4.4. *Let* $D \subseteq Y$ *be a Riemann subdomain with* $dD \neq \emptyset$, $f \in \mathrm{Lip}(\partial D; \mathbb{C})$[16], *and* $\varphi \in \mathscr{C}^0_{2m}(D \backslash \mathscr{W})$, *where* $\mathscr{W}$ *is a thin*[17] *analytic subset of* D. *A weak solution of the Dirichlet problem for the Poisson equation:*

$$dd^c u \wedge \upsilon_p^{m-1} = \varphi \quad \text{in } D \backslash \mathscr{W}, \quad u|dD = f|dD \tag{4.7}$$

is an element $u = w \in H^1_{1,(0,1,\cdots,1)}(D)$ *such that* $w \equiv f \mod (H^1_{1,(0,1,\cdots,1),c}(D))$, *and*

$$[w, v]_{D,(1,\cdots,1)} = (\varphi, v)_D, \quad \forall v \in H^1_{1,(0,1,\cdots,1),c}(D).$$

Corollary 4.5. *Let* $D \subseteq Y$ *be a Riemann subdomain with* $dD \neq \emptyset$. *Assume that* (D, p) *has the Poincaré property relative to the unit* $2m$*-weight* $(1, \cdots, 1)$. *Then for any* $f \in \mathrm{Lip}(\partial D; \mathbb{C})$ *and* $\varphi \in \mathscr{C}^0_{2m}(D \backslash \mathscr{W})$, $\mathscr{W}$ *being thin analytic in* D, *the Dirichlet problem* (4.7) *admits a weak solution* $w \in H^1_{1,(0,1,\cdots,1)}(D)$.

Proof. Consider the linear mapping $T : H^1_{1,(0,1,\cdots,1),c}(D) \to \mathbb{C}$ defined by

$$T(v) := (\varphi, \bar{v})_D - [\tilde{f}, v]_{D,(1,\cdots,1)}, \quad \forall v \in H^1_{1,(0,1,\cdots,1),c}(D).$$

where $\tilde{f}$ is a Lipschitzian extension of f to a neighborhood of $\overline{D}$. Then the Poincaré inequality (1.3) (with $\mathfrak{h}' = (1, \cdots, 1)$) implies that the Riesz representation theorem is applicable to the operator T. Therefore there exists an element $w_0 \in H^1_{1,(0,1,\cdots,1),c}(D)$ such that

$$T(v) = [w_0, v]_{D,(1,\cdots,1)}, \quad \forall v \in H^1_{1,(0,1,\cdots,1),c}(D).$$

Then $w := w_0 + \tilde{f} \in H^1_{1,(0,1,\cdots,1)}(D)$ is a weak solution to the Dirichlet problem (4.7). ∎

5 The resolvent map for an inhomogeneous Dirichlet problem

Let $\mathfrak{h}'$ be an allowable $2m$-weight on D and $H^1_{1,\underrightarrow{\mathfrak{h}'},c}(D)$ be equipped with the Dirichlet norm (defined by (1.2)). For fixed $f \in L^2(D)$, applying the Riesz's representation theorem to the bounded anti-linear functional $v \mapsto (f, \bar{v})_D$ on $H^1_{1,\underrightarrow{\mathfrak{h}'},c}(D)$, yields an element $w \in H^1_{1,\underrightarrow{\mathfrak{h}'},c}(D)$ such that

$$(f, \bar{v})_D = [w, v]_{D,\mathfrak{h}'}, \quad \forall v \in H^1_{1,\underrightarrow{\mathfrak{h}'},c}(D).$$

[16] $f \in \mathrm{Lip}(\partial D; \mathbb{C})$ means that f is (locally) Lipschitzian in a neighborhood of ∂D. As such it admits a Lipschitzian extension $\tilde{f}$ to a neighborhood of $\overline{D}$ by invoking a partition of unity.

[17] A subset T of an m-dimensional complex space Y is *thin*, if at each point $a \in T$ there is an analytic subset A of dimension $< m$ in an open neighborhood $U \subseteq Y$ of a such that $T \cap U \subseteq A$.

The association "$f \mapsto \mathscr{R}_{D,\mathfrak{h}'} f = w$" defines a continuous linear mapping $\mathscr{R}_{D,\mathfrak{h}'} : L^2(D) \to L^2(D)$ with image in $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$ satisfying the equation (1.7).

Assume now that (D, p) has the Poincaré property relative to a bounded, $\mathscr{C}^\infty$ allowable $2m$-weight $\mathfrak{h}'$ [18]. Then the Poincaré inequality (1.3) implies that the Sobolev spaces $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$ and $H^1_{0,\underset{\to}{\mathfrak{h}'},c}(D)$ are defined by equivalent norms, hence can be naturally identified with each other. The mapping $G_{0,\underset{\to}{\mathfrak{h}'}} : L^2(D) \to H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$ (the latter being identified with $H^1_{0,\underset{\to}{\mathfrak{h}'},c}(D)$) is continuous, linear and satisfies, for each fixed $\psi \in L^2(D)$, the equation

$$(\psi, \bar{v})_D = \langle G_{0,\underset{\to}{\mathfrak{h}'}} \psi, v \rangle_{0,(0,\mathfrak{h}')} = [G_{0,\underset{\to}{\mathfrak{h}'}} \psi,\, v]_{D,\mathfrak{h}'}, \quad \forall v \in H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D).$$

Hence it follows that $\mathscr{R}_{D,\mathfrak{h}'} = G_{0,\underset{\to}{\mathfrak{h}'}} : L^2(D) \to H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$. Also, since the mapping $G_{0,\underset{\to}{\mathfrak{h}'}}$ is injective, so is the mapping $\mathscr{R}_{D,\mathfrak{h}'}$. Thus $\mathscr{R}_{D,\mathfrak{h}'}$ can (justifiably) be called the *resolvent map* for the differential operator defined by the left-side of the equation (1.8).

Proof of Theorem 1.3. It follows from the hermitian symmetry of the Dirichlet product that the mapping $\mathscr{R}_{D,\mathfrak{h}'} : L^2(D) \to L^2(D)$ is self-adjoint. In the rest of this proof write "$(\mu, \mathfrak{h})$" for the pair $(1, \{0, 1, \cdots, 1\})$. The mapping $\mathscr{R}_{D,\mathfrak{h}'}$ is expressible as a composition of the (restricted) mapping $\tilde{\mathscr{R}}_{D,\mathfrak{h}'} : L^2(D) \to H^1_{\mu,\mathfrak{h},c}(D)$ and the compact Rellich embedding $\mathfrak{i} : H^1_{\mu,\mathfrak{h},c}(D) \hookrightarrow L^2(D)$. Consequently $\mathscr{R}_{D,\mathfrak{h}'} = \mathfrak{i} \circ \tilde{\mathscr{R}}_{D,\mathfrak{h}'}$ is compact. □

Remark 5.1. *The above (same) proof of the embedding Theorem 1.3 yields the following: Assume that the embedding $H^1_{\mu,\underline{\mathfrak{h}},c}(D) \hookrightarrow L^2(D)$ is compact for a (given) allowable weight $\mathfrak{h}$ for (D, μ). Then there is defined a compact mapping $\mathscr{R}^\mu_{D,\mathfrak{h}'} : L^2(D) \to L^2(D)$ (in a way similar to that for $\mathscr{R}_{D,\mathfrak{h}'} = \mathscr{R}^1_{D,\mathfrak{h}'} : L^2(D) \to L^2(D)$).*

6 Solution of an Eigenvalue Problem

Theorem 6.1. *Assume that (D, p) is a Rellich subdomain with respect to $\mathfrak{h}'$ (the latter being a given allowable $2m$-weight). Then:*

(a) *there exists a non-decreasing, unbounded sequence $\{\lambda_j\}$ of positive real numbers such that the operator equation*

$$[u,\, v]_{D,\mathfrak{h}'} = \lambda\, (u, \bar{v})_D \quad \forall v \in \mathscr{C}^{\infty,c}(D), \quad u|dD\, =\, 0, \tag{6.1}$$

admits a nontrivial solution $u \in H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$ precisely when λ is a member of the countable set $\{\lambda_j\}$;

[18] That is, each component h_j of $\mathfrak{h}'$ is bounded on D; similarly define "$\mathscr{C}^\infty$-allowable weight on D".

(b) *there exists an orthonormal basis* $\{\psi_k\}$ *of* $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$, *consisting of eigenfunctions of* $\mathscr{R}_{D,\mathfrak{h}'}$, *which is complete in both* $L^2(D)$ *and* $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$; *furthermore, if* $\mathfrak{h}$ *is of class* $\mathscr{C}^\infty$ *on* D^*, *then each* $\psi_k \in H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D) \cap \mathscr{C}^\infty(D^*)$.

Proof. By the Remark 5.1, the mapping $\mathscr{R}_{D,\mathfrak{h}'} : L^2(D) \to L^2(D)$ is compact, and self-adjoint. Hence it has real eigenvalues $\{\beta_j\}_{j\in\mathbb{N}}$ of finite multiplicity (equal to the dimension of the eigenspace $E_j := \ker(\mathscr{R}_{D,\mathfrak{h}'} - \beta_j I))$ may be arranged as a sequence $|\beta_1| \geq |\beta_2| \geq \cdots$, with no point of accumulation except possibly the origin. The members of (distinct) E_j and $E_{j'}$ are mutually orthogonal. Since the mapping $\mathscr{R}_{D,\mathfrak{h}'}$ takes values in $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$, for each $j = 1, 2 \cdots$ and $\phi_j \in E_j$, the relation

$$\mathscr{R}_{D,\mathfrak{h}'}\phi_j = \beta_j \phi_j$$

holds and implies that ϕ_j belongs to $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$. Moreover,

$$\beta_j \,[\phi_j,\, \phi_k]_{D,\mathfrak{h}'} = [\mathscr{R}_{D,\mathfrak{h}'}\phi_j,\, \phi_k]_{D,\mathfrak{h}'} = (\phi_j, \bar{\phi}_k)_D, \quad \forall j,\, k = 1, 2, \cdots .$$

This implies that each $\beta_j > 0$. By orthonormalizing a basis of each E_j and taking their union, one obtains an orthonormal basis $\{\psi_k\}$ of $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$ consisting of eigenfunctions of $\mathscr{R}_{D,\mathfrak{h}'}$. One can arrange that each ψ_k has eigenvalue β_k (by repeatedly listing the same β_j as many times as its multiplicity, namely, $\dim E_j$). Thus

$$\psi_k - \frac{1}{\beta_k}\mathscr{R}_{D,\mathfrak{h}'}\psi_k = 0 \quad \forall k = 1, 2 \cdots,$$

so that ψ_k is a solution to a Dirichlet Problem of the type (1.5) (with $g = 0$), which is equivalent to solving the functional equation (6.1) (with $\lambda_k := \frac{1}{\beta_k} > 0$ and $\lambda_k \uparrow \infty$).

To prove the completeness of the system $\{\psi_k\}$ in $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$, recall the fact that $\ker(\mathscr{R}_{D,\mathfrak{h}'}) = \{0\}$. The desired conclusion follows then from the completeness criterion of [16, p. 234]. By a standard regularity criterion, if $\mathfrak{h}$ is of class $\mathscr{C}^\infty$ on D^*, then each eigenfunction ψ_k belongs to $\mathscr{C}^\infty(Q)$ for all (open) domains $Q \Subset D^*$. Consequently ψ_k belongs to $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D) \cap \mathscr{C}^\infty(D^*)$. □

Lemma 6.2. *Given* $g \in L^2(D)$, *the Poisson problem* (1.5) *admits a solution* ψ *in* $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$ *if and only if the following functional equation on* $\mathscr{C}^{\infty,c}(D)$,

$$(I + \alpha\mathscr{R}_{D,\mathfrak{h}'})\,\psi = w \tag{6.2}$$

with $w := \mathscr{R}_{D,\mathfrak{h}'}g \in H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$, *admits a solution* ψ *in* $H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$.

Proof. The above equation (1.6) can be expressed alternatively as a functional equation on $\mathscr{C}^{\infty,c}(D)$ in the form

$$[\psi, v]_{D,\mathfrak{h}'} + \alpha\,[\mathscr{R}_{D,\mathfrak{h}'}\psi, v]_{D,\mathfrak{h}'} = [w, v]_{D,\mathfrak{h}'},$$

where $w := \mathscr{R}_{D,\mathfrak{h}'}g \in H^1_{1,\underset{\to}{\mathfrak{h}'},c}(D)$. From this the equation (6.2) follows. □

Theorem 6.3. *Assume that* (D,p) *is a Rellich subdomain with respect to* $\mathfrak{h}' = (1,\cdots,1)$. *Let* $g \in L^2(D)$. *Consider the Poisson problem*

$$-\triangle_p\psi + \alpha\,\psi \;=\; g \;\text{ a.e. in } D, \quad \psi|dD \;=\; 0. \tag{6.3}$$

(A) *If* $\alpha \notin \{-\lambda_j\}_{j=1,\cdots,\infty}$ *(the* λ_j *being the eigenvalues of* $-\triangle_p$*), then there exists a unique weak solution* $\psi \in H^1_{1,(0,1,\cdots,1),c}(D)$ *of the problem* (6.3) *with*

$$\|\psi\|_{1,(0,1,\cdots,1)} \le \text{Const.}\,\|g\|_{L^2(D)}.$$

(B) *If* $\alpha = -\lambda_j$ *(for some* λ_j *as above), then the problem* (6.3) *has a weak solution* $\psi \in H^1_{1,(0,1,\cdots,1),c}(D)$ *if and only if* $(g,\bar\psi)_D = 0$ *for each* $\psi = \psi_j^k$, $k = 1,\cdots,s$, *the latter being the associated eigenfunctions of the problem of Theorem 6.1:*

$$(I - \lambda_j \mathscr{R}_{D,(1,\cdots,1)})(\psi) = 0.$$

Each solution of the the inhomogeneous problem (6.3) *is of the form*

$$\psi \;=\; \psi_0 + \sum_{k=1}^{s} c_k \psi_j^k, \tag{6.4}$$

where ψ_0 *is a fixed solution and the* c_k *are suitable constants.*

Proof. In this proof let the allowable weight $\underline{\mathfrak{h}}$ be $(0,(1,\cdots,1))$. Since the Rellich embedding $\mathsf{j} : H^1_{1,(0,1,\cdots,1),c}(D) \hookrightarrow L^2(D)$ is compact, so is the composition $\mathscr{R}_{D,(0,1,\cdots,1)} \circ \mathsf{j} : H^1_{1,(0,1,\cdots,1),c}(D) \to H^1_{1,(0,1,\cdots,1),c}(D)$. It is known that for a constant $\alpha \neq -\lambda_j$, the operator $I + \alpha\mathscr{R}_{D,(0,1,\cdots,1)} \circ \mathsf{j}$ is invertible with a bounded inverse. Therefore the equation (6.2) has a unique solution

$$\psi \;=\; (I + \alpha\mathscr{R}_{D,(0,1,\cdots,1)})^{-1} w_1$$

with $w_1 := \mathscr{R}_{D,(0,1,\cdots,1)}\, g \in H^1_{1,\underline{\mathfrak{h}},c}(D)$, and

$$\|\psi\|_{1,\underline{\mathfrak{h}}} \le \|(I + \alpha\mathscr{R}_{D,(0,1,\cdots,1)})^{-1}\|\,\|w_1\|_{1,\underline{\mathfrak{h}}}.$$

Since $\|w_1\|_{1,\underline{\mathfrak{h}}} \le \text{Const.}\,\|g\|_{L^2(D)}$, the assertion in (A) is proved. To prove the assertion in (B), observe that the closure of the range of an operator is the orthogonal complement of the null space of its adjoint. The equation

$$\psi - \lambda_j\,\mathscr{R}_{D,(0,1,\cdots,1)}\psi \;=\; v_1, \tag{6.5}$$

where $v_1 = \mathscr{R}_{D,(0,1,\cdots,1)}\, g \in H^1_{1,\mathfrak{h},c}(D)$, has a solution if and only if $v_1 \in R(I - \lambda_j \mathscr{R}_{D,(0,1,\cdots,1)})$, which is equivalent to $v_1 \perp \ker(I - \lambda_j \mathscr{R}^*_{D,(0,1,\cdots,1)})$. The latter means that v is orthogonal (with respect to the inner product $[\;,\;]_{D,(0,1,\cdots,1)}$ on $H^1_{1,\underline{\mathfrak{h}},c}(D)$) to all the eigenfunction ψ_j^k corresponding to the eigenvalue λ_j, namely,

$$[v_1, \psi_j^k]_{D,(0,1,\cdots,1)} = (g, \bar\psi_j^k)_D = 0.$$

The expression (6.4) is a consequence of the equation (6.5). $\square$

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Weakly strongly star-Menger spaces

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ABSTRACT

A space X is called weakly strongly star-Menger space if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there is a sequence $(F_n : n \in \omega)$ of finite subsets of X such that $\overline{\bigcup_{n\in\omega} St(F_n,\mathcal{U}_n)}$ is X. In this paper, we investigate the relationship of weakly strongly star-Menger spaces with other related spaces. It is shown that a Hausdorff paracompact weakly star Menger P-space is star-Menger. We also study the images and preimages of weakly strongly star-Menger spaces under various type of maps.

RESUMEN

Un espacio X se llama débilmente fuertemente estrella-Menger si para cada sucesión $(\mathcal{U}_n : n \in \omega)$ de cubrimientos abiertos de X, existe una sucesión $(F_n : n \in \omega)$ de subconjuntos finitos de X tales que $\overline{\bigcup_{n\in\omega} St(F_n,\mathcal{U}_n)}$ es X. En este artículo, investigamos la relación entre espacios débilmente fuertemente estrella-Menger con otros espacios relacionados. Se muestra que un P-espacio Hausdorff paracompacto débilmente estrella Menger es estrella-Menger. También estudiamos las imágenes y preimágenes de espacios débilmente fuertemente estrella-Menger bajo diversos tipos de aplicaciones.

Keywords and Phrases: Stronlgy star-Menger, star-Menger, almost star-Menger, Weakly star-Menger, covering topological spaces.

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1 Introduction

The study of selection principles in topology and their relations to game theory and Ramsey theory was started by Scheepers [24] (see also [12]). In the last two decades, these have gained enough importance to become one of the most active areas of set theoretic topology. Several covering properties are defined based on these selection principles ([17, 18]). A number of results in the literature show that many topological properties can be described and characterized in terms of star covering properties ([7, 21, 22]). The method of stars has been used to study the problem of metrization of topological spaces, and for definitions of several important classical topological notions.

Let us recall that a space X is countably compact (CC) if every countable open cover of X has a finite subcover. Fleischman [10] defined a space X to be starcompact if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \phi\}$. He proved that every countably compact space is starcompact. Van Douwen in [7] showed that every T_2 starcompact space is countably compact, but this does not hold for T_1-spaces (see [26, Example 2.5]).

Matveev [20] defined a space X to be absolutely countably compact (ACC) if for each open cover $\mathcal{U}$ of X and each dense subset D of X, there exists a finite subset F of D such that $St(F,\mathcal{U}) = X$. It is clear that every T_2-absolutely countably compact space is countably compact.

Kočinac *et al.* ([1, 2, 15, 16]), defined a space X to be strongly star-Menger (SSM) if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there exists a sequence $(F_n : n \in \omega)$ of finite subsets of X such that $\{St(F_n,\mathcal{U}_n) : n \in \omega\}$ is an open cover of X. The SSM property is stronger than the star-Menger (SM) property.

Pansera [23], defined a space X to be weakly star-Menger (WSM) if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \omega)$ with $\mathcal{V}_n$ a finite subset of $\mathcal{U}_n$ for each $n \in \omega$, and $\overline{\bigcup_{n\in\omega} St(\cup\mathcal{V}_n,\mathcal{U}_n)} = X$. WSM is weaker than the SSM property.

In this paper we introduce a star property which lies between SSM and WSM called weakly strongly star-Menger (WSSM).

The paper is organized as follows. Section 2 contains some preliminaries used in the paper. In Section 3 we investigate the relationship of WSSM spaces with other related spaces. Section 4 contains the information on subspaces and product spaces of WSSM and in the last Section 5 we study the image and preimage of WSSM spaces under continuous maps.

2 Preliminaries

Throughout this paper a space means topological space. The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we define follow [9].

We make use of two of the cardinals defined in [8]. Define ω^ω as the set of all functions from ω to itself. For all $f, g \in \omega^\omega$, we say $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many n. The unbounding number, denoted by $\mathfrak{b}$, is the smallest cardinality of an unbounded subset of $(\omega^\omega, \leq^*)$. The dominating number, denoted by $\mathfrak{d}$, is the smallest cardinality of a cofinal subset of $(\omega^\omega, \leq^*)$. It is not difficult to show that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ and it is known that $\omega_1 < \mathfrak{b} = \mathfrak{c}$, $\omega_1 < \mathfrak{d} = \mathfrak{c}$ and $\omega_1 \leq \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ are all consistent with the axioms of ZFC (see [8] for details).

A space X is said to be absolutely strongly star-Menger (ASSM) [6], if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X and each dense subset D of X, there exists a sequence $(F_n : n \in \omega)$ of finite subsets of D such that $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

A space X is called star-Menger (SM) [15], if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \omega)$ with $\mathcal{V}_n$ a finite subset of $\mathcal{U}_n$ for each $n \in \omega$, and $\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is a cover of X.

A space X is called almost star-Menger (ASM) [14], if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \omega)$ with $\mathcal{V}_n$ a finite subset of $\mathcal{U}_n$ for each $n \in \omega$, and $\{\overline{St(\cup\mathcal{V}_n, \mathcal{U}_n)} : n \in \omega\}$ is a cover of X.

Definition 2.1. *A space X is called weakly strongly star-Menger (WSSM) if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there is a sequence $(F_n : n \in \omega)$ of finite subsets of X such that $\overline{\bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)} = X$.*

From the above definitions we have the following diagram of implications:

$$\begin{tikzcd}
 & & ACC \arrow[d] \arrow[ddll, dashed, "T_2-space"'] & & \\
 & & ASSM \arrow[rr] \arrow[d] & & ASM \arrow[d] \\
CC \arrow[r] & Starcompact \arrow[l, dashed, bend left, "T_2-space"] \arrow[r] & SSM \arrow[r] \arrow[d] & WSSM \arrow[r] \arrow[ur, dashed, "P-space"] & WSM \arrow[dll, dashed, "T_2+Paracompact+P-space"] \\
 & & SM & &
\end{tikzcd}$$

The purpose of this paper is to investigate the relationships of weakly strongly star-Menger spaces with other spaces. In Example 3.3, we have shown that the WSSM property need not be SSM property. Presently, we do not know that the WSM property implies WSSM and ASM property. On the other hand, there are several examples in the literature on star-selection principles showing that other reverse implications need not be true.

3 Weakly strongly star-Menger spaces and related spaces

In this section, we give some results and examples showing relationships of weakly strongly star-Menger with other properties.

A subspace (subset) Y of a space X is WSSM if Y is WSSM as a subspace.

Theorem 3.1. *If X has a dense WSSM subspace, then X is $WSSM$.*

Proof. If $D = X$ then we are done. Let D be a non-trivial dense WSSM subspace of X and $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of X. Then $(\mathcal{U}_n^{'} : n \in \omega)$ is a sequence of open covers of D, where $\mathcal{U}_n^{'} = \{U \cap D : U \in \mathcal{U}_n\}$. Therefore there exists a sequence $(F_n^{'} : n \in \omega)$ of finite subsets of D with $\overline{\bigcup_{n\in\omega} St(F_n', \mathcal{U}_n')} = D$. Hence $\overline{\bigcup_{n\in\omega} St(F_n', \mathcal{U}_n)} = X$ as D is dense in X. □

Corollary 3.2. *Every separable topological space is WSSM.*

Given an almost disjoint family $\mathcal{A}$ of infinite subsets of ω (that is, the intersection of every two distinct elements of $\mathcal{A}$ is finite) the ψ-space or the Isbell-Mrówka space associated to $\mathcal{A}$ (denoted by $\psi(\mathcal{A})$ has $\omega \cup \mathcal{A}$ as the underlying set, the points of ω being isolated, while the basic open neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$, where F ranges over all finite subsets of ω. For more details (see [3, 11]).

Example 3.3. *There exists a Tychonoff WSSM space X which is not SSM.*

Proof. Let $X = \omega \cup \mathcal{A}$ be the Isbell-Mrówka space, where $\mathcal{A}$ is the maximal almost disjoint family of infinite subsets of ω with $|\mathcal{A}| = \mathfrak{c}$. Then X is not strongly star-Menger ([25, Example 2.3]). But X is WSSM, ω being a countable dense subset of X. □

Recall that a topological space X is a P-space [13] if every intersection of countably many open subsets of X is open.

Proposition 3.4. *A WSSM P-space X is almost star-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \omega)$ be sequence of open covers of X. Then there exists a sequence $(F_n : n \in \omega)$ of finite subsets of X with $\overline{\bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)} = X$ as X is WSSM. Since X is P-space, $\bigcup_{n\in\omega} \overline{St(F_n, \mathcal{U}_n)}$ is a closed subset of X which contains $\bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)$. Hence $\bigcup_{n\in\omega} \overline{St(F_n, \mathcal{U}_n)} = \overline{\bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)}$. Then we can find a sequence $\mathcal{V}_n$ of finite subsets of $\mathcal{U}_n$ containing F_n such that $\bigcup_{n\in\omega} \overline{St(\cup\mathcal{V}_n, \mathcal{U}_n)} = X$. □

Theorem 3.5. *A Hausdorff paracompact weakly star-Menger P-space X is star-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of X. Since a Hausdorff paracompact space is regular, for each $x \in X$ there exists an open neighborhood say $V_{n,x}$ of x with $\overline{V_{n,x}} \subseteq U$ for some $U \in \mathcal{U}_n$. Let $\mathcal{V}_n$ be a locally finite open refinement of the open cover $\{V_{n,x} : x \in X\}$. Since X is WSM there exists a sequence $(\mathcal{V}_n^{'} : n \in \omega)$ such that $\mathcal{V}_n^{'}$ is a finite subset of $\mathcal{V}_n$ for each $n \in \omega$ with $\overline{\bigcup_{n\in\omega} St(\cup\mathcal{V}_n^{'}, \mathcal{V}_n)} = X$. Now for each $V \in \mathcal{V}_n$ there is a $U_V \in \mathcal{U}_n$ with $\overline{V} \subseteq U_V$. Then for each fixed $n \in \omega$, $\overline{St(\cup\mathcal{V}_n^{'}, \mathcal{V}_n)} \subseteq St(\cup\mathcal{U}_n^{'}, \mathcal{U}_n)$, where $\mathcal{U}_n^{'}$ is finite subset of $\mathcal{U}_n$ such that for every $V \in \mathcal{V}_n^{'}$ there is $U \in \mathcal{U}_n^{'}$ containg $\overline{V}$. Therefore, $X = \overline{\bigcup_{n\in\omega} St(\cup\mathcal{V}_n^{'}, \mathcal{V}_n)} = \bigcup_{n\in\omega} \overline{St(\cup\mathcal{V}_n^{'}, \mathcal{V}_n)} = \bigcup_{n\in\omega} St(\cup\mathcal{U}_n^{'}, \mathcal{U}_n)$, because X is a P-space. □

In [15], Kočinac has shown that the property strongly star Menger is equivalent to the property star-Menger in Hausdorff paracompact space X. Then we have the following corollary:

Corollary 3.6. *For a Hausdorff paracompact P-space X, the following statements are equivalent:*

(1) X is strongly star-Menger;

(2) X is weak strongly star-Menger;

(3) X is almost star-Menger;

(4) X is weakly star-Menger;

(5) X is star-Menger.

In [13], Kocev defined d-paracompact space. A space X is said to be d-paracompact if every dense family of subsets of X has a locally finite refinement.

Theorem 3.7. *A WSM and d-paracompact space X is almost star-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of X. As X is WSM, there exists a sequence $(\mathcal{V}_n : n \in \omega)$, where $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ with $\bigcup\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ dense in X. By the assumption $\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ has a locally finite refinement say, $\mathcal{W}$. Then $\cup\mathcal{W} = \bigcup_{n\in\omega} St(\cup\mathcal{V}_n, \mathcal{U}_n)$ and therefore $\overline{\cup\mathcal{W}} = \overline{\bigcup_{n\in\omega} St(\cup\mathcal{V}_n, \mathcal{U}_n)}$. As $\mathcal{W}$ is a locally finite family, we have $\overline{\cup\mathcal{W}} = \bigcup_{W\in\mathcal{W}} \overline{W}$. Since each $W \in \mathcal{W}$ is contained in $St(\cup\mathcal{V}_n, \mathcal{U}_n)$ for some $n \in \omega$, $\bigcup_{n\in\omega} \overline{St(\cup\mathcal{V}_n, \mathcal{U}_n)} = X$. □

Corollary 3.8. *For a Hausdorff paracompact d-paracompact space X, the following statements are equivalent:*

(1) X is strongly star-Menger;

(2) X is weak strongly star-Menger;

(3) X is weakly star-Menger;

(4) X is almost star-Menger;

(5) X is star-Menger.

At the end of this section, we study the relation of WSSM property to Lindelöf covering properties.

Recall, a space X is called Lindelöf if for each open cover $\mathcal{U}$ of X there is countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = \bigcup \mathcal{V}$.

Let X be a space of the Example 4.1, then X is WSSM space but it is not Lindelöf, because X has a uncountable discrete closed subset. That means WSSM property does not imply Lindelöf property.

Theorem 3.9. *Every T_2-paracompact WSSM space is Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$ there exists an open neighborhood say V_x of x such that $\overline{V_x} \subseteq U$ for some $U \in \mathcal{U}$, because a T_2-paracompact space is regular. Let $\mathcal{V}$ be a locally finite open refinement of the cover $\{V_x : x \in X\}$. Then $(\mathcal{V}_n : n \in \omega)$ be a sequence of open covers of X, where $\mathcal{V}_n = \mathcal{V}$ for each $n \in \omega$. Since X is WSSM, there exists a sequence $(F_n : n \in \omega)$ of finite subsets of X such that $\overline{\bigcup_{n\in\omega} St(F_n, \mathcal{V}_n)} = X$. Since $\mathcal{V}_n$ is locally finite family, there exist finite subset $\mathcal{V}_n^{'}$ of $\mathcal{V}_n$ such that $St(F_n, \mathcal{V}_n) \subset \cup\mathcal{V}_n^{'}$, so $X = \overline{\bigcup_{n\in\omega} St(F_n, \mathcal{V}_n)} \subset \overline{\bigcup_{n\in\omega} \cup\{V' : V' \in \mathcal{V}_n'\}} = \bigcup_{n\in\omega} \overline{\cup\{V' : V' \in \mathcal{V}_n'\}} = \bigcup_{n\in\omega} \cup\{\overline{V'} : V' \in \mathcal{V}_n'\}$. For each $V \in \mathcal{V}_n$ there is a $U_V \in \mathcal{U}$ with $\overline{V} \subseteq U_V$. Hence we can constuct a countable subset $\mathcal{U}'$ of $\mathcal{U}$ such that $\bigcup_{n\in\omega} \cup\{\overline{V'} : V' \in \mathcal{V}_n'\} \subset \bigcup \mathcal{U}'$. ☐

Definition 3.10. *[7] A space X is called strongly star-Lindelöf (in short, SSL) if for each open cover $\mathcal{U}$ of X there is a countable subset F of X such that $St(F,\mathcal{U}) = X$.*

Clearly, SSM property implies SSL property. But next we will show that WSSM property need not be SSL property.

A space X is almost star countable [28], if for each open cover $\mathcal{U}$ of X there exists a countable subset F of X such that $\bigcup_{x\in F} \overline{St(x,\mathcal{U})} = X$.

Evidently, strongly star-Lindelöf $\Rightarrow$ almost star-countable.

Example 3.11. *A WSSM space need not be SSL.*

Proof. Let D be a discrete space of cardinality ω_1, $X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$ is a subspace of the product space $\beta D \times (\omega + 1)$. Then X is WSSM by Lemma 4.4., because $\beta D \times \omega$ is a dense σ-compact (hence, σ-countably compact) subset of X. But X is not SSL, because X is not almost star countable (see [28, Example 2.5]). ☐

Theorem 3.12. *Every T_2-paracompact WSSM space is SSL.*

Proof. The proof follows the same constructions of Theorem 3.9, thus omitted. □

4 Subspaces and product spaces

In this section we study subspaces of a WSSM space and also show that product of two WSSM spaces need not be WSSM. For some relative version of star selection principles see ([4, 5, 19]).

Example 4.1. *A closed subset of WSSM space need not be WSSM.*

Proof. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{I}$ the set of irrational numbers and $\mathbb{Q}$ the set of rational numbers. For each irrational x we choose a sequence $\{x_i : i \in \omega\}$ of rational numbers converging to x in the Euclidean topology. The rational sequence topology τ (see [29, Example 65]) is then defined by declaring each rational open and selecting the sets $U_n(x) = \{x_{n,i} : i \in \omega\} \cup \{x\}$ as a basis for the irrational point x. Then the set of irrational points $\mathbb{I}$ is a closed subset of $(\mathbb{R}, \tau)$ and $\mathbb{I}$ as a subspace of the space $(\mathbb{R}, \tau)$ is not WSSM, because it is uncountable discrete subspace. On the other hand, $(\mathbb{R}, \tau)$ is WSSM, because $\mathbb{Q}$ is dense in $(\mathbb{R}, \tau)$. □

Proposition 4.2. *Every clopen subset of a WSSM space is WSSM.*

Proof. Let Y be a clopen subset of a WSSM space X and let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of Y. Then $(\mathcal{V}_n : n \in \omega)$, where $\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus Y\}$ is a sequence of open covers of X. Since X is WSSM, there exists a sequence of finite subsets F_n of X with $\overline{\bigcup_{n\in\omega} St(F_n, \mathcal{V}_n)} = X$. Put $F_n' = Y \cap F_n$. Then clearly, $\overline{\bigcup_{n\in\omega} St(F_n', \mathcal{U}_n)} = Y$. □

Song [25] gave an example showing that the product of two countably compact spaces is not strongly star compact. This example also shows that the product of two WSSM spaces need not be WSSM. We sketch it below.

Example 4.3. *There exist two countably compact (and hence WSSM) spaces X and Y such that $X \times Y$ is not WSSM.*

Proof. Let D be the discrete space of the cardinality $\mathfrak{c}$. We define $X = \bigcup_{\alpha<\omega_1} E_\alpha$, $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(D)$ which are defined inductively so as to satisfy the following three conditions:

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$.

(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$.

(3) every infinite subset of E_α (resp., F_α) has an accumulation point in $E_{\alpha+1}$ (resp, $F_{\alpha+1}$).

These sets E_α and F_α are well-defined since every infinite closed set in $\beta(D)$ has the cardinality $2^{\mathfrak{c}}$, for detail see [30]. Then, $X \times Y$ is not WSSM since the diagonal $\{< d, d >: d \in D\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\mathfrak{c}$ and the property WSSM is preserved by open and closed subsets. Hence product of WSSM spaces need not be WSSM. □

Now we give some positive results.

Recall that a subset A of a space X is said to be σ-countably compact if it is union of countably many countably compact subset of X.

Lemma 4.4. *If a space X has a σ-countably compact dense subset, then X is WSSM.*

Proof. Let $D = \bigcup_{n\in\omega} D_n$ be a dense subset of X, where each D_n is countably compact subset of X. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of X. Then for each $n \in \omega$ there exists a finite subset say F_n of D_n such that $D_n \subset St(F_n, \mathcal{U}_n)$. So $(F_n : n \in \omega)$ is a sequence of finite subsets of D such that $D = \bigcup_{n\in\omega} St(F_n, \mathcal{U}_n)$. □

Theorem 4.5. *Let $\{X_n : n \in \omega\}$ be a countable collection of countably compact subsets of space X such that $X = \bigcup_{n\in\omega} X_n$. Then X is a WSSM space.*

Corollary 4.6. *If $\{X_n : n \in \omega\}$ is a countable collection of mutually disjoint countably compact spaces, then the topological sum $\bigoplus_{n\in\omega} X_n$ is WSSM if and only if each X_n is WSSM.*

Corollary 4.7. *If X is countably compact space, then $X \times \omega$, where ω has discrete topology is WSSM.*

Proof. Since $X \times \omega$ is homeomorphic to $\bigoplus_{n\in\omega} X \times \{n\}$ and $X \times \{n\}$ is homeomorphic to X for each $n \in \omega$. Then, by Corollary 4.6, $\bigoplus_{n\in\omega} X \times \{n\}$ is WSSM. □

5 Images and Preimages

In this section we study the images and preimages of WSSM spaces under continuous maps.

Theorem 5.1. *A continuous image of a WSSM space is WSSM.*

Proof. Let $f : X \to Y$ be a continuous surjection and X be a WSSM space. Let $(\mathcal{U}_n : n \in \omega)$ be sequence of open covers of Y. Then $(\mathcal{U}_n^{'} : n \in \omega)$, where $\mathcal{U}_n^{'} = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ is a sequence of open covers of X. Thus there exists a sequence $(F_n^{'} : n \in \omega)$ of finite subsets of X such that $\overline{\bigcup_{n\in\omega} St(F_n^{'}, \mathcal{U}_n^{'})} = X$. Let $F_n = f(F_n^{'})$. Then $(F_n : n \in \omega)$ is a sequence of finite subsets of Y. Hence result follows from the fact that for an arbitrary $y \in Y$ and each neighbourhood U of y, $U \bigcap \bigcup_{n\in\omega} St(F_n, \mathcal{U}_n) \neq \phi$. □

Next we turn to consider preimages. We show that the preimage of a WSSM space under a closed 2-to-1 continuous map need not be WSSM. First we discuss examples.

Recall the Alexandroff duplicate $A(X)$ of a space X. The underlying set $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $< x, 0 > \in X \times\{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{< x, 1 >\})$, where U is a neighborhood of x in X.

Example 5.2. *Assuming $\mathfrak{d} = \mathfrak{c}$, there exists a WSSM space X such that $A(X)$ is not WSSM.*

Proof. Assume that $\mathfrak{d} = \mathfrak{c}$. Let $X = \omega \cup \mathcal{A}$ be the Isbell–Mrówka space with $|\mathcal{A}| = \omega_1$. Then X is absolutely strongly star-Menger ([27, Example 3.5]), and hence WSSM. However $A(X)$ is not WSSM. Since the set $\mathcal{A} \times \{1\}$ is an open and closed subset of $A(X)$ with $|\mathcal{A} \times \{1\}| = \omega_1$, and for each $a \in \mathcal{A}$, the point $< a, 1 >$ is isolated in $A(X)$. Hence $A(X)$ is not WSSM, since every open and closed subset of a WSSM space is WSSM, and $\mathcal{A} \times \{1\}$ is not WSSM . □

Example 5.3. *Assuming $\mathfrak{d} = \mathfrak{c}$, there exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is a WSSM space, but X is not a WSSM.*

Proof. Let Y be the space X of Example 5.2. Then Y is WSSM. Let X be the space $A(Y)$. Then X is not WSSM. Let $f : X \to Y$ be the projection. Then f is a closed 2-to-1 continuous map, which completes the proof. □

Theorem 5.4. *Let f be an open and closed, finite-to-one continuous map from a space X onto a WSSM space Y. Then X is WSSM.*

Proof. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of X and let $y \in Y$. Since $f^{-1}(y)$ is finite, for each $n \in \omega$ there exists a finite sub-collection $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subset \cup\mathcal{U}_{n_y}$ and $U \cap f^{-1}(y) \neq \phi$ for each $U \in \mathcal{U}_{n_y}$. Since f is closed, there exists an open neighbourhood V_{n_y} of y in Y such that $f^{-1}(V_{n_y}) \subseteq \cup\{U : U \in \mathcal{U}_{n_y}\}$. Since f is open, we can assume that $V_{n_y} \subseteq \cap\{f(U) : U \in \mathcal{U}_{n_y}\}$. For each $n \in \omega$, take such open set V_{n_y} for each $y \in Y$, and put $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$ of Y. Thus $(\mathcal{V}_n : n \in \omega)$ is a sequence of open covers of Y. Since Y is WSSM, there exists a sequence $(F_n : n \in \omega)$ of finite subsets of Y such that $\overline{\bigcup_{n\in\omega} St(F_n, \mathcal{V}_n)} = Y$. Since f is finite to one, the sequence $(f^{-1}(F_n) : n \in \omega)$ is a sequence of finite subsets of X. We show that $\overline{\bigcup_{n\in\omega} St(f^{-1}(F_n), \mathcal{U}_n)} = X$. Let $x \in X$ and V be an arbitrary neighbourhood of x in X, then $f(V)$ is a neighbourhood of $y = f(x)$ as f is an open map. Then there exist $n \in \omega$ and $y' \in Y$ such that $y \in f(V) \cap V_{n_{y'}}$ with $V_{n_{y'}} \cap F_n \neq \phi$. Choose $U \in \mathcal{U}_{n_{y'}}$. Then $V_{n_{y'}} \subseteq f(U)$. Hence $U \cap f^{-1}(F_n) \neq \phi$ as $V_{n_{y'}} \cap F_n \neq \phi$. Therefore, $x \in \overline{\bigcup_{n\in\omega} St(f^{-1}(F_n), \mathcal{U}_n)}$. This shows that X is WSSM. □

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Subclasses of λ-bi-pseudo-starlike functions with respect to symmetric points based on shell-like curves

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ABSTRACT

In this paper we define the subclass $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ of the class Σ of bi-univalent functions defined in the unit disk, called λ-bi-pseudo-starlike, with respect to symmetric points, related to shell-like curves connected with Fibonacci numbers. We determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions $f \in \mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$. Further we determine the Fekete-Szegö result for the function class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ and for the special cases $\alpha = 0$, $\alpha = 1$ and $\tau = -0.618$ we state corollaries improving the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

RESUMEN

En este artículo definimos la subclase $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ de la clase Σ de funciones bi-univalentes definidas en el disco unitario, llamadas λ-bi-pseudo-estrelladas, con respecto a puntos simétricos, relacionadas a curvas espirales en conexión con números de Fibonacci. Determinamos los coeficientes iniciales de Taylor-Maclaurin $|a_2|$ y $|a_3|$ para funciones $f \in \mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$. Más aún determinamos el resultado de Fekete-Szegö para la clase de funciones $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ y para los casos especiales $\alpha = 0$, $\alpha = 1$ y $\tau = -0.618$ enunciamos corolarios mejorando los coeficientes iniciales de Taylor-Maclaurin $|a_2|$ y $|a_3|$.

Keywords and Phrases: Analytic functions, bi-univalent, shell-like curve, Fibonacci numbers, starlike functions.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The Koebe one quarter theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w \qquad (|w| < r_0(f), \ r_0(f) \geq \tfrac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since $f \in \Sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots. \tag{1.2}$$

We notice that the class Σ is not empty. For example, the functions z, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of Σ. However, the Koebe function is not a member of Σ. In fact, Srivastava *et al.* [15] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by (see [2, 3, 9, 15, 16, 17]).

An analytic function f is subordinate to an analytic function F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, provided there is an analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F(\omega(z))$. It follows from Schwarz Lemma that

$$f(z) \prec F(z) \quad \Longleftrightarrow \quad f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}), \ z \in \mathbb{U}$$

(for details see [4, 8]). We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Recently, in [14], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition:

Definition 1.1. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$.

It should be observed $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^*$.

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i\arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \le r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n\tau + u_{n-1}.$$

In [11] Raina and Sokół showed that

$$\begin{aligned}\tilde{p}(z) &= \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} = \left(t+\frac{1}{t}\right)\frac{t}{1-t-t^2} \\ &= \frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t} - \frac{1}{1-\tau t}\right) \\ &= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n,\end{aligned} \tag{1.3}$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1-\sqrt{5}}{2}, \quad t = \tau z \qquad (n = 1, 2, \ldots). \tag{1.4}$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \cdots$. And they got

$$\begin{aligned}\tilde{p}(z) &= 1 + \sum_{n=1}^{\infty}\tilde{p}_n z^n \\ &= 1 + (u_0+u_2)\tau z + (u_1+u_3)\tau^2z^2 + \sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau^n z^n \\ &= 1 + \tau z + 3\tau^2z^2 + 4\tau^3z^3 + 7\tau^4z^4 + 11\tau^5z^5 + \cdots.\end{aligned} \tag{1.5}$$

Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $Re\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1.2. *[6] The function* $\tilde{p}(z) = \dfrac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$ *belongs to the class* $\mathcal{P}(\beta)$ *with* $\beta = \sqrt{5}/10 \approx 0.2236$.

Now we give the following lemma which will use in proving.

Lemma 1.3. *[10] Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *then*

$$|c_n| \leq 2, \qquad for \qquad n \geq 1. \tag{1.6}$$

2 Bi-Univalent function class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$

In this section, we introduce a new subclass of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class by subordination.

Firstly, let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}$. Then there exists an analytic function u such that $|u(z)| < 1$ in $\mathbb{U}$ and $p(z) = \tilde{p}(u(z))$. Therefore, the function

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.1}$$

is in the class $\mathcal{P}$. It follows that

$$u(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \tag{2.2}$$

and

$$\begin{aligned} \tilde{p}(u(z)) \quad &= \quad 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_1 + \frac{c_1^2}{4}\tilde{p}_2\right\} z^2 \\ &\quad + \left\{\frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_2 + \frac{c_1^3}{8}\tilde{p}_3\right\} z^3 + \cdots . \end{aligned} \tag{2.3}$$

And similarly, there exists an analytic function v such that $|v(w)| < 1$ in $\mathbb{U}$ and $p(w) = \tilde{p}(v(w))$. Therefore, the function

$$k(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + \cdots \tag{2.4}$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w) = \frac{d_1 w}{2} + \left(d_2 - \frac{d_1^2}{2}\right)\frac{w^2}{2} + \left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\frac{w^3}{2} + \cdots \tag{2.5}$$

and

$$\begin{aligned} \tilde{p}(v(w)) \quad &= \quad 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left\{\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_1 + \frac{d_1^2}{4}\tilde{p}_2\right\} w^2 \\ &\quad + \left\{\frac{1}{2}\left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_2 + \frac{d_1^3}{8}\tilde{p}_3\right\} w^3 + \cdots . \end{aligned} \tag{2.6}$$

The class $\mathcal{L}_\lambda(\alpha)$ of λ-pseudo-starlike functions of order α $(0 \leq \alpha < 1)$ were introduced and investigated by Babalola [1] whose geometric conditions satisfy

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \alpha, \quad \lambda > 0.$$

He showed that all pseudo-starlike functions are Bazilevič of type $\left(1-\frac{1}{\lambda}\right)$ order $\alpha^{\frac{1}{\lambda}}$ and univalent in open unit disk $\mathbb{U}$. If $\lambda = 1$, we have the class of starlike functions of order α, which in this context, are 1-pseudo-starlike functions of order α. A function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\mathbb{U}$ if for every r close to 1, $r < 1$ and every z_0 on $|z| = r$ the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. This class was introduced and studied by Sakaguchi [13] presented the class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points. This class consists of functions $f(z) \in \mathcal{S}$ satisfying the condition

$$\Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > 0, \quad z \in \mathbb{U}.$$

Motivated by $\mathcal{S}_s^*$, Wang *et al.* [18] introduced the class $\mathcal{K}_s$ of functions convex with respect to symmetric points, which consists of functions $f(z) \in \mathcal{S}$ satisfying the condition

$$\Re\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right) > 0, \quad z \in \mathbb{U}.$$

It is clear that, if $f(z) \in \mathcal{K}_s$, then $zf'(z) \in \mathcal{S}_s^*$. For such a function ϕ, Ravichandran [12] presented the following subclasses: A function $f \in A$ is in the class $\mathcal{S}_s^*(\phi)$ if

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \phi(z), \quad z \in \mathbb{U},$$

and in the class $\mathcal{K}_s(\phi)$ if

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \phi(z) \quad z \in \mathbb{U}.$$

Motivated by aforementioned works [1, 13, 12, 18] and recent study of Sokól [14] (also see [11]), in this paper we define the following new subclass $f \in \mathcal{PSL}_{s,\Sigma}^\lambda(\tilde{p}(z))$ of Σ named as λ-bi-pseudo-starlike functions with respect to symmetric points, related to shell-like curves connected with Fibonacci numbers, and determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Further we determine the Fekete-Szegö result for the function class $\mathcal{PSL}_{s,\Sigma}^\lambda(\tilde{p}(z))$ and the special cases are stated as corollaries which are new and have not been studied so far.

Definition 2.1. *For $0 \leq \alpha \leq 1; \lambda > 0; \lambda \neq \frac{1}{3}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}_{s,\Sigma}^\lambda(\alpha, \tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2z(f'(z))^\lambda}{f(z)-f(-z)}\right)^\alpha \left(\frac{2[(z(f'(z)))']^\lambda}{[f(z)-f(-z)]'}\right)^{1-\alpha} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{2.7}$$

and

$$\left(\frac{2w(g'(w))^\lambda}{g(w)-g(-w)}\right)^\alpha \left(\frac{2[(w(g'(w)))']^\lambda}{[g(w)-g(-w)]'}\right)^{1-\alpha} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{2.8}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Specializing the parameter $\lambda = 1$ we have the following definitions, respectively:

Definition 2.2. *For $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(\alpha, \tilde{p}(z)) \equiv \mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(z(f'(z)))'}{[f(z)-f(-z)]'}\right)^{1-\alpha} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{2.9}$$

and

$$\left(\frac{2wg'(w)}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2(w(g'(w)))'}{[g(w)-g(-w)]'}\right)^{1-\alpha} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \tag{2.10}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Further by specializing the parameter $\alpha = 1$ and $\alpha = 0$ we state the following new classes $\mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ respectively.

Definition 2.3. *A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(1, \tilde{p}(z)) \equiv \mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{2.11}$$

and

$$\frac{2wg'(w)}{g(w)-g(-w)} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \tag{2.12}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Definition 2.4. *A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(0, \tilde{p}(z)) \equiv \mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\frac{2(z(f'(z)))'}{[f(z)-f(-z)]'} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{2.13}$$

and

$$\frac{2(w(g'(w)))'}{[g(w)-g(-w)]'} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \tag{2.14}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Definition 2.5. *For $\lambda > 0; \lambda \neq \frac{1}{3}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right) \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{2.15}$$

and

$$\left(\frac{2w(g'(w))^{\lambda}}{g(w)-g(-w)}\right) \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \tag{2.16}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Definition 2.6. *For $\lambda > 0; \lambda \neq \frac{1}{3}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{GSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\left(\frac{2[(z(f'(z)))']^{\lambda}}{[f(z)-f(-z)]'}\right) \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{2.17}$$

and

$$\left(\frac{2[(w(g'(w)))']^{\lambda}}{[g(w)-g(-w)]'}\right) \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{2.18}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases.

Theorem 2.7. *Let f given by (1.1) be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau}}. \tag{2.19}$$

and

$$|a_3| \leq \frac{2\lambda|\tau|\left[2\lambda(\alpha-2)^2 - \{5\lambda(\alpha-2)^2 + 4 - 3\alpha\}\tau\right]}{(3\lambda-1)(3-2\alpha)\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau\right]} \tag{2.20}$$

where $0 \leq \alpha \leq 1; \lambda > 0$ and $\lambda \neq \frac{1}{3}$.

Proof. Let $f \in \mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ and $g = f^{-1}$. Considering (2.7) and (2.8), we have

$$\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2[(z(f'(z)))']^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha} = \tilde{p}(u(z)) \tag{2.21}$$

and

$$\left(\frac{2w(g'(w))^{\lambda}}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2[(w(g'(w)))']^{\lambda}}{[g(w)-g(-w)]'}\right)^{1-\alpha} = \tilde{p}(v(w)) \tag{2.22}$$

for some Schwarz functions u and v where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2). Since

$$\left(\frac{2z[f'(z)]^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2[(z(f'(z)))']^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha}$$
$$= 1 - 2\lambda(\alpha-2)a_2 z + \{[2\lambda^2(\alpha-2)^2 + 2\lambda(3\alpha-4)]a_2^2 + (3\lambda-1)(3-2\alpha)a_3\}z^2 + \cdots$$

and

$$\left(\frac{2w(g'(w))^{\lambda}}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2[(w(g'(w)))']^{\lambda}}{[g(w)-g(-w)]'}\right)^{1-\alpha}$$
$$= 1 + 2\lambda(\alpha-2)a_2 w + \{[2\lambda^2(\alpha-2)^2 + 2\lambda(5-3\alpha) + 2(2\alpha-3)]a_2^2 + (3\lambda-1)(2\alpha-3)a_3\}w^2 + \cdots.$$

Thus we have

$$\begin{aligned} & 1-2\lambda(\alpha-2)a_2z+\{[2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)]a_2^2+(3\lambda-1)(3-2\alpha)a_3\}z^2+\cdots \\ = \; & 1+\frac{\tilde{p}_1c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right]z^2 \\ & +\left[\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right]z^3+\cdots. \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} & 1+2\lambda(\alpha-2)a_2w+\{[2\lambda^2(\alpha-2)^2+2\lambda(5-3\alpha)+2(2\alpha-3)]a_2^2+(3\lambda-1)(2\alpha-3)a_3\}w^2 \\ = \; & 1+\frac{\tilde{p}_1d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right]w^2 \\ & +\left[\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right]w^3+\cdots. \end{aligned} \tag{2.24}$$

It follows from (1.5), (2.23) and (2.24) that

$$-2\lambda(\alpha-2)a_2=\frac{c_1\tau}{2}, \tag{2.25}$$

$$[2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)]a_2^2+(3\lambda-1)(3-2\alpha)a_3=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{3}{4}c_1^2\tau^2, \tag{2.26}$$

and

$$2\lambda(\alpha-2)a_2=\frac{d_1\tau}{2}, \tag{2.27}$$

$$[2\lambda^2(\alpha-2)^2+2\lambda(5-3\alpha)+2(2\alpha-3)]a_2^2+(3\lambda-1)(2\alpha-3)a_3=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{3}{4}d_1^2\tau^2. \tag{2.28}$$

From (2.25) and (2.27), we have

$$c_1=-d_1, \tag{2.29}$$

and

$$a_2^2=\frac{(c_1^2+d_1^2)}{32\lambda^2(\alpha-2)^2}\tau^2. \tag{2.30}$$

Now, by summing (2.26) and (2.28), we obtain

$$\left[4\lambda^2(\alpha-2)^2+2(\lambda+2\alpha-3)\right]a_2^2=\frac{1}{2}(c_2+d_2)\tau-\frac{1}{4}(c_1^2+d_1^2)\tau+\frac{3}{4}(c_1^2+d_1^2)\tau^2. \tag{2.31}$$

By putting (2.30) in (2.31), we have

$$2\left[8\lambda^2(\alpha-2)^2-\{20\lambda^2(\alpha-2)^2-2(\lambda+2\alpha-3)\}\tau\right]a_2^2=(c_2+d_2)\tau^2. \tag{2.32}$$

Therefore, using Lemma 1.3 we obtain

$$|a_2|\leq\frac{|\tau|}{\sqrt{4\lambda^2(\alpha-2)^2-\{10\lambda^2(\alpha-2)^2-\lambda-2\alpha+3\}\tau}}. \tag{2.33}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (2.26) and (2.28). So, we find

$$2(3\lambda-1)(3-2\alpha)a_3-2(3\lambda-1)(3-2\alpha)a_2^2=\frac{1}{2}\left(c_2-d_2\right)\tau. \tag{2.34}$$

Hence, we get

$$2(3\lambda-1)(3-2\alpha)|a_3| \leq 2|\tau| + 2(3\lambda-1)(3-2\alpha)|a_2|^2. \tag{2.35}$$

Then, in view of (2.33), we obtain

$$|a_3| \leq \frac{2\lambda|\tau|\left[2\lambda(\alpha-2)^2 - \{5\lambda(\alpha-2)^2+4-3\alpha\}\tau\right]}{(3\lambda-1)(3-2\alpha)\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2-\lambda-2\alpha+3\}\tau\right]}. \tag{2.36}$$

□

If we can take the parameter $\lambda = 1$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$.

Corollary 2.8. *Let f given by* (1.1) *be in the class $\mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4(\alpha-2)^2 - 2(5\alpha^2-21\alpha+21)\tau}} \tag{2.37}$$

and

$$|a_3| \leq \frac{|\tau|\left[2(\alpha-2)^2 - \{5\alpha^2-23\alpha+24\}\tau\right]}{(3-2\alpha)\left[4(\alpha-2)^2 - \{10\alpha^2-42\alpha+42\}\tau\right]}. \tag{2.38}$$

Further by taking $\alpha = 1$ and $\alpha = 0$ and $\tau = -0.618$ in Corollary 2.8, we have the following improved initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ respectively.

Corollary 2.9. *Let f given by* (1.1) *be in the class $\mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-10\tau}} \simeq 0.19369 \tag{2.39}$$

and

$$|a_3| \leq \frac{|\tau|(1-3\tau)}{2-5\tau} \simeq 0.3465. \tag{2.40}$$

Corollary 2.10. *Let f given by* (1.1) *be in the class $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{16-42\tau}} \simeq 0.0954 \tag{2.41}$$

and

$$|a_3| \leq \frac{4|\tau|(1-3\tau)}{3(8-21\tau)} \simeq 0.17647. \tag{2.42}$$

Corollary 2.11. *Let f given by* (1.1) *be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4\lambda^2 - \{10\lambda^2-\lambda+1\}\tau}} \tag{2.43}$$

and

$$|a_3| \leq \frac{2\lambda|\tau|\left[2\lambda - \{5\lambda+1\}\tau\right]}{(3\lambda-1)\left[4\lambda^2 - \{10\lambda^2-\lambda+1\}\tau\right]} \tag{2.44}$$

where $\lambda > 0$ and $\lambda \neq \frac{1}{3}$.

Corollary 2.12. *Let f given by* (1.1) *be in the class* $\mathcal{GSL}_{s,\Sigma}^{\lambda}(\tilde{p}(z))$*. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{16\lambda^2 - \{40\lambda^2 - \lambda + 3\}\tau}} \tag{2.45}$$

and

$$|a_3| \leq \frac{2\lambda|\tau|\left[8\lambda - \{20\lambda + 4\}\tau\right]}{3(3\lambda - 1)\left[16\lambda^2 - \{40\lambda^2 - \lambda + 3\}\tau\right]} \tag{2.46}$$

where $\lambda > 0$ *and* $\lambda \neq \frac{1}{3}$.

3 Fekete-Szegö inequality for the function class $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$

Fekete and Szegö [7] introduced the generalized functional $|a_3 - \mu a_2^2|$, where μ is some real number. Due to Zaprawa [19], in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$.

Theorem 3.1. *Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let f given by (1.1) be in the class $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau\right]}. \tag{3.1}$$

Proof. From (2.32) and (2.34) we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1-\mu)(c_2+d_2)\tau^2}{4\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau\right]} + \frac{\tau(c_2-d_2)}{4(3\lambda-1)(3-2\alpha)} \\ &= \left(\frac{(1-\mu)\tau^2}{4\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau\right]} + \frac{\tau}{4(3\lambda-1)(3-2\alpha)}\right) c_2 \\ &+ \left(\frac{(1-\mu)\tau^2}{4\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau\right]} - \frac{\tau}{4(3\lambda-1)(3-2\alpha)}\right) d_2. \end{aligned}$$

So we have

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{\tau}{4(3\lambda-1)(3-2\alpha)}\right) c_2 + \left(h(\mu) - \frac{\tau}{4(3\lambda-1)(3-2\alpha)}\right) d_2 \tag{3.2}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4\lambda^2(\alpha-2)^2 - \{10\lambda^2(\alpha-2)^2 - \lambda - 2\alpha + 3\}\tau\right]}.$$

Then, by taking modulus of (3.2), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda-1)(3-2\alpha)}. \end{cases} \qquad \square$$

Taking $\mu = 1$, we have the following corollary.

Corollary 3.2. *If $f \in \mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$, then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{4(3\lambda-1)(3-2\alpha)}. \tag{3.3}$$

If we can take the parameter $\lambda = 1$ in Theorem 3.1, we can state the following:

Corollary 3.3. *Let f given by (1.1) be in the class $\mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{8(3-2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{8(3-2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{8(3-2\alpha)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4(\alpha-2)^2 - \{10(\alpha-2)^2 - 2\alpha + 2\}\tau\right]}.$$

Further by fixing $\lambda = 1$ taking $\alpha = 1$ and $\alpha = 0$ in the above corollary, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}_{s,\Sigma}^{*}(\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$, respectively.

Corollary 3.4. *Let f given by (1.1) be in the class $\mathcal{SL}_{s,\Sigma}^{*}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{24}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{24} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{24} \end{cases}$$

where $h(\mu) = \dfrac{(1-\mu)\tau^2}{8\left[2 - 5\tau\right]}$.

Corollary 3.5. *Let f given by (1.1) be in the class $\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{8}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{8} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{8} \end{cases}$$

where $h(\mu) = \dfrac{(1-\mu)\tau^2}{8\left[8 - 21\tau\right]}$.

By assuming $\lambda \in \mathbb{R}; \lambda > \frac{1}{3}$ and taking $\alpha = 1$ and $\alpha = 0$ we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\tilde{p}(z))$ and $\mathcal{GSL}_{s,\Sigma}^{\lambda}(\tilde{p}(z))$, respectively.

Corollary 3.6. *Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let f given by (1.1) be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{4(3\lambda-1)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda-1)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda-1)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4\lambda^2 - \{10\lambda^2 - \lambda + 1\}\tau\right]}.$$

Corollary 3.7. *Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let f given by (1.1) be in the class $\mathcal{GSL}^{\lambda}_{s,\Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{12(3\lambda-1)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{12(3\lambda-1)} \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{12(3\lambda-1)} \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[16\lambda^2 - \{40\lambda^2 - \lambda + 3\}\tau\right]}.$$

Conclusions

Our motivation is to get many interesting and fruitful usages of a wide variety of Fibonacci numbers in Geometric Function Theory. By defining a subclass λ-bi-pseudo-starlike functions with respect to symmetric points of Σ related to shell-like curves connected with Fibonacci numbers we were able to unify and extend the various classes of analytic bi-univalent function, and new extensions were discussed in detail. Further, by specializing $\alpha = 0$ and $\alpha = 1$ and $\tau = -0.618$ we have attempted at the discretization of some of the new and well-known results. Our main results are new and better improvement to initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

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A new class of graceful graphs: k-enriched fan graphs and their characterisations

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ABSTRACT

The Graceful Tree Conjecture stated by Rosa in the mid 1960s says that every tree can be gracefully labelled. It is one of the best known open problems in Graph Theory. The conjecture has caused a great interest in the study of gracefulness of simple graphs and has led to many new contributions to the list of graceful graphs. However, it has to be acknowledged that not much is known about the structure of graceful graphs after 55 years.
Our paper adds an infinite family of classes of graceful graphs to the list of known simple graceful graphs. We introduce classes of k-enriched fan graphs kF_n for all integers $k, n \geq 2$ and we prove that these graphs are graceful. Moreover, we provide characterizations of the k-enriched fan graphs kF_n among all simple graphs via Sheppard's labelling sequences introduced in the 1970s, as well as via labelling relations and graph chessboards. These last approaches are new tools for the study of graceful graphs introduced by Haviar and Ivaška in 2015. The labelling relations are closely related to Sheppard's labelling sequences while the graph chessboards provide a nice visualization of graceful labellings. We close our paper with an open problem concerning another infinite family of extended fan graphs.

RESUMEN

La Conjetura del Árbol Amable enunciada por Rosa a mediados de los 1960s dice que cada árbol puede ser etiquetado amablemente. Es uno de los problemas abiertos mejor conocidos en Teoría de Grafos. La conjetura ha causado un gran interés en el estudio de la amabilidad de grafos simples y ha llevado a muchas contribuciones nuevas a la lista de grafos amables. De todas formas, debe reconocerse que no se sabe mucho acerca de la estructura de grafos amables tras 55 años.
Nuestro artículo añade una familia infinita de clases de grafos amables a la lista de grafos amables simples conocidos. Introducimos clases de grafos abanico k-enriquecidos kF_n para todos los enteros $k, n \geq 2$ y demostramos que estos grafos son amables. Más aún, entregamos caracterizaciones de los grafos abanico k-enriquecidos kF_n entre todos los grafos simples vía sucesiones de etiquetado de Sheppard introducidas en los 1970s, y también a través de relaciones de etiquetados y tableros de ajedrez de grafos. Estos últimos acercamientos son herramientas nuevas para el estudio de grafos amables introducidos por Haviar e Ivaška en 2015. Las relaciones de etiquetado están relacionadas cercanamente con las sucesiones de etiquetado de Sheppard mientras que los tableros de ajedrez de grafos proveen una linda visualización de etiquetados amables. Concluimos nuestro artículo con un problema abierto relacionado con otra familia infinita de grafos abanico extendidos.

Keywords and Phrases: graph, graceful labelling, graph chessboard, labelling sequence, labelling relation.

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1 Introduction

A simple graph of size m has a *graceful labelling*, and it is said to be *graceful*, when its vertices can be assigned different labels from the set $\{0, 1, ..., m\}$ such that the absolute values of the differences in vertex labels of edges form the set $\{1, ..., m\}$. *The Graceful Tree Conjecture* stated by Rosa in [9] and [10] says that every tree is graceful. The conjecture has now been open for more than a half century and is one of the most attractive open problems in Graph Theory. The best source of information on attacks of the conjecture and on the study of labellings of graphs is the electronic book by Gallian [2].

Still, not much is known about the structure of graceful graphs. One of the known results, due to Hrnčiar and Haviar [6], is that all trees of diameter five are graceful.

The Graceful Tree Conjecture has led to a much increased interest in the study of gracefulness of simple graphs. In this paper we introduce a new infinite family of classes of simple graphs, k-enriched fan graphs kF_n, for all integers $k, n \geq 2$, and we show that they are graceful. These classes of graphs have been recently considered in the second author's MSc thesis [8]. The k-enriched fan graphs kF_n are a natural generalisation of the class of fan graphs F_n, which were shown to be graceful in [5].

For a better understanding of the general k-enriched fan graphs and their characterisations, the special cases $k = 2$ and $k = 3$ of double and triple fan graphs are dealt with separately. Characterisations of the double and triple fan graphs, and then of the k-enriched fan graphs, are presented using the tools of labelling sequences, labelling relations and graph chessboards. Labelling sequences were introduced in 1976 by Sheppard in [11] while the labelling relations and graph chessboards as new tools for the study of graceful graphs were introduced and studied rather recently by Haviar and Ivaška in [5].

The basic terms and facts needed in this paper are presented in Section 2. This includes the concepts of labelling sequences, labelling relations and graph chessboards. In Section 3 we describe fan graphs and their graceful labellings from [5]. Then we generalise the concept of a fan graph to the one called a *double fan graph* resp. *triple fan graph* by connecting n paths P_2 resp. P_3 to the main path of a given fan graph F_n. We construct for these graphs their graceful labellings and describe them by the corresponding labelling sequences, the labelling relations and the graph chessboards. Finally we study the general case of the *k-enriched fan graphs kF_n* for any integers $k, n \geq 2$. These graphs contain, compared to the classic fan graphs F_n, n copies of the star S_k connected to the main path P_n. Again, we construct for these graphs their graceful labellings and characterize them by their labelling sequences, the labelling relations and the graph chessboards.

We conclude the paper with proposing another natural extension of the class of fan graphs and raising an open problem whether these extended fan graphs can be shown to be graceful and characterized in the manner presented here.

2 Preliminaries

In this section we recall the necessary basic terms concerning graph labellings as well as the concepts of labelling sequences, labelling relations and simple chessboards, which we use as our tools to describe new classes of graceful graphs. These definitions are taken primarily from [10] and [5].

In this paper we study only finite *simple graphs*, that is, finite unoriented graphs without loops and multiple edges. The following concept was called *valuation* by Rosa in his seminal paper [10].

Definition 2.1 ([10, 5]). *A vertex labelling (or labelling for short) f of a simple graph $G = (V, E)$ is a one-to-one mapping of its vertex set $V(G)$ into the set of non-negative integers assigning to the vertices so-called vertex labels.*

By the *label of an edge uv* in the labelling f we mean the number $|f(u) - f(v)|$, where $f(u), f(v)$ are the vertex labels of u, v. In this text we will denote by $f(V_G)$ the set of all vertex labels and by $f(E_G)$ the set of all edge labels in the labelling f of a graph G.

Several types of graph labellings are known, e.g. $\alpha, \beta, \sigma, \rho$, which have become very well-known since the publication of Rosa's paper [10] in 1967, and further γ, δ, p, q introduced by Rosa in his dissertation thesis [9] in 1965. In this text we study only β-labellings called *graceful labellings*.

Definition 2.2 ([10, 5]). *A graceful labelling (or β-labelling) of a graph $G = (V, E)$ of size m is a vertex labelling with the following properties:*

(1) $f(V_G) \subseteq \{0, 1, \ldots, m\}$, *and*

(2) $f(E_G) = \{1, 2, \ldots, m\}$.

The term *graceful* was given to β-labellings in 1972 by Golomb [4] and its immediate use was enhanced by a popularization by Gardner [3]. Hence a graceful labelling of a graph of size m has vertex labels among the numbers $0, 1, ..., m$ such that the induced edge labels are different and cover all values $1, 2, ..., m$. In Figure 1 we can see some graceful graphs.

Each graceful graph can be represented by a sequence of non-negative integers. This was shown by Sheppard in [11] where he introduced the concept of a *labelling sequence* as follows:

Definition 2.3 ([11, 5]). *For a positive integer m, a labelling sequence is the sequence of non-negative integers $(j_1, j_2, \ldots, j_m)$, denoted (j_i), where*

$$0 \leq j_i \leq m - i \quad \textit{for all } i \in \{1, 2, \ldots, m\}. \tag{LS}$$

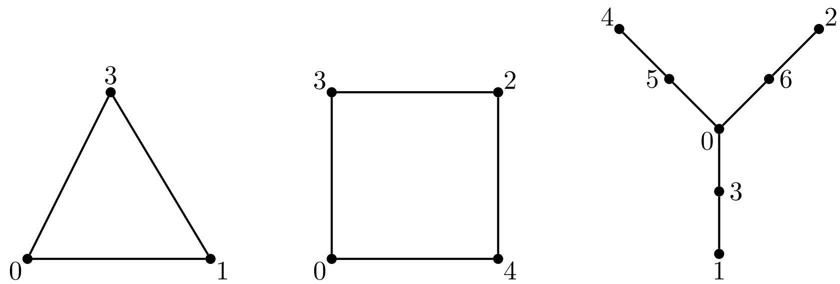


Figure 1: Some graceful graphs

Sheppard also proved that there is a one-to-one correspondence between graceful labellings of graphs (without isolated vertices) and labelling sequences. Therefore we can understand labelling sequences as a tool to encode graceful labellings of graphs. The connection is described in the following theorem.

Theorem 2.4 ([11, 5])**.** *There exists a one-to-one correspondence between graphs of size m having a graceful labelling f and between labelling sequences (j_i) of m terms. The correspondence is given by*

$$j_i = \min\{f(u), f(v)\}, \quad i \in \{1, 2, \dots, m\},$$

where u, v are the end-vertices of the edge labelled i.

Now we recall the definition of a labelling relation. It is another tool to describe gracefully labelled graphs, which is closely related to the labelling sequence. The concept of the *labelling relation* was introduced and studied in [5].

Definition 2.5 ([5])**.** *Let $L = (j_1, j_2, \dots, j_m)$ be a labelling sequence. Then the relation $A(L) = \{[j_i, j_i + i], i \in \{1, 2, \dots, m\}\}$ is called a labelling relation assigned to the labelling sequence L.*

To visualize a labelling relation and also a labelling sequence we will use a *labelling table* (see Figure 2). A table is formed by using the numbers $1, 2, ..., m$ as headers for the columns, followed by two rows. The first row contains the numbers from the labelling sequence and the second row contains the sums of the corresponding numbers from the heading and from the first row. The pairs from first and second row in each column are then the elements of the labelling relation (and correspond to the edges of the graph).

1	2	3	...	m
j_1	j_2	j_3	...	j_m
$j_1 + 1$	$j_2 + 2$	$j_3 + 3$	...	$j_m + m$

Figure 2: Displaying a labelling table

Example 2.6. *In Figure 3 we can see the labelling table assigned to the labelling sequence* $(5, 4, 3, 2, 1, 0)$ *and its graceful graph whose edges correspond to the elements of the labelling relation.*

1	2	3	4	5	6
5	4	3	2	1	0
6	6	6	6	6	6

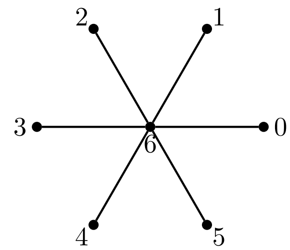


Figure 3: Example of a labelling table and its corresponding graceful graph

Every labelled simple graph of order n can be represented by a *chessboard*, i.e. a table with n rows and n columns, where every edge labelled uv is represented by a pair of dots with coordinates $[u, v]$ and $[v, u]$. This idea of visualization of vertex labellings of graphs by chessboards and independent discoveries of similar ideas are described in [5, Chapter 2]. One can also obtain such a graph chessboard by taking the adjacency matrix of a graph and placing dots in the cells corresponding to "ones" in the matrix while leaving the cells corresponding to "zeros" in the matrix empty (cf. [5, p. 25]).

Several types of graph chessboards like simple chessboards, double chessboards, M-chessboards, dual chessboards and twin chessboards were studied by Haviar and Ivaška in [5]. In this text we

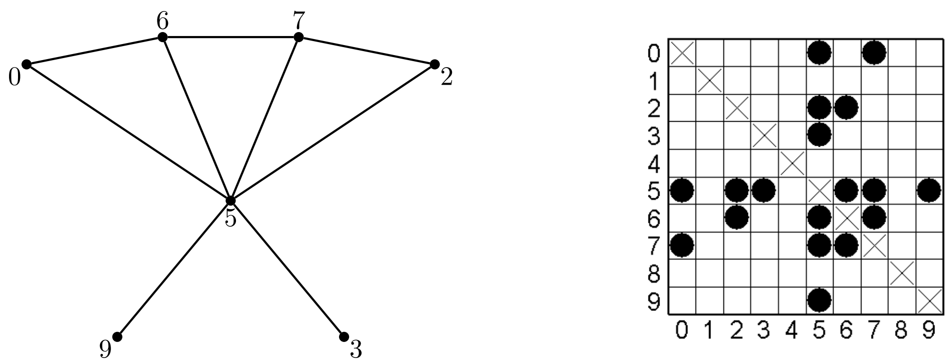


Figure 4: Example of a graph of size $m = 9$ and its corresponding simple chessboard

use only the idea of a simple chessboard, which is a useful tool in the visualization of gracefully labelled graphs.

For a positive integer m consider an $(m+1) \times (m+1)$-table. Rows are numbered by $0, 1, ..., m$ from the top to the bottom and columns are numbered by $0, 1, ..., m$ from the left to the right as shown in Figure 4. The cell with coordinates $[i, j]$ of the table will mean the cell in the i-th row and the j-th column. The *r-th diagonal* in the table is the set of all cells with coordinates $[i, j]$ where $i - j = r$ and $i \geq j$. The *main diagonal* is the 0-th diagonal in the table and all other diagonals are called *associate* diagonals.

To a graph of size m whose vertices are labelled by different numbers from the set $\{0, 1, 2, ..., m\}$

we assign its *simple chessboard* as the $(m+1) \times (m+1)$-table described above such that every edge labelled uv in the graph is represented by a pair of dots in the cells with coordinates $[u, v]$ and $[v, u]$. It follows that the simple chessboards are symmetric about the main diagonal. An illustration of the simple chessboard of such a labelled graph of size 9 is in Figure 4.

If there is exactly one dot on each of the associate diagonals, then the simple chessboard will be called *graceful* as it clearly encodes a graceful graph. We can see a gracefully labelled graph of size 8 and its graceful simple chessboard in Figure 5.

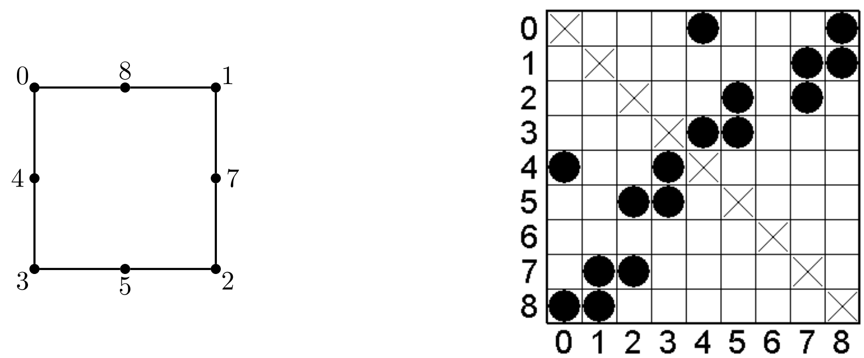


Figure 5: Gracefully labelled graph and its corresponding graceful simple chessboard

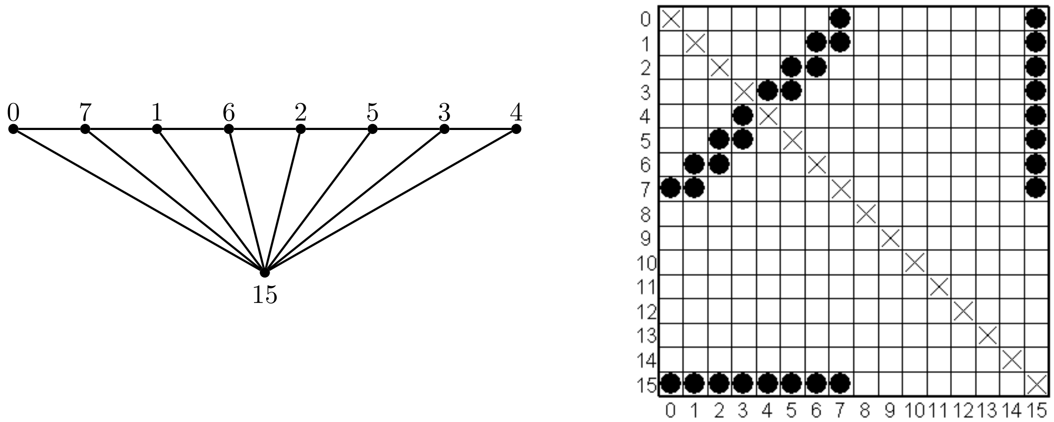
3 Fan graphs and their descriptions

A *fan graph* is a join of a path and a single vertex K_1. (See [5, Section 4.4.7] and Figure 6.) Its "bottom part" is formed by a star.

In this text we introduce the notation F_n for a fan graph whose main path is P_n. We obviously require $n \geq 2$ in order to have the shape of "a fan". The fan graph F_n thus has order $n+1$ and size $2n-1$. In Figure 6 we can see a gracefully labelled fan graph F_8 of size 15, with the main path P_8, its corresponding simple chessboard and its labelling relation.

The following two results taken from [5] characterize fan graphs via their labelling sequences, labelling relations and simple chessboards. (It is important to notice that [5, Section 4.4.7] considers the fan graphs that are F_{n+1} in our present notation. In Figure 4.10 there we have the fan graph F_7 with considering $n = 6$. That is why for our graphs G studied in [5, Section 4.4.7] we talk about their size $m = 2n+1$.) The notation $\mathbb{N}$ in the next theorem and in our subsequent results is used, as in [5], for the set of all positive integers and the notation $\lfloor x \rfloor$ is used for the largest integer not greater than x (the floor function).

Theorem 3.1 ([5]). *Let G be a graph of size $m = 2n+1$ for some $n \in \mathbb{N}$. Then G is a fan graph*



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3	2	2	1	1	0	7	6	5	4	3	2	1	0
4	5	5	6	6	7	7	15	15	15	15	15	15	15	15

Figure 6: Representations of the gracefully labelled fan graph F_8

if and only if there exists a labelling sequence $(j_1, j_2, \ldots, j_m)$ *of* G *such that*

$$j_i = \begin{cases} \left\lfloor \frac{n-i+1}{2} \right\rfloor, & \text{if } i \le n, \\ m - i, & \text{if } i > n. \end{cases} \tag{LSFG}$$

Corollary 3.2 ([5])**.** *Let* G *be a graph of size* $m = 2n+1$ *that can be gracefully labelled and let* $\mathcal{L}$ *be the set of all its labelling sequences. Then the following are equivalent:*

(a) G *is a fan graph.*

(b) There exists a labelling sequence $L \in \mathcal{L}$ *which satisfies (LSFG).*

(c) For some $L \in \mathcal{L}$*, the labelling relation is*

$$\begin{aligned} A(L) = & \{[i, n-1] \mid i \in \{0, 1, \ldots, \lfloor \tfrac{n-i}{2} \rfloor\}\} \cup \\ & \{[i, n-i+1] \mid i \in \{0, 1, \ldots, \lfloor \tfrac{n}{2} \rfloor\}\} \cup \\ & \{[i, m] \mid i \in \{0, 1, \ldots, n\}\}. \end{aligned}$$

(d) Some chessboard of G *looks like the chessboard in Figure 6.*

Example 3.3. *The sequence* $(3, 3, 2, 2, 1, 1, 0, 7, 6, 5, 4, 3, 2, 1, 0)$ *is the labelling sequence of the fan graph* F_8 *whose representations by the graph chessboard and the labelling table are seen in Figure 6.*

4 Double and triple fan graphs and their descriptions

By a *double fan graph* DF_n we will mean the fan graph F_n with extra n pendant vertices, which are attached to the main path P_n. Hence double fan graphs DF_n can be understood such that we connect n paths P_2 to a given fan graph F_n by identifying one vertex of each of the paths P_2 with one vertex of the main path P_n of the fan graph. When we connect this way n paths P_3 to a fan graph F_n, we get a *triple fan graph* TF_n.

In Figure 7 we see the double fan graph DF_5 on the left side and the triple fan graph TF_4 on the right side.

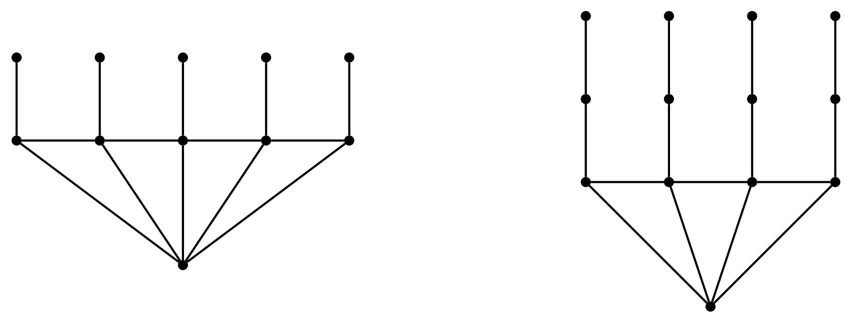
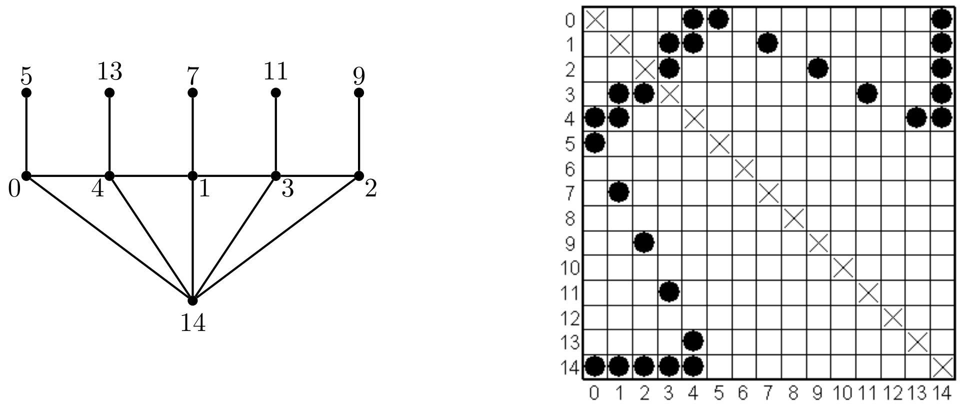


Figure 7: The double fan graph DF_5 (left) and the triple fan graph TF_4 (right)

We can divide vertices of such graphs into these groups: (i) the *root vertex*, which forms together with n adjacent edges the "bottom" star, (ii) the "middle" vertices of the main path P_n, and (iii) the "upper" pendant vertices, which together with the "middle" vertices form n paths P_2 resp. P_3 connected to the main path. Hence double fan graphs DF_n resp. triple fan graphs TF_n can be understood as certain extensions of the fan graphs by adding in the "upper part" n paths P_2 (in double fan graphs) resp. n paths P_3 (in triple fan graphs). We will often refer to the "bottom", "middle" and "upper parts" (in spite of the fact these parts are not pairwise disjoint and our naming of these parts refers only to our chosen visualization of these graphs) to assist in our proofs.

Example 4.1. *The sequence* $(2, 1, 1, 0, 0, 1, 2, 3, 4, 4, 3, 2, 1, 0)$ *is a labelling sequence of a gracefully labelled double fan graph* DF_5 *of size* 14 *and order* 11*. The corresponding graph chessboard and labelling table are in Figure 8. Because the dots in the graph chessboard representing the edges of the mentioned "bottom", "middle" and "upper parts" of the double fan graph look respectively like the "bottom part", the "head" and the "neck" of a swan, we will refer to such chessboards as* swan chessboards*.*

In the next theorems we characterise double fan graphs DF_n and triple fan graphs TF_n by their simple chessboards, labelling sequences and labelling relations. Within these characterisations we prove there are specific graceful labellings of these graphs, thus we show the graphs DF_n and TF_n



1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	1	0	0	1	2	3	4	4	3	2	1	0
3	3	4	4	5	7	9	11	13	14	14	14	14	14

Figure 8: Representations of the gracefully labelled double fan graph DF_5

are graceful. So our double and triple fan graphs are new additions to the list of graceful graphs.

Theorem 4.2. *Let G be a graph of size $m = 3n-1$ for some $n \in \mathbb{N} - \{1\}$. Then the following are equivalent:*

(1) G is the double fan graph DF_n.

(2) There is a graceful labelling of G with a swan chessboard.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{n-i}{2} \right\rfloor, & \text{if } i < n, \\ i - n, & \text{if } n \leq i < 2n, \\ m - i, & \text{if } i \geq 2n. \end{cases} \tag{$LSDFG$}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$\begin{aligned} A(L) =& \{[u,v] \mid u,v \in \{0,1,\ldots,n-1\},\ u<v,\ n-1 \leq u+v \leq n\} \cup \\ & \{[u, n+2u] \mid u \in \{0,1,\ldots,n-1\}\} \cup \\ & \{[n-1-u, m] \mid u \in \{0,1,\ldots,n-1\}\}. \end{aligned}$$

Proof. (1) $\Rightarrow$ (2) Let G be the double fan graph DF_n. We will label it in the following way. Let the labelling of the single vertex at the "bottom" (the root of DF_n) be m. Let the vertices of the middle path P_n be labelled from one end-vertex of P_n gradually by numbers $0, n-1, 1, n-2, 2, ...$

(see Figure 8). Hence the second end-vertex is labelled by $\lfloor \frac{n}{2} \rfloor$. Finally we will label n pendant vertices in the "upper part". We start from the vertex which is attached to end-vertex, at which we started the labelling of the main path P_n. We label the vertices from this vertex gradually by numbers $n, m-1, n+2, m-3, n+4, ...$ We get edges with labellings $\{0, n\}, \{n-1, m-1\}, \{1, n+2\}, \{n-2, m-3\}$, and so on, in this "upper part". Now our labelling is done and we show this labelling is graceful with a corresponding swan chessboard (see Figure 8). One can easily verify the following statements:

(i) The "bottom part" of G which is the star of size n is in the chessboard represented by n dots in the bottom row with coordinates $[m, 0], [m, 1], ..., [m, n-1]$ which form the "bottom of the swan".

(ii) The "middle part" of G which is the path P_n is in the chessboard represented by $n-1$ dots with coordinates $[n-1, 0], [n-1, 1], [n-2, 1], [n-2, 2], ...$ which form the "head of the swan".

(iii) The "upper part" of G which is the union of n paths P_2 is in the chessboard represented by n dots with coordinates $[n, 0], [n+2, 1], [n+4, 2], [n+6, 3], ...$ which form the "neck of the swan".

The described swan chessboard with three blocks of dots obviously has one dot on each diagonal. Hence G is graceful.

(2) $\Rightarrow$ (3) Assume we have a graceful labelling of G with a swan chessboard (as in Figure 8). We will verify that when we make the labelling sequence corresponding to this graceful labelling, we get exactly the labelling sequence satisfying (LSDFG). We use the distribution of dots in a swan chessboard into three blocks as in the previous part of the proof. So we consider the "head", the "neck" and the "bottom part" of a "swan". One can verify that these three blocks of dots in our chessboard can be assigned to the corresponding integers in the labelling sequence. The first block of dots, the "head", is represented in the corresponding labelling sequence by the integers j_i of the form $\lfloor \frac{n-i}{2} \rfloor$ for $i < n$. The second block of dots, the "neck", is represented in the corresponding labelling sequence by the integers j_i of the form $i - n$ for $n \leq i < 2n$. The third block of dots, the "bottom part", is represented in the corresponding labelling sequence by the integers j_i of the form $m - i$ for $i \geq 2n$.

We have shown that there exists a labelling sequence of G satisfying the formula (LSDFG).

(3) $\Rightarrow$ (4) Assume we have a labelling sequence L of G which satisfies (LSDFG). We will show that the labelling relation $A(L)$ from this labelling sequence consists of the pairs as described in (4). Indeed, one can verify that the non-negative integers j_i from the labelling sequence, which have the form $\lfloor \frac{n-i}{2} \rfloor$ for $i < n$, correspond in $A(L)$ to the pairs $[u, v]$ where $u, v \in \{0, 1, \ldots, n-1\}$, $u < v$ and $n - 1 \leq u + v \leq n$. The next j_i from the labelling sequence, which have the form $i - n$ for

$n \leq i < 2n$, correspond to the pairs $[u, n+2u]$ for $u \in \{0,1,\ldots,n-1\}$. Finally, j_i from the labelling sequence, which have the form $m-i$ for $i \geq 2n$, correspond to the pairs $[n-1-u, m]$ for $u \in \{0,1,\ldots,n-1\}$.

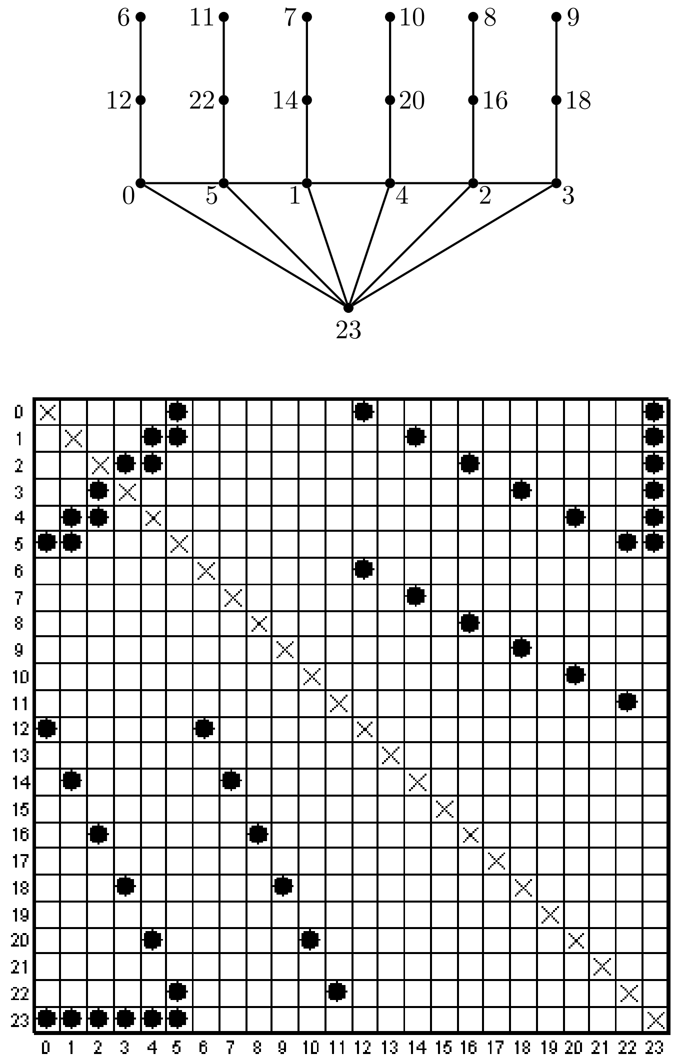
(4) $\Rightarrow$ (1) Assume that there exists a labelling sequence L of G with the labelling relation $A(L)$ as in (4). From the definition we know the labelled edges of G correspond to the pairs in $A(L)$. One can verify that the pairs $[n-1-u, m]$ from $A(L)$ for $u \in \{0,1,\ldots,n-1\}$ correspond to the edges in "the bottom part" of graph G which therefore is a star of size n. The pairs $[u,v]$ from $A(L)$ for $u,v \in \{0,1,\ldots,n-1\}$, $u < v$ and such that $n-1 \leq u+v \leq n$ correspond to the edges in "the middle part" of G which therefore is the path P_n. Finally, the pairs $[u, n+2u]$ for $u \in \{0,1,\ldots,n-1\}$ correspond to the edges in "the upper part" of G which therefore form n paths P_2 connected to the main path P_n. So the three parts in $A(L)$ correspond to the three parts of the double fan graph DF_n. □

Hence a double fan graph can always be gracefully labelled so that its chessboard has the "head", the "neck" and the "bottom part" of a "swan". In the rest of this section we focus on a description of triple fan graphs.

Example 4.3. *The sequence* $(2,2,1,1,0,6,7,8,9,10,11,0,1,2,3,4,5,5,4,3,2,1,0)$ *is a labelling sequence of a triple fan graph of size* 23 *and order* 19*. The corresponding graph diagram, graph chessboard and labelling table are in Figure 9.*

Notice that while the swan chessboard of a double fan graph had one "neck" connecting the "head" and the "bottom part" of the "swan", the simple chessboard of a triple fan graph has 3 blocks of dots that look like a "swan without head", then "the head separated from the neck", and one extra "separated neck". Hence we will refer to such simple chessboards representing the triple fan graphs as 3-part swan chessboards*.*

The following characterisations of triple fan graphs can be proven using a method similar to that used in the previous theorem, and will be covered by the general case in the subsequent section.



1	2	3	4	5	6	7	8	9	10	11
2	2	1	1	0	6	7	8	9	10	11
3	4	4	5	5	12	14	16	18	20	22

12	13	14	15	16	17	18	19	20	21	22	23
0	1	2	3	4	5	5	4	3	2	1	0
12	14	16	18	20	22	23	23	23	23	23	23

Figure 9: Representations of the gracefully labelled triple fan graph TF_6

Theorem 4.4. *Let G be a graph of size $m = 4n - 1$ for some $n \in \mathbb{N} - \{1\}$. Then the following are equivalent:*

(1) G is the triple fan graph TF_n.

(2) There is a graceful labelling of G with a 3-part swan chessboard.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{n-i}{2} \right\rfloor, & \text{if } i < n, \\ i, & \text{if } n \leq i < 2n, \\ i - 2n, & \text{if } 2n \leq i < 3n, \\ m - i, & \text{if } i \geq 3n. \end{cases} \qquad (LSTFG)$$

(4) There exists a labelling sequence L of G with the labelling relation

$$\begin{aligned} A(L) = & \{[u, v] \mid u, v \in \{0, 1, \ldots, n-1\},\ u < v,\ n-1 \leq u+v \leq n\} \cup \\ & \{[u, 2u] \mid u \in \{n, n+1, \ldots, 2n-1\}\} \cup \\ & \{[u, 2n+2u] \mid u \in \{0, 1, \ldots, n-1\}\} \cup \\ & \{[n-1-u, m] \mid u \in \{0, 1, \ldots, n-1\}\}. \end{aligned}$$

5 General case: k-enriched fan graphs kF_n and their descriptions

According to the previous section we would assume that the more "separated necks" we have in the graph chessboard, the longer the paths in the "upper part" of the graph will be. Our work with Graph processor introduced in [7] and much used also in [5] has led us to a surprising observation that the "necks" in the graph chessboard do not represent in the corresponding graph the paths but the stars. We use for these graphs the term *k-enriched fan graphs kF_n*, where k represents the order of the stars S_k in the "upper part" of the graph and n represents the order of the "middle" path P_n. Now we formally define this new term.

Definition 5.1. *The k-enriched fan graph kF_n, for fixed integers $k, n \geq 2$, is the graph of size $(k+1)n - 1$ obtained by connecting n copies of the star S_k of order k to the fan graph F_n such that one vertex of each copy of the star S_k is identified with one vertex of the main path P_n of F_n.*

We notice that here the stars S_k are connected to the main path P_n of the fan graph F_n exactly as in the previous section the paths P_2 (which are the stars S_2) resp. P_3 (the stars S_3) were connected to the main path P_n of F_n in the case of the double fan graphs DF_n resp. triple fan graphs TF_n.

Example 5.2. *In Figure 10 we see a gracefully labelled 4-enriched fan graph $4F_6$ obtained by connecting 6 copies of stars S_4 of order 4 to the fan graph F_6 as described above. The corresponding simple chessboard and labelling table are also in Figure 10. The labelling sequence of this graph is*

$$(2, 2, 1, 1, 0, 12, 13, 14, 15, 16, 17, 6, 7, 8, 9, 10, 11, 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, 1, 0).$$

We notice that we can divide the vertices of the k-enriched fan graph kF_n with any $k \geq 2$ similarly as in the case of double and triple fan graphs. Hence the graph consists of the "bottom part" (the root vertex and the adjacent edges), the "middle part" (the main path P_n) and the "upper part" (the disjoint union of n stars S_k). The corresponding simple chessboard of the k-enriched fan graph kF_n for $k \geq 3$ has k blocks of dots that form a "swan without head", then "the separated head", and $k-2$ extra "separated necks". Hence we will refer to such simple chessboards representing the k-enriched fan graphs kF_n as k-part swan chessboards.

Theorem 5.3. *Let G be a graph of size $m = (k+1)n - 1$ for some fixed integers $k, n \geq 2$. Then the following are equivalent:*

(1) G is the k-enriched fan graph kF_n.

(2) There is a graceful labelling of G with a k-part swan chessboard.

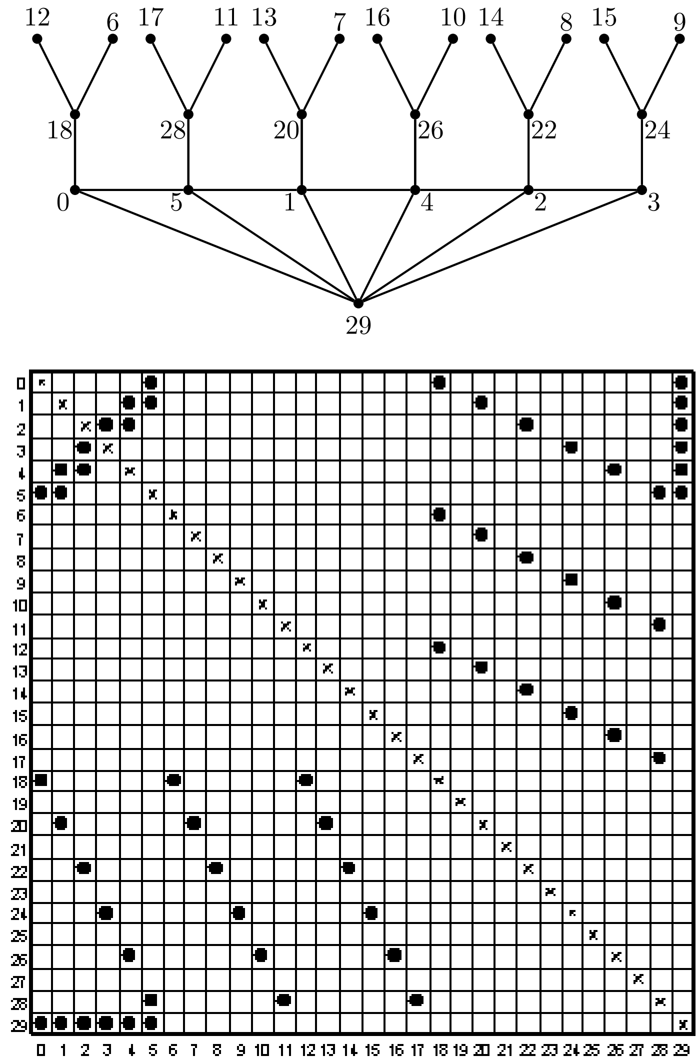
(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{n-i}{2} \right\rfloor, & \text{if } i < n, \\ i - n + (k-2)n, & \text{if } n \leq i < 2n, \\ i - n + (k-4)n, & \text{if } 2n \leq i < 3n, \\ \vdots & \vdots \\ i - n + (k - 2(k-1))n, & \text{if } (k-1)n \leq i < kn, \\ m - i, & \text{if } i \geq kn. \end{cases} \qquad (LSKFG)$$

(4) There exists a labelling sequence L of G with the labelling relation $A(L)$ of the form

$$\begin{aligned} &\{[\lfloor \tfrac{n-i}{2} \rfloor, \lfloor \tfrac{n-i}{2} \rfloor + i] \mid i < n\} \cup \\ &\{[i - n + (k-2)n, 2i - n + (k-2)n] \mid n \leq i < 2n\} \cup \\ &\{[i - n + (k-4)n, 2i - n + (k-4)n] \mid 2n \leq i < 3n\} \cup \\ &\vdots \\ &\{[i - n + (k - 2(k-1))n, 2i - n + (k - 2(k-1))n] \mid (k-1)n \leq i < kn\} \cup \\ &\{[m - i, m] \mid i \geq kn\}. \end{aligned}$$

Proof. (1) $\Rightarrow$ (2) Let G be the k-enriched fan graph kF_n. We will label the graph G in a way similar to those labellings for the double and triple fan graphs. Let the labelling of the root of kF_n be m. Let the vertices of the main path P_n be labelled from one end-vertex of P_n (we see it in Figure 10 from the left side) gradually by numbers $0, n-1, 1, n-2, ...$ Hence the second end-vertex



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	1	1	0	12	13	14	15	16	17	6	7	8	9
3	4	4	5	5	18	20	22	24	26	28	18	20	22	24

16	17	18	19	20	21	22	23	24	25	26	27	28	29
10	11	0	1	2	3	4	5	5	4	3	2	1	0
26	28	18	20	22	24	26	28	29	29	29	29	29	29

Figure 10: Representations of the gracefully labelled 4-enriched fan graph $4F_6$

of P_n is labelled by $\lfloor \frac{n}{2} \rfloor$. We start to label the central vertices of the n stars S_k in the "upper part" of the graph G from the central vertex of S_k attached to the vertex of the main path P_n labelled by $n-1$. We label this vertex by $m-1$, the next central vertex of S_k attached to the vertex $n-2$ will be labelled by $m-3$, the central vertex of S_k attached to the vertex $n-3$ will be labelled by $m-5$ and so on, we end by labelling the central vertex of S_k attached to the vertex 0 by $m-2n+1$ (see Figure 10). To label the remaining vertices of the n stars S_k, we begin with the star, whose central vertex is labelled by $m-1$. The remaining vertices of this star are labelled by $((m-1)-(2n-1)), ((m-1)-(3n-1))$,... It does not matter in what 'direction' we label these vertices. We proceed in the same way with labelling the remaining vertices of the other stars in the "upper part" of G, the role of the previous number $m-1$ is always played by the labelling of the central vertex of the given star (see Figure 10). Now our labelling of G is done and we show that this labelling is graceful with a k-part swan chessboard.

For this we use a visualization via the corresponding simple chessboard of G (see Figure 10). One can easily verify that:

(i) The "bottom part" of G which is the star of size n is in the simple chessboard represented by n dots with coordinates $[m, 0], [m, 1], ..., [m, n-1]$ in the bottom row of the chessboard. (Exactly as for the double and triple fan graphs.)

(ii) The "middle part" of G which is the main path P_n is in the chessboard represented by $n-1$ dots with coordinates $[n-1, 0], [n-1, 1], [n-2, 1], [n-2, 2], ...$ which form a certain "separated head" in the case $k \geq 3$. (Exactly as for the triple fan graphs.)

(iii) The "upper part" of G which is the union of n stars S_k is in the simple chessboard represented by $k-1$ "necks" each consisting of n dots.

Hence the graph chessboard corresponding to our labelling is a k-part swan chessboard (as chessboard in Figure 10) and obviously it has exactly one dot on each diagonal. So the described labelling of G is graceful.

(2) $\Rightarrow$ (3) Assume we have a graceful labelling of G with a k-part swan chessboard. Consider the following k blocks of dots of this chessboard: the "separated head", $k-1$ swan "necks" and the bottom row of the chessboard. One can verify that the "separated head" is represented in the corresponding labelling sequence by the integers j_i having the form $\left\lfloor \frac{n-i}{2} \right\rfloor$ for $i < n$. The first "neck" from the right below the main diagonal is represented in the corresponding labelling sequence by the integers j_i having the form $i-n+(k-2)n$ for $n \leq i < 2n$. The second "neck" from the right below the main diagonal is represented in the corresponding labelling sequence by the integers j_i having the form $i-n+(k-4)n$ for $2n \leq i < 3n$, and so on. Finally, the last "neck" (the first one from the left) is represented in the corresponding labelling sequence by the integers j_i having the form $i-n+(k-2(k-1))n$ for $(k-1)n \leq i < kn$. The bottom row of the chessboard is represented in the corresponding labelling sequence by the integers j_i having the form $m-i$ for $i \geq kn$.

So we have shown that the labelling sequence corresponding to our k-part swan chessboard satisfies the formula (LSKFG).

(3) $\Rightarrow$ (4) Assume there is a labelling sequence L of G which satisfies (LSKFG). One can verify that the corresponding labelling relation $A(L)$ consists of the pairs as described in (4). Indeed, the non-negative integers j_i from the labelling sequence having the form $\lfloor \frac{n-i}{2} \rfloor$ for $i < n$ correspond in $A(L)$ to the pairs $[\lfloor \frac{n-i}{2} \rfloor, \lfloor \frac{n-i}{2} \rfloor + i]$. The next integers j_i from the labelling sequence, which have the form $i-n+(k-2)n$ for $n \leq i < 2n$, correspond in $A(L)$ to pairs $[i-n+(k-2)n, 2i-n+(k-2)n]$. Further, the integers j_i from the labelling sequence, which have the form $i - n + (k - 4)n$ for $2n \leq i < 3n$, correspond in $A(L)$ to pairs $[i - n + (k - 4)n, 2i - n + (k - 4)n]$, and so on for the next parts of the labelling sequence, which correspond to the "necks". Finally, the integers j_i from the labelling sequence, which have the form $m - i$ for $i \geq kn$, correspond to the pairs $[m - i, m]$.

(4) $\Rightarrow$ (1) Let L be a labelling sequence of G with the labelling relation $A(L)$ as in (4). The pairs $[m - i, m]$ in $A(L)$ for $i \geq kn$ correspond to the edges of the "bottom part" of G, which therefore is a star of order n. The pairs $[\lfloor \frac{n-i}{2} \rfloor, \lfloor \frac{n-i}{2} \rfloor + i]$ in $A(L)$ for $i < n$ correspond in the graph G to the edges which form the "middle path" P_n. The remaining pairs in $A(L)$ correspond to the edges in "the upper part" of G. More precisely, each of the pairs $[i - n + (k - 2)n, 2i - n + (k - 2)n]$ for $n \leq i < 2n$ corresponds to one edge in each of the n "upper" stars S_k, each of the pairs $[i - n + (k - 4)n, 2i - n + (k - 4)n]$ for $2n \leq i < 3n$ corresponds to one of the (other) edges in each of the n stars S_k in the "upper part" of G, and so on, and finally, each of the pairs $[i-n+(k-2(k-1))n, 2i-n+(k-2(k-1))n]$ for $(k-1)n \leq i < kn$ corresponds to the remaining edges in each of the n stars S_k in the "upper part" of G. So we get n stars S_k in the "upper part" of G. Thus we have in G the "bottom" star of size n, the "middle" path P_n and the n "upper" stars S_k connected to the vertices of the "middle" path P_n. Hence the graph G is the k-enriched fan graph kF_n. $\square$

6 Conclusion

Studies of other extended fan graphs can be found in the literature. Some of them are seen in Figure 11 below. The first two graphs are taken from [1]. The author named the first one as a double fan graph, but it is different from our double fan graph as defined in this paper. It consists of two fan graphs that have a common main path. The second graph was obtained by adding some edges to a vertex of the main path.

One of the possibilities for other extensions of fan graphs would be adding paths of the same length P_k (not stars S_k as in our case) to the main path P_n of the fan graph F_n. These graphs and our k-enriched fan graphs kF_n would be the same for the cases $k = 2$ and $k = 3$. Such *extended fan graphs*, let us denote them as P_kF_n (our k-enriched fan graphs kF_n in this more universal notation would be denoted S_kF_n) can look like the third graph in Figure 11. This graph would be the extended fan graph P_4F_3 as the paths P_4 are connected to the main path of the fan graph F_3. We conclude our paper with the following open problem:

Problem 6.1. *Are the extended fan graphs P_kF_n (obtained by connecting in the described way n paths P_k to the main path P_n of the fan graph F_n) graceful? And if so, is there a characterization of them via a certain "canonical" graph chessboard and the corresponding labelling sequence and the labelling relation like the characterizations of the graceful graphs presented in this paper?*

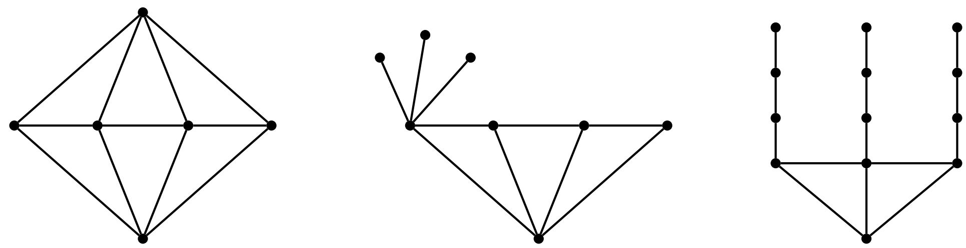


Figure 11: Other extended fan graphs

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On the conformally k-th Gauduchon condition and the conformally semi-Kähler condition on almost complex manifolds

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ABSTRACT

We introduce the k-th Gauduchon condition on almost complex manifolds. We show that if both the conformally k-th Gauduchon condition and the conformally semi-Kähler condition are satisfied, then it becomes conformally quasi-Kähler.

RESUMEN

Introducimos la k-ésima condición de Gauduchon en variedades casi complejas. Mostramos que si la k-ésima condición de Gauduchon conforme y la condición semi-Kähler conforme se satisfacen ambas, entonces la variedad es cuasi-Kähler conforme.

Keywords and Phrases: Almost Hermitian manifold, k-th Gauduchon metric, semi-Kähler metric.

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1 Introduction

S. Ivanov and G. Papadopoulous introduced the conditions on the Hermitian form such that $\omega^l \wedge \partial\bar{\partial}\omega^k = 0$ for $1 \leq k + l \leq n - 1$, which is called the $(l|k)$-SKT condition. They have proven that every compact conformally balanced $(l|k)$-SKT manifold, $k < n - 1$, $n > 2$, is Kähler (cf. [5]). J. Fu, Z. Wang and D. Wu introduced and investigated the generalization of Gauduchon metrics, which is called k-th Gauduchon. The k-th Gauduchon condition is the case $l = n - k - 1$, $1 \leq k \leq n-1$ of the $(l|k)$-SKT condition. By definition, $(n-1)$-th Gauduchon metrics are the usual Gauduchon metrics, astheno-Kähler metrics are examples of $(n - 2)$-th Gauduchon metrics, and pluriclosed metrics are in particular 1-st Gauduchon. They proved that there exists a non-Kähler 3-fold which can support a 1-Gauduchon metric and a balanced metric simultaneously (cf. [2]). Since K. Liu and X. Yang have shown that if a compact complex manifold is k-th Gauduchon for $1 \leq k \leq n - 2$ and also balanced, then it must be Kähler, a 1-Gauduchon metric and a balanced metric on a non-Kähler 3-fold which Fu, Wang and Wu discovered must be different Hermitian metrics. Liu and Yang also have shown that the conformally Kählerianity is equivalent to that both the conformally k-th Gauduchon for $1 \leq k \leq n - 2$, and the conformally balancedness are satisfied (cf. [7]). Our aim in this paper is to generalize the Liu-Yang's equivalence [7, Corollary 1.17] to almost Hermitian geometry.

Let (M^{2n}, J) be an almost complex manifold with $n \geq 3$ and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1, 0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1, 1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}g_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric. We introduce the definition of a Gauduchon metric and we define a k-th Gauduchon metric as follows.

Definition 1.1. *Let (M^{2n}, J) be an almost complex manifold. A metric g is called a Gauduchon metric on M if g is an almost Hermitian metric whose associated real $(1, 1)$-form $\omega = \sqrt{-1}g_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ satisfies $d^*(Jd^*\omega) = 0$, where d^* is the adjoint of d with respect to g, which is equivalent to $d(Jd(\omega^{n-1})) = 0$, or $\partial\bar{\partial}(\omega^{n-1}) = 0$. When an almost Hermitian metric g is Gauduchon, the triple (M^{2n}, J, g) will be called a Gauduchon manifold. For $1 \leq k \leq n - 1$, an almost Hermitian metric ω is called k-th Gauduchon if it satisfies that $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$.*

Notice that the condition $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$ for $1 \leq k \leq n - 2$ is not equivalent to $d(Jd(\omega^k)) \wedge \omega^{n-k-1} = 0$ for $1 \leq k \leq n-2$ since there exist A and $\bar{A}$ parts of the exterior differential operator d in the almost complex setting (Note that these conditions are equivalent in the case of $k = n-1$ as we confirmed in Definition 1.1 since then we have $A(\omega^{n-1}) = \bar{A}(\omega^{n-1}) = 0$.). Hence the condition $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$ for $1 \leq k \leq n - 1$ can be regarded as a natural extension of the Gauduchon condition on almost complex manifolds.

We next introduce the definition of a semi-Kähler metric.

Definition 1.2. *Let (M^{2n}, J) be an almost complex manifold. A metric g is called a semi-Kähler metric on M if g is an almost Hermitian metric whose associated real $(1,1)$-form $\omega = \sqrt{-1}g_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ satisfies $d(\omega^{n-1}) = 0$. When an almost Hermitian metric g is semi-Kähler, the triple (M^{2n}, J, ω) will be called a semi-Kähler manifold.*

Recall that on an almost Hermitian manifold (M, J, g), a quasi-Kähler structure is an almost Hermitian structure whose real $(1,1)$-form ω satisfies $(d\omega)^{(1,2)} = \bar{\partial}\omega = 0$, which is equivalent to the original definition of quasi-Kählerianity: $D_X J(Y) + D_{JX} J(JY) = 0$ for all vector fields X, Y (cf. [4]), where D is the Levi-Civita connection associated to g. It is important for us to study quasi-Kähler manifolds since they include the classes of almost Kähler manifolds and nearly Kähler manifolds. An almost Kähler or quasi-Kähler manifold with J integrable is a Kähler manifold. We define some conformally conditions.

Definition 1.3. *Let (M, J, ω) be an almost Hermitian manifold. We say ω is conformally k-th Gauduchon (resp. semi-Kähler, quasi-Kähler) if there exist a k-th Gauduchon (resp. semi-Kähler, quasi-Kähler) metric $\tilde{\omega}$ and a smooth function $F \in C^\infty(M, \mathbb{R})$ such that $\omega = e^F \tilde{\omega}$.*

Our main result is as follows.

Theorem 1.4. *On a compact almost Hermitian manifold (M, J, ω), the following are equivalent:*

(1) (M, J, ω) *is conformally quasi-Kähler.*

(2) (M, J, ω) *is conformally k-th Gauduchon for $1 \leq k \leq n-2$, and conformally semi-Kähler.*

In particular, the following are also equivalent:

(a) (M, J, ω) *is quasi-Kähler.*

(b) (M, J, ω) *is k-th Gauduchon for $1 \leq k \leq n-2$, and conformally semi-Kähler.*

This paper is organized as follows: in the second section, we recall some basic definitions and computations. In the last section, we will give a proof of the main result. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in all this paper.

2 Preliminaries

2.1 The Nijenhuis tensor of the almost complex structure

Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{2.1}$$

which is the Nijenhuis tensor of J. The Nijenhuis tensor N satisfies $N(X,Y) = -N(Y,X)$, $N(JX,Y) = -JN(X,Y)$, $N(X,JY) = -JN(X,Y)$, $N(JX,JY) = -N(X,Y)$. For any $(1,0)$-vector fields W and V, $N(V,W) = -[V,W]^{(0,1)}$, $N(V,\bar{W}) = N(\bar{V},W) = 0$ and $N(\bar{V},\bar{W}) = -[\bar{V},\bar{W}]^{(1,0)}$ since we have $4N(V,W) = -2([V,W] + \sqrt{-1}J[V,W])$, $4N(\bar{V},\bar{W}) = -2([\bar{V},\bar{W}] - \sqrt{-1}J[\bar{V},\bar{W}])$. An almost complex structure J is called integrable if $N = 0$ everywhere on M. Giving a complex structure on a differentiable manifold M is equivalent to giving an integrable almost complex structure on M. Let (M, J) be an almost complex manifold. A Riemannian metric g on M is called J-invariant if J is compatible with g, *i.e.*, for any $X, Y \in \Gamma(TM)$, $g(X,Y) = g(JX, JY)$. In this case, the pair (J, g) is called an almost Hermitian structure. The fundamental 2-form ω associated to a J-invariant Riemannian metric g, *i.e.*, an almost Hermitian metric, is determined by, for $X, Y \in \Gamma(TM)$, $\omega(X,Y) = g(JX,Y)$. Indeed we have, for any $X, Y \in \Gamma(TM)$,

$$\omega(Y,X) = g(JY,X) = g(J^2Y, JX) = -g(JX,Y) = -\omega(X,Y) \tag{2.2}$$

and $\omega \in \Gamma(\bigwedge^2 T^*M)$. We will also refer to the associated real fundamental $(1,1)$-form ω as an almost Hermitian metric. The form ω is related to the volume form dV_g by $n!dV_g = \omega^n$. Let a local $(1,0)$-frame $\{Z_r\}$ on (M, J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, i.e., $\zeta^i(Z_j) = \delta^i_j$ for $i, j = 1, \dots, n$. Since g is almost Hermitian, its components satsfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$.

We write $T^{\mathbb{R}}M$ for the real tangent space of M. Then its complexified tangent space is given by $T^{\mathbb{C}}M = T^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$. By extending J $\mathbb{C}$-linearly and g, ω, $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent space $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \big| X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \big| X \in TM\}. \tag{2.3}$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q}M$ for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms, where $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$,

$$\Lambda^{1,0}M = \{\alpha + \sqrt{-1}J\alpha \big| \alpha \in \Lambda^1 M\}, \quad \Lambda^{0,1}M = \{\alpha - \sqrt{-1}J\alpha \big| \alpha \in \Lambda^1 M\} \tag{2.4}$$

and $\Lambda^1 M$ denotes the dual of TM. For any $\alpha \in \Lambda^1 M$, we define $J\alpha(X) = -\alpha(JX)$ for $X \in TM$.

Let (M^{2n}, J, g) be an almost Hermitian manifold. An affine connection D on TM is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (*cf.* [3, 9, 11]).

Lemma 2.1. *Let (M,J,g) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. Then for any given vector valued $(1,1)$-form $\Theta = (\Theta^i)_{1\le i\le n}$, there exists a unique almost Hermitian connection D on (M,J,g) such that the $(1,1)$-part of the torsion is equal to the given Θ.*

If the $(1,1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connction is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇.

Note that for any p-form ψ, there holds that

$$\begin{aligned} d\psi(X_1,\ldots,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}X_i(\psi(X_1,\ldots,\widehat{X_i},\ldots,X_{p+1})) \\ &\quad+\sum_{i<j}(-1)^{i+j}\psi([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{p+1}) \end{aligned} \tag{2.5}$$

for any vector fields $X_1,\ldots,X_{p+1}$ on M (*cf.* [11]). We directly compute that

$$d\zeta^s = -\frac{1}{2}B^s_{kl}\zeta^k\wedge\zeta^l - B^s_{k\bar l}\zeta^k\wedge\zeta^{\bar l} + \frac{1}{2}N^s_{\bar k\bar l}\zeta^{\bar k}\wedge\zeta^{\bar l}. \tag{2.6}$$

According to the direct computation above, we may split the exterior differential operator $d : \Lambda^pM\otimes_{\mathbb{R}}\mathbb{C}\to\Lambda^{p+1}M\otimes_{\mathbb{R}}\mathbb{C}$, into four components

$$d = A + \partial + \bar\partial + \bar A \tag{2.7}$$

with

$$\partial : \Lambda^{p,q}M \to \Lambda^{p+1,q}M, \quad \bar\partial : \Lambda^{p,q}M\to\Lambda^{p,q+1}M, \tag{2.8}$$

$$A : \Lambda^{p,q}M\to\Lambda^{p+2,q-1}M, \quad \bar A:\Lambda^{p,q}M\to\Lambda^{p-1,q+2}M, \tag{2.9}$$

since we have

$$d(\Gamma(\Lambda^{r,s}M)) \subseteq \Gamma(\Lambda^{r+2,s-1}M\oplus\Lambda^{r+1,s}M\oplus\Lambda^{r,s+1}M\oplus\Lambda^{r-1,s+2}M). \tag{2.10}$$

In terms of these components, the condition $d^2=0$ can be written as

$$\begin{gathered} A^2=0,\quad \partial A + A\partial = 0,\quad \bar\partial\bar A+\bar A\bar\partial = 0,\quad \bar A^2=0,\\ A\bar\partial+\partial^2+\bar\partial A = 0,\quad A\bar A+\partial\bar\partial+\bar\partial\partial+\bar AA=0,\quad \partial\bar A+\bar\partial^2+\bar A\partial=0. \end{gathered} \tag{2.11}$$

Notice that J is integrable if and only if $A=0$, equivalently, if and only if $\bar\partial^2=0$.

For any real $(1,1)$-form $\sigma=\sqrt{-1}\sigma_{i\bar j}\zeta^i\wedge\zeta^{\bar j}$, we have

$$\bar\partial\sigma = \frac{\sqrt{-1}}{2}\Big(Z_{\bar j}(\sigma_{k\bar i}) - Z_{\bar i}(\sigma_{k\bar j}) - B^s_{k\bar i}\sigma_{s\bar j} + B^s_{k\bar j}\sigma_{s\bar i} + B^{\bar s}_{\bar i\bar j}\sigma_{k\bar s}\Big)\zeta^k\wedge\zeta^{\bar i}\wedge\zeta^{\bar j},$$

$$\partial\sigma = \frac{\sqrt{-1}}{2}\Big(Z_i(\sigma_{j\bar k}) - Z_j(\sigma_{i\bar k}) - B^s_{ij}\sigma_{s\bar k} - B^{\bar s}_{i\bar k}\sigma_{j\bar s} + B^{\bar s}_{j\bar k}\sigma_{i\bar s}\Big)\zeta^i\wedge\zeta^j\wedge\zeta^{\bar k}.$$

From these computations above, we have

$$\bar{\partial}\omega = \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(g_{k\bar{i}}) - Z_{\bar{i}}(g_{k\bar{j}}) - B^{s}_{k\bar{i}}g_{s\bar{j}} + B^{s}_{k\bar{j}}g_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}g_{k\bar{s}}\Big)\zeta^{k}\wedge\zeta^{\bar{i}}\wedge\zeta^{\bar{j}} = \frac{\sqrt{-1}}{2}T_{\bar{j}\bar{i}k}\zeta^{k}\wedge\zeta^{\bar{i}}\wedge\zeta^{\bar{j}} \tag{2.12}$$

and

$$\partial\omega = \frac{\sqrt{-1}}{2}\Big(Z_{i}(g_{j\bar{k}}) - Z_{j}(g_{i\bar{k}}) - B^{s}_{ij}g_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}g_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}g_{i\bar{s}}\Big)\zeta^{i}\wedge\zeta^{j}\wedge\zeta^{\bar{k}} = \frac{\sqrt{-1}}{2}T_{ij\bar{k}}\zeta^{i}\wedge\zeta^{j}\wedge\zeta^{\bar{k}}, \tag{2.13}$$

where T is the torsion of the Chern connection. For any $\varphi \in C^{\infty}(M,\mathbb{R})$, a direct computation yields

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{1}{2}(dJd\varphi)^{(1,1)} = \sqrt{-1}(Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\varphi)\zeta^{i}\wedge\zeta^{\bar{j}}, \tag{2.14}$$

so we write locally

$$\partial_i\partial_{\bar{j}}\varphi = (Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi. \tag{2.15}$$

For basic definitions and computations about the torsion and the curvature on almost Hermitian manifolds, see [6, Section 2].

3 Proof of Theorem 1.4

We need the following lemmas in order to prove Theorem 1.4. Here we introduce the following characterizations of quasi-Kählerianity and semi-Kählerianity.

Lemma 3.1 (*cf.* [8, Lemma 2.4]). *An almost Hermitian manifold (M^{2n}, g, J) is quasi-Kähler if and only if $T^{k}_{\bar{i}\bar{j}} = 0$ for all i, j and k when a local unitary $(1,0)$-frame is fixed, where T is the torsion of the Chern connection ∇.*

Here, we define $w_r := T^{i}_{ri}$ and the torsion $(1,0)$-form $\eta := T^{i}_{ir}\zeta^{r} = -w_r\zeta^{r}$ (*cf.* [10]), where $T = (T^{i})$ is the torsion of the Chern connection ∇.

Lemma 3.2 (*cf.* [6, Lemma 4.3]). *An almost Hermitian manifold (M^{2n}, J, ω) is semi-Kähler if and only if $\eta = 0$.*

Proof. We have $\partial\omega = \frac{\sqrt{-1}}{2}T_{ij\bar{k}}\zeta^{i}\wedge\zeta^{j}\wedge\zeta^{\bar{k}}$ as we see (2.12), (2.13). Then a direct calculation shows that

$$\partial\omega^{n-1} = (n-1)\partial\omega\wedge\omega^{n-2} = -\eta\wedge\omega^{n-1}, \tag{3.1}$$

where we used that $\eta = -w_i\zeta^{i} = -(n-1)\dfrac{\partial\omega\wedge\omega^{n-2}}{\omega^{n-1}}$. Similarly, we obtain that

$$\bar{\partial}\omega^{n-1} = (n-1)\bar{\partial}\omega\wedge\omega^{n-2} = -\bar{\eta}\wedge\omega^{n-1}, \tag{3.2}$$

since we have $\bar{\partial}\omega = \frac{\sqrt{-1}}{2}T_{\bar{j}\bar{i}k}\zeta^{k}\wedge\zeta^{\bar{i}}\wedge\zeta^{\bar{j}}$ and $\bar{\eta} = -w_{\bar{i}}\zeta^{\bar{i}} = -(n-1)\dfrac{\bar{\partial}\omega\wedge\omega^{n-2}}{\omega^{n-1}}$.

Recall that the metric g is said to be semi-Kähler if ω^{n-1} is closed. These identities (3.1), (3.2) show that g is semi-Kähler if and only if $\eta = 0$. $\square$

Proof of Theorem 1.4. Assume that ω is conformally k-th Gauduchon and conformally semi-Kähler. Then since ω is conformally semi-Kähler, there exist an semi-Kähler metric ω_B and a smooth function $F \in C^\infty(M, \mathbb{R})$ such that $\omega = e^F \omega_B$. By the conformally k-th Gauduchon condition, there exist a k-th Gauduchon ω_G and a smooth function $\tilde{F} \in C^\infty(M, \mathbb{R})$ such that $\omega = e^{\tilde{F}} \omega_G$. Set $f := F - \tilde{F}$, then we have $\omega_G = e^f \omega_B$. Since ω_G is k-th Gauduchon, we get

$$(e^f \omega_B)^{n-k-1} \wedge \partial\bar{\partial}(e^f \omega_B)^k = 0,$$

and then

$$\omega_B^{n-k-1} \wedge \partial\bar{\partial}(e^f \omega_B)^k = 0. \tag{3.3}$$

Since ω_B is semi-Kähler, $0 = d(\omega_B^{n-1}) = (\partial + A + \bar{A} + \bar{\partial})(\omega_B^{n-1}) = (\partial + \bar{\partial})(\omega_B^{n-1})$, which tells us that $\partial(\omega_B^{n-1}) = \bar{\partial}(\omega_B^{n-1}) = 0$, where we have used that $A(\omega_B^{n-1}) = \bar{A}(\omega_B^{n-1}) = 0$. Hence we see that

$$\omega_B^{n-k-1} \wedge \partial(\omega_B^k) = k\omega_B^{n-2} \wedge \partial\omega_B = \frac{k}{n-1}\partial(\omega_B^{n-1}) = 0. \tag{3.4}$$

Then from (3.3), we have

$$e^{kf}\omega_B^{n-k-1} \wedge \partial\bar{\partial}\omega_B^k + \omega_B^{n-1} \wedge \partial\bar{\partial}(e^{kf}) = 0. \tag{3.5}$$

Therefore, we obtain

$$\begin{aligned}
\int_M e^{kf}\omega_B^{n-k-1} \wedge \partial\bar{\partial}(\omega_B^k) &= -\int_M \omega_B^{n-1} \wedge \partial\bar{\partial}(e^{kf}) \\
&= -\frac{1}{n}\int_M n \cdot \frac{\partial\bar{\partial}(e^{kf}) \wedge \omega_B^{n-1}}{\omega_B^n}\omega_B^n \\
&= -\frac{1}{n}\int_M \Delta_B(e^{kf})\omega_B^n = 0.
\end{aligned} \tag{3.6}$$

Applying (3.4) and (3.6), we obtain

$$\begin{aligned}
0 &= \int_M e^{kf}\omega_B^{n-k-1} \wedge \partial\bar{\partial}(\omega_B^k) \\
&= \int_M \partial(e^{kf}\omega_B^{n-k-1} \wedge \bar{\partial}(\omega_B^k)) - \partial(e^{kf}) \wedge \omega_B^{n-k-1} \wedge \bar{\partial}(\omega_B^k) - e^{kf}\partial(\omega_B^{n-k-1}) \wedge \bar{\partial}(\omega_B^k) \\
&= \int_M d(e^{kf}\omega_B^{n-k-1} \wedge \bar{\partial}(\omega_B^k)) - e^{kf}\partial(\omega_B^{n-k-1}) \wedge \bar{\partial}(\omega_B^k) \\
&= -k(n-k-1)\int_M e^{kf}\omega_B^{n-3} \wedge \partial\omega_B \wedge \bar{\partial}\omega_B \\
&= -k(n-k-1)\int_M e^{kf}\frac{T'_B \wedge \bar{T}'_B \wedge \omega_B^{n-3}}{\omega_B^n}\omega_B^n \\
&= -\frac{k(n-k-1)}{6n(n-1)(n-2)}\int_M e^{kf}\mathrm{Tr}(T'_B \wedge \bar{T}'_B)\omega_B^n \\
&= -\frac{k(n-k-1)}{6n(n-1)(n-2)}\int_M e^{kf}(6|w_B|^2 - 3|T'_B|^2)\omega_B^n,
\end{aligned} \tag{3.7}$$

which gives that $2\int_M e^{kf}|w_B|^2\omega_B^n = \int_M e^{kf}|T'_B|^2\omega_B^n$, where we used $\omega_B^{n-k-1}\wedge\bar{\partial}(\omega_B^k) = 0$ from (3.4), $(\bar{\partial}+A+\bar{A})(e^{kf}\omega_B^{n-k-1}\wedge\bar{\partial}(\omega_B^k)) = 0$, and that $\partial\omega_B = T'_B$ since we have $(\partial\omega_B)_{jl\bar{k}} = \partial_j(g_B)_{l\bar{k}} - \partial_l(g_B)_{j\bar{k}} = (T_B)_{jl\bar{k}}$ from (2.13). Note that as in [1, Chapter 2],

$$\begin{aligned}(T'_B\wedge\bar{T}'_B)_{ikm\bar{j}\bar{l}\bar{n}} &= (T_B)_{im\bar{j}}(T_B)_{\bar{l}\bar{n}k} + (T_B)_{im\bar{l}}(T_B)_{\bar{n}\bar{j}k} + (T_B)_{im\bar{n}}(T_B)_{\bar{j}\bar{l}k}\\ &\quad +(T_B)_{mk\bar{j}}(T_B)_{\bar{l}\bar{n}i} + (T_B)_{mk\bar{l}}(T_B)_{\bar{n}\bar{j}i} + (T_B)_{mk\bar{n}}(T_B)_{\bar{j}\bar{l}i}\\ &\quad +(T_B)_{ki\bar{j}}(T_B)_{\bar{l}\bar{n}m} + (T_B)_{ki\bar{l}}(T_B)_{\bar{n}\bar{j}m} + (T_B)_{ki\bar{n}}(T_B)_{\bar{j}\bar{l}m}\end{aligned}$$

and

$$\mathrm{Tr}(T'_B\wedge\bar{T}'_B) = g^{i\bar{j}}g^{k\bar{l}}g^{m\bar{n}}(T'_B\wedge\bar{T}'_B)_{ikm\bar{j}\bar{l}\bar{n}} = 6|w_B|^2 - 3|T'_B|^2,$$

where $(w_B)_r = (T_B)^i_{ri}$ and $\eta_B = (T_B)^i_{ir}\zeta^r = -(w_B)_r\zeta^r$ is the torsion $(1,0)$-form of ω_B. Since the metric ω_B is semi-Kähler, which is equivalent to that $\eta_B = 0$ from Lemma 3.2. Since $\eta_B = 0$ implies that $(w_B)_r = 0$ for all $r = 1,\ldots,n$, we get

$$\int_M e^{kf}|T'_B|^2\omega_B^n = 2\int_M e^{kf}|w_B|^2\omega_B^n = 0.$$

Hence we have $T'_B = 0$, which is equivalent to the quasi-Kählerianity from Lemma 3.1. Notice that since ω_B is now quasi-Kähler, we have that from (3.5),

$$\Delta_B(e^{kf}) = n\cdot\frac{\partial\bar{\partial}(e^{kf})\wedge\omega_B^{n-1}}{\omega_B^n} = 0,$$

which implies that f is constant. The converse is obvious. The equivalence of (a) and (b) in the statement of Theorem 1.4 follows by the same argument under the condition $\omega = \omega_G$ and $f = F$. □

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