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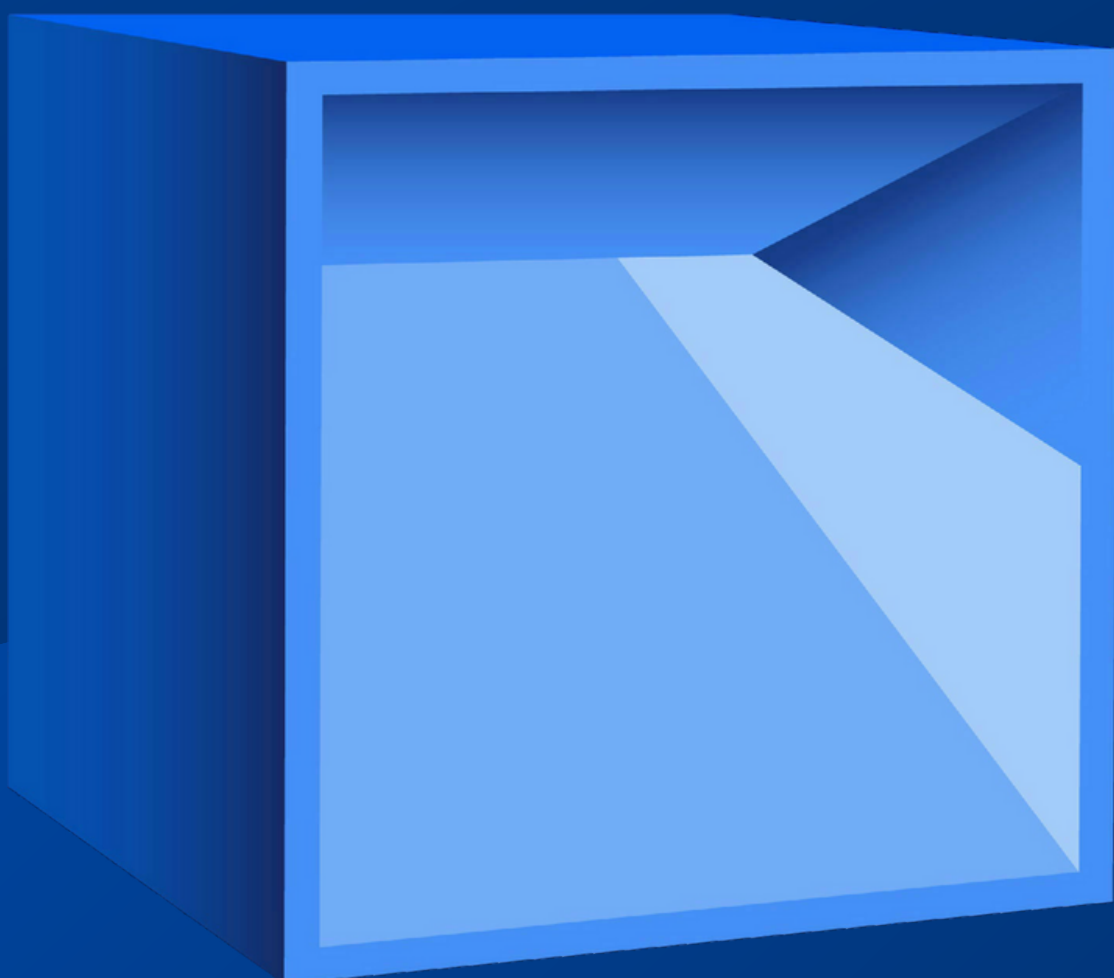

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SUMMARY

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Quasi bi-slant submersions in contact geometry

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ABSTRACT

The aim of the paper is to introduce the concept of quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds as a generalization of semi-slant and hemi-slant submersions. We mainly focus on quasi bi-slant submersions from cosymplectic manifolds. We give some non-trivial examples and study the geometry of leaves of distributions which are involved in the definition of the submersion. Moreover, we find some conditions for such submersions to be integrable and totally geodesic.

RESUMEN

El objetivo de este artículo es introducir el concepto de submersiones cuasi bi-inclinadas desde variedades casi contacto métricas hacia variedades Riemannianas, como una generalización de submersiones semi-inclinadas y hemi-inclinadas. Principalmente nos enfocamos en submersiones cuasi bi-inclinadas desde variedades cosimplécticas. Damos algunos ejemplos no triviales y estudiamos la geometría de hojas de distribuciones que están involucradas en la definición de la submersión. Más aún, encontramos algunas condiciones para que estas submersiones sean integrables y totalmente geodésicas.

Keywords and Phrases: Riemannian submersion, semi-invariant submersion, bi-slant submersion, quasi bi-slant submersion, horizontal distribution.

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1 Introductions

In differential geometry, there are so many important applications of immersions and submersions *both in mathematics and in physics.* The properties of slant submersions became an interesting subject in differential geometry, both in complex geometry and in contact geometry.

In 1966 and 1967, the theory of Riemannian submersions was initiated by O'Neill [17] and Gray [11] respectively. Nowadays, *Riemannian submersions are of great interest not only in mathematics, but also in theoretical pyhsics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories* (see [7, 8, 10, 13, 14]). In 1976, the almost complex type of Riemannian submersions was studied by Watson [29]. He also introduced almost Hermitian submersions between almost Hermitian manifolds requiring that such Riemannian submersions are almost complex maps. In 1985, D. Chinea [9] extended the notion of almost Hermitian submersion to several kinds of sub-classes of almost contact manifolds. In [4] and [5], there are so many important and interesting results about Riemannian and almost Hermitian submersions. In 2010, B. Şahin introduced anti invariant submersions from almost Hermitian manifolds onto Riemannian manifolds [25]. Inspired by B. Şahin's article, many geometers introduced several new types of Riemannian submersions in different ambient spaces such as semi-invariant submersion [21, 23], generic submersion [27], slant submersion [12, 22], hemi-slant submersion [28], semi-slant submersion [18], bi-slant submersion [26], quasi hemi-slant submersion [16], quasi bi-slant submersion [19, 20], conformal anti-invariant submersion [1], conformal slant submersion [2] and conformal semi-slant submersion [3, 15]. Also, these kinds of submersions were considered in different kinds of structures such as cosymplectic, Sasakian, Kenmotsu, nearly Kaehler, almost product, para-contact, etc. Recent developments in the theory of submersions can be found in the book [24]. Inspired from the good and interesting results of above studies, we introduce the notion of quasi bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds.

The paper is organized as follows: In the second section, we gather some basic definitions related to quasi bi-slant Riemannian submersion. In the third section, we obtain some results on quasi bi-slant Riemannian submersions from a cosymplectic manifold onto a Riemannian manifold. We also study the geometry of the leaves of the distributions involved in the considered submersions and discuss their totally geodesicity. We obtain conditions for the fibres or the horizontal distribution to be totally geodesic. In the last section, we provide some examples for such submersions.

2 Preliminaries

An $n-$dimensional smooth manifold M is said to have an almost contact structure, if there exist on M, a tensor field ϕ of type $(1,1)$, a vector field ξ and $1-$form η such that:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \; \eta \circ \phi = 0, \tag{2.1}$$

$$\eta(\xi) = 1. \tag{2.2}$$

There exists a Riemannian metric g on an almost contact manifold M satisfying the next conditions:

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.3}$$

$$g(U, \xi) = \eta(U), \tag{2.4}$$

where U, V are vector fields on M.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $M \times \mathbb{R}$ is given by

$$J\left(U, \; \alpha\frac{d}{dt}\right) = \left(\phi U - \alpha\xi, \; \eta(U)\frac{d}{dt}\right) \tag{2.5}$$

and α is the differentiable function on $M \times \mathbb{R}$ has no torsion, *i.e.*, J is integrable. The condition for normality in terms of ϕ, ξ, and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on M, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally, the fundamental $2-$form Φ is defined by $\Phi(U, V) = g(U, \phi V)$.

An almost contact metric manifold with almost contact structure (ϕ, ξ, η, g) is said to be cosymplectic if

$$(\nabla_U \phi)V = 0, \tag{2.6}$$

for any U, V on M.

It is both normal and closed and the structure equation of a cosymplectic manifold is given by

$$\nabla_U \xi = 0, \tag{2.7}$$

for any U on M, where ∇ denotes the Riemannian connection of the metric g on M.

Example 2.1 ([6]). $\mathbb{R}^{2n+1}$ *with Cartesian coordinates* $(x_i, y_i, z)(i = 1, \ldots, n)$ *and its usual contact form*

$$\eta = dz.$$

The characteristic vector field ξ *is given by* $\frac{\partial}{\partial z}$ *and its Riemannian metric* g *and tensor field* ϕ *are given by*

$$g = \sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2) + (dz)^2, \qquad \phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, n.$$

This gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. The vector fields $e_i = \frac{\partial}{\partial y_i}$, $e_{n+i} = \frac{\partial}{\partial x_i}$, ξ form a ϕ-basis for the cosymplectic structure.

Before giving our definition, we recall the following definition:

Definition 2.2 ([28])**.** *Let M be an almost Hermitian manifold with Hermitian metric g_M and almost complex structure J, and let N be a Riemannian manifold with Riemannian metric g_N. A Riemannian submersion $f : (M, g_M, J) \to (N, g_N)$ is called a hemi-slant submersion if the vertical distribution $\ker f_*$ of f admits two orthogonal complementary distributions D^θ and $D^\perp$ such that D^θ is slant with angle θ and $D^\perp$ is anti-invariant, i.e, we have*

$$\ker f_* = D^\theta \oplus D^\perp.$$

In this case, the angle θ is called the hemi-slant angle of the submersion.

Definition 2.3. *Let $(M, \phi, \xi, \eta, g_M)$ be an almost contact metric manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion*

$$f : (M, \phi, \xi, \eta, g_M) \to (N, g_N),$$

is called a quasi bi-slant submersion if there exist four mutually orthogonal distributions D, D_1, D_2 and $< \xi >$ such that

(i) $\ker f_* = D \oplus D_1 \oplus D_2 \oplus < \xi >$,

(ii) $\phi(D) = D$ *i.e., D is invariant,*

(iii) $\phi(D_1) \perp D_2$ *and* $\phi(D_2) \perp D_1$,

(iv) for any non-zero vector field $U \in (D_1)_p$, $p \in M$, the angle θ_1 between ϕU and $(D_1)_p$ is constant and independent of the choice of the point p and U in $(D_1)_p$,

(v) for any non-zero vector field $U \in (D_2)_q$, $q \in M$, the angle θ_2 between ϕU and $(D_2)_q$ is constant and independent of the choice of point q and U in $(D_2)_q$,

These angles θ_1 and θ_2 are called the slant angles of the submersion.

We easily observe that

(a) If $\dim D \neq 0$, $\dim D_1 = 0$ and $\dim D_2 = 0$, then f is an invariant submersion.

(b) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is proper semi-slant submersion.

(c) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is slant submersion with slant angle θ_1.

(d) If $\dim D = 0$, $\dim D_1 = 0$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then f is slant submersion with slant angle θ_2.

(e) If $\dim D = 0$, $\dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is an anti-invariant submersion.

(f) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is an semi-invariant submersion.

(g) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then f is a hemi-slant submersion.

(h) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then f is a bi-slant submersion.

(i) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then we may call f is an quasi-hemi-slant submersion.

(j) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then f is proper quasi bi-slant submersion.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.8}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.9}$$

for any vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions.

From equations (2.8) and (2.9) we have

$$\nabla_U V = \mathcal{T}_U V + \mathcal{V}\nabla_U V, \tag{2.10}$$

$$\nabla_U X = \mathcal{T}_U X + \mathcal{H}\nabla_U X, \tag{2.11}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U, \tag{2.12}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.13}$$

for $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$, where $\mathcal{H}\nabla_U Y = \mathcal{A}_Y U$, if Y is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

It is seen that for $q \in M$, $U \in \mathcal{V}_q$ and $X \in \mathcal{H}_q$ the linear operators

$$\mathcal{A}_X,\ \mathcal{T}_U : T_qM \to T_qM$$

are skew-symmetric, that is

$$g_M(\mathcal{A}_X E, F) = -g_M(E, \mathcal{A}_X F) \text{ and } g_M(\mathcal{T}_U E, F) = -g_M(E, \mathcal{T}_U F) \tag{2.14}$$

for each $E, F \in T_qM$. Since $\mathcal{T}_U$ is skew-symmetric, we observe that f has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

Let $(M, \phi, \xi, \eta, g_M)$ be a cosymplectic manifold, (N, g_N) be a Riemannian manifold and $f : M \to N$ a smooth map. Then the second fundamental form of f is given by

$$(\nabla f_*)(Y, Z) = \nabla^f_Y f_* Z - f_*(\nabla_Y Z), \quad \text{for } Y, Z \in \Gamma(T_pM), \tag{2.15}$$

where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N and ∇^f is the pullback connection.

We recall that a differentiable map f between two Riemannian manifolds is totally geodesic if

$$(\nabla f_*)(Y, Z) = 0, \text{ for all } Y, Z \in \Gamma(TM).$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

3 Quasi bi-slant submersions

Let f be quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have

$$TM = \ker f_* \oplus (\ker f_*)^{\perp}. \tag{3.1}$$

Now, for any vector field $U \in \Gamma(\ker f_*)$, we put

$$U = PU + QU + RU + \eta(U)\xi, \tag{3.2}$$

where P, Q and R are projection morphisms of $\ker f_*$ onto D, D_1 and D_2, respectively. For any $U \in \Gamma(\ker f_*)$, we set

$$\phi U = \psi U + \omega U, \tag{3.3}$$

where $\psi U \in \Gamma(\ker f_*)$ and $\omega U \in \Gamma(\ker f_*)^{\perp}$.

Now, let U_1, U_2 and U_3 be vector fields in D, D_1 and D_2 respectively. Since D is invariant, *i.e.* $\phi D = D$, we get $\omega U_1 = 0$. For any $U_2 \in \Gamma(D_1)$ we get $\omega U_2 \in \Gamma(\omega D_1)$ and for any $U_3 \in \Gamma(D_2)$ we get $\omega U_3 \in \Gamma(\omega D_2)$, hence $\omega U_2 \oplus \omega U_3 \in \Gamma(\omega D_1 \oplus \omega D_2) \subseteq \Gamma(\ker f_*)^{\perp}$.

From equations (3.2) and (3.3), we have

$$\begin{aligned} \phi U &= \phi(PU) + \phi(QU) + \phi(RU), \\ &= \psi(PU) + \omega(PU) + \psi(QU) + \omega(QU) + \psi(RU) + \omega(RU). \end{aligned}$$

Since $\phi D = D$, we get $\omega PU = 0$.

Hence above equation reduces to

$$\phi U = \psi PU + \psi QU + \omega QU + \psi RU + \omega RU. \tag{3.4}$$

Thus we have the following decomposition according to equation (3.4)

$$\phi(\ker f_*) = (\psi D) \oplus (\psi D_1 \oplus \psi D_2) \oplus (\omega D_1 \oplus \omega D_2), \tag{3.5}$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $U \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$. Then

$$g_M(U, V) = 0.$$

From Definition 2.3 (iii), we have

$$g_M(\phi U, V) = g_M(U, \phi V) = 0.$$

Now, consider

$$g_M(\psi U, V) = g_M(\phi U - \omega U, V) = g_M(\phi U, V) = 0.$$

Similarly, we have

$$g_M(U, \psi V) = 0.$$

Let $W \in \Gamma(D)$ and $U \in \Gamma(D_1)$. Then we have

$$g_M(\psi U, W) = g_M(\phi U - \omega U, W) = g_M(\phi U, W) = -g(U, \phi W) = 0,$$

as D is invariant, *i.e.*, $\phi W \in \Gamma(D)$.

Similarly, for $W \in \Gamma(D)$ and $V \in \Gamma(D_2)$, we obtain

$$g_M(\psi V, W) = 0,$$

From above equations, we have

$$g_M(\psi U, \psi V) = 0,$$

and

$$g_M(\omega U, \omega V) = 0,$$

for all $U \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$.

So, we can write

$$\psi D_1 \cap \psi D_2 = \{0\}, \quad \omega D_1 \cap \omega D_2 = \{0\}.$$

If $\theta_2 = \frac{\pi}{2}$, then $\psi R = 0$ and D_2 is anti-invariant, *i.e.*, $\phi(D_2) \subseteq (\ker f_*)^{\perp}$.

We also have

$$\phi(\ker f_*) = \psi D \oplus \psi D_1 \oplus \omega D_1 \oplus \omega D_2. \tag{3.6}$$

Since $\omega D_1 \subseteq (\ker f_*)^\perp$, $\omega D_2 \subseteq (\ker f_*)^\perp$. So we can write

$$(\ker f_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V},$$

where $\mathcal{V}$ is invariant and orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker f_*)^\perp$.

Also for any non-zero vector field $W \in \Gamma(\ker f)^\perp$, we have

$$\phi W = BW + CW, \tag{3.7}$$

where $BW \in \Gamma(\ker f)$ and $CW \in \Gamma(\mathcal{V})$.

Lemma 3.1. *Let f be a quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\psi^2 U + B\omega U = -U + \eta(U)\xi, \quad \omega\psi U + C\omega U = 0,$$

$$\omega BW + C^2 W = -W, \quad \psi BW + BCW = 0,$$

for all $U \in \Gamma(\ker f_*)$ and $W \in \Gamma(\ker f_*)^\perp$.

Lemma 3.2. *Let f be a quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(i) $\psi^2 U = -(\cos^2\theta_1)U$,

(ii) $g_M(\psi U, \psi V) = \cos^2\theta_1 g_M(U, V)$,

(iii) $g_M(\omega U, \omega V) = \sin^2\theta_1 g_M(U, V)$,

for all $U, V \in \Gamma(D_1)$.

Lemma 3.3. *Let f be a quasi bi-slant submersion from an contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(i) $\psi^2 W = -(\cos^2\theta_2)W$,

(ii) $g_M(\psi W, \psi Z) = \cos^2\theta_2 g_M(W, Z)$,

(iii) $g_M(\omega W, \omega Z) = \sin^2\theta_2 g_M(W, Z)$,

for all $W, Z \in \Gamma(D_2)$.

Lemma 3.4. *Let f be a quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\mathcal{V}\nabla_U \psi V + \mathcal{T}_U \omega V = \psi \mathcal{V}\nabla_U V + B\mathcal{T}_U V, \tag{3.8}$$

$$\mathcal{T}_U \psi V + \mathcal{H}\nabla_U \omega V = \omega \mathcal{V}\nabla_U V + C\mathcal{T}_U V, \tag{3.9}$$

$$\mathcal{V}\nabla_X BY + \mathcal{A}_X CY = \psi \mathcal{A}_X Y + B\mathcal{H}\nabla_X Y, \tag{3.10}$$

$$\mathcal{A}_X BY + \mathcal{H}\nabla_X CY = \omega \mathcal{A}_X Y + C\mathcal{H}\nabla_X Y, \tag{3.11}$$

$$\mathcal{V}\nabla_U BX + \mathcal{T}_U CX = \psi \mathcal{T}_U X + B\mathcal{H}\nabla_U X, \tag{3.12}$$

$$\mathcal{T}_U BX + \mathcal{H}\nabla_U CX = \omega \mathcal{T}_U X + C\mathcal{H}\nabla_U X, \tag{3.13}$$

$$\mathcal{V}\nabla_Y \psi U + \mathcal{A}_Y \omega U = B\mathcal{A}_Y U + \psi \mathcal{V}\nabla_Y U, \tag{3.14}$$

$$\mathcal{A}_Y \psi U + \mathcal{H}\nabla_Y \omega U = C\mathcal{A}_Y U + \omega \mathcal{V}\nabla_Y U, \tag{3.15}$$

for any $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$.

Now, we define

$$(\nabla_U \psi)V = \mathcal{V}\nabla_U \psi V - \psi \mathcal{V}\nabla_U V, \tag{3.16}$$

$$(\nabla_U \omega)V = \mathcal{H}\nabla_U \omega V - \omega \mathcal{V}\nabla_U V, \tag{3.17}$$

$$(\nabla_X C)Y = \mathcal{H}\nabla_X CY - C\mathcal{H}\nabla_X Y, \tag{3.18}$$

$$(\nabla_X B)Y = \mathcal{V}\nabla_X BY - B\mathcal{H}\nabla_X Y, \tag{3.19}$$

for any $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$.

Lemma 3.5. *Let f be a quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$(\nabla_U \psi)V = B\mathcal{T}_U V - \mathcal{T}_U \omega V,$$

$$(\nabla_U \omega)V = C\mathcal{T}_U V - \mathcal{T}_U \psi V,$$

$$(\nabla_X C)Y = \omega \mathcal{A}_X Y - \mathcal{A}_X BY,$$

$$(\nabla_X B)Y = \psi \mathcal{A}_X Y - \mathcal{A}_X CY,$$

for any vectors $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$.

The proofs of above Lemmas follow from straightforward computations, so we omit them.

If the tensors ψ and ω are parallel with respect to the linear connection ∇ on M respectively, then

$$B\mathcal{T}_U V = \mathcal{T}_U \omega V,$$

and

$$C\mathcal{T}_U V = \mathcal{T}_U \psi V,$$

for any $U, V \in \Gamma(TM)$.

Lemma 3.6. *Let f be a quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(i) $g_M(\nabla_X Y, \xi) = 0$ *for all* $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$,

(ii) $g_M([X, Y], \xi) = 0$ *for all* $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$.

Proof. Let $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$, consider

$$\nabla_X\{g_M(Y, \xi)\} = (\nabla_X g_M)(Y, \xi) + g_M(\nabla_X Y, \xi) + g_M(Y, \nabla_X \xi).$$

Since X and Y are orthogonal to ξ ie.

$$g_M(\nabla_X Y, \xi) = -g_M(Y, \nabla_X \xi),$$

using equation (2.7) and the property that metric tensor is $\nabla-$parallel, we have both results of this lemma. ☐

Theorem 3.7. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the invariant distribution D is integrable if and only if*

$$g_M(\mathcal{T}_V \psi U - \mathcal{T}_U \psi V, \omega Q W + \omega R W) = g_M(\mathcal{V}\nabla_U \psi V - \mathcal{V}\nabla_V \psi U, \psi Q W + \psi R W), \tag{3.20}$$

for $U, V \in \Gamma(D)$ *and* $W \in \Gamma(D_1 \oplus D_2)$.

Proof. For $U, V \in \Gamma(D)$, and $W \in \Gamma(D_1 \oplus D_2)$, using equations (2.1)–(2.4), (2.6), (2.7), (2.10), (3.2), (3.3) and Lemma 3.6 we have

$$\begin{aligned} g_M([U, V], W) &= g_M(\nabla_U \phi V, \phi W) + \eta(W)\eta(\nabla_U V) - g_M(\nabla_V \phi U, \phi W) - \eta(W)\eta(\nabla_V U), \\ &= g_M(\nabla_U \psi V, \phi W) - g_M(\nabla_V \psi U, \phi W), \\ &= g_M(\mathcal{T}_U \psi V - \mathcal{T}_V \psi U, \omega Q W + \omega R W) - g_M(\mathcal{V}\nabla_U \psi V - \mathcal{V}\nabla_V \psi U, \psi Q W + \psi R W), \end{aligned}$$

which completes the proof. ☐

Theorem 3.8. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the slant distribution D_1 is integrable if and only if*

$$\begin{aligned} g_M(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, X) &= g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, \phi P X + \psi R X) \\ &\quad + g_M(\mathcal{H}\nabla_W \omega Z - \mathcal{H}\nabla_Z \omega W, \omega R X), \end{aligned} \tag{3.21}$$

for all $W, Z \in \Gamma(D_1)$ *and* $X \in \Gamma(D \oplus D_2)$.

Proof. For all $W, Z \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$, we have

$$g_M([W,Z],X) = g_M(\nabla_W Z, X) - g_M(\nabla_Z W, X).$$

Using equations (2.1)–(2.4), (2.6), (2.7), (2.11), (3.2), (3.3) and Lemma 3.2 we have

$$\begin{aligned} g_M([W,Z],X) &= g_M(\nabla_W \phi Z, \phi X) - g_M(\nabla_Z \phi W, \phi X), \\ &= g_M(\nabla_W \psi Z, \phi X) + g_M(\nabla_W \omega Z, \phi X) - g_M(\nabla_Z \psi W, \phi X) - g_M(\nabla_W \omega Z, \phi X), \\ &= \cos^2\theta_1 g_M(\nabla_W Z, X) - \cos^2\theta_1 g_M(\nabla_Z W, X) - g_M(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, X) \\ &\quad + g_M(\mathcal{H}\nabla_W \omega Z + \mathcal{T}_W \omega Z, \phi PX + \psi RX + \omega RX) \\ &\quad - g_M(\mathcal{H}\nabla_Z \omega W + \mathcal{T}_Z \omega W, \phi PX + \psi RX + \omega RX). \end{aligned}$$

Now, we obtain

$$\begin{aligned} \sin^2\theta_1 g_M([W,Z],X) &= g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, \phi PX + \psi RX) + g_M(\mathcal{H}\nabla_W \omega Z - \mathcal{H}\nabla_Z \omega W, \omega RX) \\ &\quad - g_M(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, X), \end{aligned}$$

which completes the proof. □

Theorem 3.9. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the slant distribution D_2 is integrable if and only if*

$$\begin{aligned} g_M(\mathcal{T}_U \omega\psi V - \mathcal{T}_V \omega\psi U, Y) &= g_M(\mathcal{H}\nabla_U \omega V - \mathcal{H}\nabla_V \omega U, \omega QY) \\ &\quad + g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi PY + \psi QY), \end{aligned} \tag{3.22}$$

for all $U, V \in \Gamma(D_2)$ and $Y \in \Gamma(D \oplus D_1)$.

Proof. For all $U, V \in \Gamma(D_2)$ and $Y \in \Gamma(D \oplus D_1)$, using equations (2.1)–(2.4), (2.6), (2.7), (3.3) and Lemma 3.6 we have

$$g_M([U,V],Y) = g_M(\nabla_U \psi V, \phi Y) + g_M(\nabla_U \omega V, \phi Y) - g_M(\nabla_V \psi U, \phi Y) - g_M(\nabla_V \omega U, \phi Y).$$

From equations (2.9), (3.2) and Lemma 3.3 we have

$$\begin{aligned} g_M([U,V],Y) &= \cos^2\theta_2 g_M([U,V],Y) + g_M(\mathcal{H}\nabla_U \omega V - \mathcal{H}\nabla_V \omega U, \omega QY) \\ &\quad + g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi PY + \psi QY) - g_M(\mathcal{T}_U \omega\psi V - \mathcal{T}_V \omega\psi U, Y). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2\theta_2 g_M([U,V],Y) &= g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi PY + \psi QY) - g_M(\mathcal{T}_U \omega\psi V - \mathcal{T}_V \omega\psi U, Y) \\ &\quad + g_M(\mathcal{H}\nabla_U \omega V - \mathcal{H}\nabla_V \omega U, \omega QY), \end{aligned}$$

which the proof follows from the above equations. □

Theorem 3.10. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the horizontal distribution $(\ker f_*)^\perp$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g_M(\mathcal{A}_U V, PW + \cos^2\theta_1 QW + \cos^2\theta_2 RW) &= g_M(\mathcal{H}\nabla_U V, \omega\psi PW + \omega\psi QW + \omega\psi RW) \\ &\quad - g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega W), \end{aligned} \tag{3.23}$$

for all $U, V \in \Gamma(\ker f_*)^\perp$ and $W \in \Gamma(\ker f_*)$.

Proof. For $U, V \in \Gamma(\ker f_*)^\perp$ and $W \in \Gamma(\ker f_*)$, we have

$$g_M(\nabla_U V, W) = g_M(\nabla_U V, PW + QW + RW + \eta(W)\xi).$$

Using equations (2.1)–(2.4), (2.6), (2.7), (2.12), (2.13), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3 we have

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\phi\nabla_U V, \phi PW) + g_M(\phi\nabla_U V, \phi QW) + g_M(\phi\nabla_U V, \phi RW), \\ &= g_M(\mathcal{A}_U V, PW + \cos^2\theta_1 QW + \cos^2\theta_2 RW) \\ &\quad - g_M(\mathcal{H}\nabla_U V, \omega\psi PW + \omega\psi QW + \omega\psi RW) \\ &\quad + g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega PW + \omega QW + \omega RW). \end{aligned}$$

Taking into account $\omega PW + \omega QW + \omega RW = \omega W$ and $\omega PW = 0$ in the above, one obtains

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\mathcal{A}_U V, PW + \cos^2\theta_1 QW + \cos^2\theta_2 RW) \\ &\quad - g_M(\mathcal{H}\nabla_U V, \omega\psi PW + \omega\psi QW + \omega\psi RW) \\ &\quad + g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega W). \end{aligned}$$

□

Theorem 3.11. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the vertical distribution $(\ker f_*)$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g_M(\mathcal{T}_X PY + \cos^2\theta_1 \mathcal{T}_X QY + \cos^2\theta_2 \mathcal{T}_X RY, U) &= g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\mathcal{T}_X \omega Y, BU) + g_M(\mathcal{H}\nabla_X \omega Y, CU), \end{aligned} \tag{3.24}$$

for all $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^\perp$.

Proof. For all $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^\perp$, by using equations (2.1)–(2.4), (2.6) and (2.7) we have

$$g_M(\nabla_X Y, U) = g_M(\nabla_X \phi PY, \phi U) + g_M(\nabla_X \phi QY, \phi U) + g_M(\nabla_X \phi RY, \phi U).$$

Taking into account of (2.10), (2.11), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3 we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X PY, U) + \cos^2\theta_1 g_M(\mathcal{T}_X QY, U) + \cos^2\theta_2 g_M(\mathcal{T}_X RY, U) \\ &\quad - g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\nabla_X \omega PY + \nabla_X \omega QY + \nabla_X \omega RY, \phi U). \end{aligned}$$

Since $\omega PY + \omega QY + \omega RY = \omega Y$ and $\omega PY = 0$, we derive

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X PY + \cos^2\theta_1 \mathcal{T}_X QY + \cos^2\theta_2 \mathcal{T}_X RY, U) \\ &\quad - g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\mathcal{T}_X \omega Y, BU) + g_M(\mathcal{H}\nabla_X \omega Y, CU), \end{aligned}$$

which completes the proof. □

From Theorems 3.10 and 3.11 we also have the following decomposition results.

Theorem 3.12. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the total space is locally a product manifold of the form $M_{\ker f_*} \times M_{(\ker f_*)^\perp}$, where $M_{\ker f_*}$ and $M_{(\ker f_*)^\perp}$ are leaves of $\ker f_*$ and $(\ker f_*)^\perp$ respectively if and only if*

$$\begin{aligned} g_M(\mathcal{A}_U V, PY + \cos^2\theta_1 QY + \cos^2\theta_2 RY) &= g_M(\mathcal{H}\nabla_U V, \omega\psi PY + \omega\psi QY + \omega\psi RY) \\ &\quad + g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega Y), \end{aligned}$$

and

$$\begin{aligned} g_M(\mathcal{T}_X Y + \cos^2\theta_1 \mathcal{T}_X QY + \cos^2\theta_2 \mathcal{T}_X RY, U) &= g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\mathcal{T}_X \omega Y, BU) + g_M(\mathcal{H}\nabla_X \omega Y, CU), \end{aligned}$$

for all $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Theorem 3.13. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution D defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_U \phi PV, \omega QW + \omega RW) = -g_M(\mathcal{V}\nabla_U \phi PV, \psi QW + \psi RW), \tag{3.25}$$

and

$$g_M(\mathcal{T}_U \phi PV, CY) = -g_M(\mathcal{V}\nabla_U \phi PV, BY), \tag{3.26}$$

for all $U, V \in \Gamma(D), W \in \Gamma(D_1 \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^\perp$.

Proof. For all $U, V \in \Gamma(D)$, $W \in \Gamma(D_1 \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^\perp$, using equations (2.1)–(2.4), (2.6), (2.7), (3.2), (3.3) and Lemma 3.6 we have

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\nabla_U \phi V, \phi W), \\ &= g_M(\nabla_U \phi PV, \phi QW + \phi RW), \\ &= g_M(\mathcal{T}_U \phi PV, \omega QW + \omega RW) + g_M(\mathcal{V}\nabla_U \phi PV, \psi QW + \psi RW). \end{aligned}$$

Now, again using equations (2.10), (3.2), (3.3) and (3.7) we have

$$\begin{aligned} g_M(\nabla_U V, Y) &= g_M(\nabla_U \phi V, \phi Y), \\ &= g_M(\nabla_U \phi PV, BY + CY), \\ &= g_M(\mathcal{V}\nabla_U \phi PV, BY) + g_M(\mathcal{T}_U \phi PV, CY), \end{aligned}$$

which completes the proof. □

Theorem 3.14. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution D_1 defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_W \omega\psi Z, U) = g_M(\mathcal{T}_W \omega QZ, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_W \omega QZ, \omega RU), \tag{3.27}$$

and

$$g_M(\mathcal{H}\nabla_W \omega\psi Z, Y) = g_M(\mathcal{H}\nabla_W \omega Z, CY) + g_M(\mathcal{T}_W \omega Z, BY), \tag{3.28}$$

for all $W, Z \in \Gamma(D_1), U \in \Gamma(D \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^\perp$.

Proof. For all $W, Z \in \Gamma(D_1)$, $U \in \Gamma(D \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^\perp$, using equations (2.1)–(2.4), (2.6), (2.7), (2.11), (3.2), (3.3) and Lemma 3.2, we have

$$\begin{aligned} g_M(\nabla_W Z, U) &= g_M(\nabla_W \phi Z, \phi U) \\ &= g_M(\nabla_W \psi Z, \phi U) + g_M(\nabla_W \omega Z, \phi U), \\ &= \cos^2\theta_1 g_M(\nabla_W Z, U) - g_M(\mathcal{T}_W \omega\psi Z, U) \\ &\quad + g_M(\mathcal{T}_W \omega QZ, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_W \omega QZ, \omega RU). \end{aligned}$$

Now, we obtain

$$\sin^2\theta_1 g_M(\nabla_W Z, U) = -g_M(\mathcal{T}_W \omega\psi Z, U) + g_M(\mathcal{T}_W \omega QZ, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_W \omega QZ, \omega RZ)$$

Next, from equations (2.1)–(2.4), (2.6), (2.7), (2.12), (3.3), (3.7) and Lemma 3.2, we have

$$\begin{aligned} g_M(\nabla_W Z, Y) &= g_M(\nabla_W \phi Z, \phi Y), \\ &= g_M(\nabla_W \psi Z, \phi Y) + g_M(\nabla_W \omega Z, \phi Y), \\ &= \cos^2\theta_1 g_M(\nabla_W Z, Y) - g_M(\mathcal{H}\nabla_W \omega\psi Z, Y) \\ &\quad + g_M(\mathcal{H}\nabla_W \omega Z, CY) + g_M(\mathcal{T}_W \omega Z, BY). \end{aligned}$$

Now, we arrive

$$\sin^2\theta_1 g_M(\nabla_W Z, Y) \;=\; -g_M(\mathcal{H}\nabla_W\omega\psi Z, Y) + g_M(\mathcal{H}\nabla_W\omega Z, CY) + g_M(\mathcal{T}_W\omega Z, BY),$$

which completes the proof. □

Theorem 3.15. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution D_2 defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_U\omega\psi V, W) = g_M(\mathcal{T}_U\omega QV, \phi PW + \phi RW) + g_M(\mathcal{H}\nabla_U\omega QV, \omega RW), \tag{3.29}$$

and

$$g_M(\mathcal{H}\nabla_U\omega\psi V, Y) = g_M(\mathcal{H}\nabla_U\omega V, CY) + g_M(\mathcal{T}_U\omega V, BY), \tag{3.30}$$

for all $U, V \in \Gamma(D_2), W \in \Gamma(D \oplus D_1)$ and $Y \in \Gamma(\ker f_*)^\perp$.

Proof. For all $U, V \in \Gamma(D_2), W \in \Gamma(D \oplus D_1)$ and $Y \in \Gamma(\ker f_*)^\perp$, by using equations (2.1)–(2.4), (2.6), (2.7), (2.10), (3.3) and from Lemma 3.2 and Lemma 3.6, we have

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\nabla_U\psi V, \phi W) + g_M(\nabla_U\omega V, \phi W), \\ &= \cos^2\theta_2 g_M(\nabla_U V, W) - g_M(\mathcal{T}_U\omega\psi V, W) \\ &\quad + g_M(\mathcal{T}_U\omega QV, \phi PW + \psi RW) + g_M(\mathcal{H}\nabla_U\omega QV, \omega RW). \end{aligned}$$

Now, we get

$$\sin^2\theta_2 g_M(\nabla_U V, W) = -g_M(\mathcal{T}_U\omega\psi V, W) + g_M(\mathcal{T}_U\omega QV, \phi PW + \psi RW) + g_M(\mathcal{H}\nabla_U\omega QV, \omega RW).$$

Next, from equations (2.1)–(2.4), (2.6), (2.7), (2.12), (3.2) (3.3), (3.7) and Lemma 3.2, we have

$$\begin{aligned} g_M(\nabla_U V, Y) &= g_M(\nabla_U\psi V, \phi Y) + g_M(\nabla_U\omega V, \phi Y), \\ &= \cos^2\theta_2 g_M(\nabla_U V, Y) - g_M(\mathcal{H}\nabla_U\omega\psi V, Y) \\ &\quad + g_M(\mathcal{H}\nabla_U\omega V, CY) + g_M(\mathcal{T}_U\omega V, BY). \end{aligned}$$

Now, we obtain

$$\sin^2\theta_1 g_M(\nabla_U V, Y) \;=\; -g_M(\mathcal{H}\nabla_U\omega\psi V, Y) + g_M(\mathcal{H}\nabla_U\omega V, CY) + g_M(\mathcal{T}_U\omega V, BY),$$

which completes the proof. □

We recall that a differentiable map f between two Riemannian manifolds is totally geodesic if

$$(\nabla f_*)(Y, Z) = 0, \text{ for all } Y, Z \in \Gamma(TM).$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Theorem 3.16. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the map f is totally geodesic if and only if*

$$\begin{aligned} &g_M(\mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi RV - \cos^2\theta_1 \mathcal{T}_U QV - \cos^2\theta_2 \mathcal{T}_U RV, W) \\ &\quad = g_M(\mathcal{V}\nabla_U \phi PV + \mathcal{T}_U \omega QV + \mathcal{T}_U \omega RV, BW) + g_M(\mathcal{T}_U \phi PV + \mathcal{H}\nabla_U \omega QV + \mathcal{H}\nabla_U \omega RV, CW), \end{aligned}$$

and

$$\begin{aligned} &g_M(\mathcal{H}\nabla_W \omega\psi QU + \mathcal{H}\nabla_W \omega\psi RU - \cos^2\theta_1 \mathcal{A}_W QU - \cos^2\theta_2 \mathcal{A}_W RU, Z) \\ &= g_M(\mathcal{V}\nabla_W \phi PU + \mathcal{A}_W \omega QU + \mathcal{A}_W \omega RU, BZ) + g_M(\mathcal{A}_W \phi PU + \mathcal{H}\nabla_W \omega QU + \mathcal{H}\nabla_W \omega RU, CZ), \end{aligned}$$

for all $U, V \in \Gamma(\ker f_*)$ and $W, Z \in \Gamma(\ker f_*)^\perp$.

Proof. For all $U, V \in \Gamma(\ker f_*)$ and $W, Z \in \Gamma(\ker f_*)^\perp$, making use of (2.1)–(2.4), (2.6), (2.7), (2.10), (2.11), (3.2), (3.3), (3.7) and from Lemma 3.2 and 3.3, we derive

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\nabla_U \phi V, \phi W) \\ &= g_M(\nabla_U \phi PV, \phi W) + g_M(\nabla_U \phi QV, \phi W) + g_M(\nabla_U \phi RV, \phi W), \\ &= g_M(\nabla_U \phi PV, \phi W) + g_M(\nabla_U \psi QV, \phi W) + g_M(\nabla_U \psi RV, \phi W) \\ &\quad + g_M(\nabla_U \omega QV, \phi W) + g_M(\nabla_U \omega RV, \phi W), \\ &= g_M(\mathcal{V}\nabla_U \phi PV + \mathcal{T}_U \omega QV + \mathcal{T}_U \omega RV, W) \\ &\quad + g_M(\mathcal{T}_U \phi PV + \mathcal{H}\nabla_U \omega QV + \mathcal{H}\nabla_U \omega RV, CW) \\ &\quad + g_M(\cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV - \mathcal{H}\nabla_U \omega\psi QV - \mathcal{H}\nabla_U \omega\psi RV, W). \end{aligned}$$

Next, taking account of (2.1)–(2.4), (2.6), (2.7), (2.10), (2.12), (2.13), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3, we have

$$\begin{aligned} g_M(\nabla_W U, Z) &= g_M(\phi \nabla_W U, \phi Z) \\ &= g_M(\nabla_W \phi U, \phi Z), \\ &= g_M(\nabla_W \phi PU, \phi Z) + g_M(\nabla_W \phi QU, \phi Z) + g_M(\nabla_W \phi RU, \phi Z), \\ &= g_M(\nabla_W \phi PU, \phi Z) + g_M(\nabla_W \psi QU, \phi Z) + g_M(\nabla_W \psi RU, \phi Z) \\ &\quad + g_M(\nabla_W \omega QU, \phi Z) + g_M(\nabla_W \omega RU, \phi Z), \\ &= g_M(\mathcal{V}\nabla_W \phi PU + \mathcal{A}_W \omega QU + \mathcal{A}_W \omega RU, BZ) \\ &\quad + g_M(\mathcal{A}_W \phi PU + \mathcal{H}\nabla_W \omega QU + \mathcal{H}\nabla_W \omega RU, CZ) \\ &\quad + g_M(\cos^2\theta_1 \mathcal{A}_W QU + \cos^2\theta_2 \mathcal{A}_W RU - \mathcal{H}\nabla_W \omega\psi QU - \mathcal{H}\nabla_W \omega\psi RU, Z), \end{aligned}$$

which completes the proof. $\square$

4 Examples

In this section, we are going to give some non-trivial examples. We will use the notation mentioned in Example 2.1.

Example 4.1. *Define a map*

$$\begin{aligned} \pi \quad &: \quad \mathbb{R}^{15} \to \mathbb{R}^6 \\ \pi(x_1, x_2, \dots, x_7, y_1, y_2, \dots, y_7, z) \quad &= \quad (x_2 \cos\theta_1 - y_3 \sin\theta_1, y_2, x_4 \sin\theta_2 - y_5 \cos\theta_2, x_5, x_7, y_7), \end{aligned}$$

which is a quasi bi-slant submersion such that

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \frac{\partial}{\partial x_2} \sin\theta_1 + \frac{\partial}{\partial y_3} \cos\theta_1, \quad X_4 = \frac{\partial}{\partial x_3},$$

$$X_5 = \frac{\partial}{\partial x_4} \cos\theta_2 + \frac{\partial}{\partial y_5} \sin\theta_2, \quad X_6 = \frac{\partial}{\partial y_4}, \quad X_7 = \frac{\partial}{\partial x_6}, \quad X_8 = \frac{\partial}{\partial y_6},$$

$$X_9 = \xi = \frac{\partial}{\partial z},$$

$$(\ker \pi_*) = (D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle),$$

where

$$D = \left\langle X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial y_1}, X_7 = \frac{\partial}{\partial x_6}, X_8 = \frac{\partial}{\partial y_6} \right\rangle,$$

$$D_1 = \left\langle X_3 = \frac{\partial}{\partial x_2} \sin\theta_1 + \frac{\partial}{\partial y_3} \cos\theta_1, X_4 = \frac{\partial}{\partial x_3} \right\rangle,$$

$$D_2 = \left\langle X_5 = \frac{\partial}{\partial x_4} \cos\theta_2 + \frac{\partial}{\partial y_5} \sin\theta_2, X_6 = \frac{\partial}{\partial y_4} \right\rangle,$$

$$\langle \xi \rangle = \left\langle X_9 = \frac{\partial}{\partial z} \right\rangle,$$

and

$$(\ker \pi_*)^\perp = \left\langle \frac{\partial}{\partial x_2} \cos\theta_1 - \frac{\partial}{\partial y_3} \sin\theta_1, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_4} \sin\theta_2 - \frac{\partial}{\partial y_5} \cos\theta_2, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial y_7} \right\rangle,$$

with bi-slant angles θ_1 and θ_2. Thus the above example verifies the Lemmas 3.1, 3.2, 3.3 and 3.6.

Example 4.2. *Define a map*

$$\begin{aligned} \pi \quad &: \quad \mathbb{R}^{13} \to \mathbb{R}^6 \\ \pi(x_1, x_2, \dots, x_6, y_1, y_2, \dots, y_6, z) \quad &= \quad \left(\frac{x_1 - x_2}{\sqrt{2}}, y_1, \frac{\sqrt{3}x_4 - x_5}{2}, y_5, x_6, y_6 \right), \end{aligned}$$

which is a quasi bi-slant submersion such that

$$X_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), \quad X_2 = \frac{\partial}{\partial y_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial y_3},$$

$$X_5 = \frac{1}{2} \left(\frac{\partial}{\partial x_4} + \sqrt{3} \frac{\partial}{\partial x_5} \right), \quad X_6 = \frac{\partial}{\partial y_4},$$

$$X_7 = \xi = \frac{\partial}{\partial z},$$
$$(\ker \pi_*) = (D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle),$$

where

$$D = \left\langle X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial y_3} \right\rangle,$$
$$D_1 = \left\langle X_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), X_2 = \frac{\partial}{\partial y_2} \right\rangle,$$
$$D_2 = \left\langle X_5 = \frac{1}{2} \left(\frac{\partial}{\partial x_4} + \sqrt{3} \frac{\partial}{\partial x_5} \right), X_6 = \frac{\partial}{\partial y_4} \right\rangle,$$
$$\langle \xi \rangle = \left\langle X_7 = \frac{\partial}{\partial z} \right\rangle,$$

and

$$(\ker \pi_*)^{\perp} = \left\langle \frac{\partial}{\partial y_1}, \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), \frac{1}{2} \left(\sqrt{3} \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} \right), \frac{\partial}{\partial y_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial y_6} \right\rangle,$$

with bi-slant angles $\theta_1 = \frac{\pi}{4}$ *and* $\theta_2 = \frac{\pi}{3}$. *Therefore, the above example verifies the Lemmas 3.1, 3.2, 3.3 and 3.6.*

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Infinitely many positive solutions for an iterative system of singular BVP on time scales

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ABSTRACT

In this paper, we consider an iterative system of singular two-point boundary value problems on time scales. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of infinitely many positive solutions. Finally, we provide an example to check the validity of our obtained results.

RESUMEN

En este artículo, consideramos un sistema iterativo de problemas de valor en la frontera singulares de dos puntos en escalas de tiempo. Aplicando la desigualdad de Hölder y el teorema de punto fijo cónico de Krasnoselskii en un espacio de Banach, derivamos condiciones suficientes para la existencia de una cantidad infinita de soluciones positivas. Finalmente, entregamos un ejemplo para verificar la validez de nuestros resultados.

Keywords and Phrases: Iterative system, time scales, singularity, cone, Krasnoselskii's fixed point theorem, positive solutions.

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1 Introduction

The theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. A time scale is any closed and nonempty subset of the real numbers. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and the monographs of Bohner and Peterson [6, 7]. There is a great deal of research activity devoted to existence of solutions to the dynamic equations on time scales, see for example [8, 9, 13, 16–19] and references therein.

In [14], Liang and Zhang studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales,

$$\begin{aligned} &\big(\varphi(\upsilon^{\Delta}(t))\big)^{\nabla} + a(t)f\big(\upsilon(t)\big) = 0,\ t \in [0,\mathfrak{T}]_{\mathbb{T}},\\ &\upsilon(0) = \sum_{i=1}^{m-2} a_i\upsilon(\xi_i),\ \upsilon^{\Delta}(\mathfrak{T}) = 0, \end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [12], Khuddush, Prasad and Vidyasagar considered second order n-point boundary value problem on time scales,

$$\begin{aligned} \upsilon_i^{\Delta\nabla}(t) + \lambda(t)\mathrm{g}_\ell\big(\upsilon_{i+1}(t)\big) = 0,\ 1 \le i \le n,\ t \in (0,\sigma(a)]_{\mathbb{T}},\\ \upsilon_{n+1}(t) = \upsilon_1(t),\ t \in (0,\sigma(a)]_{\mathbb{T}},\\ \upsilon_i^{\Delta}(0) = 0,\ \upsilon_i(\sigma(a)) = \sum_{k=1}^{n-2} c_k\upsilon_i(\zeta_k),\ 1 \le i \le n, \end{aligned}$$

and established existence of positive solutions by applying Krasnoselskii's fixed point theorem.

Inspired by the aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of infinitely many positive solutions for the iterative system of two-point boundary value problems with n–singularities on time scales,

$$\left.\begin{aligned} \upsilon_\ell^{\Delta\Delta}(t) + \lambda(t)\mathrm{g}_\ell\big(\upsilon_{\ell+1}(t)\big) = 0,\ 1 \le \ell \le m,\ t \in (0,\mathfrak{T})_{\mathbb{T}},\\ \upsilon_{m+1}(t) = \upsilon_1(t),\ t \in (0,\mathfrak{T})_{\mathbb{T}}, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \upsilon_\ell(0) = \upsilon_\ell^{\Delta}(0),\ 1 \le \ell \le m,\\ \upsilon_\ell(\mathfrak{T}) = -\upsilon_\ell^{\Delta}(\mathfrak{T}),\ 1 \le \ell \le m, \end{aligned}\right\} \tag{1.2}$$

where $m \in \mathbb{N}$, $\lambda(t) = \prod_{i=1}^{k} \lambda_i(t)$ and each $\lambda_i(t) \in L_{\Delta}^{p_i}([0,\mathfrak{T}]_{\mathbb{T}})$ $(p_i \geq 1)$ has n–singularities in the interval $(0,\mathfrak{T})_{\mathbb{T}}$.

We assume the following conditions are true throughout the paper:

(H_1) $\mathsf{g}_\ell : [0,+\infty) \to [0,+\infty)$ is continuous.

(H_2) $\lim\limits_{t\to t_i} \lambda_i(t) = \infty$, where $0 < t_n < t_{n-1} < \cdots < t_1 < \mathfrak{T}$.

2 Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions.

Definition 2.1 ([6]). *A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to [0,+\infty)$ are defined by*

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\},$$

$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\},$$

and

$$\mu(t) = \sigma(t) - t,$$

respectively.

- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.*
- *If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$.*
- *If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T},\mathbb{R})$.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T},\mathbb{R})$.*
- *By an interval time scale, we mean the intersection of a real interval with a given time scale, i.e., $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$. Other intervals can be defined similarly.*

Definition 2.2 ([5,11])**.** *Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$measure and the Lebesgue $\nabla-$measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.*

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that P holds on $A \backslash \Gamma_1$, then P is said to hold Δ–a.e. on A.

(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that P holds on $A \backslash \Gamma_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3 ([4, 5])**.** *Let $E \subset \mathbb{T}$ be a $\Delta-$measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be a $\Delta-$measurable function. We say that f belongs to $L^p_\Delta(E)$ provided that either*

$$\int_E |f|^p(s)\Delta s < \infty \quad \text{if} \quad p \in [1, +\infty),$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \quad \Delta - a.e. \text{ on } E \quad \text{if} \quad p = +\infty.$$

Lemma 2.4 ([20])**.** *Let $E \subset \mathbb{T}$ be a $\Delta-$measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is $\Delta-$integrable on E, then*

$$\int_E f(s)\Delta s = \int_E f(s)ds + \sum_{i \in I_E} \big(\sigma(t_i) - t_i\big) f(t_i) + r(f, E),$$

where

$$r(f, E) = \begin{cases} \mu_{\mathtt{N}}(E) f(M), & \text{if } \mathtt{N} \in \mathbb{T}, \\ \\ 0, & \text{if } \mathtt{N} \notin \mathbb{T}, \end{cases}$$

$I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}$, $I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$.

Lemma 2.5. *For any $\mathtt{y}(t) \in \mathcal{C}_{rd}([0, \mathfrak{T}]_{\mathbb{T}})$, the boundary value problem,*

$$\upsilon_1^{\Delta\Delta}(t) + \mathtt{y}(t) = 0, \ t \in (0, \mathfrak{T})_{\mathbb{T}}, \tag{2.1}$$

$$\upsilon_1(0) = \upsilon_1^{\Delta}(0), \ \upsilon_1(\mathfrak{T}) = -\upsilon_1^{\Delta}(\mathfrak{T}), \tag{2.2}$$

has a unique solution

$$\upsilon_1(t) = \int_0^{\mathfrak{T}} \aleph(t, \tau)\mathtt{y}(\tau)\Delta\tau, \tag{2.3}$$

where

$$\aleph(t, \tau) = \frac{1}{2 + \mathfrak{T}} \begin{cases} (\mathfrak{T} - t + 1)(\sigma(\tau) + 1), & \text{if} \ \ \sigma(\tau) < t, \\ (\mathfrak{T} - \sigma(\tau) + 1)(t + 1), & \text{if} \ \ t < \tau. \end{cases} \tag{2.4}$$

Proof. Suppose υ_1 is a solution of (2.1), then

$$\begin{aligned}\upsilon_1(t) &= -\int_0^t \int_0^\tau \mathrm{y}(\tau_1)\Delta\tau_1\Delta\tau + A_1 t + A_2\\ &= -\int_0^t (t-\sigma(\tau))\mathrm{y}(\tau)\Delta\tau + A_1 t + A_2,\end{aligned}$$

where $A_1 = \upsilon_1^\Delta(0)$ and $A_2 = \upsilon_1(0)$. By the conditions (2.2), we get

$$A_1 = A_2 = \frac{1}{2+\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T}-\sigma(\tau)+1)\mathrm{y}(\tau)\Delta\tau.$$

So, we have

$$\begin{aligned}\upsilon_1(t) &= \int_0^t (t-\sigma(\tau))\mathrm{y}(\tau)\Delta\tau + \frac{1}{2+\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T}-\sigma(\tau)+1)(1+t)\mathrm{y}(\tau)\Delta\tau\\ &= \int_0^{\mathfrak{T}} \aleph(t,\tau)\mathrm{y}(\tau)\Delta\tau.\end{aligned}$$

This completes the proof. □

Lemma 2.6. *Suppose (H_1)–(H_2) hold. For $\varepsilon \in (0, \frac{\mathfrak{T}}{2})_{\mathbb{T}}$, let $\mathcal{G}(\varepsilon) = \dfrac{\varepsilon+1}{\mathfrak{T}+1} < 1$. Then $\aleph(t,\tau)$ has the following properties:*

(i) $0 \le \aleph(t,\tau) \le \aleph(\tau,\tau)$ *for all* $t, \tau \in [0,1]_{\mathbb{T}}$,

(ii) $\mathcal{G}(\varepsilon)\aleph(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\varepsilon, \mathfrak{T}-\varepsilon]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$.

Proof. *(i)* is evident. To prove *(ii)*, let $t \in [\varepsilon, \mathfrak{T}-\varepsilon]_{\mathbb{T}}$ and $t \le \tau$. Then

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)} = \frac{t+1}{\tau+1} \ge \frac{\varepsilon+1}{\mathfrak{T}+1} = \mathcal{G}(\varepsilon).$$

For $\tau \le t$,

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)} = \frac{\mathfrak{T}-t+1}{\mathfrak{T}-\tau+1} \ge \frac{\varepsilon+1}{\mathfrak{T}+1} = \mathcal{G}(\varepsilon).$$

This completes the proof. □

Notice that an $m-$tuple $(\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \ldots, \upsilon_m(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\begin{aligned}\upsilon_\ell(t) &= \int_0^1 \aleph(t,\tau)\lambda(\tau)\mathrm{g}_\ell(\upsilon_{\ell+1}(\tau))\Delta\tau,\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ 1 \le \ell \le m,\\ \upsilon_{m+1}(t) &= \upsilon_1(t),\ t \in (0,\mathfrak{T})_{\mathbb{T}},\end{aligned}$$

i.e.,

$$\begin{aligned}\upsilon_1(t) = \int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathrm{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathrm{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ \times \mathrm{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Let $\mathtt{B}$ be the Banach space $C_{rd}((0,\mathfrak{T})_{\mathbb{T}},\mathbb{R})$ with the norm $\|\upsilon\| = \max_{t\in(0,\mathfrak{T})_{\mathbb{T}}} |\upsilon(t)|$. For $\varepsilon \in \left(0,\frac{\mathfrak{T}}{2}\right)_{\mathbb{T}}$, we define the cone $\mathtt{K}_\varepsilon \subset \mathtt{B}$ as

$$\mathtt{K}_\varepsilon = \left\{\upsilon \in \mathtt{B} : \upsilon(t) \text{ is nonnegative and } \min_{t\in[\varepsilon,\,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \upsilon(t) \geq \mathcal{G}(\varepsilon)\|\upsilon(t)\|\right\}.$$

For any $\upsilon_1 \in \mathtt{K}_\varepsilon$, define an operator $\Omega : \mathtt{K}_\varepsilon \to \mathtt{B}$ by

$$\begin{aligned}(\Omega\upsilon_1)(t) = &\int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ &\times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Lemma 2.7. *Assume that (H_1)–(H_2) hold. Then for each $\varepsilon \in \left(0,\frac{\mathfrak{T}}{2}\right)_{\mathbb{T}}$, $\Omega(\mathtt{K}_\varepsilon) \subset \mathtt{K}_\varepsilon$ and $\Omega : \mathtt{K}_\varepsilon \to \mathtt{K}_\varepsilon$ are completely continuous.*

Proof. From Lemma 2.6, $\aleph(t,\tau) \geq 0$ for all $t,\tau \in (0,\mathfrak{T})_{\mathbb{T}}$. So, $(\Omega\upsilon_1)(t) \geq 0$. Also, for $\upsilon_1 \in \mathtt{K}_\varepsilon$, we have

$$\begin{aligned}\|\Omega\upsilon_1\| = &\max_{t\in(0,\mathfrak{T})_{\mathbb{T}}} \int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ &\times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1\\ \leq &\int_0^1 \aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ &\times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Again from Lemma 2.6, we get

$$\begin{aligned}\min_{t\in[\varepsilon,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \big\{(\Omega\upsilon_1)(t)\big\} \geq \mathcal{G}(\varepsilon) &\int_0^1 \aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ &\times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

It follows from the above two inequalities that

$$\min_{t\in[\varepsilon,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \big\{(\Omega\upsilon_1)(t)\big\} \geq \mathcal{G}(\varepsilon)\|\Omega\upsilon_1\|.$$

So, $\Omega\upsilon_1 \in \mathtt{K}_\varepsilon$ and thus $\Omega(\mathtt{K}_\varepsilon) \subset \mathtt{K}_\varepsilon$. Next, by standard methods and the Arzela-Ascoli theorem, it can be proved easily that the operator Ω is completely continuous. The proof is complete. □

3 Infinitely many positive solutions

For the existence of infinitely many positive solutions for iterative system of boundary value problem (1.1)–(1.2), we apply following theorems.

Theorem 3.1 ([10]). *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and let $\mathtt{M}_1, \mathtt{M}_2$ be open sets with $0 \in \mathtt{M}_1, \overline{\mathtt{M}}_1 \subset \mathtt{M}_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\mathtt{M}}_2 \backslash \mathtt{M}_1) \to \mathcal{E}$ be a completely continuous operator such that* *(a)* $\|\mathcal{A}v\| \leq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_1$, *and* $\|\mathcal{A}v\| \geq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_2$, *or*

(b) $\|\mathcal{A}v\| \geq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_1$, *and* $\|\mathcal{A}v\| \leq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_2$.

Then $\mathcal{A}$ *has a fixed point in* $\mathcal{E} \cap (\overline{\mathtt{M}}_2 \backslash \mathtt{M}_1)$.

Theorem 3.2 ([7,15])**.** *Let* $f \in L^p_\nabla(J)$ *with* $p > 1$, $g \in L^q_\Delta(J)$ *with* $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in L^1_\Delta(J)$ *and* $\|fg\|_{L^1_\Delta} \leq \|f\|_{L^p_\Delta}\|g\|_{L^q_\Delta}$, *where*

$$\|f\|_{L^p_\Delta} := \begin{cases} \left[\int_J |f|^p(s)\Delta s\right]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ \inf\left\{M \in \mathbb{R} \,/\, |f| \leq M \ \ \Delta - a.e. \ \ on \ J\right\}, & p = \infty, \end{cases}$$

and $J = [a, b)_\mathbb{T}$.

Theorem 3.3 (Hölder's inequality [3,4,15])**.** *Let* $f \in L^{p_i}_\Delta(J)$ *with* $p_i > 1$, *for* $i = 1, 2, \dots, n$ *and* $\sum_{i=1}^n \frac{1}{p_i} = 1$. *Then* $\prod_{i=1}^k \mathtt{g}_i \in L^1_\Delta(J)$ *and* $\left\|\prod_{i=1}^k \mathtt{g}_i\right\|_1 \leq \prod_{i=1}^k \|\mathtt{g}_i\|_{p_i}$. *Further, if* $f \in L^1_\Delta(J)$ *and* $g \in L^\infty_\Delta(J)$, *then* $fg \in L^1_\Delta(J)$ *and* $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$.

We need the following condition in the sequel:

(H_3) There exists $\delta_i > 0$ such that $\lambda_i(t) > \delta_i$ $(i = 1, 2, \dots, n)$ for $t \in [0, \mathfrak{T}]_\mathbb{T}$.

Consider the following three possible cases for $\lambda_i \in L^{p_i}_\Delta(0, \mathfrak{T})_\mathbb{T}$:

$$\sum_{i=1}^n \frac{1}{p_i} < 1, \quad \sum_{i=1}^n \frac{1}{p_i} = 1, \quad \sum_{i=1}^n \frac{1}{p_i} > 1.$$

Firstly, we seek infinitely many positive solutions for the case $\displaystyle\sum_{i=1}^n \frac{1}{p_i} < 1$.

Theorem 3.4. *Suppose* (H_1)–(H_3) *hold, let* $\{\varepsilon_r\}_{r=1}^\infty$ *be such that* $0 < \varepsilon_1 < \mathfrak{T}/2$, $\varepsilon \downarrow t^*$ *and* $0 < t^* < t_n$. *Let* $\{\Gamma_r\}_{r=1}^\infty$ *and* $\{\Lambda_r\}_{r=1}^\infty$ *be such that*

$$\Gamma_{r+1} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r < \theta\Lambda_r < \Gamma_r, \ r \in \mathbb{N},$$

where

$$\theta = \max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^k \delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau, \tau)\Delta\tau\right]^{-1}, \ 1\right\}.$$

Assume that $\mathtt{g}_\ell$ *satisfies*

(C_1) $\mathtt{g}_\ell(\upsilon) \leq \mathfrak{N}_1\Gamma_r \ \forall \ t \in (0, \mathfrak{T})_\mathbb{T}$, $0 \leq \upsilon \leq \Gamma_r$, *where*

$$\mathfrak{N}_1 < \left[\|\aleph\|_{L^q_\Delta} \prod_{i=1}^k \|\lambda_i\|_{L^{p_i}_\Delta}\right]^{-1},$$

(C_2) $\mathtt{g}_\ell(\upsilon) \geq \theta\Lambda_r \ \forall \ t \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_\mathbb{T}$, $\mathcal{G}(\varepsilon_r)\Lambda_r \leq \upsilon \leq \Lambda_r$.

Then the iterative boundary value problem (1.1)–(1.2) *has infinitely many solutions* $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \dots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ *such that* $\upsilon_\ell^{[r]}(t) \geq 0$ *on* $(0, \mathfrak{T})_{\mathbb{T}}$, $\ell = 1, 2, \dots, m$ *and* $r \in \mathbb{N}$.

Proof. Let

$$\mathtt{M}_{1,r} = \{\upsilon \in \mathtt{B} : \|\upsilon\| < \Gamma_r\}, \quad \mathtt{M}_{2,r} = \{\upsilon \in \mathtt{B} : \|\upsilon\| < \Lambda_r\},$$

be open subsets of $\mathtt{B}$. Let $\{\varepsilon_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \varepsilon_r < t_r < \frac{\mathfrak{T}}{2},$$

for all $r \in \mathbb{N}$. For each $r \in \mathbb{N}$, we define the cone $\mathtt{K}_{\varepsilon_r}$ by

$$\mathtt{K}_{\varepsilon_r} = \Big\{\upsilon \in \mathtt{B} : \upsilon(t) \geq 0, \min_{t \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_{\mathbb{T}}} \upsilon(t) \geq \mathcal{G}(\varepsilon_r)\|\upsilon(t)\|\Big\}.$$

Let $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial \mathtt{M}_{1,r}$. Then, $\upsilon_1(\tau) \leq \Gamma_r = \|\upsilon_1\|$ for all $\tau \in (0, \mathfrak{T})_{\mathbb{T}}$. By (C_1) and for $\tau_{m-1} \in (0, \mathfrak{T})_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\leq \int_0^{\mathfrak{T}} \aleph(\tau_m, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m \\ &\leq \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_m, \tau_m) \prod_{i=1}^{k} \lambda_i(\tau_m)\Delta\tau_m.\end{aligned}$$

There exists a $q > 1$ such that $\dfrac{1}{q} + \displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$. So,

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\leq \mathfrak{N}_1\Gamma_r \big\|\aleph\big\|_{L_\Delta^q} \left\|\prod_{i=1}^{k} \lambda_i\right\|_{L_\Delta^{p_i}} \\ &\leq \mathfrak{N}_1\Gamma_r \|\aleph\|_{L_\Delta^q} \prod_{i=1}^{k} \|\lambda_i\|_{L_\Delta^{p_i}} \leq \Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{m-2} \in (0, \mathfrak{T})_{\mathbb{T}}$), that

$$\begin{aligned}&\int_0^{\mathfrak{T}} \aleph(\tau_{m-2}, \tau_{m-1})\lambda(\tau_{m-1})\mathrm{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\Delta\tau_{m-1} \\ &\qquad \leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-2}, \tau_{m-1})\lambda(\tau_{m-1})\mathrm{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ &\qquad \leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_{m-1})\lambda(\tau_{m-1})\mathrm{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ &\qquad \leq \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_{m-1}) \prod_{i=1}^{k} \lambda_i(\tau_{m-1})\Delta\tau_{m-1} \\ &\qquad \leq \mathfrak{N}_1\Gamma_r \|\aleph\|_{L_\Delta^q} \prod_{i=1}^{k} \|\lambda_i\|_{L_\Delta^{p_i}} \leq \Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)\mathsf{g}_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathsf{g}_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots\\ &\quad\times \mathsf{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1\\ &\leq \Gamma_r.\end{aligned}$$

Since $\Gamma_r = \|\upsilon_1\|$ for $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{1,r}$, we get

$$\|\Omega\upsilon_1\| \leq \|\upsilon_1\|. \tag{3.1}$$

Let $t \in [\varepsilon_r, \mathfrak{T}-\varepsilon_r]_{\mathbb{T}}$. Then,

$$\Lambda_r = \|\upsilon_1\| \geq \upsilon_1(t) \geq \min_{t\in[\varepsilon_r,\mathfrak{T}-\varepsilon_r]_{\mathbb{T}}} \upsilon_1(t) \geq \mathcal{G}(\varepsilon_r)\,\|\upsilon_1\| \geq \mathcal{G}(\varepsilon_r)\Lambda_r.$$

By (C_2) and for $\tau_{m-1} \in [\varepsilon_r, \mathfrak{T}-\varepsilon_r]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\geq \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\\ &\geq \mathcal{G}(\varepsilon_r)\theta\Lambda_r \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_m,\tau_m)\lambda(\tau_m)\Delta\tau_m\\ &\geq \mathcal{G}(\varepsilon_r)\theta\Lambda_r \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_m,\tau_m)\prod_{i=1}^{k}\lambda_i(\tau_m)\Delta\tau_m\\ &\geq \mathcal{G}(\varepsilon_1)\theta\Lambda_r \prod_{i=1}^{k}\delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau_m,\tau_m)\Delta\tau_m\\ &\geq \Lambda_r.\end{aligned}$$

Continuing with the bootstrapping argument, we get

$$\begin{aligned}(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)\mathsf{g}_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathsf{g}_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots\\ &\quad\times \mathsf{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1\\ &\geq \Lambda_r.\end{aligned}$$

Thus, if $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{K}_{2,r}$, then

$$\|\Omega\upsilon_1\| \geq \|\upsilon_1\|. \tag{3.2}$$

It is evident that $0 \in \mathtt{M}_{2,k} \subset \overline{\mathtt{M}}_{2,k} \subset \mathtt{M}_{1,k}$. From (3.1)–(3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\upsilon_1^{[r]} \in \mathtt{K}_{\varepsilon_r} \cap \big(\overline{\mathtt{M}}_{1,r}\backslash\mathtt{M}_{2,r}\big)$ such that $\upsilon_1^{[r]}(t) \geq 0$ on $(0,\mathfrak{T})_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\upsilon_{m+1} = \upsilon_1$, we obtain infinitely many positive solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \ldots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\upsilon_\ell(t) = \int_0^{\mathfrak{T}} \aleph(t,\tau)\lambda(\tau)\mathsf{g}_\ell(\upsilon_{\ell+1}(\tau))\Delta\tau,\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ \ell = m, m-1, \ldots, 1.$$

The proof is completed. $\square$

For $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i}=1$, we have the following theorem.

Theorem 3.5. *Suppose (H_1)–(H_3) hold, let $\{\varepsilon_r\}_{r=1}^{\infty}$ be such that $0<\varepsilon_1<\mathfrak{T}/2$, $\varepsilon\downarrow t^*$ and $0<t^*<t_n$. Let $\{\Gamma_r\}_{r=1}^{\infty}$ and $\{\Lambda_r\}_{r=1}^{\infty}$ be such that*

$$\Gamma_{r+1}<\mathcal{G}(\varepsilon_r)\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r,\ r\in\mathbb{N},$$

where

$$\theta=\max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1}\aleph(\tau,\tau)\Delta\tau\right]^{-1},\ 1\right\}.$$

Assume that $\mathtt{g}_\ell$ satisfies (C_2) and

(C_3) $\mathtt{g}_j(\upsilon)\le\mathfrak{N}_2\Gamma_r\ \forall\ t\in(0,\mathfrak{T})_{\mathbb{T}}$, $0\le\upsilon\le\Gamma_r$, *where*

$$\mathfrak{N}_2<\min\left\{\left[\|\aleph\|_{L^{\infty}_{\Delta}}\prod_{i=1}^{k}\|\lambda_i\|_{L^{p_i}_{\Delta}}\right]^{-1},\theta\right\}.$$

Then the iterative boundary value problem (1.1)–(1.2) *has infinitely many solutions $\{(\upsilon_1^{[r]},\upsilon_2^{[r]},\dots,\upsilon_m^{[r]})\}_{r=1}^{\infty}$ such that $\upsilon_\ell^{[r]}(t)\ge 0$ on $(0,\mathfrak{T})_{\mathbb{T}}$, $\ell=1,2,\dots,m$ and $r\in\mathbb{N}$.*

Proof. For a fixed r, let $\mathtt{M}_{1,r}$ be as in the proof of Theorem 3.4 and let $\upsilon_1\in\mathtt{K}_{\varepsilon_r}\cap\partial\mathtt{M}_{2,r}$. Again

$$\upsilon_1(\tau)\le\Gamma_r=\|\upsilon_1\|,$$

for all $\tau\in(0,\mathfrak{T})_{\mathbb{T}}$. By (C_3) and for $\tau_{\ell-1}\in(0,\mathfrak{T})_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}}\aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m&\le\int_0^{\mathfrak{T}}\aleph(\tau_m,\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\\&\le\mathfrak{N}_1\Gamma_r\int_0^{\mathfrak{T}}\aleph(\tau_m,\tau_m)\prod_{i=1}^{k}\lambda_i(\tau_m)\Delta\tau_m\\&\le\mathfrak{N}_1\Gamma_r\big\|\aleph\big\|_{L^{\infty}_{\Delta}}\left\|\prod_{i=1}^{k}\lambda_i\right\|_{L^{p_i}_{\Delta}}\\&\le\mathfrak{N}_1\Gamma_r\|\aleph\|_{L^{\infty}_{\Delta}}\prod_{i=1}^{k}\|\lambda_i\|_{L^{p_i}_{\Delta}}\le\Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{m-2} \in [0,1]_{\mathbb{T}}$), that

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-2},\tau_{m-1})\lambda(\tau_{m-1})\mathtt{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\Delta\tau_{m-1} \\ \leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-2},\tau_{m-1})\lambda(\tau_{m-1})\mathtt{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ \leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_{m-1})\lambda(\tau_{m-1})\mathtt{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ \leq \mathfrak{N}_1\Gamma_r\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_{m-1})\prod_{i=1}^{k}\lambda_i(\tau_{m-1})\Delta\tau_{m-1} \\ \leq \mathfrak{N}_1\Gamma_r\|\aleph\|_{L^\infty_\Delta}\prod_{i=1}^{k}\|\lambda_i\|_{L^{p_i}_\Delta} \leq \Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\upsilon_1)(t) = \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots \\ \times \mathtt{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1 \\ \leq \Gamma_r.\end{aligned}$$

Since $\Gamma_r = \|\upsilon_1\|$ for $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{1,r}$, we get

$$\|\Omega\upsilon_1\| \leq \|\upsilon_1\|. \tag{3.3}$$

Now define $\mathtt{M}_{2,r} = \{\upsilon_1 \in \mathtt{B} : \|\upsilon_1\| < \Lambda_r\}$. Let $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{2,r}$ and let $\tau \in [\varepsilon_r, \mathfrak{T}-\varepsilon_r]_{\mathbb{T}}$. Then, the argument leading to (3.2) can be done to the present case. Hence, the theorem. □

Lastly, the case $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i} > 1$.

Theorem 3.6. *Suppose (H_1)–(H_3) hold, let $\{\varepsilon_r\}_{r=1}^\infty$ be such that $0 < \varepsilon_1 < \mathfrak{T}/2$, $\varepsilon \downarrow t^*$ and $0 < t^* < t_n$. Let $\{\Gamma_r\}_{r=1}^\infty$ and $\{\Lambda_r\}_{r=1}^\infty$ be such that*

$$\Gamma_{r+1} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r < \theta\Lambda_r < \Gamma_r,\ r \in \mathbb{N},$$

where

$$\theta = \max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1}\aleph(\tau,\tau)\Delta\tau\right]^{-1},\ 1\right\}.$$

Assume that $\mathtt{g}_\ell$ satisfies (C_2) and

(C_4) $\mathtt{g}_j(\upsilon) \leq \mathfrak{N}_3\Gamma_r\ \forall\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ 0 \leq \upsilon \leq \Gamma_r,$ *where* $\mathfrak{N}_3 < \min\left\{\left[\|\aleph\|_{L^\infty_\Delta}\prod_{i=1}^{k}\|\lambda_i\|_{L^1_\Delta}\right]^{-1}, \theta\right\}$.

Then the iterative boundary value problem (1.1)–(1.2) has infinitely many solutions $\{(\upsilon_1^{[r]},\upsilon_2^{[r]},\ldots,\upsilon_m^{[r]})\}_{r=1}^\infty$ such that $\upsilon_\ell^{[r]}(t) \geq 0$ on $(0,\mathfrak{T})_{\mathbb{T}}$, $\ell = 1,2,\ldots,m$ and $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.4. So, we omit the details here. □

4 Example

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following boundary value problem on $\mathbb{T} = \mathbb{R}$.

$$\left.\begin{aligned} \upsilon_\ell''(t) + \lambda(t)\mathrm{g}_\ell(\upsilon_{\ell+1}(t)) = 0,\ \ell = 1, 2, \\ \upsilon_3(t) = \upsilon_1(t), \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \upsilon_\ell(0) = \upsilon_\ell'(0), \\ \upsilon_\ell(1) = -\upsilon_\ell'(1), \end{aligned}\right\} \tag{4.2}$$

where

$$\lambda(t) = \lambda_1(t)\lambda_2(t)$$

in which

$$\lambda_1(t) = \frac{1}{|t - \frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \lambda_2(t) = \frac{1}{|t - \frac{3}{4}|^{\frac{1}{2}}},$$

$$\mathrm{g}_1(\upsilon) = \mathrm{g}_2(\upsilon) = \begin{cases} \frac{1}{5} \times 10^{-4}, & \upsilon \in (10^{-4}, +\infty), \\ \frac{25\times 10^{-(4r+3)} - \frac{1}{5}\times 10^{-4r}}{10^{-(4r+3)} - 10^{-4r}}(\upsilon - 10^{-4r}) + & \\ \frac{1}{5} \times 10^{-8r}, & \upsilon \in \left[10^{-(4r+3)}, 10^{-4r}\right], \\ 25 \times 10^{-(4r+3)}, & \upsilon \in \left(\frac{1}{5} \times 10^{-(4r+3)}, 10^{-(4r+3)}\right), \\ \frac{25\times 10^{-(4r+3)} - \frac{1}{5}\times 10^{-8r}}{\frac{1}{5}\times 10^{-(4r+3)} - 10^{-(4r+4)}}(\upsilon - 10^{-(4r+4)}) + & \\ \frac{1}{5} \times 10^{-8r}, & \upsilon \in \left(10^{-(4r+4)}, \frac{1}{5} \times 10^{-(4r+3)}\right], \\ 0, & \upsilon = 0. \end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4}, \quad \varepsilon_r = \frac{1}{2}(t_r + t_{r+1}), \quad r = 1, 2, 3, \dots,$$

then

$$\varepsilon_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32},$$

and

$$t_{r+1} < \varepsilon_r < t_r, \quad \varepsilon_r > \frac{1}{5}.$$

Therefore,

$$\mathcal{G}(\varepsilon_r) = \frac{\varepsilon_r + 1}{\mathfrak{T} + 1} = \frac{\varepsilon_r + 1}{2} > \frac{1}{5}, \quad r = 1, 2, 3, \dots$$

It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \quad t_r - t_{r+1} = \frac{1}{4(r+2)^4}, \quad r = 1, 2, 3, \dots$$

Since $\displaystyle\sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{\pi^4}{90}$ and $\displaystyle\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.4637941914,$$

$$\lambda_1, \lambda_2 \in L^p[0,1] \quad \text{for all} \quad 0 < p < 2, \quad \text{and} \quad \delta_1 = \delta_2 = (4/3)^{1/4},$$

$$\mathcal{G}(\varepsilon_1) = 0.7336033951.$$

$$\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau,\tau)\Delta\tau = \int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}} \frac{(2-\tau)(1+\tau)}{3} d\tau = 0.04918197801.$$

Thus, we get

$$\begin{aligned}
\theta &= \max\left\{ \left[\mathcal{G}(\varepsilon_1) \prod_{i=1}^{k} \delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau,\tau)\nabla\tau \right]^{-1}, \ 1 \right\} \\
&= \max\left\{ \frac{1}{0.04166167167}, 1 \right\} \\
&= 24.00287746.
\end{aligned}$$

Next, let $0 < \mathfrak{a} < 1$ be fixed. Then $\lambda_1, \lambda_2 \in L^{1+\mathfrak{a}}[0,1]$ and $\frac{2}{1+\mathfrak{a}} > 1$ for $0 < \mathfrak{a} < 1$. It follows that

$$\prod_{i=1}^{k} \|\lambda_i\|_{L_\Delta^{p_i}} \approx \pi - \ln(7 - 4\sqrt{3}),$$

and also $\|\aleph\|_\infty = \frac{2}{3}$. So, for $0 < \mathfrak{a} < 1$, we have

$$\mathfrak{N}_1 < \left[\|\aleph\|_\infty \prod_{i=1}^{k} \|\lambda_i\|_{L_\Delta^{p_i}} \right]^{-1} \approx 0.2597173925.$$

Taking $\mathfrak{N}_1 = \frac{1}{4}$. In addition if we take

$$\Gamma_r = 10^{-4r}, \quad \Lambda_r = 10^{-(4r+3)},$$

then

$$\Gamma_{r+1} = 10^{-(4r+4)} < \frac{1}{5} \times 10^{-(4r+3)} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r = 10^{-(4r+3)} < \Gamma_r = 10^{-4r},$$

$\theta\Lambda_r = 24.00287746 \times 10^{-(4r+3)} < \frac{1}{4} \times 10^{-4r} = \mathfrak{N}_1\Gamma_r$, $r = 1, 2, 3, \ldots$, and $\mathsf{g}_1, \mathsf{g}_2$ satisfy the following growth conditions:

$$\begin{aligned}
\mathsf{g}_1(\upsilon) = \mathsf{g}_2(\upsilon) &\leq \mathfrak{N}_1\Gamma_r = \frac{1}{4} \times 10^{-4r}, \quad \upsilon \in \left[0, 10^{-4r}\right], \\
\mathsf{g}_1(\upsilon) = \mathsf{g}_2(\upsilon) &\geq \theta\Lambda_r = 24.00287746 \times 10^{-(4r+3)}, \quad \upsilon \in \left[\frac{1}{5} \times 10^{-(4r+3)}, 10^{-(4r+3)}\right].
\end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (1.1) has infinitely many solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]})\}_{r=1}^{\infty}$ such that $\upsilon_\ell^{[r]}(t) \geq 0$ on $[0,1]$, $\ell = 1, 2$ and $r \in \mathbb{N}$.

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Smooth quotients of abelian surfaces by finite groups that fix the origin

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ABSTRACT

Let A be an abelian surface and let G be a finite group of automorphisms of A fixing the origin. Assume that the analytic representation of G is irreducible. We give a classification of the pairs (A, G) such that the quotient A/G is smooth. In particular, we prove that $A = E^2$ with E an elliptic curve and that $A/G \simeq \mathbb{P}^2$ in all cases. Moreover, for fixed E, there are only finitely many pairs (E^2, G) up to isomorphism. This fills a small gap in the literature and completes the classification of smooth quotients of abelian varieties by finite groups fixing the origin started by the first two authors.

RESUMEN

Sea A una superficie abeliana y sea G un grupo finito de automorfismos de A fijando el origen. Se asume que la representación analítica de G es irreducible. Damos una clasificación de los pares (A, G) tales que el cociente A/G es suave. En particular, probamos que $A = E^2$ con E una curva elíptica y que $A/G \simeq \mathbb{P}^2$ en todos los casos. Más aún, para E fija, hay solo una cantidad finita de pares (E^2, G), salvo isomorfismo. Esto llena una pequeña brecha en la literatura y completa la clasificación de cocientes suaves de variedades abelianas por grupos finitos fijando el origen comenzado por los dos primeros autores.

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1 Introduction

The purpose of this paper is to give a complete classification of all smooth quotients of abelian surfaces by finite groups that fix the origin, and is to be seen as the completion of the classification given in [2] of smooth quotients of abelian varieties that fix the origin. This kind of quotients of abelian surfaces has already been studied by Tokunaga and Yoshida in [6], where infinite 2-dimensional complex reflection groups, which are extensions of a finite complex reflection group G by a G-invariant lattice, are classified. However, these do not cover all possible G-invariant lattices and hence not all possible group actions on abelian surfaces. Moreover, there seem to be some complex reflection groups that the authors missed, as can be seen by looking at Popov's classification of the same groups in [3].

The techniques used in this paper are similar, but not exactly the same, to the methods used in [2]. Indeed, the ideas used in this last paper have been modified in order to apply them to the two-dimensional case. Moreover our approach is far different from that used in [6].

Our main theorem states the following:

Theorem 1.1. *Let A be an abelian surface and let G be a (non-trivial) finite group of automorphisms of A that fix the origin. Then the following conditions are equivalent:*

(1) *A/G is smooth and the analytic representation of G is irreducible.*

(2) *$A/G \simeq \mathbb{P}^2$.*

(3) *There exists an elliptic curve E such that $A \simeq E^2$ and (A, G) satisfies exactly one of the following:*

  (a) $G \simeq C^2 \rtimes S_2$ where C is a non-trivial (cyclic) subgroup of automorphisms of E that fix the origin; here the action of C^2 is coordinate-wise and S_2 permutes the coordinates.

  (b) $G \simeq S_3$ and acts on

$$A \simeq \{(x_1, x_2, x_3) \in E^3 : x_1 + x_2 + x_3 = 0\},$$

  by permutations.

  (c) $E = \mathbb{C}/\mathbb{Z}[i]$ and G is the order 16 subgroup of $\mathrm{GL}_2(\mathbb{Z}[i])$ generated by:

$$\left\{ \begin{pmatrix} -1 & 1+i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix} \right\},$$

  acting on $A \simeq E^2$ in the obvious way.

The first two cases found in item (3) of the above theorem were studied in detail in [1] (in arbitrary dimension), where it was proven that both examples give the projective plane as a quotient.

Throughout the paper we will refer to these two examples as Example (a) and Example (b), respectively. The equivalent assertion for Example (c) is Proposition 4.1 in this paper. Note that, aside from Examples (a) and (b) which belong to infinite families, Example (c) is the only new case of an action of G on an abelian variety satisfying condition (1) from Theorem 1.1, cf. [2, Thm. 1.1].

Remark 1.2. *If A/G is smooth and the analytic representation of G is* reducible*, then the results in [2] imply that A is isogenous to a product of two elliptic curves. The quotient is then either $\mathbb{P}^1 \times \mathbb{P}^1$ (in which case $A = E_1 \times E_2$) or a bielliptic surface.*

In [7], Yoshihara introduces the notion of a Galois embedding of a smooth projective variety. If X is a smooth projective variety of dimension n and D is a very ample divisor that induces an embedding $X \hookrightarrow \mathbb{P}^N$, then the embedding is said to be *Galois* if there exists an $(N-n-1)$-dimensional linear subspace W of $\mathbb{P}^N$ such that $X \cap W = \varnothing$ and the restriction of the linear projection $\pi_W : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ to X is Galois. Yoshihara specifically studies when abelian surfaces have a Galois embedding. He gives a classification of abelian surfaces having a Galois embedding, along with their Galois groups, and proves that after taking the quotient of the original abelian variety by the translations of the Galois group, the abelian variety must be isomorphic to the self-product of an elliptic curve. Unfortunately, his results were incomplete since they depended on a classification of smooth quotients like the one given in this paper, which Yoshihara attributed to Tokunaga and Yoshida in [6]. But as stated before, Tokunaga and Yoshida's results do not imply such a classification. Nevertheless, we can now safely say, thanks to Theorem 1.1, that Yoshihara's results remain correct.

The structure of this paper is as follows: in Section 2 we fix notations and give some preliminary results that will be needed in the proofs of Theorem 1.1. The implication (2) $\Rightarrow$ (1) is obvious and (3) $\Rightarrow$ (2) was already treated in [1] in the case of Examples (a) and (b). Thus, we are mainly concerned with (1) $\Rightarrow$ (3), which we treat in Section 3. Finally, in Section 4 we treat (3) $\Rightarrow$ (2) for Example (c), which is a construction of a different nature that only exists in the 2-dimensional case.

2 Preliminaries on group actions on abelian varieties

We recall here some elementary results that were proved in [2]. Let A be an abelian surface and let G be a group of automorphisms of A that fix the origin, such that the quotient variety A/G is smooth. By the Chevalley-Shephard-Todd Theorem, the stabilizer in G of each point in A must be generated by pseudoreflections; that is, elements that fix pointwise a divisor (i.e. a curve) containing the point. In particular, $G = \mathrm{Stab}_G(0)$ is generated by pseudoreflections and G acts on the tangent space at the origin $T_0(A)$ (this is the analytic representation). In this context, a pseudoreflection is an element that fixes a line pointwise. We will often abuse notation and display

G as either acting on A or $T_0(A)$; it will be clear from the context which action we are considering.

In what follows, let $\mathcal{L}$ be a fixed G-invariant polarization on A (take the pullback of an ample class on A/G, for example). For σ a pseudoreflection in G of order r, define

$$D_\sigma := \mathrm{im}(1+\sigma+\cdots+\sigma^{r-1}), \quad E_\sigma := \mathrm{im}(1-\sigma).$$

These are both abelian subvarieties of A. The following result corresponds to [2, Lem. 2.1].

Lemma 2.1. *We have the following:*

1. *D_σ is the connected component of $\ker(1-\sigma)$ that contains 0 and E_σ is the complementary abelian subvariety of D_σ with respect to $\mathcal{L}$. In particular, D_σ and E_σ are elliptic curves.*

2. *$F_\sigma := D_\sigma \cap E_\sigma$ consists of 2-torsion points for $r = 2, 4$, of 3-torsion points for $r = 3$ and $D_\sigma \cap E_\sigma = 0$ for $r = 6$.*

We will consider now a new abelian surface B equipped with a G-equivariant isogeny to A, which we will call G-isogeny from now on. Let Λ_A denote the lattice in $\mathbb{C}^2$ such that $A = \mathbb{C}^2/\Lambda_A$. Let $\Lambda_B \subset \Lambda_A$ be a G-invariant sublattice, and let $B := \mathbb{C}^2/\Lambda_B$ be the induced abelian surface, along with the G-isogeny

$$\pi : B \to A,$$

whose analytic representation is the identity. Note that this implies that $\sigma \in G$ is a pseudoreflection of B if and only if it is a pseudoreflection of A. We may then consider the subvarieties $E_\sigma, D_\sigma, F_\sigma \subset A$ defined as above, which we will denote by $E_{\sigma,A}, D_{\sigma,A}$ and $F_{\sigma,A}$. We do similarly for B. Note that, by definition, π sends $E_{\sigma,B}$ to $E_{\sigma,A}$ and $D_{\sigma,B}$ to $D_{\sigma,A}$, hence $F_{\sigma,B}$ to $F_{\sigma,A}$. The following result was proved in [2, Prop. 2.4].

Proposition 2.2. *Let $\sigma \in G$ be a pseudoreflection and let L be the line defining both $E_{\sigma,A}$ and $E_{\sigma,B}$. Assume that the map $F_{\sigma,B} \to F_{\sigma,A}$ is surjective and that $\Lambda_A \cap L = \Lambda_B \cap L$. Then $\ker(\pi)$ is contained in $D_{\tau\sigma\tau^{-1},B}$ for* every *$\tau \in G$.* $\square$

Define $\Delta := \ker(\pi)$. Since π is G-equivariant, G acts on Δ and hence we may consider the group $\Delta \rtimes G$. This group acts on B in the obvious way: Δ acts by translations and G by automorphisms that fix the origin. In particular, we see that the quotient $B/(\Delta \rtimes G)$ is isomorphic to A/G. We conclude this section by recalling a result on pseudoreflections in $\Delta \rtimes G$ (cf. [2, Lem. 2.5]).

Lemma 2.3. *Let $\sigma \in \Delta \rtimes G$ be a pseudoreflection. Then $\sigma = (t, \tau)$ with $\tau \in G$ a pseudoreflection and $t \in \Delta \cap E_{\tau,B}$.*

3 Proof of $(1) \Rightarrow (3)$

Assume (1), that is, we have an abelian surface A with an action of a finite group G that fixes the origin and such that A/G is smooth and the analytic representation of G is irreducible. Under these conditions, we see that G is an irreducible finite complex reflection group in the sense of Shephard-Todd [4]. These groups were completely classified by Shephard and Todd in [4]. In the particular case of dimension 2, we get that G is either one of 19 sporadic cases or it is isomorphic to a semidirect product $G(m,p) := H(m,p) \rtimes S_2$, where $p|m$, $m \geq 2$, and

$$H = H(m,p) = \{(\zeta_m^{a_1}, \zeta_m^{a_2}) \mid a_1 + a_2 \equiv 0 \pmod{p}\} \subset \mu_m^2,$$

with ζ_m denoting a primitive m-th root of unity. The action of S_2 on H is the obvious one. The case $G = G(2,2)$ is excluded since G is then a Klein group and thus the representation is not irreducible. The action of G on $\mathbb{C}^2$ is given as follows: H acts on $\mathbb{C}^2$ coordinate-wise while S_2 permutes the coordinates.

Emulating the work done in [2], we wish to describe which of these actions actually appear on abelian surfaces and give smooth quotients. The sporadic cases were already treated in [2] and were proven not to give a smooth quotient (cf. [2, §3.3]), so we may and will assume henceforth that $G = G(m,p)$ as above. This fixes a G-equivariant isomorphism of $T_0(A)$ with $\mathbb{C}^2$. We denote by e_1 and e_2 the canonical basis of $T_0(A)$ thus obtained.

Lemma 3.1. *Assume that G acts on A as above. Then $m \in \{2,3,4,6\}$.*

Proof. We have that (ζ_m, ζ_m^{-1}) acts on A, and so the characteristic polynomial of $(\zeta_m, \zeta_m^{-1}) \oplus \overline{(\zeta_m, \zeta_m^{-1})}$ must have integer coefficients, and so must be the k-th power of the m-th cyclotomic polynomial Φ_m, where $k = 2$ if $m \geq 2$ and $k = 4$ if $m = 2$. Looking at the degrees, we get that $4 = k\varphi(m)$, where φ is Euler's totient function. Therefore, if $m \neq 2$ then $\varphi(m) = 2$ and so $m \in \{3,4,6\}$. ☐

Having proved this result, we see that there is a finite list of cases to be analyzed, that is:

$$(m,p) \in \{(2,1),(2,2),(3,1),(4,1),(6,1),(3,3),(4,2),(4,4),(6,2),(6,3),(6,6)\}.$$

Recall that we have already eliminated the case $(2,2)$ since the analytic representation of $G(2,2)$ is not irreducible. Moreover, it is well-known that there is an exceptional isomorphism of complex reflection groups between $G(4,4)$ and $G(2,1)$. We will prove then the following:

- If $G = G(m,1)$ and A/G is smooth, then the pair (A,G) corresponds to Example (a) (Sections 3.1, 3.3, 3.4, 3.5);
- If $G = G(3,3)$ and A/G is smooth, then the pair (A,G) corresponds to Example (b) (Section 3.8);

- If $G = G(4,2)$ and A/G is smooth, then the pair (A, G) corresponds to Example (c) (Section 3.2);
- If $G = G(6,p)$ with $p \geq 2$, then A/G cannot be smooth (Sections 3.6, 3.7, 3.9).

In order to do this, we will construct a G-isogeny $B \to A$ such that the action of G on B is "well-known". Let us concentrate first on the cases where $m \neq p$. Then we obtain B as follows:

Let E_i be the image of $\mathbb{C}e_i$ in A via the exponential map. We claim that it corresponds to an elliptic curve. Indeed, consider the non-trivial element $\tau = (\zeta_m^p, 1) \in H$. Then a direct computation shows that $\mathrm{im}(1-\tau) = \mathbb{C}e_1$. This tells us that $E_1 = (1-\tau)(A)$ and hence it corresponds to an elliptic curve. The same proof works for E_2.

Now, let Λ_A be a lattice for A in $\mathbb{C}^2$. Then $\mathbb{C}e_i \cap \Lambda_A$ corresponds to the lattice of E_i in $\mathbb{C} = \mathbb{C}e_i$. We can thus define the G-stable sublattice of Λ_A

$$\Lambda_B := (\mathbb{C}e_1 \cap \Lambda_A) \oplus (\mathbb{C}e_2 \cap \Lambda_A).$$

As in Section 2, this defines a G-isogeny $\pi : B \to A$. Moreover, we see that $B \simeq E_1 \times E_2 \simeq E^2$ and that $\pi|_{E_i}$ is an injection. Let Δ be the kernel of π. We will study the different possible quotients A/G by studying the possible quotients $B/(\Delta \rtimes G)$ and thus by studying the possible Δ's. Our first result is the following:

Lemma 3.2. *Assume that $m \neq p$. Then the coordinates of every element in Δ are invariant by ζ_m^p, so in particular these elements are*

- *2-torsion if $(m,p) \in \{(2,1),(4,1),(4,2),(6,3)\}$;*
- *3-torsion if $(m,p) \in \{(3,1),(6,2)\}$;*
- *trivial if $(m,p) = (6,1)$.*

Proof. Let $\bar{t} = (t_1, t_2) \in \Delta$. Then, since Δ is G-stable, we have that, for $\tau_1 = (\zeta_m^p, 1) \in H$,

$$(1-\tau_1)(\bar{t}) = ((1-\zeta_m^p)t_1, 0) \in \Delta.$$

But, by construction, there are no elements of the form $(x, 0)$ in Δ. We deduce then that t_1 is ζ_m^p-invariant. The same proof works for t_2. The assertion on the torsion of t_1 and t_2 follows immediately. $\square$

Let us study now pseudoreflections in $\Delta \rtimes G$. Define the elements

$$\rho := (\zeta_m, \zeta_m^{-1}) \in H \subset G; \quad \sigma := (1\,2) \in S_2 \subset G; \quad \tau := (\zeta_m^p, 1) \in H \subset G.$$

Then there are two types of pseudoreflections in G:

- conjugates of $\rho^a\sigma$ for $0 \leq a < \frac{m}{p}$;
- conjugates of powers of τ;

and the corresponding elliptic curves in B are respectively:

$$E_{\rho^a\sigma} = \{(x, -\zeta_m^a x) \mid x \in E\}; \quad E_\tau = \{(x, 0) \mid x \in E\}.$$

Recall that elements of the form $(x, 0)$ are not in Δ by construction of the isogeny $\pi : B \to A$. Using Lemmas 2.3 and 3.2, we obtain immediately the following result:

Lemma 3.3. *Every pseudoreflection in $\Delta \rtimes G$ that is not in G is a conjugate of $(\bar{t}, \rho^a\sigma)$, where $0 \leq a < \frac{m}{p}$, $\bar{t} = (t, -\zeta_m^a t) \in \Delta$ and t is ζ_m^p-invariant.* □

With these considerations, we can start a case by case study of the non-trivial Δ's. We recall that the main tool will be the Chevalley-Shephard-Todd Theorem, which states that $A/G = B/(\Delta \rtimes G)$ is smooth if and only if the stabilizer in $\Delta \rtimes G$ of each point in B is generated by pseudoreflections.

3.1 The case $G = G(2, 1)$

By Lemma 3.2, we know that Δ is 2-torsion. Since we also know that there are no elements of the form $(t, 0)$ for $t \in E$, we get the following possible options for Δ:

(1) $\Delta = \{0\}$;

(2) $\Delta = \langle (t, t) \rangle$ with $t \in E[2]$;

(3) $\Delta = \{(t, t) \mid t \in E[2]\}$;

(4) $\Delta = \{(0, 0), (t_1, t_2), (t_2, t_1), (t_1 + t_2, t_1 + t_2)\}$ with $t_1, t_2 \in E[2]$, $t_1 \neq t_2$.

Case (1) clearly corresponds to Example (a) (which gives a smooth quotient, cf. [2, Prop. 3.4]). Case (2) cannot give a smooth quotient and this follows directly from [2, Prop. 3.7].[1]

In case (3), we claim that the pair (A, G) is isomorphic to the pair (B, G). This will reduce us to the case with trivial Δ, which was already dealt with. To prove the claim, consider the canonical basis of $T_0(A) = T_0(B) = \mathbb{C}^2$. Then the analytic representation of G is given by the following values in its generators:

$$\rho_a((1, -1)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_a((1\,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[1] The proof of this proposition only uses two variables and thus it works in dimension 2 as well.

Now, with this basis and this Δ, we can view the G-isogeny $B \to A$ as the morphism $E^2 \to E^2$ given by the following matrix:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{*}$$

for which one can check that its kernel is precisely the elements in Δ. In order to prove that the pairs (A, G) and (B, G) are isomorphic, it suffices thus to prove that the image of this representation of G under conjugation by M is G once again. Direct computations give:

$$M\rho_a((1,-1))M^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho_a((1\,2)), \quad M\rho_a((1\,2))M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho_a((1,-1)).$$

And these clearly generate the same group G.

In case (4), consider the element $\bar{t} = (t'_1, t'_2)$ where $2t'_i = t_i$. Note that G cannot fix $\bar{t}$ as t'_1 and t'_2 lie in different orbits by the action of μ_2. Now, it is easy to see that there is no way the action of Δ can compensate the action of G except in the case when we add the element (t_1, t_2). A direct computation tells us then that the only element fixing $\bar{t}$ is $((t_1, t_2), (-1, -1)) \in \Delta \rtimes G$ and since this stabilizer is not generated by pseudoreflections by Lemma 3.3, we see that A/G is not smooth.

3.2 The case $G = G(4, 2)$

Since $G(4, 2)$ contains $G(2, 1)$, we may start from the precedent list of possible non-trivial Δ's. However, these must also be stable by the new element $(i, i) \in H(4, 2)$ (where $i = \zeta_4$). Note that such an element acts on each component E of B by multiplication by i, which implies in particular that $E = \mathbb{C}/\mathbb{Z}[i]$. Defining by t_0 the only non-trivial i-invariant element in E, we get the following possibilities:

(1) $\Delta = \{0\}$;

(2) $\Delta = \langle (t_0, t_0) \rangle$;

(3) $\Delta = \{(t, t) \mid t \in E[2]\}$;

(4) $\Delta = \{(0, 0), (t, t + t_0), (t + t_0, t), (t_0, t_0)\}$ with $t \in E[2]$, $t \neq t_0$.

Case (1) does not give a smooth quotient A/G, cf. [2, Prop. 3.4]. Case (2) corresponds to Example (c) (and it actually gives a smooth quotient A/G as we prove in section 4). Indeed, the G-isogeny $B \to A$ corresponds in this case to the morphism $E^2 \to E^2$ with $E = \mathbb{C}/\mathbb{Z}[i]$ given by the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & i - 1 \end{pmatrix},$$

and the generators given in Example (c) correspond to the conjugates by this matrix of the following respective matrices:

$$\left\{\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}.$$

But these are clearly the matrix expressions of the generators $(-1,1),(-i,i)\in H$ and $(1\,2)\in S_2$ of $G=H\rtimes S_2$.

Remark 3.4. *Since the first and third matrices above generate the subgroup $G(2,1)\subset G(4,2)$, we see that if we take F to be the subgroup of G spanned by the pseudoreflections*

$$\begin{pmatrix}-1 & i+1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-1 & 0\\ i-1 & 1\end{pmatrix},$$

then F is isomorphic to $G(2,1)$ and $A/F\simeq\mathbb{P}^2$. In particular, the pair (A,F) is isomorphic to Example (a) with C cyclic of order 2.

In cases (3) and (4), we claim that the pair (A,G) is isomorphic to the pair (B,G). This will reduce us to the case with trivial Δ, which was already dealt with. To prove the claim, we consider as for $G=G(2,1)$ the canonical basis of $T_0(A)=T_0(B)=\mathbb{C}^2$. Then the analytic representation of G is given by the following values in its generators:

$$\rho_a((i,-i))=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \quad \rho_a((-1,1))=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad \rho_a((1\,2))=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Now, with this basis and the Δ from case (2), we already know that $B\to A$ looks like $E^2\to E^2$ with matrix M from (*). It suffices to check then that the new generator $\rho_a((i,-i))$ falls into $\rho_a(G)$ after conjugation by M. And indeed we have that $M\rho_a((i,-i))M^{-1}=\rho_a((i,i))\rho_a((1\,2))$.

With the Δ from case (3), the corresponding matrix for $B\to A$ is:

$$N=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}.$$

And once again, direct computations give:

$$N\rho_a((i,-i))N^{-1}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\rho_a((-1,1))\rho_a((1\,2)),$$

$$N\rho_a((-1,1))N^{-1}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}=\rho_a((1\,2))\rho_a((i,-i)),$$

$$N\rho_a((1\,2))N^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\rho_a((1\,2)).$$

And these clearly generate the same group G.

3.3 The case $G = G(4, 1)$

Since $G(4, 1)$ contains $G(4, 2)$, we may start from the precedent list of possible non-trivial Δ's. Now, by Lemma 3.2, we know that the coordinates of the elements in Δ are i-invariant. We get then that there are only two options for Δ, that is the trivial case and $\Delta = \langle (t_0, t_0) \rangle$.

In the trivial case, we immediately see that (A, G) corresponds to Example (a). Assume then that Δ is non-trivial and consider the element $(s, t) \in B$ with $s \in E[2]$, s *not* i-invariant and $2t = t_0$. Since clearly these elements have different order, we see that the orbits of these elements by the action of $\langle t_0 \rangle \times \mu_4$ are different. Thus no action of an element in $\Delta \times H \subset \Delta \rtimes G$ can compensate the action of $(1\,2) \in G$ in order to fix (s, t). In other words, the stabilizer of $\bar{t}$ must be contained in $\Delta \times H$. It is easy to see then that it corresponds to $\langle ((t_0, t_0), (i, -1)) \rangle$. By Lemma 3.3, this stabilizer is not generated by pseudoreflections and hence A/G is not smooth in this case.

3.4 The case $G = G(3, 1)$

By Lemma 3.2, we know that the coordinates of the elements in Δ are ζ_3-invariant. Now, there are only two such non-trivial elements that we will denote by s_0 and $-s_0$. Since we also know that there are no elements of the form $(t, 0)$ for $t \in E$, we get the following possible options for a non-trivial Δ:

(1) $\Delta = \{0\}$;

(2) $\Delta = \langle (s_0, s_0) \rangle$;

(3) $\Delta = \langle (s_0, -s_0) \rangle$.

We immediately see that the trivial case gives us Example (a). In case (2), Lemma 3.3 tells us that the only pseudoreflections in $\Delta \rtimes G$ are those coming from G. In particular, in order to prove that A/G cannot be smooth, it suffices to exhibit an element in B such that its stabilizer in $\Delta \rtimes G$ has elements that are not in G. Let $\tau = (\zeta_3, \zeta_3) \in H \subset G$, then $1 - \tau$ is surjective. Then there exists an element $\bar{z} \in B$ such that $\bar{z} - \tau(\bar{z}) = (s_0, s_0)$. This implies that $((s_0, s_0), \tau) \in \Delta \rtimes G$ stabilizes z, proving thus that A/G is not smooth in this case.

In case (3), consider the element $\bar{s} = (s, -s) \in B$ with $s \in E[3]$ and s *not* ζ_3-invariant. Note that $\langle s_0 \rangle \times \mu_3$ acts on $E[3]$ and a direct computation tells us that the orbit of s is $\{s, s + s_0, s - s_0\}$. In particular, we see that s and $-s$ lie in different orbits for this action. The same argument used in the case of $G(4, 1)$ tells us then that the stabilizer of $\bar{s}$ must be contained in $\Delta \times H$. It is easy to see then that, up to changing $\bar{s}$ by $-\bar{s}$, it corresponds to $\langle ((s_0, -s_0), (\zeta_3, \zeta_3)) \rangle$. Since this stabilizer is not generated by pseudoreflections by Lemma 3.3, we see that A/G is not smooth in this case as well.

3.5 The case $G = G(6,1)$

By Lemma 3.2, we know that the only possibility is a trivial Δ. This clearly corresponds to Example (a).

3.6 The case $G = G(6,2)$

Since $G(6,2)$ contains $G(3,1)$, we may start from the possible non-trivial Δ's for that case. Note that these are all 3-torsion subgroups. Thus, if $\bar{x} \in B$ denotes a 2-torsion element, we see that its stabilizer in $\Delta \rtimes G$ can only contain elements in G. Consider then the element $\bar{t} = (t, 0)$ where t is a non-trivial 2-torsion element. As it is proven in [2, Prop. 3.4], the stabilizer of this element in G is not generated by pseudoreflections. This implies that $A/G = B/(\Delta \rtimes G)$ cannot be smooth regardless of the choice of possible Δ.

3.7 The case $G = G(6,3)$

Since $G(6,3)$ contains $G(2,1)$, we may start from the possible non-trivial Δ's for that case. Note that these are all 2-torsion subgroups. Thus, like we noticed in the previous case, if $\bar{x} \in B$ denotes a 3-torsion element, its stabilizer in $\Delta \rtimes G$ only contains elements in G. Consider then the element $\bar{s} = (s_0, 0)$ where s_0 is a ζ_3-invariant element (hence 3-torsion). Once again, as proven in [2, Prop. 3.4], the stabilizer of this element in G is not generated by pseudoreflections, which implies that A/G cannot be smooth in any case of Δ.

This finishes the study of the cases where $m \neq p$. We are left thus with the cases $G(3,3)$ and $G(6,6)$. In these particular cases we forget all the constructions done before and start from scratch.

3.8 The case $G = G(3,3)$

The group $G(3,3)$ is easily seen to be isomorphic *as a complex reflection group* to S_3 acting on $\mathbb{C}^2$ via the standard representation. As such, it has already been treated by the first two authors in [2, §3.1] and we know that in that case we get a smooth quotient if and only if we are in Example (b).

3.9 The case $G = G(6,6)$

Note that $G(6,6)$ is isomorphic to the direct product $G(3,3) \times \{\pm 1\}$. Since the actions of S_3 and $\mu_2 = \{\pm 1\}$ commute, we may follow the approach taken by [2] for S_3 and we will prove the following:

Proposition 3.5. *Let $G(6,6) = S_3 \times \mu_2$ act on an abelian surface A in such a way that its action on $T_0(A)$ is the standard one for S_3 and the obvious one for μ_2. Then A/G is not smooth.*

Proof. Let $\sigma = (1\,2) \in S_3$ and $E = E_\sigma$ be induced by a line $L_\sigma \subset T_0(A)$, and define the lattice

$$\Lambda_B := \sum_{\tau \in S_3} \tau(L_\sigma \cap \Lambda_A).$$

Since clearly all lattices are μ_2-invariant, this gives us a G-invariant sublattice of Λ_A. Therefore, we get a G-equivariant isogeny $\pi : B \to A$ with kernel Δ. Applying this construction to Example (b), to which we can naturally add the action of μ_2 in order to get an action of G, we see that it gives the whole lattice. We can thus see B as

$$B = \{(x_1, x_2, x_3) \in E^3 \mid x_1 + x_2 + x_3 = 0\},$$

where S_3 and μ_2 act in their respective natural ways. Using the notations from Section 2, we see by inspection that $F_{\sigma,B} = E_{\sigma,B}[2] \simeq E[2]$, hence the map $\pi : F_{\sigma,B} \to F_{\sigma,A}$ is surjective since by Lemma 2.1, case 2., we have $F_{\sigma,A} \subset E_{\sigma,A}[2] \simeq E[2]$. By Proposition 2.2, we have that Δ is contained in the fixed locus of all the conjugates of σ, which clearly generate S_3. Thus, Δ consists of elements of the form $(x, x, x) \in E^3$ such that $3x = 0$. In particular, Δ is isomorphic to a subgroup of $E[3]$ and hence of order 1, 3 or 9.

Assume that Δ is trivial, that is, that $A = B$. Then the action of $G = S_3 \times \mu_2$ on $B \simeq E^2$ induces an action of μ_2 on $B/S_3 \simeq \mathbb{P}^2$ (recall that the action of S_3 on B is that of Example (b)). We only need to notice then that any quotient of $\mathbb{P}^2$ by a non trivial action of the group μ_2 is not smooth. This is well-known.

Assume now that Δ has order 3 and let $\bar{t} = (t, t, t) \in \Delta$ be a non-trivial element (thus $t \in E[3]$). Let $x \in E[3]$ be a non-trivial element different from $\pm t$ and consider $\bar{x} = (x, x+t, x-t)$. It is then easy to see that the element $(\bar{t}, (1\,2\,3)) \in \Delta \rtimes G$ fixes $\bar{x}$ and that $\mathrm{Stab}_G(\bar{x}) = \{1\}$, so that every pseudoreflection fixing $\bar{x}$ must lie outside G. Let $(\bar{s}, \sigma)$ be such a pseudoreflection. Using Lemma 2.3, we see that $\sigma \in \{-(1\,2), -(2\,3), -(1\,3)\}$, where $-\tau$ denotes $(\tau, -1) \in S_3 \times \mu_2 = G$. Now, for any such σ, direct computations tell us that $(\bar{s}, \sigma)$ fixes $\bar{x}$ if and only $\bar{s} = (s, s, s)$ with $s = a_\sigma x + b_\sigma t$ for some $a_\sigma \neq 0$. Since $x \notin \langle t \rangle \subset E[3]$, we see that $\bar{s} \notin \Delta$ and hence these pseudoreflections do not exist. We get then that $\mathrm{Stab}_{\Delta \rtimes G}(\bar{x})$ is not generated by pseudoreflections and hence A/G cannot be smooth.

Assume finally that Δ has order 9. We claim that in this case the pair (A, G) is isomorphic to the pair (B, G). This will reduce us to the case with trivial Δ, which was already dealt with. To prove the claim, fix the basis $\{(1, 0, -1), (0, 1, -1)\}$ of $T_0(B) = T_0(A) \subset \mathbb{C}^3$. Then the analytic representation of G is given by the following values in its generators:

$$\rho_a((1\,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_a(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho_a((1\,2\,3)) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

Now, with this basis and this Δ, the analytic representation of $B \to A$ is given by the inverse of the following matrix:

$$M = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Indeed, this corresponds to the morphism that sends $(x, y, -x-y) \in B \subset E^3$ to $(-x-2y, 2x+y, -x+y) \in A \subset E^3$ and thus its kernel is precisely the elements of the form $(x, x, x) \in E[3]^3 \subset B$, that is, Δ. In order to prove that the pairs (A, G) and (B, G) are isomorphic, it suffices thus to prove that the image of this representation of G under conjugation by M is G once again. Direct computations give:

$$M\rho_a(-1)M^{-1} = \rho_a(-1), \qquad M\rho_a((1\,2\,3))M^{-1} = \rho_a((1\,2\,3)),$$
$$M\rho_a((1\,2))M^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \rho_a((1\,2))\rho_a(-1).$$

And these clearly generate the same group G. $\square$

4 Proof of (3) $\Rightarrow$ (2)

The only thing left to prove is that Example (c) satisfies property (2) from Theorem 1.1 (the other two are proved in [1]). Let us then study this example in detail.

Recall that in section 3.2 we proved that the pair (A, G) from Example (c) can be obtained as follows. Let $G = G(4, 2)$ and let $B = E^2$ with $E = \mathbb{C}^2/\mathbb{Z}[i]$. Denote by t_0 the i-invariant element in E and denote by q_0 the quotient morphism $E \to E/\langle t_0 \rangle \simeq E$. Then $A = B/\Delta$ with $\Delta = \langle (t_0, t_0) \rangle \in E^2 = B$ and the action of G on A is the one induced by $B \to A$.

Note now that G has an index 2 subgroup $G_1 := G(2, 1) = H_1 \rtimes S_2$, which is thus normal in G (here, $H_1 = \{\pm 1\}^2$). Moreover, the pair (B, G_1) corresponds to Example (a), so that $B/G_1 \simeq \mathbb{P}^2$. Finally, note that Δ is an order 2 subgroup of B and thus G acts *trivially* on it. In particular, the actions of G and Δ on B commute and hence we have a commutative diagram of Galois covers

$$\begin{tikzcd}
B \arrow[r, "\Delta"] \arrow[d, "G_1"] \arrow[dd, bend right, "G"'] & A \arrow[d] \arrow[dd, bend left, "G"] \\
\mathbb{P}^2 \arrow[r] \arrow[d, "G/G_1"] & A/G_1 \arrow[d] \\
B/G \arrow[r] & A/G,
\end{tikzcd}$$

where parallel arrows have the same Galois group. Since Δ and G/G_1 have both order 2, we see then that A/G is a quotient of $\mathbb{P}^2$ by the action of a Klein group.

Proposition 4.1. *The quotient A/G is isomorphic to $\mathbb{P}^2$.*

This proposition finishes the proof of (3) $\Rightarrow$ (2) in Theorem 1.1.

Remark 4.2. *This example was already known to Tokunaga and Yoshida (cf. [6, §5, Table II]). However, in order to prove that $A/G \simeq \mathbb{P}^2$, they cite an article by Švarcman which contains no proofs (cf. [5]).*

Proof. Since A/G is a quotient of $\mathbb{P}^2$ by the action of a Klein group K, the only thing we need to check is that this action gives $\mathbb{P}^2$ as a quotient. Note first that the action is faithful since it comes from the faithful action of $G \times \Delta$ on B. Consider then K as a subgroup of $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ and let K_1 be its preimage in SL_3. This is an order 12 group and hence any 2-Sylow subgroup of K_1 gives a lift of K to a subgroup of GL_3. This implies that the action lifts to $\mathbb{C}^3$ and it can thus be seen as a linear representation of K. Since there are exactly four irreducible representations of K, all of dimension 1, a direct check tells us that any choice of three different representations gives the same faithful action on $\mathbb{P}^2$ up to conjugation, whereas any other choice gives a non-faithful action. We may assume then that the nontrivial elements $x_i \in K$ for $i = 1, 2, 3$ act on $\mathbb{P}^2$ via the diagonal matrices with 1 on the i-th coordinate and -1 elsewhere. The quotient of $\mathbb{P}^2$ by such a group is the weighted projective space $\mathbb{P}(2, 2, 2)$, which is well-known to be isomorphic to $\mathbb{P}(1, 1, 1) = \mathbb{P}^2$. This concludes the proof. □

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On graphs that have a unique least common multiple

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ABSTRACT

A graph G without isolated vertices is a least common multiple of two graphs H_1 and H_2 if G is a smallest graph, in terms of number of edges, such that there exists a decomposition of G into edge disjoint copies of H_1 and there exists a decomposition of G into edge disjoint copies of H_2. The concept was introduced by G. Chartrand *et al.* and they proved that every two nonempty graphs have a least common multiple. Least common multiple of two graphs need not be unique. In fact two graphs can have an arbitrary large number of least common multiples. In this paper graphs that have a unique least common multiple with $P_3 \cup K_2$ are characterized.

RESUMEN

Un grafo G sin vértices aislados es un mínimo común múltiplo de dos grafos H_1 y H_2 si G es uno de los grafos más pequeños, en términos del número de ejes, tal que existe una descomposición de G en copias de H_1 disjuntas por ejes y existe una descomposición de G en copias de H_2 disjuntas por ejes. El concepto fue introducido por G. Chartrand *et al.* donde ellos demostraron que cualquera dos grafos no vaciós tienen un mínimo común múltiplo. El mínimo común múltiplo de dos grafos no es necesariamente único. De hecho, dos grafos pueden tener un número arbitrariamente grande de mínimos comunes múltiplos. En este artículo caracterizamos los grafos que tienen un único mínimo común múltiplo con $P_3 \cup K_2$.

Keywords and Phrases: Graph decomposition, common multiple of graphs.

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1 Introduction

All graphs considered in this paper are assumed to be simple and to have no isolated vertices. The number of edges of a graph G denoted by $e(G)$, is called the size of G. $\delta(G)$ and $\Delta(G)$ respectively denote the minimum and maximum of the degrees of all vertices in G. $\chi'(G)$ denotes the edge chromatic number of G, the minimum number of colors needed to color the edges of G, so that no two adjacent edges in G have the same color. An odd component of a graph is a maximal connected subgraph of the graph with odd number of edges. Two graphs G and H are said to be isomorphic, denoted as $G \cong H$ if there exists a bijection between the vertex sets of G and H, $f\colon V(G) \to V(H)$ such that two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. For graphs G_1 and G_2, their union $G_1 \cup G_2$ is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set consisting of all the edges in G_1 together with all the edges in G_2. If k is a positive integer, then kG is the union of k disjoint copies of G.

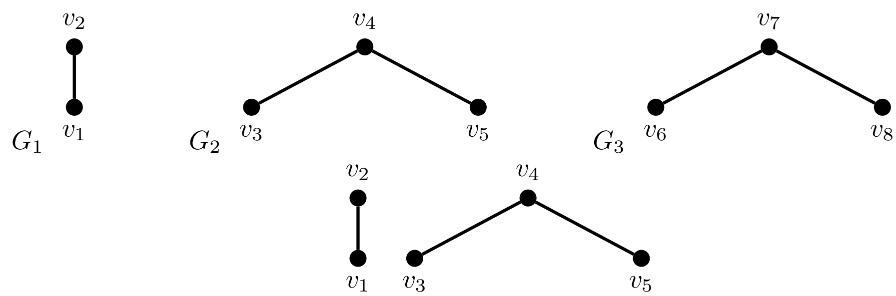


Figure 1: $G_1 \cup G_2$

Let $G = G_2$. Then $G \cong G_3$ and $2G$ is shown in Figure 2.

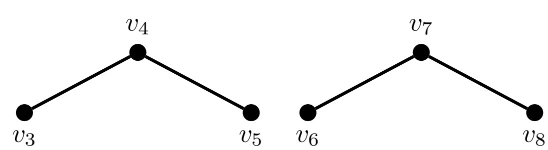


Figure 2: $2G$

A vertex u of a graph G is said to cover an edge e of G or e is covered by u, if e is incident with u. Let u, w be two vertices of a graph G and take two copies of G : G_1, G_2. The graph H obtained by identifying the vertex u in G_1 with the vertex w in G has vertex set $V(H) = V(G_1) \cup V(G_2) - \{w\}$ and edge set $E(H) = E(G_1) \cup E(G_2)$, where the edges in G_2 incident with w are now incident with u.

A graph H is said to divide a graph G if there exists a set of subgraphs of G, each isomorphic to H, whose edge sets partition the edge set of G. Such a set of subgraphs is called an *H-decomposition* of G. If G has an *H-decomposition*, we say that G is *H-decomposable* and write $H|G$.

A graph is called a common multiple of two graphs H_1 and H_2 if both $H_1|G$ and $H_2|G$. A graph G is a least common multiple of H_1 and H_2 if G is a common multiple of H_1 and H_2 and no other common multiple has a smaller positive number of edges. Several authors have investigated the problem of finding least common multiples of pairs of graphs H_1 and H_2; that is graphs of minimum size which are both H_1 and H_2 decomposable. The problem was introduced by Chartrand *et al.* in [5] and they showed that every two nonempty graphs have a least common multiple. The problem of finding the size of least common multiples of graphs has been studied for several pairs of graphs: cycles and stars [5, 13, 14], paths and complete graphs [11], pairs of cycles [10], pairs of complete graphs [4], complete graphs and a 4-cycle [1], pairs of cubes [2] and paths and stars [8]. Least common multiple of digraphs were considered in [7].

If G is a common multiple of H_1 and H_2 and G has q edges, then we call G a *(q, H_1, H_2) graph*. An obvious necessary condition for the existence of a (q, H_1, H_2) graph is that $e(H_1)|q$ and $e(H_2)|q$. This obvious necessary condition is not always sufficient. Therefore, we may ask: Given two graphs H_1 and H_2, for which value of q does there exist a (q, H_1, H_2) graph? Adams, Bryant and Maenhaut [1] gave a complete solution to this problem in the case where H_1 is the 4-cycle and H_2 is a complete graph; Bryant and Maenhaut [4] gave a complete solution to this problem when H_1 is the complete graph K_3 and H_2 is a complete graph. The problem to find least common multiples of two graphs H_1 and H_2 is to find all (q, H_1, H_2) graphs G of minimum size q. We denote the set of all least common multiples of H_1 and H_2 by $LCM(H_1, H_2)$. The size of a least common multiple of H_1 and H_2 is denoted by $lcm(H_1, H_2)$. Since every two nonempty graphs have a least common multiple, $LCM(H_1, H_2)$ is nonempty. For many pairs of graphs the number of elements of $LCM(H_1, H_2)$ is greater than one. For example both P_7 and C_6 are least common multiples of P_4 and P_3. In fact Chartrand *et al.* [6] proved that for every positive integer n there exist two graphs having exactly n least common multiples. In [11] it was shown that every least common multiple of two connected graphs is connected and that every least common multiple of two 2-connected graphs is 2-connected. But this is not the case for disconnected graphs. For example if we take $H_1 = 2K_2$, $H_2 = C_5$, then $G_1 = 2C_5$ and G_2 - the graph obtained by identifying two vertices in two copies of C_5, are in $LCM(H_1, H_2)$ of which G_1 is disconnected while G_2 is connected.

As two graphs can have several least common multiples, it is interesting to search for pairs of graphs that have a unique least common multiple. Pairs of graphs having a unique least common multiple were investigated by G. Chartrand *et al.* in [6] and they proved the following results.

Theorem 1.1. *A graph G of order p without isolated vertices and the graph P_3 have a unique least common multiple if and only if every component of G has even size or $G \cong K_p$, where $p \equiv 2$ or 3 (mod 4).*

Theorem 1.2. *A nonempty graph G without isolated vertices and the graph $2K_2$ have a unique* *least common multiple if and only if* $G \cong K_2, G \cong K_3$ *or* $2K_2|G$.

Theorem 1.3. *Let* r *and* s *be integers with* $2 \leq r \leq s$. *Then the stars* $K_{1,r}$ *and* $K_{1,s}$ *have a unique least common multiple if and only if* $gcd(r, s) \neq 1$.

A result proved by N. Alon [3] on tK_2-decomposition of a graph is used to find those graphs that have a unique least common multiple with tK_2.

Theorem 1.4. *For every graph* G *and every* $t > 1$, $tK_2|G$ *if and only if* $t|e(G)$ *and* $\chi'(G) \leq \dfrac{e(G)}{t}$.

We will also make use of a result proved by O. Favaron, Z. Lonc and M. Truszczynski [9] to characterize those graphs that have a unique least common multiple with $P_3 \cup K_2$.

Theorem 1.5. *If* G *is none of the six graphs* G_1 *to* G_6 *listed below, then* G *is* $P_3 \cup K_2$ *decomposable if and only if*

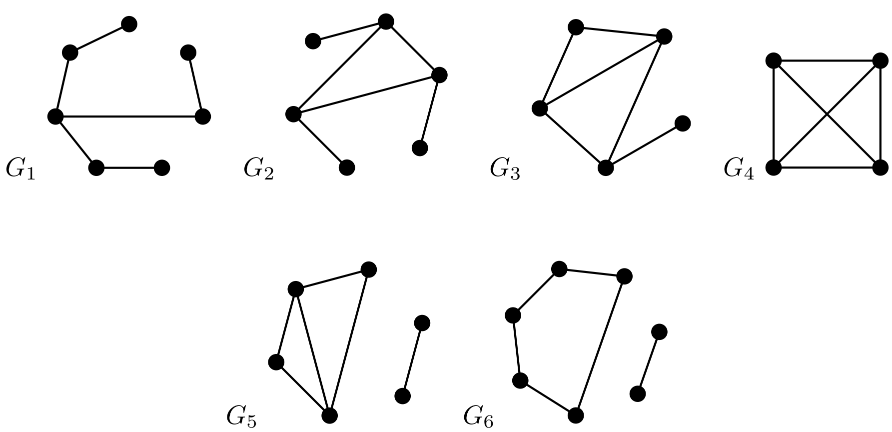
(1) $e(G) \equiv 0 \pmod 3$,

(2) $\Delta(G) \leq \frac{2}{3}e(G)$,

(3) $c(G) \leq \frac{1}{3}e(G)$, *where* $c(G)$ *denote the number of odd components of* G,

(4) *the edges of* G *cannot be covered by two adjacent vertices;*

where,



2 Main results

In this section we are characterizing those graphs that have a unique least common multiple with tK_2 and $P_3 \cup K_2$.

2.1 On graphs that have a unique least common multiple with tK_2

Theorem 2.1. *A nonempty graph G without isolated vertices and the graph tK_2 have a unique least common multiple if and only if $tK_2|G$ or $\delta(G) > \dfrac{lcm(tK_2, G)}{2t}$.*

Proof. Consider the graph tG. Clearly tG is both G and tK_2 decomposable. Let $q = e(G)$. Since $e(tG) = tq$, we have $lcm(tK_2, G) \leq tq$. But $lcm(tK_2, G)$ is a multiple of q. So $lcm(tK_2, G) = ql$, where $l \leq t$. This implies $\frac{lcm(tK_2,G)}{t} = \frac{ql}{t}$. Let H be a least common multiple of G and tK_2.

Case 1. $l > 1$.

Since H is tK_2-decomposable, by Theorem 1.4, $\chi'(H) \leq \frac{ql}{t}$. Since $G|H$, $\chi'(G) \leq \chi'(H) \leq \frac{ql}{t}$. Thus $\Delta(G) \leq \frac{ql}{t}$.

Subcase (i): $\delta(G) \leq \frac{ql}{2t}$.

Consider the graph $G \circ G$, which is obtained by identifying two vertices of least degree in G. In this subcase $\Delta(G \circ G) \leq \frac{ql}{t}$, since $\Delta(G) \leq \frac{ql}{t}$. $\chi'(G) \leq \frac{ql}{t}$ implies $\chi'(G \circ G) \leq \frac{ql}{t}$. Color G_1, a copy of G in $G \circ G$, with $k \leq \frac{ql}{t}$ colors. This is possible, since $\chi'(G) \leq \frac{ql}{t}$. Let v be the identified vertex in $G \circ G$. Since $\delta(G) \leq \frac{ql}{2t}$, the edges adjacent to v in G_1 are colored using at most $\frac{ql}{2t}$ colors. Color G_2, the copy of G in $G \circ G$ other than G_1, with the same k colors as follows. Color the edges adjacent to v in G_2 using colors different from those which were used to color the edges adjacent to v in G_1. The remaining colors used in the coloring of G_1 can be used to color other edges of G_2. Thus $\chi'(G \circ G) = k \leq \frac{ql}{t}$.

Let $H_1 = lG$, the union of l disjoint copies of G and $H_2 = G \circ G \cup (l-2)G$. Clearly H_1 and H_2 are divisible by G. Since $\chi'(H_1) = \chi'(G) \leq \frac{ql}{t}$, H_1 is tK_2-decomposable. $\chi'(H_2) = \chi'(G \circ G) \leq \frac{ql}{t}$, H_2 is tK_2-decomposable by Theorem 1.4. Thus $H_1, H_2 \in LCM(tK_2, G)$. $e(H_1) = e(H_2) = ql$, where $q = e(G)$. Since $lcm(tK_2, G) = ql$, H_1 and H_2 are two non-isomorphic least common multiples of tK_2 and G.

Subcase (ii): $\delta(G) > \frac{ql}{2t}$.

In this case $l > 1$ and $lcm(tK_2, G) = ql$, where $q = e(G)$. Thus $H \in LCM(tK_2, G)$, should be decomposed into at least two copies of G. If H is different from lG, then $\Delta(H) > \frac{ql}{t}$ which implies $\chi'(H) > \frac{ql}{t}$ and hence by Theorem 1.4, H is not tK_2-decomposable. Thus lG is the unique least common multiple of tK_2 and G.

Case 2. $l = 1$.

In this case $lcm(tK_2, G) = q$. Thus $tK_2|G$ and G is the unique least common multiple. □

Remark 2.2. *The result in the above theorem, Theorem 2.1, appeared in [12]. We are giving the proof of this result here since the result is needed for proving Theorem 2.3. The result was proved* *by the first author of this manuscript.*

2.2 On graphs that have a unique least common multiple with $P_3 \cup K_2$

Theorem 2.3. *A nonempty graph G without isolated vertices and the graph $P_3 \cup K_2$ have a unique least common multiple if and only if $G = K_2$ or $P_3 \cup K_2 \mid G$.*

Proof. Let $q = e(G)$.

Case 1. G is a connected graph.

If $G = K_2$, then $G \mid P_3 \cup K_2$. Thus $LCM(P_3 \cup K_2, K_2) = \{P_3 \cup K_2\}$ and hence their least common multiple is unique. So we are going to analyse the case where $G \neq K_2$.

Consider the graph $3G$, a union of three disjoint copies of G. Then

(1) $e(3G) \equiv 0 \pmod 3$.

(2) $\Delta(3G) = \Delta(G) \leq q = \frac{1}{3}(3q) \leq \frac{2}{3}(3q) = \frac{2}{3}e(3G)$.

(3) $c(3G) \leq 3 \leq \frac{1}{3}(3q) = \frac{1}{3}e(3G)$, if $e(G) \geq 3$. If $e(G) = 2$, then $c(3G) = 0 \leq \frac{1}{3}e(3G)$.

(4) The edges of $3G$ cannot be covered by two adjacent vertices, since the graph is disconnected.

Thus by Theorem 1.5, $3G$ is $P_3 \cup K_2$-decomposable. Clearly $3G$ is G-decomposable. Hence $lcm(P_3 \cup K_2, G) \leq 3q$.

Subcase (i): $lcm(P_3 \cup K_2, G) = 3q$.

Consider the graph $H = G \circ G \cup G$, where $G \circ G$ is the graph obtained by identifying a least degree vertex in two copies of G. Then

(1) $e(H) \equiv 0 \pmod 3$.

(2) $\Delta(H) \leq 2q = \frac{2}{3}(3q) = \frac{2}{3}e(H)$.

(3) $c(H) \leq 1 \leq \frac{1}{3}(3q) = \frac{1}{3}e(H)$.

(4) Since H is disconnected, edges of H cannot be covered by two adjacent vertices.

Thus by Theorem 1.5, H is $P_3 \cup K_2$- decomposable. Clearly H is G- decomposable. Hence in this case both H and $3G$ are elements of $LCM(P_3 \cup K_2, G)$ and hence their least common multiple is not unique.

Subcase (ii): $lcm(P_3 \cup K_2, G) = 2q$.

In this case there exists a graph H such that $e(H) = 2q$ and $H \in LCM(P_3 \cup K_2, G)$. Consider the graph $2G$.

(1) Since $H \in LCM(P_3 \cup K_2, G)$ we get $3 \mid e(H) = 2q = e(2G)$ and hence $e(2G) \equiv 0 \pmod 3$.

(2) Since H is G-decomposable and $\Delta(G) = \Delta(2G)$, $\Delta(2G) \le \Delta(H)$. H is $P_3 \cup K_2$-decomposable and so by Theorem 1.5, $\Delta(H) \le \frac{2}{3}e(H) = \frac{2}{3}e(2G)$. Thus $\Delta(2G) \le \frac{2}{3}e(2G)$.

(3) In this case $q \ge 3$ $\big($if $q = 1$, then $G = K_2$ and if $q = 2$, then $e(2G) = 4 \not\equiv 0 \pmod 3\big)$. So $c(2G) \le 2 \le \frac{1}{3}2q = \frac{1}{3}e(2G)$.

(4) Since $2G$ is disconnected, the edges of $2G$ cannot be covered by two adjacent vertices.

By applying Theorem 1.5, $2G$ is $P_3 \cup K_2$-decomposable. $2G$ is clearly G-decomposable. Thus $2G \in LCM(P_3 \cup K_2, G)$.

We can also prove that $G \circ G \in LCM(P_3 \cup K_2, G)$.

(1) $e(G \circ G) = e(2G) \equiv 0 \pmod 3$.

(2) In order to prove that $\Delta(G \circ G) \le \frac{2}{3}e(G \circ G)$ it is enough to prove that $\Delta(G)$ and $2\delta(G)$ are less than or equal to $\frac{2}{3}e(G \circ G)$, since $G \circ G$ is obtained by identifying a vertex of least degree in two copies of G.

Since $H \in LCM(P_3 \cup K_2, G)$, $\Delta(G) \le \Delta(H) \le \frac{2}{3}e(H) = \frac{2}{3}e(G \circ G)$.

$2\delta(G) \le \frac{2}{3}e(G \circ G) \iff \delta(G) \le \frac{2q}{3}$. Suppose $\delta(G) > \frac{2q}{3}$. Then $2q = \sum_{v \in V(G)} d(v) \ge \sum_{v \in V(G)} \delta(G) = n\delta(G) > n\frac{2q}{3}$, where $n = |V(G)|$. This implies $n < 3$. G is a connected graph without isolated vertices and $G \ne K_2$. Thus $n \ge 3$ and so $\delta(G) \le \frac{2q}{3}$.

(3) $c(G \circ G) = 0 < \frac{1}{3}e(G \circ G)$.

(4) The edges of $G \circ G$ cannot be covered by two adjacent vertices. Suppose the edges of $G \circ G$ can be covered by two adjacent vertices, then the identified vertex is one such vertex, since in $G \circ G$, no two vertices are adjacent except the identified vertex. This implies using the identified vertex and one other vertex it is possible to cover all the edges of $G \circ G$. This is possible only if G is a star with the identified vertex as the center of the star. This is a contradiction, since to construct $G \circ G$ a vertex of least degree in G is identified.

Applying Theorem 1.5, $G \circ G$ is $P_3 \cup K_2$- decomposable and it is clearly G-decomposable. So $G \circ G \in LCM(P_3 \cup K_2, G)$.

We have proved that $2G$ and $G \circ G \in LCM(P_3 \cup K_2, G)$ and hence their least common multiple is not unique.

Subcase (iii): $lcm(P_3 \cup K_2, G) = q$.

In this subcase G is the unique least common multiple, since $q = e(G)$.

Case 2. G is disconnected.

As in the first case, assume that $G \neq tK_2$. Then at least one component of G has more than one edge. We construct a graph of size $3q$, which is a $(3q, G, P_3 \cup K_2)$-graph, where $q = e(G)$. The construction is as follows. Take a least degree vertex from each component of G. Let H be the connected graph obtained by identifying all these vertices together. Take a least degree vertex in H. Denote by $H \circ H \circ H$, the graph obtained by identifying this least degree vertex in three copies of H.

(1) $e(H \circ H \circ H) = e(3H) = 3e(G) \equiv 0 \pmod 3$.

(2) $\Delta(H \circ H \circ H) \leq 2\Delta(H) \leq 2e(G) = \frac{2}{3}e(3G) = \frac{2}{3}e(H \circ H \circ H)$.

(3) $c(H \circ H \circ H) \leq 1 \leq \frac{1}{3}e(H \circ H \circ H)$.

(4) As in Subcase *(ii)* of the previous case, the edges of $H \circ H \circ H$ cannot be covered by two adjacent vertices.

By Theorem 1.5, $H \circ H \circ H$ is $P_3 \cup K_2$-decomposable and obviously it is G-decomposable. Thus $lcm(P_3 \cup K_2, G) \leq 3q$.

Subcase (i): $lcm(P_3 \cup K_2, G) = 3q$.

From the above discussion $H \circ H \circ H$ is a least common multiple in this subcase. Consider the graph $H \circ H \cup H$.

(1) $e(H \circ H \cup H) = 3e(G) \equiv 0 \pmod 3$.

(2) $\Delta(H \circ H \cup H) \leq 2\Delta(H) \leq 2e(G) = \frac{2}{3}e(3G) = \frac{2}{3}e(H \circ H \cup H)$.

(3) $c(H \circ H \cup H) \leq 1 = \frac{1}{3}e(H \circ H \cup H)$.

(4) Since $H \circ H \cup H$ is disconnected, the edges of $H \circ H \cup H$ cannot be covered by two adjacent vertices.

Applying Theorem 1.5, $H \circ H \cup H$ is $P_3 \cup K_2$-decomposable and by construction it is G-decomposable. Thus both $H \circ H \circ H$ and $H \circ H \cup H$ are in $LCM(P_3 \cup K_2, G)$ and hence their least common multiple is not unique.

Subcase (ii): $lcm(P_3 \cup K_2, G) = 2q$.

In this subcase there exists a graph H' of size $2q$ which is both G and $P_3 \cup K_2$ decomposable. We will prove that $H \circ H$ and $H \cup H$ are in $LCM(P_3 \cup K_2, G)$.

(1) $e(H \circ H) = 2q \equiv 0 \pmod 3$, since $e(H') = 2q$ and H' is $P_3 \cup K_2$- decomposable.

(2) In order to prove that $\Delta(H \circ H) \leq \frac{2}{3}e(H \circ H)$, it is enough to prove that $2\delta(H) \leq \frac{2}{3}e(H \circ H)$. That is we need to prove $\delta(H) \leq \frac{1}{3}(2q)$, where $q = e(H) = e(G)$.

Suppose $\delta(H) > \frac{2q}{3}$. Then $2q = \sum_{v\in V(H)} d(v) \geq \sum_{v\in V(H)} \delta(H) = n\delta(H) > n(\frac{2q}{3}) \Rightarrow n < 3$. Since G is a disconnected graph without isolated vertices, $n < 3$ is not possible. Hence $\delta(H) \leq \frac{2q}{3}$. Thus $\Delta(H \circ H) \leq \frac{2}{3}e(H \circ H)$.

(3) $c(H \circ H) = 0 < \frac{1}{3}e(H \circ H)$.

(4) By the construction of $H \circ H$, the edges of $H \circ H$ cannot be covered by two adjacent vertices.

By Theorem 1.5, $H \circ H$ is $P_3 \cup K_2$-decomposable and by construction, $H \circ H$ is G-decomposable and so $H \circ H \in LCM(P_3 \cup K_2, G)$.

Also $H \cup H \in LCM(P_3 \cup K_2, G)$, since

(1) $e(H \cup H) = 2q \equiv 0 \pmod 3$, since $lcm(P_3 \cup K_2) = 2q$, where $q = e(G) = e(H)$.

(2) $\Delta(H \cup H) \leq \Delta(H \circ H) \leq \frac{2}{3}e(H \circ H) = \frac{2}{3}e(H \cup H)$.

(3) Here $c(H \cup H) \leq 2$. Thus $c(H \cup H) \leq \frac{1}{3}e(H \cup H)$ if $2 \leq \frac{2q}{3}$, that is if $q \geq 3$, where, $q = e(G) = e(H)$. Since G is a disconnected graph without isolated vertices, $q \neq 1$. Also if $q = 2$, then $2q = 4 \not\equiv 0 \pmod 3$. Thus in this subcase, $q \geq 3$ and hence $c(H \cup H) \leq \frac{1}{3}e(H \cup H)$.

Thus $H \circ H$ and $H \cup H$ belong to $LCM(P_3 \cup K_2, G)$ and hence their least common multiple is not unique. □

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The topological degree methods for the fractional $p(\cdot)$-Laplacian problems with discontinuous nonlinearities

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ABSTRACT

In this article, we use the topological degree based on the abstract Hammerstein equation to investigate the existence of weak solutions for a class of elliptic Dirichlet boundary value problems involving the fractional $p(x)$-Laplacian operator with discontinuous nonlinearities. The appropriate functional framework for this problems is the fractional Sobolev space with variable exponent.

RESUMEN

En este artículo, usamos el grado topológico basado en la ecuación abstracta de Hammerstein para investigar la existencia de soluciones débiles para una clase de problemas elípticos de valor en la frontera de Dirichlet que involucran el operador $p(x)$-Laplaciano fraccional con no linealidades discontinuas. El marco funcional apropiado para estos problemas es el espacio de Sobolev fraccional con exponente variable.

Keywords and Phrases: Fractional $p(x)$-Laplacian, weak solution, discontinuous nonlinearity, topological degree theory.

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1 Introduction and main result

The study of fractional Sobolev spaces and the corresponding nonlocal equations has received a tremendous popularity in the last two decades considering their intriguing structure and great application in many fields, such as social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory, and Levy process, see [34, 35] and references therein for more details.

On the other hand, in recent years, a great deal of attention has been paid to the study of differential equations and variational problems involving $p(x)$-growth conditions since they can be used to model a variety of physical phenomena that occur in the fields of elastic mechanics, electro-rheological fluids (”smart fluids”), and image processing, etc. The readers are guided to [19, 20, 27] and its references.

It is only normal to wonder what results can be obtained when the fractional $p(\cdot)$-Laplacian is used instead of the $p(\cdot)$-Laplacian. The fractional $p(\cdot)$-Laplacian has also recently been investigated in elliptic problems; see [8, 10, 25, 26]. U. Kaufmann *et al.* [26] presented a new class of fractional Sobolev spaces with variable exponents in a recent paper. The authors in [8, 9] showed some additional basic properties on this function space as well as the associated nonlocal operator.

They used the critical point theory in [4] to prove the existence of solutions for fractional $p(\cdot)$-Laplacian equations. K. Ho and Y.-H. Kim [25] managed to obtain fundamental imbeddings for a new fractional Sobolev space with variable exponents, which is a generalization of previously defined fractional Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with Lipschitz boundary and let $p : \overline{\Omega} \times \overline{\Omega} \to (1,\ +\infty)$ be a continuous bounded function. The purpose of this paper is to establish the existence of nontrivial weak solutions for the following fractional $p(x)$-Laplacian problems with discontinuous nonlinearities.

$$\begin{cases} (-\triangle_{p(x)})^s u(x) + |u(x)|^{q(x)-2}u(x) + \lambda H(x,u) \in -\left[\underline{\psi}(x,u), \overline{\psi}(x,u)\right] & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $ps < N$ with $0 < s < 1$ and $(-\triangle_{p(x)})^s$ is the fractional $p(x)$-Laplacian operator defined by

$$(-\Delta)^s_{p(x)} u(x) = p.v. \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}} dy, \quad x \in \mathbb{R}^N \tag{1.2}$$

$\forall x \in \Omega$, where $p.v.$ is a commonly used abbreviation in the principal value sense and let $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,\ y) \leq p(x,\ y) \leq p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,\ y) < +\infty, \tag{1.3}$$

p is symmetric *i. e.*

$$p(x,\ y) = p(y,\ x),\ \forall (x,\ y) \in \overline{\Omega} \times \overline{\Omega}; \tag{1.4}$$

and $B_\varepsilon(x) := \{y \in \mathbb{R}^N \ : \ |x-y| < \varepsilon\}$.
Let as denote by:

$$\widetilde{p}(x) = p(x,\ x),\ \forall x \in \overline{\Omega}.$$

Furthermore, the Carathéodory's functions H satisfy only the growth condition, for all $s \in \mathbb{R}$ and a. e. $x \in \Omega$.

$$(H_0) \qquad |H(x,s)| \leq \varrho(e(x) + |s|^{q(x)-1}),$$

where ϱ is a positive constant, $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$.

In the simplest case $p = 2$, we have the well-known fractional Laplacian, a large amount of papers were written on this direction see [6, 15]. Moreover, if $s = 1$, we get the classic Laplacian. Some related results can be found in [21, 39, 40, 41, 42]. Notice that when $s = 1$, the problems like (1.1) have been studied in many papers, we refer the reader to [1, 5, 24], in which the authors have used various methods to get the existence of solutions for (1.1). In the case when $p = p(x)$ is a continuous function, problem (1.1) has also been studied by many authors. For more information, see [11, 23].

In order to prove the existence of nontrivial weak solutions, the main difficulties are reflected in the following aspect, we cannot directly use the topological degree methods in a natural way because the nonlinear term ψ is discontinuous. In order to overcome the discontinuous difficulty, we will transform this Dirichlet boundary value problem involving the fractional p-Laplacian operator with discontinuous nonlinearities into a new one governed by a Hammerstein equation. Then, we shall employ the topological degree theory developed by Kim in [29, 28] for a class of weakly upper semi-continuous locally bounded set-valued operators of (S_+) type in the framework of real reflexive separable Banach spaces, based on the Berkovits-Tienari degree [12]. The topological degree theory was constructed for the first time by Leray-Schauder [31] in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Furthermore, Browder [14] has developed a topological degree for operators of class (S_+) in reflexive Banach spaces, see also [37, 38]. Among many examples, we refer the reader to the classical works [2, 3, 18, 45] for more details.

To this end, we always assume that $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a possibly discontinuous function, we "fill the discontinuity gaps" of ψ, replacing ψ by an interval $[\underline{\psi}(x,u), \overline{\psi}(x,u)]$, where

$$\underline{\psi}(x,s) = \liminf_{\eta \to s} \psi(x,\eta) = \lim_{\delta \to 0^+} \inf_{|\eta - s| < \delta} \psi(x,\eta),$$

$$\overline{\psi}(x,s) = \limsup_{\eta \to s} \psi(x,\eta) = \lim_{\delta \to 0^+} \sup_{|\eta - s| < \delta} \psi(x,\eta).$$

Such that

(H_1) $\overline{\psi}$ and $\underline{\psi}$ are super-positionally measurable (*i. e.*, $\overline{\psi}(\cdot, u(\cdot))$ and $\underline{\psi}(\cdot, u(\cdot))$ are measurable on Ω for every measurable function $u : \Omega \to \mathbb{R}$).

(H_2) ψ satisfies the growth condition:

$$|\psi(x,s)| \leq b(x) + c(x)|s|^{\gamma(x)-1},$$

for almost all $x \in \Omega$ and all $s \in \mathbb{R}$, where $b \in L^{\gamma'(x)}(\Omega)$, $c \in L^{\infty}(\Omega)$, where $1 < \gamma(x) < p(x)$ for all $x \in \overline{\Omega}$.

First of all, we define the operator $\mathcal{N}$ acting from $W_0^{s,p(x,y)}(\Omega)$ into $2^{\left(W_0^{s,p(x,y)}(\Omega)\right)^*}$ by

$$\begin{aligned}\mathcal{N}u = \Big\{\varphi \in \left(W_0^{s,p(x,y)}(\Omega)\right)^* \setminus \exists\, h \in L^{p'(x)}(\Omega);& \\ \underline{\psi}(x,u(x)) \leq h(x) \leq \overline{\psi}(x,u(x)) \text{ a. e. } x \in \Omega& \\ \text{and } \langle \varphi, v\rangle = \int_\Omega hv dx \quad \forall v \in W_0^{s,p(x,y)}(\Omega)\Big\}&.\end{aligned}$$

In this spirit, we consider $F : W_0^{s,p(x,y)}(\Omega) \longrightarrow \left(W_0^{s,p(x,y)}(\Omega)\right)^*$ such that

$$\langle Fu, v\rangle = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}} dxdy, \tag{1.5}$$

for all $v \in W_0^{s,p(x,y)}(\Omega)$ and the operator $A : W_0 \to W_0^*$ setting by

$$\langle Au, v\rangle = \int_\Omega |u(x)|^{q(x)-2}(u(x)v(x) + \lambda H(x,u))v(x)dx, \quad \forall u, v \in W_0,$$

where the spaces $W_0^{s,p(x,y)}(\Omega) := W_0$ will be introduced in Section 2.

Next, we give the definition of weak solutions for problem (1.1).

Definition 1.1. *A function $u \in W_0^{s,p(x,y)}(\Omega)$ is called a weak solution to problem* (1.1)*, if there exists an element $\varphi \in \mathcal{N}u$ verifying*

$$\langle Fu, v\rangle + \langle Au, v\rangle + \langle \varphi, v\rangle = 0, \quad \textit{for all} \quad v \in W_0^{s,p(x,y)}(\Omega).$$

Now we are in a position to present our main result.

Theorem 1.2. *Assume that ψ satisfies $(H_1), (H_2)$ and H satisfies (H_0). Then, the problem* (1.1) *has a weak solution u in $W_0^{s,p(x,y)}(\Omega)$.*

2 Preliminaries

2.1 Lebesgue and fractional Sobolev spaces with variable exponent

In this subsection, we first recall some useful properties of the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$. For more details we refer the reader to [22, 30, 44].

Denote

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) | \inf_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ := \max\{h(x),\ x \in \overline{\Omega}\}, \quad h^- := \min\{h(x),\ x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue spaces

$$L^{p(x)}(\Omega) = \left\{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\}.$$

Endowed with *Luxemburg norm*

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 \,|\, \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

where

$$\rho_{p(\cdot)}\ (u) = \int_\Omega |u(x)|^{p(x)} dx,\ \forall u \in L^{p(x)}$$

$(L^{p(x)}(\Omega), ||.||_{p(x)})$ is a Banach space, separable and reflexive. Its conjugate space is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. We have also the following result

Proposition 2.1. *([22]) For any $u \in L^{p(x)}(\Omega)$ we have*

(i) $\|u\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$,

(ii) $\|u\|_{p(x)} \geq 1 \Rightarrow \|u\|^{p^-}_{p(x)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{p(x)}$,

(iii) $\|u\|_{p(x)} \leq 1 \Rightarrow \|u\|^{p^+}_{p(x)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{p(x)}$.

From this proposition, we can deduce the inequalities

$$\|u\|_{p(x)} \leq \rho_{p(\cdot)}(u) + 1, \tag{2.1}$$

$$\rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{p(x)} + \|u\|^{p^+}_{p(x)}. \tag{2.2}$$

If $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{q(x)}(\Omega) \to L^{p(x)}(\Omega)$.

Next, we present the definition and some results on fractional Sobolev spaces with variable exponent that was introduced in [8, 26]. Let s be a fixed real number such that $0 < s < 1$, and let $q : \overline{\Omega} \to (0, \infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (0, \infty)$ be two continuous functions. Furthermore, we suppose that the assumptions (1.3) and (1.4) be satisfied, we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$W = W^{s,,q(x),p(x,y)}(\Omega) = \left\{u \in L^{q(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < +\infty,\right.$$
$$\left.\text{for some } \lambda > 0\right\}.$$

We equip the space W with the norm

$$\|u\|_W = \|u\|_{q(x)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = \inf\left\{\lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \leq 1\right\}.$$

The space $(W,\ \|\cdot\|_W)$ is a Banach space (see [17]), separable and reflexive (see [8, Lemma 3.1]).

We also define W_0 as the subspace of W which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||_W$. From [7, Theorem 2.1 and Remark 2.1]

$$\|\cdot\|_{W_0} := [\cdot]_{s,p(x,y)}$$

is a norm on W_0 which is equivalent to the norm $\|\cdot\|_W$, and we have the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}$. So the space $(W_0,\ \|\cdot\|_{W_0})$ is a Banach space separable and reflexive.

We defne the modular $\rho_{p(\cdot,\cdot)} : W_0 \to \mathbb{R}$ by

$$\rho_{p(\cdot,\cdot)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.$$

The modular ρ_p checks the following results, which is similar to Proposition 2.1 (see [43, Lemma 2.1])

Proposition 2.2. *([30]) For any $u \in W_0$ we have*

(i) $\|u\|_{W_0} \geq 1 \Rightarrow \|u\|_{W_0}^{p^-} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^+}$,

(ii) $\|u\|_{W_0} \leq 1 \Rightarrow \|u\|_{W_0}^{p^+} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^-}$.

2.2 Some classes of operators and an outline of Berkovits degree

Now, we introduce the theory of topological degree which is the major tool for our results. We start by defining some classes of mappings. Let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\cdot\,,\,\cdot\rangle$ between X^* and X in this order. The symbol $\rightharpoonup$ stands for weak convergence. Let Y be another real Banach space.

Definition 2.3.

(1) *We say that the set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is bounded, if* F *maps bounded sets into bounded sets;*

(2) *we say that the set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is locally bounded at the point $u \in \Omega$, if there is a neighborhood V of u such that the set* $\mathrm{F(V)} = \bigcup_{u\in V} \mathrm{F}u$ *is bounded.*

Definition 2.4. *The set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is called*

(1) *upper semicontinuous (u.s.c.) at the point* u*, if, for any open neighborhood* V *of the set* $\mathrm{F}u$*, there is a neighbhorhood* U *of the point* u *such that* $\mathrm{F}(U) \subseteq V$*. We say that* F *is upper semicontinuous (u.s.c) if it is u.s.c at every* $u \in X$*;*

(2) *weakly upper semicontinuous (w.u.s.c.), if* $\mathrm{F}^{-1}(U)$ *is closed in* X *for all weakly closed set* U *in* Y.

Definition 2.5. *Let* Ω *be a nonempty subset of* X*,* $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $F : \Omega \subset X \to 2^{X^*} \setminus \emptyset$*. Then, the set-valued operator* F *is*

(1) *of type* (S_+)*, if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (h_n) *in* X^* *with* $h_n \in Fu_n$ *such that*

$$\limsup_{n\to\infty} \langle h_n, u_n - u\rangle \leq 0,$$

we get $u_n \to u$ *in* X*;*

(2) *quasi-monotone, if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (w_n) *in* X^* *such that* $w_n \in Fu_n$ *yield*

$$\liminf_{n\to\infty} \langle w_n, u_n - u\rangle \geq 0.$$

Definition 2.6. *Let* Ω *be a nonempty subset of* X *such that* $\Omega \subset \Omega_1$*,* $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $T : \Omega_1 \subset X \to X^*$ *be a bounded operator. Then, the set-valued operator* $F : \Omega \subset X \to 2^X \setminus \emptyset$ *is of type* $(S_+)_T$*, if*

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \\ Tu_n \rightharpoonup y \text{ in } X^*, \end{cases}$$

and for any sequence (h_n) *in* X *with* $h_n \in Fu_n$ *such that*

$$\limsup_{n\to\infty} \langle h_n, Tu_n - y\rangle \leq 0,$$

we have $u_n \to u$ *in* X.

Next, we consider the following sets :

$$\mathcal{F}_1(\Omega) := \{F : \Omega \to X^* | F \text{ is bounded, demicontinuous and of type } (S_+)\},$$
$$\mathcal{F}_T(\Omega) := \{F : \Omega \to 2^X | F \text{ is locally bounded, w.u.s.c. and of type } (S_+)_T\},$$

for any $\Omega \subset D_F$ and each bounded operator $T : \Omega \to X^*$, where D_F denotes the domain of F.

Remark 2.7. *We say that the operator* T *is an essential inner map of* F*, if* $T \in \mathcal{F}_1(\overline{G})$.

Lemma 2.8. *([29, Lemma 1.4]) Let* X *be a real reflexive Banach space and* $G \subset X$ *is a bounded open set. Assume that* $T \in \mathcal{F}_1(\overline{G})$ *is continuous and* $S : D_S \subset X^* \to 2^X$ *weakly upper semicontinuous and locally bounded with* $T(\overline{G}) \subset D_s$*. Then the following alternative holds:*

(1) *If S is quasi-monotone, yield $I + S \circ T \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*

(2) *If S is of type (S_+), yield $S \circ T \in \mathcal{F}_T(\overline{G})$.*

Definition 2.9. *([29]) Let $T : \overline{G} \subset X \to X^*$ be a bounded operator, a homotopy $H : [0,1]\times\overline{G} \to 2^X$ is called of type $(S_+)_T$, if for every sequence (t_k, u_k) in $[0,1]\times\overline{G}$ and each sequence (a_k) in X with $a_k \in H(t_k, u_k)$ such that*

$$u_k \rightharpoonup u \ \in X, \quad t_k \to t \in [0,1], \quad Tu_k \rightharpoonup y \quad in \quad X^* \ and \quad \limsup_{k\to\infty}\langle a_k, Tu_k - y\rangle \leq 0,$$

we get $u_k \to u$ in X.

Lemma 2.10. *([29]) Let X be a real reflexive Banach space and $G \subset X$ is a bounded open set, $T : \overline{G} \to X^*$ is continuous and bounded. If F, S are bounded and of class $(S_+)_T$, then an affine homotopy $H : [0,1] \times \overline{G} \to 2^X$ given by*

$$H(t,u) := (1-t)Fu + tSu, \quad for\ (t,u) \in [0,1] \times \overline{G},$$

is of type $(S_+)_T$.

Now, we introduce the topological degree for a class of locally bounded, w.u.s.c. and satisfies condition $(S_+)_T$ for more details see [29].

Theorem 2.11. *Let*

$$L = \big\{(F,G,g)|G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ g \notin F(\partial G)\big\},$$

where $\mathcal{O}$ denotes the collection of all bounded open sets in X. There exists a unique (Hammerstein type) degree function

$$d\ :\ L \longrightarrow \mathbb{Z}$$

such that the following alternative holds:

(1) *(*Normalization*) For each $g \in G$, we have $d(I, G, g) = 1$.*

(2) *(*Domain Additivity*) Let $F \in \mathcal{F}_T(\overline{G})$. We have*

$$d(F,G,g) = d(F,G_1,g) + d(F,G_2,g),$$

with G_1, $G_2 \subseteq G$ disjoint open such that $g \notin F(\overline{G}\backslash(G_1 \cup G_2))$.

(3) *(*Homotopy invariance*) If $H : [0,1] \times \overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and g: $[0,1] \to X$ is a continuous path in X such that $g(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,\cdot), G, g(t))$ is constant for any $t \in [0,1]$.*

(4) *(*Solution Property*) If $d(F,G,g) \neq 0$, then the equation $g \in Fu$ has a solution in G.*

3 Proof of Theorem 1.2

In the present section, following compactness methods (see [18, 32]), we prove the existence of weak solutions for the problem (1.1) in fractional Sobolev spaces. In doing so, we transform this elliptic Dirichlet boundary value problem involving the fractional p-Laplacian operator with discontinuous nonlinearities into a new problem governed by a Hammerstein equation. More precisely, by means of the topological degree theory introduced in section 2, we establish the existence of weak solutions to the state problem, which holds under appropriate assumptions. First, we give several lemmas.

Lemma 3.1. *Let* $0 < s < 1$ *and* $1 < p(x,y) < +\infty$, *(or* $sp_+ < N$*) the operator* F *defined in* (1.5) *is*

(i) bounded and strictly monotone operator.

(ii) of type (S_+).

Proof. *(i)* It is clear that F is a bounded operator. For all $\xi, \eta \in \mathbb{R}^N$, we have the Simon inequality (see [36]) from which we can obtain the strictly monotonicity of F:

$$\begin{cases} |\xi-\eta|^p \leq c_p\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)(\xi-\eta); & p \geq 2 \\ \\ |\xi-\eta|^p \leq C_p\Big[\Big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\Big)(\xi-\eta)\Big]^{\frac{p}{2}}(|\xi|^p+|\eta|^p)^{\frac{2-p}{2}}; & 1 < p < 2, \end{cases}$$

where $c_p = \left(\frac{1}{2}\right)^{-p}$ and $C_p = \dfrac{1}{p-1}$.

(ii) Let $(u_n) \in W_0^{s,p(x,y)}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n - Fu, u_n - u\rangle \leq 0$. In view of (i), we get

$$\lim_{n\to\infty}\langle Fu_n - Fu, u_n - u\rangle = 0.$$

Thanks to Proposition 2.1, we obtain

$$u_n(x) \to u(x), \text{ a.e. } x \in \Omega. \tag{3.1}$$

In the sequel, we denote by $L(x,y) = |x-y|^{-N-sp(x,y)}$.

By Fatou's lemma and (3.1), we get

$$\liminf_{n\to+\infty}\int_{\Omega\times\Omega}|u_n(x)-u_n(y)|^{p(x,y)}L(x,y)dxdy \geq \int_{\Omega\times\Omega}|u(x)-u(y)|^{p(x,y)}L(x,y)dxdy. \tag{3.2}$$

On the other hand, from $u_n \rightharpoonup u$ we have

$$\lim_{n\to+\infty}\langle Fu_n,\ u_n - u\rangle = \lim_{n\to+\infty}\langle Fu_n - Fu, u_n - u\rangle = 0. \tag{3.3}$$

Now, by using Young's inequality, there exists a positive constant c such that

$$\begin{aligned}\langle Fu_n, u_n - u\rangle &= \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} L(x,y)dxdy \\ &- \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))(u(x) - u(y))L(x,y)dxdy \\ &\geq \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} L(x,y)dxdy \\ &- \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)-1}|u(x) - u(y)|L(x,y)dxdy \\ &\geq c\int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} L(x,y)dxdy \\ &- c\int_{\Omega\times\Omega} |u(x) - u(y)|^{p(x,y)} L(x,y)dxdy,\end{aligned} \tag{3.4}$$

combining (3.2), (3.3) and (3.4), we obtain

$$\lim_{n\to+\infty}\int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} L(x,y)dxdy = \int_{\Omega\times\Omega} |u(x) - u(y)|^{p(x,y)} L(x,y)dxdy. \tag{3.5}$$

According to (3.1), (3.5) and the Brezis-Lieb lemma [13], our result is proved. □

Proposition 3.2. *([16, Proposition 1]) For any fixed $x \in \Omega$, the functions $\overline{\psi}(x,s)$ and $\underline{\psi}(x,s)$ are upper semicontinuous (u.s.c.) functions on $\mathbb{R}^N$.*

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary. The operator $A : W_0^{s,p(x,y)}(\Omega) \to \left(W_0^{s,p(x,y)}(\Omega)\right)^*$ defined by*

$$\langle Au, v\rangle = \int_\Omega (|u(x)|^{q(x)-2}u(x) + \lambda H(x,u))vdx, \quad \forall u, v \in W_0$$

is compact.

Proof. The proof is broken down into three sections.

Step 1. Let $\phi : W_0 \to L^{q'(x)}(\Omega)$ be the operator defined by

$$\phi\, u(x) := -|u(x)|^{q(x)-2}u(x) \quad \text{for} \quad u \in W_0 \quad \text{and} \quad x \in \Omega.$$

It is obvious that ϕ is continuous. Next we show that ϕ is bounded. For every $u \in W_0$, we have by the inequalities (2.1) and (2.2) that

$$\|\phi u\|_{q'(x)} \leq \rho_{q'(\cdot)}(\phi u) + 1 = \int_\Omega \left||u|^{q(x)-1}\right|^{q'(x)} dx + 1 = \rho_{q(\cdot)}(u) \leq \|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + 1.$$

By the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ we have

$$\|\phi u\|_{q'(x)} \leq \text{const}\left(\|u\|_{W_0}^{q^-} + \|u\|_{W_0}^{q^+}\right) + 1.$$

This implies that ϕ is bounded on W_0.

Step 2. We show that the operator ψ defined from W_0 into $L^{p'(x)}(\Omega)$ by

$$\psi u(x) := -\lambda H(x,u) \quad \text{for} \quad u \in W_0 \quad \text{and} \quad x \in \Omega$$

is bounded and continuous. Let $u \in W_0$, by using the growth condition (H_0) we obtain

$$\begin{aligned}
\|\psi\, u\|_{p'(x)}^{p'(x)} &\leq \int_\Omega |\lambda H(x,u)|^{p'(x)} dx \\
&\leq (\varrho\lambda)^{p'(x)} \int_\Omega \big(|e(x)|^{p'(x)} + |u|^{(q(x)-1)p'(x)}\big) dx \\
&\leq (\varrho\lambda)^{p'(x)} \int_\Omega \big(|e(x)|^{p'(x)} + |u|^{(p(x)-1)p'(x)}\big) dx \qquad (3.6) \\
&\leq (\varrho\lambda)^{p'(x)} \int_\Omega |e(x)|^{p'(x)} dx + (\varrho\lambda)^{p'(x)} \int_\Omega |u|^{p(x)}\, dx \\
&\leq (\varrho\lambda)^{p'(x)} (\|e\|_{p'(x)}^{p'+} + \|e\|_{p'(x)}^{p'-}) + (\varrho\lambda)^{p'(x)} (\|u\|_{p(x)}^{p+} + \|u\|_{p(x)}^{p-}) \\
&\leq C_m (\|u\|_{W_0}^{p+} + \|u\|_{W_0}^{p-} + 1),
\end{aligned}$$

where $C_m = \max\big((\varrho\lambda)^{p'(x)}(\|e\|_{p'(x)}^{p'+} + \|e\|_{p'(x)}^{p'-}), (\varrho\lambda)^{p'(x)}\big)$. (Due to $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$).

Therefore ψ is bounded on $W^{s,q(x),p(x,y)}(\Omega)$.

Next, we show that ψ is continuous, let $u_n \to u$ in $W^{s,q(x),p(x,y)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Thus there exists a subsequence still denoted by (u_n) and measurable function φ in $L^{p(x)}(\Omega)$ such that

$$u_n(x) \to u(x),$$

$$|u_n(x)| \leq \varphi(x),$$

for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$. Since H satisfies the Carathéodory condition, we obtain

$$H(x,u_n(x)) \to H(x,u(x)) \quad \text{a.e. } x \in \Omega. \qquad (3.7)$$

Thanks to (H_0) we obtain

$$|H(x,u_n(x))| \leq \varrho\big(e(x) + |\varphi(x)|^{q(x)-1}\big)$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since

$$e(x) + |\varphi(x)|^{p(x)-1} \in L^{p'(x)}(\Omega),$$

and from (3.7), we get

$$\int_\Omega |H(x,u_k(x)) - H(x,u(x))|^{p'(x)} dx \longrightarrow 0,$$

by using the dominated convergence theorem we have

$$\psi u_k \to \psi u \quad \text{in} \quad L^{p'(x)}(\Omega).$$

Thus the entire sequence (ψu_n) converges to ψu in $L^{p'(x)}(\Omega)$ and then ψ is continuous.

Step 3. Since the embedding $I : W_0 \to L^{q(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{q'(x)}(\Omega) \to W_0^*$ is also compact. Therefore, the compositions $I^* \circ \phi$ and $I^* \circ \psi : W_0 \to W_0^*$ are compact. We conclude that $S = I^* \circ \phi + I^* \circ \psi$ is compact. □

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary. If the hypotheses (H_1) and (H_2) hold, then the set-valued operator $\mathcal{N}$ defined above is bounded, upper semicontinuous (u.s.c.) and compact.*

Proof. Let $\Lambda : L^{p(x)}(\Omega) \to 2^{L^{p'(x)}(\Omega)}$ be a set-valued operator defined as follows

$$\Lambda u = \left\{ h \in L^{p'(x)}(\Omega) | \quad \underline{\psi}(x, u(x)) \leq h(x) \leq \overline{\psi}(x, u(x)) \quad \text{a. e.} \quad x \in \Omega \right\}.$$

Let $u \in W_0$, by the assumption (H_2) we obtain

$$\max \left\{ \, |\underline{\psi}(x, s)| \, ; \, |\overline{\psi}(x, s)| \, \right\} \leq b(x) + c(x)|s|^{\gamma(x)-1}.$$

for all $(x, t) \in \Omega \times \mathbb{R}$ where $1 < \gamma(x) < p(x)$ for all $x \in \overline{\mathbb{R}}$.

As a result

$$\int_\Omega |\, \overline{\psi}(x, u(x))|^{p'(x)} dx \leq 2^{p'^+ +1} \Big(\int_\Omega |b(x)|^{p'(x)} dx + \int_\Omega |c|^{p'(x)} |u(x)|^{p(x)} dx \Big).$$

A same inequality is shown for $\underline{\psi}(x, s)$, it follows that the set-valued operator Λ is bounded on $W_0(\Omega)$. It remains to prove that Λ is upper semi-continuous (u.s.c.), *i. e.*,

$$\forall \varepsilon > 0, \; \exists \delta > 0, \quad \|u - u_0\|_p < \delta \; \Rightarrow \; \Lambda u \subset \Lambda u_0 + B_\varepsilon,$$

where B_ε is the ε-ball in $L^{p'(x)}(\Omega)$.

To come to an end, given $u_0 \in L^{p(x)}(\Omega)$, let us consider the sets

$$G_{m,\varepsilon} = \bigcap_{t \in \mathbb{R}^N} K_t,$$

where

$$K_t = \left\{ x \in \Omega, \text{ if } |t - u_0(x)| < \frac{1}{m}, \text{ then } [\underline{\psi}(x, t), \overline{\psi}(x, t)] \subset \Big] \underline{\psi}(x, u_0(x)) - \frac{\varepsilon}{R}, \overline{\psi}(x, u_0(x)) + \frac{\varepsilon}{R} \Big[\right\},$$

m being an integer, $|t| = \max\limits_{1 \leq i \leq N} |t_i|$ and R is a constant to be determined in the following pages. In view of Proposition 3.2, we define the sets of points as follows

$$G_{m,\varepsilon} = \bigcap_{r \in \mathbb{R}^N_a} K_r,$$

where $\mathbb{R}^N_a$ denotes the set of all rational grids in $\mathbb{R}^N$. For any $r = (r_1, \dots, r_N) \in \mathbb{R}^N_a$,

$$K_r = \left\{ x \in \Omega \mid u_0(x) \in C \prod_{i=1}^N \Big] r_i - \frac{1}{m}, r_i + \frac{1}{m} \Big[\right\} \cup \left\{ x \in \Omega \mid u_0(x) \in \prod_{i=1}^N \Big] r_i - \frac{1}{m}, r_i + \frac{1}{m} \Big[\right\}$$
$$\cap \left\{ x \in \Omega \mid \overline{\psi}(x, r) < \overline{\psi}(x, u_0(x)) + \frac{\varepsilon}{R} \text{ and } \underline{\psi}(x, r) > \underline{\psi}(x, u_0(x)) - \frac{\varepsilon}{R} \right\},$$

so that K_r and therefore $G_{m,\varepsilon}$ are measurable. It is obvious that

$$G_{1,\varepsilon} \subset G_{2,\varepsilon} \subset \cdots$$

In light of Proposition 3.2, we have

$$\bigcup_{m=1}^{\infty} G_{m,\varepsilon} = \Omega,$$

therefore there exists $m_0 \in \mathbb{N}$ such that

$$m(G_{m_0,\varepsilon}) > m(\Omega) - \frac{\varepsilon}{R}. \tag{3.8}$$

But for each $\varepsilon > 0$, there is $\eta = \eta(\varepsilon) > 0$, such that $m(T) < \eta$ yields

$$2^{p'^{+}-1} \int_T 2|b(x)|^{p'(x)} + c^{p'(x)}(x)(2^{p'^{+}-1}+1)|u_0(x)|^{p(x)} dx < \left(\frac{\varepsilon}{3}\right)^{p'^{+}}, \tag{3.9}$$

because of $b \in L^{p'(x)}(\Omega)$ and $u_0 \in L^{p(x)}(\Omega)$.

Let now

$$0 < \delta < \min\left\{ \frac{1}{m_0} \left(\frac{\eta}{2}\right)^{\frac{1}{p^-}}, \frac{1}{2^{p^+-2}} \left(\frac{\varepsilon}{6C}\right)^{\frac{p'^+}{\theta}} \right\}, \tag{3.10}$$

$$R > \max\left\{ \frac{2\varepsilon}{\eta}, 3\left(m(\Omega)\right)^{\frac{1}{p'^-}} \right\}, \tag{3.11}$$

where

$$\theta = \begin{cases} p^+ & \text{if} \quad \|u - u_0\|_{p(x)} \leq 1 \\ p^- & \text{if} \quad \|u - u_0\|_{p(x)} \geq 1. \end{cases}$$

Suppose that $\|u - u_0\|_{p(x)} < \delta$ and define the set $G = \left\{ x \in \Omega \setminus |u(x) - u_0(x)| \geq \dfrac{1}{m_0} \right\}$, we get

$$m(G) < (m_0\delta)^{p(x)} < \frac{\eta}{2}. \tag{3.12}$$

If $x \in G_{m_0,\varepsilon} \backslash G$, then, for any $h \in \Lambda u$,

$$|u(x) - u_0(x)| < \frac{1}{m_0}$$

and

$$h(x) \in \left] \underline{\psi}(x, u_0(x)) - \frac{\varepsilon}{R}, \overline{\psi}(x, u_0(x)) + \frac{\varepsilon}{R} \right[.$$

Let

$$\begin{aligned} K^0 &= \left\{ x \in \Omega; \quad h(x) \in \left[\underline{\psi}(x, u_0(x)), \overline{\psi}(x, u_0(x)) \right] \right\}, \\ K^- &= \left\{ x \in \Omega; \quad h(x) < \underline{\psi}(x, u_0(x)) \right\}, \\ K^+ &= \left\{ x \in \Omega; \quad h(x) > \overline{\psi}(x, u_0(x)) \right\}, \end{aligned}$$

and

$$w(x) = \begin{cases} \overline{\psi}(x, u_0(x)), & \text{for} \quad x \in K^+; \\ h(x) \ , & \text{for} \quad x \in K^0; \\ \underline{\psi}(x, u_0(x)), & \text{for} \quad x \in K^-. \end{cases}$$

Hence $w \in \Lambda u_0$ and

$$|w(x) - h(x)| < \frac{\varepsilon}{R} \quad \text{for all} \quad x \in G_{m_0,\varepsilon} \setminus G. \tag{3.13}$$

From (3.11) and (3.13), we have

$$\int_{G_{m_0,\varepsilon} \setminus G} |w(x) - h(x)|^{p'(x)} dx < \Big(\frac{\varepsilon}{R}\Big)^{p'^+} m(\Omega) < \Big(\frac{\varepsilon}{3}\Big)^{p'^+}. \tag{3.14}$$

Assume that V is a coset in Ω of $G_{m_0,\varepsilon} \setminus G$, then $V = (\Omega \setminus G_{m_0,\varepsilon}) \cup (G_{m_0,\varepsilon} \cap G)$ and

$$m(V) \le m(\Omega \setminus G_{m_0,\varepsilon}) + m(G_{m_0,\varepsilon} \cap G) < \frac{\varepsilon}{R} + m(G) < \eta.$$

According to (3.8), (3.11) and (3.12). From (H_2), (3.9) and (3.10), we obtain

$$\begin{aligned}
\int_V |w(x) - h(x)|^{p'(x)} dx &\le \int_V |w(x)|^{p'(x)} + |h(x)|^{p'(x)} dx \\
&\le 2^{p'^+ - 1} \left(\int_V |b(x)|^{p'(x)} + c^{p'(x)}(x) |u_0(x)|^{p(x)} + |b(x)|^{p'(x)} + c^{p'}(x) |u(x)|^{p(x)} dx \right) \\
&\le 2^{p'^+ - 1} \Big(\int_V 2|b(x)|^{p'(x)} + c^{p'(x)}(x) (2^{p^+ - 1} + 1) |u_0(x)|^{p(x)} dx \Big) \\
&\quad + 2^{p'^+ - 1} \Big(\int_V 2^{p^+ - 1} c^{p'(x)}(x) |u(x) - u_0(x)|^{p(x)} dx \Big) \\
&\le 2^{p'^+ - 1} \int_V 2|b(x)|^{p'(x)} + c^{p'(x)}(x) (2^{p^+ - 1} + 1) |u_0(x)|^{p(x)} dx \\
&\quad + 2^{p^+ + p'^+ - 2} \, \|c^{p'^+}\|_{L^\infty(\Omega)} \int_V |u(x) - u_0(x)|^{p(x)} dx \\
&\le \Big(\frac{\varepsilon}{3}\Big)^{p'^+} + 2^{p^+ + p'^+ - 2} \, \|c^{p'^+}\|_{L^\infty(\Omega)} \delta^\theta \le 2 \Big(\frac{\varepsilon}{3}\Big)^{p'^+} \le \varepsilon^{p'^+}.
\end{aligned} \tag{3.15}$$

Thanks to (3.14), (3.15) and (2.1), we get $\|w - h\|_{p'(x)} \le \int_\Omega |w(x) - h(x)|^{p'(x)} dx + 1 < \varepsilon$.

Hence Λ is upper semicontinuous (u.s.c.). Hence $\mathcal{N} = I^* \circ \Lambda \circ I$ is clearly bounded, upper semicontinuous (u.s.c.) and compact. □

Next, we give the proof of Theorem 1.2. Let $S := A + \mathcal{N} : W_0^{s,p(x,y)}(\Omega) \to 2^{\left(W_0^{s,p(x,y)}(\Omega)\right)^*}$, where A and $\mathcal{N}$ were defined in Lemma 3.3 and in section 2 respectively. This means that a point $u \in W_0^{s,p(x,y)}(\Omega)$ is a weak solution of (1.1) if and only if

$$Fu \in -Su, \tag{3.16}$$

with F defined in (1.5). By the properties of the operator F given in Lemma 3.1 and the Minty-Browder's Theorem on monotone operators in [45, Theorem 26 A], we guarantee that the inverse operator $T := F^{-1} : \big(W_0^{s,p(x,y)}(\Omega)\big)^* \to W_0^{s,p(x,y)}(\Omega)$ is continuous, of type (S_+) and bounded. Moreover, thanks to Lemma 3.3 the operator S is quasi-monotone, upper semicontinuous (u.s.c.) and bounded. As a result, the equation (3.16) is equivalent to the abstract Hammerstein equation

$$u = Tv \quad \text{and} \quad v \in -S \circ Tv. \tag{3.17}$$

We will apply the theory of degrees introduced in section 3 to solve the equations (3.17). For this, we first show the following Lemma.

Lemma 3.5. *The set*

$$B := \Big\{\ v \in \big(W_0\big)^* \ \text{ such that } v \in -tS \circ Tv \quad \text{for some} \quad t \in [0,1]\Big\}$$

is bounded.

Proof. Let $v \in B$, so, $v + ta = 0$ for every $t \in [0,1]$, with $a \in S \circ Tv$. Setting $u := Tv$, we can write $a = Au + \varphi \in Su$, where $\varphi \in \mathcal{N}u$, namely,

$$\langle \varphi, u\rangle = \int_\Omega h(x)u(x)dx,$$

for each $h \in L^{p'(x)}(\Omega)$ with $\underline{\psi}(x,u(x)) \le h(x) \le \overline{\psi}(x,u(x))$ for almost all $x \in \Omega$.

If $\|u\|_{W_0} \le 1$, then $\|Tv\|_{W_0}$ is bounded.

If $\|u\|_{W_0} > 1$, then we get by the implication (i) in Proposition 2.1 and the inequality (2.2) and using (H_0), the Young inequality, the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$, the estimate

$$\begin{aligned}
\|Tv\|_{W_0}^{p^-} &= \|u\|_{W_0}^{p^-} \\
&\le \rho_{p(\cdot,\cdot)}(u) \\
&\le t|\langle a, Tv\rangle| \\
&\le t\int_\Omega |u|^{q(x)}\,dx + t\int_\Omega \lambda|H(x,u)|u dx + t\int_\Omega |hu|dx \\
&\le t\int_\Omega |u|^{q(x)} + tC_{p'}\int_\Omega |\lambda H(x,u)|^{q'(x)}dx + tC_p\int_\Omega |u|^{q(x)}\,dx \\
&\qquad\qquad + C_\gamma t\left(\int_\Omega |u|^{\gamma(x)}dx\right) + C_{\gamma'}t\left(\int_\Omega |h|^{\gamma'(x)}dx\right) \\
&\le \text{Const}\Big(\|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + \|u\|_{\gamma(x)}^{\gamma^-} + \|u\|_{\gamma(x)}^{\gamma+} + 1\Big) \\
&\le \text{Const}\Big(\|u\|_{W_0}^{q^-} + \|u\|_{W_0}^{q^+} + \|u\|_{W_0}^{\gamma^-} + \|u\|_{W_0}^{\gamma+} + 1\Big) \\
&\le \text{Const}\Big(\|Tv\|_{W_0}^{q^+} + \|Tv\|_{W_0}^{\gamma^+} + 1\Big).
\end{aligned}$$

Hence it is obvious that $\{\ Tv \ \mid\ v \in B\ \}$ is bounded.

As the operator S is bounded and from (3.17), we deduce the set B is bounded in $\big(W_0\big)^*$. ∎

Thanks to Lemma 3.5, we can find a positive constant R such that

$$\|v\|_{\left(W_0\right)^*} < R \quad \text{for any} \quad v \in B.$$

This says that

$$v \in -tS \circ Tv \quad \text{for each} \quad v \in \partial B_R(0) \quad \text{and each} \quad t \in [0,1].$$

Under the Lemma 2.8, we get

$$I + S \circ T \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = F \circ T \in \mathcal{F}_T(\overline{B_R(0)}).$$

Now, we are in a position to consider the affine homotopy $H : [0,1] \times \overline{B_R(0)} \to 2^{\left(W_0\right)^*}$ defined by

$$H(t,v) := (1-t)Iv + t(I + S \circ T)v \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

By applying the normalization and homotopy invariance property of the degree d fixed in Theorem 2.11, we have

$$d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1.$$

It follows that, we can get a function $v \in B_R(0)$ such that

$$v \in -S \circ Tv.$$

Which implies that $u = Tv$ is a weak solution of (1.1). This completes the proof.

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Existence, uniqueness, continuous dependence and Ulam stability of mild solutions for an iterative fractional differential equation

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ABSTRACT

In this work, we study the existence, uniqueness, continuous dependence and Ulam stability of mild solutions for an iterative Caputo fractional differential equation by first inverting it as an integral equation. Then we construct an appropriate mapping and employ the Schauder fixed point theorem to prove our new results. At the end we give an example to illustrate our obtained results.

RESUMEN

En este trabajo, estudiamos la existencia, unicidad, dependencia continua y estabilidad de Ulam de soluciones mild para una ecuación diferencial fraccionaria de Caputo iterativa, invirtiéndola primero como ecuación integral. Luego construimos una aplicación apropiada y empleamos el teorema del punto fijo de Schauder para demostrar nuestros nuevos resultados. Finalmente damos un ejemplo para ilustrar los resultados obtenidos.

Keywords and Phrases: Iterative fractional differential equations, fixed point theorem, existence, uniqueness, continuous dependence, Ulam stability.

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1 Introduction

Fractional differential equations have gained considerable importance due to their applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas *et al.* [10], Miller and Ross [12], Podlubny [14]. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[4], [6]–[16], [18] and the references therein.

Recently, iterative functional differential equations of the form

$$x'(t) = H\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right),$$

have appeared in several papers, where

$$x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[2]}(t) = x(x(t)), \ldots,\ x^{[n]}(t) = x^{[n-1]}(x(t))$$

are the iterates of the state $x(t)$.

Iterative differential equations often arise in the modeling of a wide range of natural phenomena such as disease transmission models in epidemiology, two-body problem of classical electrodynamics, population models, physical models, mechanical models and other numerous models. This kind of equations which relates an unknown function, its derivatives and its iterates, is a special type of the so-called differential equations with state-dependent delays, see [5, 9, 19] and the references therein.

In this paper, inspired and motivated by the references [1]–[16], [18, 19], we concentrate on the existence, uniqueness, continuous dependence and Ulam stability of mild solutions for the nonlinear iterative fractional differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha} x(t) = f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right),\ t \in J, \\ x(0) = x'(0) = 0, \end{cases} \tag{1.1}$$

where $J = [0, T]$, ${}^{C}D_{0^+}^{\alpha}$ is the standard Caputo fractional derivative of order $\alpha \in (1, 2)$ and f is a positive continuous function with respect to its arguments and satisfies some other conditions that will be specified later. To reach our desired end we have to transform (1.1) into an integral equation and then use the Schauder fixed point theorem to show the existence and uniqueness of mild solutions.

The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas, and state some preliminary results needed in later sections. Also, we present the inversion of (1.1) and state the Schauder fixed point theorem. For details on the Schauder theorem we refer the reader to [17]. In Section 3, we present our main results on the existence, uniqueness, continuous dependence and Ulam stability of mild solutions for the problem (1.1) and provide an example to illustrate our results.

2 Preliminaries

Let $C(J,\mathbb{R})$ be the Banach space of all real-valued continuous functions defined on the compact interval J, endowed with the norm

$$\|x\| = \sup_{t\in J} |x(t)| .$$

For $0 < L \leq T$ and $M > 0$, define the sets

$$C(J,L) = \{x \in C(J,\mathbb{R}) : 0 \leq x(t) \leq L,\ \forall t \in J\},$$

and

$$C_M(J,L) = \{x \in C(J,L) : |x(t_2) - x(t_1)| \leq M |t_2 - t_1|,\ \forall t_1, t_2 \in J\}.$$

Then, $C_M(J,L)$ is a closed convex and bounded subset of $C(J,\mathbb{R})$.

Furthermore, we suppose that the positive function f is globally Lipschitz in x_i, that is, there exist positive constants c_1, c_2, $\ldots$, c_n such that

$$|f(t,x_1,x_2,\ldots,x_n) - f(t,y_1,y_2,\ldots,y_n)| \leq \sum_{i=1}^{n} c_i |x_i - y_i| . \tag{2.1}$$

We introduce the constants

$$\begin{aligned} \rho &= \sup_{t\in J} \{f(t,0,0,\ldots,0)\}, \\ \zeta &= \rho + L\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j, \end{aligned}$$

where $M^j = M \times M^{j-1}$.

Definition 2.1 ([10]). *The fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is given by*

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$, where Γ is the gamma function.

For instance, $I_{0^+}^{\alpha} x$ exists for all $\alpha > 0$, when $x \in C(\mathbb{R}^+)$ then $I_{0^+}^{\alpha} x \in C(\mathbb{R}^+)$ and moreover $I_{0^+}^{\alpha} x(0) = 0$.

Definition 2.2 ([10]). *The Caputo fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is given by*

$${}^{C}D_{0^+}^{\alpha} x(t) = I_{0^+}^{n-\alpha} x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3 ([10])**.** *Suppose that $x \in C^{n-1}([0,+\infty))$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$\left(I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

In particular, when $\alpha \in (1,2)$, $\left(I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} x\right)(t) = x(t) - x(0) - x'(0)\, t$.

Definition 2.4. *A function $x \in C_M(J,L)$ is a mild solution of the problem (1.1) if x satisfies the corresponding integral equation of (1.1).*

From Lemma 2.3, we deduce the following lemma.

Lemma 2.5. *Let $x \in C_M(J,L)$ is a mild solution of (1.1) if x satisfies*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right) ds, \ t \in J. \tag{2.2}$$

Lemma 2.6 ([19])**.** *If $\varphi, \psi \in C_M(J,L)$, then*

$$\left\|\varphi^{[m]} - \psi^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^j \|\varphi - \psi\|, \ m = 1, 2, \ldots$$

Theorem 2.7 (Schauder fixed point theorem [17])**.** *Let $\mathbb{M}$ be a nonempty compact convex subset of a Banach space $(\mathbb{B}, \|\cdot\|)$ and $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is a continuous mapping. Then $\mathcal{A}$ has a fixed point.*

3 Main results

In this section, we use Theorem 2.7 to prove the existence of mild solutions for (1.1). Moreover, we will introduce the sufficient conditions of the uniqueness of mild solutions of (1.1).

To transform (2.2) to be applicable to the Schauder fixed point, we define an operator $\mathcal{A} : C_M(J,L) \to C(J,\mathbb{R})$ by

$$(\mathcal{A}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds, \ t \in J. \tag{3.1}$$

Since $C_M(J,L)$ is a compact set as a uniformly bounded, equicontinuous and closed subset of the space $C(J,\mathbb{R})$. To prove that operator $\mathcal{A}$ has at least one fixed point, we will prove that $\mathcal{A}$ is well defined, continuous and $\mathcal{A}(C_M(J,L)) \subset C_M(J,L)$, *i. e.*

$$\mathcal{A}\varphi \in C_M(J,L) \text{ for all } \varphi \in C_M(J,L).$$

Lemma 3.1. *Suppose that (2.1) holds. Then the operator $\mathcal{A} : C_M(J,L) \to C(J,\mathbb{R})$ given by (3.1) is well defined and continuous.*

Proof. Let $\mathcal{A}$ be defined by (3.1). Clearly, $\mathcal{A}$ is well defined. To show the continuity of $\mathcal{A}$. Let $\varphi, \psi \in C_M(J, L)$, we have

$$\begin{aligned}|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) \\ - f\left(\psi^{[0]}(s), \psi^{[1]}(s), \psi^{[2]}(s), \ldots, \psi^{[n]}(s)\right)\Big| \, ds.\end{aligned}$$

By (2.1), we obtain

$$|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{n} c_i \left\|\varphi^{[i]} - \psi^{[i]}\right\| ds.$$

It follows from Lemma 2.6 that

$$\begin{aligned}|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\| \, ds \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\|,\end{aligned}$$

which proves that the operator $\mathcal{A}$ is continuous. □

Lemma 3.2. *Suppose that (2.1) holds. If*

$$\frac{\zeta T^\alpha}{\Gamma(\alpha+1)} \leq L, \tag{3.2}$$

and

$$\frac{\zeta T^{\alpha-1}}{\Gamma(\alpha)} \leq M, \tag{3.3}$$

then $\mathcal{A}(C_M(J, L)) \subset C_M(J, L)$.

Proof. For $\varphi \in C_M(J, L)$, we get

$$|(\mathcal{A}\varphi)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| ds.$$

But

$$\begin{aligned}&\left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| \\ &= \left| f\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) - f(s, 0, 0, \ldots, 0) + f(s, 0, 0, \ldots, 0)\right| \\ &\leq \left| f\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) - f(s, 0, 0, \ldots, 0)\right| + |f(s, 0, 0, \ldots, 0)| \\ &\leq \rho + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi\| \\ &\leq \rho + L \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j = \zeta,\end{aligned}$$

then

$$|(\mathcal{A}\varphi)(t)| \leq \frac{\zeta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds \leq \frac{\zeta T^\alpha}{\Gamma(\alpha+1)} \leq L.$$

From (3.2), we have

$$0 \leq (\mathcal{A}\varphi)(t) \leq |(\mathcal{A}\varphi)(t)| \leq L.$$

Let $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| \left|f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left|f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| ds \\
&\leq \frac{\zeta}{\Gamma(\alpha)} \left(\int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\, ds\right) \\
&\leq \frac{\zeta}{\Gamma(\alpha+1)} \left(t_2^\alpha - t_1^\alpha\right) \\
&\leq \frac{\zeta T^{\alpha-1}}{\Gamma(\alpha)} |t_2 - t_1|.
\end{aligned}$$

Using (3.3), we obtain

$$|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq M |t_2 - t_1|.$$

Therefore, $\mathcal{A}\varphi \in C_M(J, L)$ for all $\varphi \in C_M(J, L)$. So, we conclude that $\mathcal{A}(C_M(J, L)) \subset C_M(J, L)$. □

Theorem 3.3. *Suppose that conditions (2.1), (3.2) and (3.3) hold. Then (1.1) has at least one mild solution x in $C_M(J, L)$.*

Proof. From Lemma 2.5, the problem (1.1) has a mild solution x on $C_M(J, L)$ if and only if the operator $\mathcal{A}$ defined by (3.1) has a fixed point. From Lemmas 3.1 and 3.2, all conditions of the Schauder fixed point theorem are satisfied. Consequently, $\mathcal{A}$ has at least one fixed point on $C_M(J, L)$ which is a mild solution of (1.1). □

Theorem 3.4. *In addition to the assumptions of Theorem 3.3, if we suppose that*

$$\frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j < 1, \tag{3.4}$$

then (1.1) has a unique mild solution in $C_M(J, L)$.

Proof. Let φ and ψ be two distinct fixed points of the operator $\mathcal{A}$. Similarly as in the proof of Lemma 3.1 we have

$$|\varphi(t) - \psi(t)| = |(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\|.$$

It follows from (3.4) that

$$\|\varphi - \psi\| < \|\varphi - \psi\|.$$

Therefore, we arrive at a contradiction. We conclude that $\mathcal{A}$ has a unique fixed point which is the unique mild solution of (1.1). □

Theorem 3.5. *Suppose that the conditions of Theorem 3.4 hold. The unique mild solution of (1.1) depends continuously on the function f.*

Proof. Let $f_1, f_2 : J \times \mathbb{R}^n \to [0, +\infty)$ two continuous functions with respect to their arguments. From Theorem 3.4, it follows that there exist two unique corresponding functions x_1 and x_2 in $C_M(J, L)$ such that

$$x_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)\right) ds,$$

and

$$x_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s)\right) ds.$$

We get

$$\begin{aligned} |x_2(t) - x_1(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| & f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s)\right) \\ & - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)\right)\Big| ds. \end{aligned}$$

But

$$\begin{aligned} & \left| f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s)\right) - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)\right)\right| \\ & = \Big| f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s)\right) - f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)\right) \\ & + f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)\right) - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)\right)\Big|. \end{aligned}$$

Using (2.1) and Lemma 2.6, we arrive at

$$\begin{aligned} & \left| f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s)\right) - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)\right)\right| \\ & \leq \|f_2 - f_1\| + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|x_2 - x_1\|. \end{aligned}$$

Hence

$$\|x_2 - x_1\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f_2 - f_1\| + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|x_2 - x_1\|.$$

Therefore

$$\|x_2 - x_1\| \leq \frac{\frac{T^\alpha}{\Gamma(\alpha+1)}}{1 - \frac{T^\alpha}{\Gamma(\alpha+1)} \sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j} \|f_2 - f_1\|.$$

This completes the proof. □

Now, we investigate the Ulam-Hyers stability and generalized Ulam-Hyers stability for the problem (1.1).

Definition 3.6 ([18]). *The problem (1.1) is said to be Ulam-Hyers stable if there exists a real number $K_f > 0$ such that for each $\epsilon > 0$ and for each mild solution $y \in C_M(J, L)$ of the inequality*

$$\left|{}^C D_{0^+}^{\alpha} y(t) - f\left(y^{[0]}(t), y^{[1]}(t), y^{[2]}(t), \dots, y^{[n]}(t)\right)\right| \leq \epsilon, \; t \in J, \tag{3.5}$$

with $y(0) = y'(0) = 0$, there exists a mild solution $x \in C_M(J, L)$ of the problem (1.1) with

$$|y(t) - x(t)| \leq K_f \epsilon, \; t \in J.$$

Definition 3.7 ([18]). *The problem (1.1) is generalized Ulam-Hyers stable if there exists $\psi \in C(J, \mathbb{R}^+)$ with $\psi(0) = 0$ such that for each $\epsilon > 0$ and for each mild solution $y \in C_M(J, L)$ of the inequality (3.5) with $y(0) = y'(0) = 0$, there exists a mild solution $x \in C_M(J, L)$ of the problem (1.1) with*

$$|y(t) - x(t)| \leq \psi(\epsilon), \; t \in J.$$

Theorem 3.8. *Assume that the assumptions of Theorem 3.4 hold. Then the problem (1.1) is Ulam-Hyers stable.*

Proof. Let $y \in C_M(J, L)$ be a mild solution of the inequality (3.5) with $y(0) = y'(0) = 0$, i.e.

$$\begin{cases} \left|{}^C D_{0^+}^{\alpha} y(t) - f\left(y^{[0]}(t), y^{[1]}(t), y^{[2]}(t), \dots, y^{[n]}(t)\right)\right| \leq \epsilon, \; t \in J, \\ y(0) = y'(0) = 0. \end{cases} \tag{3.6}$$

Let us denote by $x \in C_M(J, L)$ the unique mild solution of the problem (1.1). By using Lemma 2.5, we get

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds, \; t \in J.$$

By integration of (3.6), we have

$$\left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(y^{[0]}(s), y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)\right) ds \right| \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \epsilon \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \epsilon.$$

On the other hand, we obtain, for each $t \in J$

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds \right| \\ &\leq \left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(y^{[0]}(s), y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)\right) ds \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(y^{[0]}(s), y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)\right) \right. \\ &\left. - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) \right| ds \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \epsilon + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|y - x\|. \end{aligned}$$

Thus, in view of (3.4)

$$\|y-x\| \leq \frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}}{1-\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j}\epsilon.$$

Then, there exists a real number $K_f = T^{\alpha}/\left(\Gamma(\alpha+1) - T^{\alpha}\sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j\right) > 0$ such that

$$|y(t)-x(t)| \leq K_f \epsilon, \; t \in J. \tag{3.7}$$

Thus, the problem (1.1) is Ulam-Hyers stable, which completes the proof. □

Corollary 3.9. *Suppose that all the assumptions of Theorem 3.8 are satisfied. Then the problem (1.1) is generalized Ulam-Hyers stable.*

Proof. Let $\psi(\epsilon) = K_f \epsilon$ in (3.7) then $\psi(0) = 0$ and the problem (1.1) is generalized Ulam-Hyers stable. □

Example 3.10. *Let us consider the following nonlinear fractional initial value problem*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2}} x(t) = \frac{1}{4} + \frac{1}{4}\cos t + \frac{1}{18}\cos^2(t)\, x^{[1]}(t) + \frac{1}{19}\sin^2(t)\, x^{[2]}(t), \; t \in [0,1], \\ x(0) = x'(0) = 0, \end{cases} \tag{3.8}$$

where $T = 1$, $J = [0,1]$ and

$$f(t,x,y) = \frac{1}{4} + \frac{1}{4}\cos t + \frac{1}{18}x\cos^2(t) + \frac{1}{19}y\sin^2(t).$$

We have

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \frac{1}{18}|x_1-y_1| + \frac{1}{19}|x_2-y_2|,$$

then

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \sum_{i=1}^{2} c_i \|x_i - y_i\|.$$

with $c_1 = \frac{1}{18}$, $c_2 = \frac{1}{19}$. Furthermore, if $L = 1$ and $M = 4$ in the definition of $C_M(J,L)$, then f is positive, $\rho = \sup\limits_{t\in J}\{f(t,0,0)\} = \frac{1}{2}$ and $\zeta = 0.5 + \left(\frac{1}{18} + \frac{4}{19}\right) \simeq 0.766$. For $\alpha = \frac{3}{2}$, we get

$$\frac{\zeta T^{\alpha}}{\Gamma(\alpha+1)} = \frac{0.766}{\Gamma\left(\frac{5}{2}\right)} \simeq 0.576 \leq L = 1,$$

and

$$\frac{\zeta T^{\alpha-1}}{\Gamma(\alpha)} = \frac{0.766}{\Gamma\left(\frac{3}{2}\right)} \simeq 0.864 \leq M = 4.$$

So,

$$\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j = \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{18} + \frac{4}{19}\right) \simeq 0.2 < 1.$$

Then, by Theorems 3.4 and 3.5, (3.8) has a unique mild solution which depends continuously on the function f. Also, from Theorem 3.8, (3.8) is Ulam-Hyers stable, and from Corollary 3.9, (3.8) is generalized Ulam-Hyers stable.

4 Conclusion

In the current paper, under some sufficient conditions on the nonlinearity, we established the existence, uniqueness, continuous dependence and Ulam stability of a mild solution for an iterative Caputo fractional differential equation. The main tool of this work is the Schauder fixed point theorem. The obtained results have a contribution to the related literature.

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A characterization of $\mathbb{F}_q$-linear subsets of affine spaces $\mathbb{F}_{q^2}^n$

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ABSTRACT

Let q be an odd prime power. We discuss possible definitions over $\mathbb{F}_{q^2}$ (using the Hermitian form) of circles, unit segments and half-lines. If we use our unit segments to define the convex hulls of a set $S \subset \mathbb{F}_{q^2}^n$ for $q \notin \{3, 5, 9\}$ we just get the $\mathbb{F}_q$-affine span of S.

RESUMEN

Sea q una potencia de primo impar. Discutimos posibles definiciones sobre $\mathbb{F}_{q^2}$ (usando la forma Hermitiana) de círculos, segmentos unitarios y semi-líneas. Si usamos nuestros segmentos unitarios para definir las cápsulas convexas de un conjunto $S \subset \mathbb{F}_{q^2}^n$ para $q \notin \{3, 5, 9\}$ simplemente obtenemos el $\mathbb{F}_q$-generado afín de S.

Keywords and Phrases: Finite field, Hermitian form.

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1 Introduction

Fix a prime p and a p-power q. There is a unique (up to isomorphism) field $\mathbb{F}_q$ with $\#\mathbb{F}_q = q$. The field $\mathbb{F}_{q^2}$ is a degree 2 Galois extension of $\mathbb{F}_q$ and the Frobenius map $t \mapsto t^q$ is a generator of the Galois group of this extension. This map allows the definition of the Hermitian product $\langle\ ,\ \rangle : \mathbb{F}_{q^2}^n \times \mathbb{F}_{q^2}^n \longrightarrow \mathbb{F}_{q^2}$ in the following way: if $u = (u_1, \dots, u_n) \in \mathbb{F}_{q^2}^n$ and $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^2}^n$, then set $\langle u, v\rangle = \sum_{i=1}^n u_i^q v_i$. The degree $q+1$ hypersurface $\{\langle (x_1, \dots, x_n),\ (x_1, \dots, x_n)\rangle = 0\}$ is the famous full rank Hermitian hypersurface ([11, Ch. 23]).

In the quantum world the classical Hermitian product over the complex numbers is fundamental. The Hermitian product $\langle\ ,\ \rangle$ is one of the tools used to pass from a classical code over a finite field to a quantum code ([17, pp. 430–431], [14, Introduction], [20, §2.2]).

The Hermitian product was used to define the numerical range of a matrix over a finite field ([1, 2, 3, 4, 8]) by analogy with the definition of numerical range for complex matrices ([9, 12, 13, 21]). Over $\mathbb{C}$ a different, but equivalent, definition of numerical range is obtained as the intersection of certain disks ([5, §15, Lemma 1]). It is an important definition, because it was used to extend the use of numerical ranges to rectangular matrices ([7]) and to tensors ([16]). This different definition immediately gives the convexity of the numerical range of complex matrices. Motivated by that definition we look at possible definitions of the unit disk of $\mathbb{F}_{q^2}$. It should be a union of circles with center at 0 and with squared-radius in the unit interval $[0, 1] \subset \mathbb{F}_q$.

For any $c \in \mathbb{F}_q$ and any $a \in \mathbb{F}_{q^2}$ set

$$C(0, c) := \{z \in \mathbb{F}_{q^2} \mid z^{q+1} = c\}, \quad C(a, c) := a + C(0, c).$$

We say that $C(a, c)$ is the *circle of* $\mathbb{F}_{q^2}$ *with center* a *and squared-radius* c. Note that $C(a, 0) = \{a\}$ and $\#C(a, c) = q + 1$ for all $c \in \mathbb{F}_q \setminus \{0\}$.

Circles occur in the description of the numerical range of many 2×2 matrices over $\mathbb{F}_{q^2}$ ([8, Lemmas 3.4 and 3.5]). Other subsets of $\mathbb{F}_{q^2}$ (seen as a 2-dimensional vector space of $\mathbb{F}_q$) appear in [6] and are called ellipses, hyperbolas and parabolas, because they are affine conics whose projective closure have 0, 2 or 1 points in the line at infinity.

All these constructions are inside $\mathbb{F}_{q^2}$ seen as a plane over $\mathbb{F}_q$. Restricting to planes we get the following definition for $\mathbb{F}_{q^2}^n$.

Definition 1.1. *A set* $E \subset \mathbb{F}_{q^2}^n$ *is said to be a circle with center* $0 \in \mathbb{F}_{q^2}^n$ *and squared-radius* c *if there is an* $\mathbb{F}_q$*-linear embedding* $f : \mathbb{F}_{q^2} \longrightarrow \mathbb{F}_{q^2}^n$ *such that* $E = f(C(0, c))$. *A set* $E \subset \mathbb{F}_{q^2}^n$ *is said to be a circle with center* $a \in \mathbb{F}_{q^2}^n$ *and squared-radius* c *if* $E - a$ *is a circle with center* 0 *and squared-radius* c. *A set* $S \subseteq \mathbb{F}_{q^2}^n$, $S \neq \emptyset$, *is said to be circular with respect to* $a \in \mathbb{F}_{q^2}^n$ *if it contains all circles with center* a *which meet* S.

In the classical theory of numerical range over $\mathbb{C}$ the numerical range of a square matrix which is the orthogonal direct sum of the square matrices A and B is obtained taking the union of all segments $[a, b] \subset \mathbb{C}$ with a in the numerical range of A and b in the numerical range of B ([21, p. 3]). For the numerical range of matrices over $\mathbb{F}_{q^2}$ instead of segments $[a, b]$ one has to use the affine $\mathbb{F}_q$-span of $\{a, b\}$ ([1, Lemma 1], [8, Proposition 3.1]). We wonder if in other linear algebra constructions something smaller than $\mathbb{F}_q$-linear span occurs. A key statement for square matrices over $\mathbb{C}$ (due to Toeplitz and Hausdorff) is that their numerical range is convex ([9, Th. 1.1-2], [21, §3]). Convexity is a property over $\mathbb{R}$ and to define it one only needs the unit interval $[0, 1] \subset \mathbb{R}$. Obviously $[0, 1] = [0, +\infty) \cap (-\infty, 1]$ and $(-\infty, 1] = 1 - [0, +\infty)$. As a substitute for the unit interval $[0, 1] \subset \mathbb{R}$ (resp. the half-line $[0, +\infty) \subset \mathbb{R}$) we propose the following sets I_q and I'_q (resp. E_q).

Definition 1.2. *Assume q odd. Set $E_q := \{a^2\}_{a\in\mathbb{F}_q} \subset \mathbb{F}_q$, $I_q := E_q \cap (1 - E_q)$, $I''_q := E_q \cap (1 + xE_q)$ with $x \in \mathbb{F}_q \setminus E_q$, and $I'_q := I''_q \cup \{0\}$.*

Note that $I'_q = \{0, 1\} \cup (E_q \cap (1 + (\mathbb{F}_q \setminus E_q))$. In the first version of this note we only used I_q, but a referee suggested that it is more natural to consider I''_q. We use I_q and I'_q because $\{0, 1\} \subseteq I_q \cap I'_q$, while $0 \in I''_q$ if and only if -1 is not a square in $\mathbb{F}_q$, *i. e.* if and only if $q \equiv 3 \pmod 4$ ([10, (ix) and (x) at p. 5], [22, p. 22]). In all statements for odd q we handle both I_q and I'_q.

In the case q even we propose to use $\{a(a + 1)\}_{\{a\in\mathbb{F}_q\}}$ as E_q, *i. e.* $E_q := \mathrm{Tr}^{-1}_{\mathbb{F}_q/\mathbb{F}_2}(0)$. Thus E_q is a subgroup of $(\mathbb{F}_q, +)$ of index 2. If q is even we do not have a useful definition of I_q.

Thus we restrict to odd prime powers, except for Propositions 1.8, 2.9 and Remarks 2.1 and 2.2.

We see I_q or I'_q (resp. E_q) as the *unit segment* $[0, 1]$ (resp. *positive half-line starting at* 0) of $\mathbb{F}_q \subset \mathbb{F}_{q^2}$. In most of the proofs we only use that $\{0, 1\} \subseteq I_q$ and that $\#I_q$ is large, say $\#I_q > (q - 1)/4$.

Remark 1.3. *Note that $\#E_q = (q + 1)/2$ for all odd prime powers q.*

We prove that $\#I_q = \#I'_q - 1 = (q + 3)/4$ if $q \equiv 1 \mod 4$ and $\#I_q = \#I'_q = (q + 5)/4$ if $q \equiv 3 \pmod 4$ (Proposition 2.3).

We only use the case $A = E_q$, $A = I_q$ and $A = I'_q$ of the following definition.

Definition 1.4. *Fix $S \subseteq \mathbb{F}_{q^2}^n$, $S \neq \emptyset$, and $A \subseteq \mathbb{F}_q$ such that $0 \in A$. We say that S is A-closed if $a + (b - a)A \subseteq S$ for all $a, b \in S$.*

In the set-up of Definition 1.4 for any $a, b \in \mathbb{F}_{q^2}^n$ the A-segment $[a, b]_A$ of $\{a, b\}$ is the set $a+(b-a)A$. Note that $[a, a]_A = \{a\}$ and that if $b \neq a$ then $b \in [a, b]_A$ if and only if $1 \in A$. If S is a subset of a real vector space and A is the unit interval $[0, 1] \subset \mathbb{R}$, Definition 1.4 gives the usual notion of convexity, because $a + (b - a)t = (1 - t)a + tb$ for all $t \in [0, 1]$.

Remark 1.5. *Take any $A \subseteq \mathbb{F}_q$ such that $0 \in A$. Any translate by an element of $\mathbb{F}_{q^2}^n$ of an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^2}^n$ is A-closed. In particular $\mathbb{F}_q^n$ and $\mathbb{F}_{q^2}^n$ are A-closed. The intersection of A-closed sets is A-closed, if non-empty. Hence we may define the A-closure of any $S \subseteq \mathbb{F}_{q^2}^n$, $S \neq \emptyset$, as the intersection of all A-closed subsets of $\mathbb{F}_{q^2}^n$ containing S.*

In most cases I_q is not I_q-closed. We prove the following result.

Theorem 1.6. *Assume q odd. Then:*

(a) If $q \notin \{3, 5, 9\}$ (resp. $q \neq 3$), then $\mathbb{F}_q$ is the I_q-closure of I_q (resp. the I'_q-closure of I'_q).

(b) If $q \notin \{3, 5, 9\}$ (resp. $q \neq 3$), then the I_q-closed (resp. I'_q-closed) subsets of $\mathbb{F}_{q^2}^n$ are the translations of the $\mathbb{F}_q$-linear subspaces.

Remark 1.7. *Fix $A \subseteq \mathbb{F}_q$ such that $0 \in A$. Assume that $\mathbb{F}_q$ is the A-closure of $\mathbb{F}_q$. Then $S \subseteq \mathbb{F}_{q^2}^n$, $S \neq \emptyset$, is A-closed if and only if it is the translation of an $\mathbb{F}_q$-linear subspace by an element of $\mathbb{F}_{q^2}^n$. Thus part (b) of Theorem 1.6 follows at once from part (a) and similar statements are true for the A-closures for any A whose A-closure is $\mathbb{F}_q$.*

As suggested by one of the referees a key part of one of our proofs may be stated in the following general way.

Proposition 1.8. *Let A, B be subsets of $\mathbb{F}_q$ containing 0. Assume $A \neq \{0\}$ and let G be the subgroup of the multiplicative group $\mathbb{F}_q \setminus \{0\}$ generated by $A \setminus \{0\}$. Assume that B is A-closed. Then $B \setminus \{0\}$ is a union of cosets of G.*

Fix $S \subset \mathbb{F}_{q^2}^n$ and a set $A \subset \mathbb{F}_q$ such that $\{0, 1\} \subseteq A$. Instead of the A-closure of S the following sets $S_{i,A}$, $i \geq 1$, seem to be better. In particular both circles and $S_{1,A}$ appear in some proofs on the numerical range. Let $S_{1,A}$ be the set of all $a + (b - a)A$, $a, b \in S$. For all $i \geq 1$ set $S_{i+1,A} := (S_{1,A})_{1,A}$. Obviously $S_{i,A}$ is A-closed for $i \gg 0$. Note that $\{0, 1\}_A = A$ and hence if we start with $S = \{0, 1\}$ we obtain the A-closure of A after finitely many steps.

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2 The proofs and related observations

We assume q odd, except in Remarks 2.1 and 2.2, Proposition 2.9 and the proof of Proposition 1.8.

Remark 2.1. *The notions of E_q-closed, I_q-closed and I'_q-closed subsets of $\mathbb{F}_{q^2}^n$ are invariant by translations of elements of $\mathbb{F}_{q^2}^n$ and by the action of $GL(n, \mathbb{F}_q)$.*

Remark 2.2. *Fix any $A \subseteq \mathbb{F}_q$ such that $0 \in A$. Any translate by an element of $\mathbb{F}_{q^2}^n$ of an A-closed set is A-closed. The $\mathbb{F}_q$-closed subsets of $\mathbb{F}_{q^2}^n$ are the translates by an element of $\mathbb{F}_{q^2}^n$ of the $\mathbb{F}_q$-linear subspaces. If $A \subseteq \{0,1\}$, then any nonempty subset of $\mathbb{F}_{q^2}^n$ is A-closed.*

Proof of Proposition 1.8: Since $\mathbb{F}_q \setminus \{0\}$ is cyclic, G is cyclic. Let $a \in A \setminus \{0\}$ be a generator of G. Fix $c \in B \setminus \{0\}$ and take $t \in \mathbb{F}_q \setminus \{0\}$ such that $c = ta^z$ for some positive integer z. We need to prove that $B \setminus \{0\}$ contains all ta^k, $k \in \mathbb{Z}$. Since $b \in B$, B is A-closed, $a \in A$ and $a = 0 + (a - 0)$, we get $ta^{z+1} \in B$. Iterating this trick we get that B contains all ta^k for large k and hence the coset tG, because G is cyclic. □

Proposition 2.3. *We have $\#I_q = \#I'_q - 1 = (q+3)/4$ if $q \equiv 1 \pmod 4$ and $\#I_q = \#I'_q = (q+5)/4$ if $q \equiv 3 \pmod 4$.*

Proof. Since $A := \{x^2+y^2 = 1\} \subset \mathbb{F}_q^2$ is a smooth affine conic, its projectivization $B := \{x^2+y^2 = z^2\} \subset \mathbb{P}^2(\mathbb{F}_q)$ has cardinality $q+1$ ([10, th. 5.1.8]). Note that the line $z = 0$ is not tangent to B and hence $B \cap \{z = 0\}$ has 2 points over $\mathbb{F}_{q^2}$. It has 2 points over $\mathbb{F}_q$ if and only if -1 is a square in $\mathbb{F}_q$, *i. e.* if and only if $q \equiv 1 \pmod 4$ ([10, (ix) and (x) at p. 5], [22, p. 22]). Hence $\#A = q+1$ if $q \equiv 3 \pmod 4$ and $\#A = q-1$ if $q \equiv 1 \pmod 4$. Note that $a \in I_q$ if and only if there is $(e,f) \in \mathbb{F}_q^2$ such that $e^2+f^2 = 1$ and $a = e^2$. Note that $(e,f) \in A$ and that conversely for each $(e,f) \in A$, $e^2 \in I_q$. Obviously $0 \in I_q$ and $(0,f) \in A$ if and only if either $f = 1$ or $f = -1$. Thus $0 \in I_q$ comes from 2 points of A. Obviously $1 \in I_q$. If either $e = 1$ or $e = -1$, then $(e,f) \in A$ if and only if $f = 0$. Thus $1 \in I_q$ comes from 2 points of A. If $e^2 \notin \{0,1\}$ and $e^2 \in I_q$, then e^2 comes from 4 points of A.

Fix a non-square $c \in \mathbb{F}_q$ and set $A' := \{x^2 - cy^2 = 1\} \subset \mathbb{F}_q^2$. Let $B' := \{x^2 - cy^2 = z^2\} \subset \mathbb{P}^2(\mathbb{F}_q)$ be the smooth conic which is the projectivization of A'. The line $\{z = 0\}$ is not tangent to B' and $\{z = 0\} \cap A' = \emptyset$. Thus $\#A' = q+1$. Note that $a \in I''_q$ if and only if there is $(e,f) \in \mathbb{F}_q^2$ such that $a = e^2$ and $e^2 - cf^2 = 1$. The element $1 \in I''_q$ comes from two elements of A'. If $0 \in I''_q$, then it comes from two elements of A'. If $0 \notin I''_q$, *i. e.* if $q \equiv 3 \pmod 4$, we get $\#I''_q = (q+1)/4$ and $\#I'_q = (q+5)/4$. If $0 \in I''_q$ we get $\#I''_q = \#I'_q = (q+7)/4$. □

Remark 2.4. *If $q \in \{3,5\}$, then $I_q = \{0,1\}$ and hence each non-empty subset of $\mathbb{F}_{q^2}^n$ is I_q-closed if $q \in \{3,5\}$. Since $\{0,1\} \subseteq I'_q$, Proposition 2.3 gives $I'_3 = I_3$. We have $I'_5 = \{0,1,4\} = E_5$, because 3 is not a square in $\mathbb{F}_5$.*

Remark 2.5. *Fix any $t \in \mathbb{F}_q \setminus E_q$. Then $\mathbb{F}_q \setminus E_q = t(E_q \setminus \{0\})$. Obviously $E_qE_q = E_q$.*

The following result characterizes E_{q^2} and hence characterizes all E_r with r a square odd prime power.

Proposition 2.6. *The set of $E_{q^2} \setminus \{0\}$ of all squares of $\mathbb{F}_{q^2} \setminus \{0\}$ is the set of all ab such that $a \in \mathbb{F}_q \setminus \{0\}$ and $b^{q+1} = 1$. We have $ab = a_1b_1$ if and only if $(a_1,b_1) \in \{(a,b),(-a,-b)\}$.*

Proof. Fix $z \in \mathbb{F}_{q^2} \setminus \{0\}$. Hence $z^{q^2-1} = 1$. Thus $z^{(q-1)^{q+1}} = 1$ (and so $z^{(1-q)^{q+1}} = 1$) and $z^{(q+1)^{q-1}} = 1$, *i. e.* $z^{q+1} \in \mathbb{F}_q \setminus \{0\}$. Note that $z^2 = z^{q+1}z^{1-q}$. Assume $ab = a_1b_1$ with a, $a_1 \in \mathbb{F}_q \setminus \{0\}$ *(i.e., with* $a^{q-1} = {a_1}^{q-1} = 1$*) and* $b^{q+1} = {b_1}^{q+1} = 1$*. Taking* $a{a_1}^{-1}$ *and* $b{b_1}^{-1}$ *instead of* a *and* b *we reduce to the case* $a_1 = b_1 = 1$ *and hence* $ab = 1$*. Thus* $a^{q+1}b^{q+1} = 1$*. Hence* $a^2 = 1$*. Since* q *is odd and* $a \neq 1$*, then* $a = -1$*. Thus* $b = -1$*.* □

Proposition 2.7. *Take* $S \subseteq \mathbb{F}_{q^2}^n$*. The set* S *is* E_q*-closed if and only if it is a translation of an* $\mathbb{F}_q$*-linear subspace.*

Proof. Remark 2.2 gives the "if" part. Assume that S is not a translation of an $\mathbb{F}_q$-linear subspace and fix $a, b \in S$ such that $a \neq b$ and the affine $\mathbb{F}_q$-line L spanned by $\{a, b\}$ is not contained in S. By Remark 2.1 it is sufficient to find a contradiction in the case $n = 1$ and $L = \mathbb{F}_q$ with $a = 0$ and $b = 1$. Thus $E_q \subseteq S$. Since S is E_q-closed and $0 \in S$, $c + (-c)E_q \subseteq S$ for all $c \in E_q$. First assume $-1 \in E_q$. In this case $-cE_q = E_q$. Thus S contains all sums of two squares. Thus $S = \mathbb{F}_q$. Now assume $-1 \notin E_q$. In this case we obtained that S contains all differences of two squares. Thus $-E_q \subset S$. Since $-1 \notin E_q$, $-E_q = \{0\} \cup (\mathbb{F}_q \setminus E_q)$ (Remark 2.5). Thus $S \supseteq L$. □

The cases of I_q-closures and I'_q-closures are more complicated, because $I_q = I'_q = \{0, 1\}$ if $q = 3, 5$ and hence all subsets of $\mathbb{F}_{q^2}^n$ are I_q-closed if $q = 3, 5$. The following observation shows that the I_9-closed subsets of $\mathbb{F}_{81}^n$ are exactly the translations of the $\mathbb{F}_3$-linear subspaces and gives many examples with $I_q \nsubseteq I'_q$.

Remark 2.8. *We always have* $2 \notin 1 + cE_q$*,* c *a non-square, because* 1 *is a square. If* q *is a square, say* $q = s^2$*, then obviously* $\mathbb{F}_s \subseteq E_q \cap (1 - E_q) = I_q$ *and hence* $2 \in I_q$*. Take* $q = 9$*. We get* $\mathbb{F}_3 \subseteq I_9$*. Since* $\#I_9 = 3$ *(Proposition 2.3), we get* $I_q = \mathbb{F}_3$*. Thus the* I_9*-closed subsets of* $\mathbb{F}_{81}^n$ *are exactly the translations of the* $\mathbb{F}_3$*-linear subspaces. Now assume that* q *is not a square. We have* $2 \in 1 - E_q$ *if and only if* -1 *is a square, i. e. if and only if* $q \equiv 1 \pmod 4$*. Since* q *is not a square, we have* $2 \in E_q$ *if and only if* 2 *is a square in* $\mathbb{F}_p$*, i. e. if and only if* $p \equiv -1, 1 \pmod 8$ *([15, Proposition 5.1.3]). Thus for a non-square* q *holds:* $2 \in I_q$ *if and only if* $p \equiv 1 \pmod 8$*.*

Proof of Theorem 1.6: Let Y be the I_q-closure of I_q. By Proposition 1.8, $Y' := Y \setminus \{0\}$ is a union of the cosets of $H := \langle I_q \setminus \{0\} \rangle$. Since $\#(I_q \setminus \{0\}) \geq (q-1)/4$ with equality if and only if $q \equiv 1 \pmod 4$, H is either $\mathbb{F}_q^*$, the set of non-zero squares, the set of non-zero cubes or (only if $q \equiv 1 \mod 4$), the set of all non-zero 4-powers. Since $I_q \subseteq E_q$, $H \neq \mathbb{F}_q^*$. If H is the set of cubes, then, as all elements of I_q are squares, it would be the set of 6-th powers, contradicting the inequality $\#I_q > (q-1)/4$.

(a) Assume that $H = E_q \setminus \{0\}$. It suffices to show that the I_q-closure of the set of squares contains a non-square. Suppose otherwise. Take an element $a \in I_q$ with $a \notin \{0, 1\}$. Then we obtain that for all squares x, y, $x + (y - x)a$ is also a square. Since a is a non-zero square, this is the same as the statement that for all squares x, z the element $z + (1-a)x$ is a square. If $1-a$ is a square we deduce that the set of all squares is closed under addition, a contradiction. If $1-a$ is not a square we may take $x = 1$, $z = 0$ to obtain a contradiction.

(b) Assume $q \equiv 1 \pmod 4$, $q \neq 9$, and that H is the set of all non-zero 4-powers. We also saw that $H = I_q \setminus \{0\}$. The proof of step (a) works using the word "4-power" instead of "square" with a a 4-power. We get that the set of all 4-powers is closed under taking differences. Thus I_q is closed under taking differences and, since it contains 0, under the multiplication by -1. H is obviously closed under taking products. Thus I_q is a subfield of order $(q+3)/4$, which is absurd if $q \neq 9$.

(c) Now we consider I'_q and set $H' := \langle I_q \setminus \{0\}\rangle$. The cases in which H' is the set of all squares or all cubes are excluded as above. Since $\#(I'_q \setminus \{0\}) > (q-1)/4$, Y is not the set of all 4-th powers. □

Proposition 2.9. *Assume q even and set $E_q := \{a(a+1)\}_{a \in \mathbb{F}_q}$.*

(1) *If $q = 2, 4$, then E_q is the E_q-closure of itself.*

(2) *If $q \geq 8$, then $\mathbb{F}_q$ is the E_q-closure of E_q.*

Proof. We have $E_2 = \{0\}$ and $E_4 = \{0, 1\}$.

Now assume $q \geq 8$ and call B the E_q-closure of E_q. Let G be the subgroup of the multiplicative group $\mathbb{F}_q \setminus \{0\}$ generated by $E_q \setminus \{0\}$. By Proposition 1.8 it is sufficient to prove that $G = \mathbb{F}_q \setminus \{0\}$. Since $\#E_q = q/2$, $E_q \setminus \{0\} \neq \emptyset$. Fix $a \in E_q \setminus \{0\}$ and a positive integer k. The E_q-closure of $\{0, a^k\}$ contains a^{k+1}. Thus B contains the multiplicative subgroup of $\mathbb{F}_q \setminus \{0\}$ generated by $E_q \setminus \{0\}$. Since $q \geq 8$, $\#(\mathbb{F}_q \setminus \{0\}) = q-1$ is odd and $q - 1 < 3(q/2 - 1) = 3\#(\mathbb{F}_q \setminus \{0\})$, we get $G = \mathbb{F}_q \setminus \{0\}$. □

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Some results on the geometry of warped product CR-submanifolds in quasi-Sasakian manifold

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ABSTRACT

The present paper deals with a study of warped product submanifolds of quasi-Sasakian manifolds and warped product CR-submanifolds of quasi-Sasakian manifolds. We have shown that the warped product of the type $M = D_{\perp} \times_y D_T$ does not exist, where $D_{\perp}$ and D_T are invariant and anti-invariant submanifolds of a quasi-Sasakian manifold $\bar{M}$, respectively. Moreover we have obtained characterization results for CR-submanifolds to be locally CR-warped products.

RESUMEN

El presente artículo trata de un estudio de subvariedades producto alabeadas de variedades cuasi-Sasakianas y CR-subvariedades producto alabeadas de variedades cuasi-Sasakianas. Hemos mostrado que el producto alabeado de tipo $M = D_{\perp} \times_y D_T$ no existe, donde $D_{\perp}$ y D_T son subvariedades invariantes y anti-invariantes de una variedad cuasi-Sasakiana $\bar{M}$, respectivamente. Más aún, hemos obtenido resultados de caracterización para que CR-subvariedades sean localmente CR-productos alabeados.

Keywords and Phrases: Warped product, CR-submanifolds, quasi Sasakian manifold, canonical structure.

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1 Introduction

If (D, g_D) and (E, g_E) are two semi-Riemannian manifolds with metrics g_D and g_E respectively and y a positive differentiable function on D, then the warped product of D and E is the manifold $D\times_y E = (D\times E, g)$, where $g = g_D + y^2 g_E$. Further, let T be tangent to $M = D\times E$ at (p, q). Then we have

$$\|T\|^2 = \|d\pi_1 T\|^2 + y^2\|d\pi_2 T\|^2$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $D\times E$ onto D and E.

A warped product manifold $D\times_y E$ is said to be trivial if the warping function y is constant. In a warped product manifold, we have

$$\nabla_U V = \nabla_V U = (U \ln y)V \tag{1.1}$$

for any vector fields U tangent to D and V tangent to E [5].

The idea of a warped product manifold was introduced by Bishop and O'Neill [5] in 1969. Chen [2] has studied the geometry of warped product submanifolds in Kaehler manifolds and showed that the warped product submanifold of the type $D_\perp\times_y D_T$ is trivial where D_T and $D_\perp$ are ϕ-invariant and anti-invariant submanifolds of a Sasakian manifold, respectively. Many research articles appeared exploring the existence or nonexistence of warped product submanifolds in different spaces [1, 10, 6]. The idea of CR-submanifolds of a Kaehlerian manifold was introduced by A. Bejancu [9]. Later, A. Bejancu and N. Papaghiue [11], introduced and studied the notion of semi-invariant submanifolds of a Sasakian manifold. These submanifolds are closely related to CR-submanifolds in a Kaehlerian manifold. However the existence of the structure vector field implies some important changes. Later on, Binh and De [4] studied CR-warped product submanifolds of a quasi-Saskian manifold. The purpose of this paper is to study the notion of a warped product submanifold of quasi-Sasakian manifolds. In the second section we recall some results and formulae for later use. In the third section, we prove that the warped product in the form $M = D_\perp\times_y D_T$ does not exist except for the trivial case, where D_T and $D_\perp$ are invariant and anti-invariant submanifolds of a quasi-Sasakian manifold $\bar{M}$, respectively. Also, we obtain a characterization result of the warped product CR-submanifolds of the type $M = D_\perp\times_y D_T$.

2 Preliminaries

If $\bar{M}$ is a real $(2n+1)$ dimensional differentiable manifold, endowed with an almost contact metric structure (f, ξ, η, g), then

$$f^2U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad f(\xi) = 0, \quad \eta(fU) = 0, \tag{2.1}$$

$$\eta(U) = g(U, \xi), \quad g(fU, fV) = g(U, V) - \eta(U)\eta(V), \tag{2.2}$$

for any vector fields U, V tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$.

Throughout the paper, all manifolds and maps are differentiable of class C^∞. We denote by $\mathcal{F}\bar{M}$ the algebra of the differentiable functions on $\bar{M}$ and by $\Gamma(E)$ the $\mathcal{F}\bar{M}$ module of the sections of a vector bundle E over $\bar{M}$.

The Nijenhuis tensor field, denoted by N_f, with respect to the tensor field f, is given by

$$N_f(U, V) = [fU, fV] + f^2[U, V] - f[fU, V] + f[U, fV],$$

and the fundamental 2-form Λ is given by

$$\Lambda(U, V) = g(U, fV), \quad \forall U, V \in \Gamma(T\bar{M}).$$

The curvature tensor field of $\bar{M}$, denoted by $\bar{R}$ with respect to the Levi-Civita connection $\bar{\nabla}$, is defined by

$$\bar{R}(U, V)W = \bar{\nabla}_U \bar{\nabla}_V W - \bar{\nabla}_V \bar{\nabla}_U W - \bar{\nabla}_{[U,V]} W, \quad \forall U, V \in \Gamma(T\bar{M}),$$

Definition 2.1.

(a) An almost contact metric manifold $\bar{M}(f, \xi, \eta, g)$ is called normal if

$$N_f(U, V) + 2d\eta(U, V)\xi = 0, \quad \forall U, V \in \Gamma(T\bar{M}),$$

or equivalently

$$(\bar{\nabla}_{fU} f)V = f(\bar{\nabla}_U f)V - g(\bar{\nabla}_U \xi, V)\xi, \quad \forall U, V \in \Gamma(T\bar{M}).$$

(b) The normal almost contact metric manifold $\bar{M}$ is called cosympletic if $d\Lambda = d\eta = 0$.

If $\bar{M}$ is an almost contact metric manifold, then $\bar{M}$ is a quasi-Sasakian manifold if and only if ξ is a Killing vector field [7] and

$$(\bar{\nabla}_U f)V = g(\bar{\nabla}_{fU}\xi, V)\xi - \eta(V)\bar{\nabla}_{fU}\xi, \quad \forall U, V \in \Gamma(T\bar{M}). \tag{2.3}$$

Next we define a tensor field F of type $(1, 1)$ by

$$FU = -\bar{\nabla}_U \xi, \quad \forall U \in \Gamma(T\bar{M}). \tag{2.4}$$

Lemma 2.1. *For a quasi-Sasakian manifold $\bar{M}$, we have*

(i) $(\bar{\nabla}_\xi f)U = 0, \quad \forall U \in \Gamma(T\bar{M})$,

(ii) $f \circ F = F \circ f$,

(iii) $F\xi = 0$,

(iv) $g(FU, V) + g(U, FV) = 0$,

(v) $\eta \circ F = 0$,

(vi) $(\bar{\nabla}_U F)V = \bar{R}(\xi, U)V$,

for all $U, V \in \Gamma(T\bar{M})$.

The tensor field f defines on $\bar{M}$ an f-structure in sense of K. Yano [12], that is

$$f^3 + f = 0.$$

If M is a submanifold of a quasi-Sasakian manifold $\bar{M}$ and denote by N the unit vector field normal to M. Denote by the same symbol g the induced tensor metric on M, by ∇ the induced Levi-Civita connection on M and by $TM^\perp$ the normal vector bundle to M. The Gauss and Weingarten methods are

$$\bar{\nabla}_U V = \nabla_U V + \sigma(U, V), \tag{2.5}$$

$$\bar{\nabla}_U \lambda = -A_\lambda U + \nabla^\perp_U \lambda, \quad \forall U, V \in \Gamma(TM), \tag{2.6}$$

where $\nabla^\perp$ is the induced connection in the normal bundle, σ is the second fundamental form of M and A_λ is the Weingarten endomorphism associated with λ. The second fundamental form σ and the shape operator A are related by

$$g(A_\lambda U, V) = g(h(U, V), \lambda), \tag{2.7}$$

where g denotes the metric on $\bar{M}$ as well as the induced metric on M [7].

For any $U \in TM$, we write

$$fU = rU + sU, \tag{2.8}$$

where rU is the tangential component of fU and sU is the normal component of fU, respectively. Similarly, for any vector field λ normal to M, we put

$$f\lambda = J\lambda + K\lambda \tag{2.9}$$

where $J\lambda$ and $K\lambda$ are the tangential and normal components of $f\lambda$, respectively.

For all $U, V \in \Gamma(TM)$ the covariant derivatives of the tensor fields r and s are defined as

$$(\bar{\nabla}_U r)V = \nabla_U rV - r\nabla_U V, \tag{2.10}$$

$$(\bar{\nabla}_U s)V = \nabla^\perp_U sV - s\nabla_U V. \tag{2.11}$$

3 Warped Product Submanifolds

If D_T and $D_\perp$ are invariant and anti-invariant submanifolds of a quasi-Sasakian manifold $\bar{M}$, then their warped product CR-submanifolds are one of the following forms:

(i) $M = D_\perp \times_y D_T$,

(ii) $M = D_T \times_y D_\perp$.

For case (i), when $\xi \in TD_T$, we have the following theorem.

Theorem 3.1. *There do not exist warped product CR-submanifolds $M = D_\perp \times_y D_T$ in a quasi-Sasakian manifold $\bar{M}$ such that D_T is an invariant submanifold, $D_\perp$ is an anti-invariant submanifold of $\bar{M}$ and ξ is tangent to M.*

Proof. If $M = D_\perp \times_y D_T$ is a warped product CR-submanifold of a quasi-Sasakian manifold $\bar{M}$ such that D_T is an invariant submanifold tangent to ξ and $D_\perp$ is an anti-invariant submanifold of $\bar{M}$, then from (1.1), we have

$$\nabla_U W = \nabla_W U = (W \ln y)U,$$

for any vector fields W and U tangent to $D_\perp$ and D_T, respectively.
In particular,

$$\nabla_W \xi = (W \ln y)\xi, \tag{3.1}$$

using (2.4), (2.5) and ξ is tangent to $D_\perp$, we have

$$\nabla_W \xi = -FW, \quad h(W, \xi) = 0. \tag{3.2}$$

It follows from (3.1) and (3.2) that $W \ln y = 0$, for all $W \in TD_\perp$, *i. e.*, y is constant for all $W \in TD_\perp$. □

Now, the other case, when ξ tangent to $D_\perp$ is dealt in the following two results.

Lemma 3.1. *Let $M = D_\perp \times_y D_T$ be a warped product CR-submanifold of a quasi-Sasakian manifold such that ξ is tangent to $D_\perp$, where $D_\perp$ and D_T are any Riemannian submanifolds of $\bar{M}$. Then*

(i) $\xi \ln y = -F$,

(ii) $g(\sigma(U, fU), sW) = -\{\eta(W)F + (W \ln y)\}\|U\|^2$,

for any $U \in TD_T$ and $W \in TD_\perp$.

Proof. Let $\xi \in TD_\perp$ then for any $U \in TD_T$, we have

$$\nabla_U \xi = (\xi \ln y)U, \tag{3.3}$$

From (2.4) and the fact that ξ is tangent to $D_\perp$, we have $\bar{\nabla}_U \xi = -FU$. With the help of (2.5), we have

$$\nabla_W \xi = -FW, \quad h(W, \xi) = 0. \tag{3.4}$$

From (3.3) and (3.4), we have $\xi \ln y = -F$. Now, for any $U \in TD_T$ and $W \in TD_\perp$, we have $\bar{\nabla}_U fW = (\bar{\nabla}_U f)W + f(\bar{\nabla}_U W)$. Using (2.3), (2.6), (2.8), (2.9) and by the orthogonality of the two distributions, we derive

$$-\eta(W)\bar{\nabla}_{fU}\xi = -A_{sW}U + \nabla_U^\perp sW - r\nabla_U W - s\nabla_U W - Jh(U, W) - Kh(U, W).$$

Equating the tangential components, we get

$$-\eta(W)FfU = A_{sW}U + r\nabla_U W + Jh(U, W).$$

Taking the product with fU and using (2.2) and (2.3), we derive

$$\begin{aligned}-\eta(W)Fg(fU, fU) &= g(A_{sW}U, fU) + (W \ln y)g(rU, fU) + g(Jh(U, W), fU) \\ &= g(h(fU, fU), sW) + (W \ln y)g(fU, fU) + g(fh(U, W), fU).\end{aligned}$$

Using (2.2), we obtain

$$g(\sigma(U, fU), sW) = -\{\eta(W)F + (W \ln y)\}\|U\|^2. \tag{3.5}$$

□

Theorem 3.2. *If $M = D_\perp \times_y D_T$ is a warped product CR-submanifold of a quasi-Sasakian manifold $\bar{M}$ such that ξ is tangent to $D_\perp$ and if $\sigma(U, fU) \in \mu$ the invariant normal subbundle of M, then $W \ln y = -\eta(W)F$, for all $U \in TD_T$ and $Z \in TN_\perp$.*

Proof. The affirmation follows from formula (3.5) by means of the known truth. □

The warped product $M = D_T \times_y D_\perp$, we have the following theorem.

Theorem 3.3. *There do not exist warped product CR-submanifolds $M = D_T \times_y D_\perp$ in a quasi-Sasakian manifold $\bar{M}$ such that ξ is tangent to $D_\perp$.*

Proof. If $\xi \in TN_\perp$, then from (1.1), we have

$$\nabla_U \xi = (U \ln y)\xi, \tag{3.6}$$

for any $U \in TD_T$. While using (2.4), (2.5) and $\xi \in TD_\perp$, we have

$$\nabla_U \xi = -FU, \quad h(U, \xi) = 0. \tag{3.7}$$

From (3.6) and (3.7), it follows that $U \ln y = 0$, for all $U \in TD_T$, and this means that y is constant on N_T. □

The remaining case, when $\xi \in TD_T$ is dealt with the following two theorems.

Theorem 3.4. *Let $M = D_T \times_y D_\perp$ be a warped product CR-submanifold of a quasi-Sasakian manifold $\bar{M}$ such that ξ is tangent to D_T. Then $(\bar{\nabla}_U F)W \in \mu$, for each $U \in TD_T$ and $W \in TD_\perp$, where μ is an invariant normal subbundle of TM.*

Proof. For any $U \in TD_T$ and $W \in TD_\perp$, we have

$$g(f\bar{\nabla}_U W, fW) = g(\bar{\nabla}_U W, W) = g(\nabla_U W, W).$$

Using (1.1), we get

$$g(f\bar{\nabla}_U W, fW) = (U \ln y)\|W\|^2. \tag{3.8}$$

On the other hand, we have

$$\bar{\nabla}_U fW = (\bar{\nabla}_U f)W + f(\bar{\nabla}_U W),$$

for any $U \in TD_T$ and $W \in TD_\perp$. Using (2.3) and the fact that ξ is tangent to D_T, the left-hand side of the above equation is identically zero, that is

$$\bar{\nabla}_U fW = f(\bar{\nabla}_U W). \tag{3.9}$$

Taking the product with fW in (3.9) and making use of formula (2.6), we obtain

$$g(f\bar{\nabla}_U W, fW) = g(\nabla^\perp_U sW, sW).$$

Then from (2.10), we derive $g(f\bar{\nabla}_U W, fW) = g((\bar{\nabla}_U s)W, sW) + g(s\nabla_U W, sW)$.

From (1.1) we have

$$\begin{aligned} g(f\bar{\nabla}_U W, fW) &= (U \ln y)g(sW, sW) + g((\bar{\nabla}_U s)W, sW) \\ &= (U \ln y)g(fW, fW) + g((\bar{\nabla}_U s)W, sW). \end{aligned}$$

Therefore by (2.2), we obtain

$$g(f\bar{\nabla}_U W, fW) = (U \ln y)\|W\|^2 + g((\bar{\nabla}_U s)W, sW). \tag{3.10}$$

Thus (3.8) and (3.9) imply

$$g((\bar{\nabla}_U s)W, sW) = 0. \tag{3.11}$$

Also, as D_T is an invariant submanifold then $fQ \in TD_T$, for any $Q \in TD_T$, thus on using (2.11) and the fact that the product of tangential components with normal is zero, we obtain

$$g((\bar{\nabla}_U s)W, fQ) = 0. \tag{3.12}$$

Hence from (3.11) and (3.12), it follows that $(\bar{\nabla}_U s)W \in \mu$, for all $U \in TD_T$ and $W \in TD_\perp$. $\square$

Theorem 3.5. *A CR-submanifold M of a quasi-Sasakian manifold $(\bar{M}, f, \xi, g)$ is a CR-warped product if and only if the shape operator of M satisfies*

$$A_{fW}U = (fU\mu)W, \quad U \in B \oplus \langle\xi\rangle, \quad W \in B^{\perp}, \tag{3.13}$$

for some function μ on M, fulfilling $C(\mu) = 0$, for each $C \in B^{\perp}$.

Proof. If $M = D_T \times_y D_{\perp}$ is a CR-warped product submanifold of a quasi-Sasakian manifold $\bar{M}$, with $\xi \in TD_T$, then for any $U \in TD_T$ and $W, Q \in TD_{\perp}$, we have

$$\begin{aligned} g(A_{fW}U, Q) &= g(\sigma(U, Q), fW) = g(\bar{\nabla}_Q U, fW) = g(f\bar{\nabla}_Q U, W) \\ &= g(\bar{\nabla}_Q fU, W) - g((\bar{\nabla}_Q f)U, W). \end{aligned}$$

By equations (1.1), (2.3) and the fact that ξ is tangent to D_T, we derive

$$g(A_{fW}U, Q) = (fU \ln y)g(W, Q). \tag{3.14}$$

On the other hand, we have $g(\sigma(U, V), sW) = g(f\bar{\nabla}_U V, W) = -g(fV, \bar{\nabla}_U W)$, for each $U, V \in TD_T$ and $W \in TN_{\perp}$. Using (1.1), we obtain $g(\sigma(U, V), sW) = 0$. Taking into account this fact in (3.14), we obtain (3.13).

Conversely, suppose that M is a proper CR-submanifold of a quasi-Sasakian manifold M satisfying (3.13), then for any $U, V \in B \oplus \langle\xi\rangle$,

$$g(\sigma(U, V), fW) = g(A_{fW}U, V) = 0.$$

This implies that $g(\bar{\nabla}_U fV, W) = 0$, that is, $g(\nabla_U V, W) = 0$. This means $B \oplus \langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. Now, for any $W, Q \in B^{\perp}$ and $U \in B \oplus \langle\xi\rangle$, we have

$$g(\nabla_W Q, fU) = g(\bar{\nabla}_W Q, fU) = g(f\bar{\nabla}_W Q, U) = g(\bar{\nabla}_W fQ, U) - g((f\bar{\nabla}_W f)Q, U).$$

By equations (2.3) and (2.6), it follows that $g(\nabla_W Q, fU) = -g(A_{fQ}W, U)$. Thus from (2.6), we arrive at $g(\nabla_W Q, fU) = -g(\sigma(W, U), fQ)$. Again using (2.7) and (3.13), we obtain

$$g(\nabla_W Q, fU) = -g(A_{fQ}U, W) = -(fU\mu)g(W, Q). \tag{3.15}$$

If $N_{\perp}$ is a leaf of $B^{\perp}$ and $\sigma^{\perp}$ is the second fundamental form of the immersion of $D_{\perp}$ into M, then for any $W, Q \in B^{\perp}$, we have

$$g(\sigma^{\perp}(W, Q), fU) = g(\nabla_W Q, fU). \tag{3.16}$$

Hence, from (3.15) and (3.16), we find that

$$g(\sigma^{\perp}(W, Q), fU) = -(fU\mu)g(W, Q).$$

This means that the integral manifold $D_\perp$ of $B^\perp$ is totally umbilical in M. Since $C(\mu) = 0$ for each $C \in B^\perp$, which implies that the integral manifold of $B^\perp$ is an extrinsic sphere in M, this means that the curvature vector field is nonzero and parallel along $N_\perp$. Hence by virtue of a result in [7], M is locally a warped product $D_T \times_y D_\perp$, where D_T and $N_\perp$ denote the integral manifolds of the distributions $B \oplus \langle \xi \rangle$ and $B^\perp$, respectively and y is the warping function. $\square$

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Optimality of constants in power-weighted Birman–Hardy–Rellich-Type inequalities with logarithmic refinements

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ABSTRACT

The principal aim of this paper is to establish the optimality (*i.e.*, sharpness) of the constants $A(m,\alpha)$ and $B(m,\alpha)$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, of the form

$$A(m,\alpha) = 4^{-m} \prod_{j=1}^{m} (2j-1-\alpha)^2,$$
$$B(m,\alpha) = 4^{-m} \sum_{k=1}^{m} \prod_{\substack{j=1 \\ j\neq k}}^{m} (2j-1-\alpha)^2,$$

in the power-weighted Birman–Hardy–Rellich-type integral inequalities with logarithmic refinement terms recently proved in [41], namely,

$$\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} \big|f(x)\big|^2$$
$$+ B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} \big|f(x)\big|^2,$$
$$f \in C_0^\infty((0,\rho)),\ m,N \in \mathbb{N},\ \alpha \in \mathbb{R},\ \rho,\gamma \in (0,\infty),\ \gamma \geqslant e_N \rho.$$

Here the iterated logarithms are given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{j+1}(\,\cdot\,) = \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N},$$

and the iterated exponentials are defined via

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Moreover, we prove the analogous sequence of inequalities on the exterior interval (r,∞) for $f \in C_0^\infty((r,\infty))$, $r \in (0,\infty)$, and once again prove optimality of the constants involved.

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RESUMEN

El objetivo principal de este artículo es establecer la optimalidad (*i.e.* la precisión) de las constantes $A(m,\alpha)$ y $B(m,\alpha)$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, de la forma

$$A(m,\alpha) = 4^{-m} \prod_{j=1}^{m} (2j-1-\alpha)^2,$$
$$B(m,\alpha) = 4^{-m} \sum_{k=1}^{m} \prod_{\substack{j=1 \\ j\neq k}}^{m} (2j-1-\alpha)^2,$$

en las desigualdades integrales de tipo Birman–Hardy–Rellich pesadas por potencias con términos de refinamiento logarítmicos recientemente demostradas en [41], es decir,

$$\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} \big|f(x)\big|^2$$
$$+ B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} \big|f(x)\big|^2,$$
$$f \in C_0^\infty((0,\rho)),\ m,N \in \mathbb{N},\ \alpha \in \mathbb{R},\ \rho,\gamma \in (0,\infty),\ \gamma \geqslant e_N \rho.$$

Acá los logaritmos iterados están dados por

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{j+1}(\,\cdot\,) = \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N},$$

y las exponenciales iteradas están definidas por

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Más aún, probamos la secuencia análoga de desigualdades en el intervalo exterior (r,∞) para $f \in C_0^\infty((r,\infty))$, $r \in (0,\infty)$, y una vez más probamos la optimalidad de las constantes involucradas.

Keywords and Phrases: Birman-Hardy-Rellich inequalities, logarithmic refinements.

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1 Introduction and notations employed

Given the notation introduced in (1.4)–(1.8) we will prove in this paper that the constants $A(m,\alpha)$ and the N constants $B(m,\alpha)$ appearing in the power-weighted Birman–Hardy–Rellich-type integral inequalities with logarithmic refinement terms,

$$\begin{aligned} \int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} \big|f(x)\big|^2 \\ + B(m,\alpha) \sum_{k=1}^N \int_0^\rho dx\, x^{\alpha-2m} \prod_{p=1}^k [\ln_p(\gamma/x)]^{-2} \big|f(x)\big|^2, \end{aligned} \tag{1.1}$$

$$f \in C_0^\infty((0,\rho)),\ m,N \in \mathbb{N},\ \alpha \in \mathbb{R},\ \rho,\gamma \in (0,\infty),\ \gamma \geqslant e_N \rho,$$

recently proved in [41], are optimal (*i.e.*, sharp). Moreover, we prove optimality of $A(m,\alpha)$ and the N constants $B(m,\alpha)$ for the analogous sequence of inequalities on the exterior interval (r,∞), that is,

$$\begin{aligned} \int_r^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha) \int_r^\infty dx\, x^{\alpha-2m} \big|f(x)\big|^2 \\ + B(m,\alpha) \sum_{k=1}^N \int_r^\infty dx\, x^{\alpha-2m} \prod_{p=1}^k [\ln_p(x/\Gamma)]^{-2} \big|f(x)\big|^2, \end{aligned} \tag{1.2}$$

$$f \in C_0^\infty((r,\infty)),\ m,N \in \mathbb{N},\ \alpha \in \mathbb{R},\ r,\Gamma \in (0,\infty),\ r \geqslant e_N \Gamma.$$

Of course, (1.1) (resp., (1.2)) extends to $N=0$, $\rho=\infty$ (resp., to $N=0$, $r=0$) upon disregarding all logarithmic terms (*i.e.*, upon putting $B(m,\alpha)=0$).

In their simplest (*i.e.*, unweighted) form, the Birman–Hardy–Rellich inequalities, as recorded by Birman in 1961, and in English translation in 1966 [19] (see also [45, pp. 83–84]), are given by

$$\int_0^\rho dx\, \big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\rho dx\, x^{-2m} |f(x)|^2, \tag{1.3}$$

$$f \in C_0^m((0,\rho)),\ m \in \mathbb{N}, \quad 0 < \rho \leqslant \infty.$$

The case $m=1$ in (1.3) represents Hardy's celebrated inequality [51], [52, Sect. 9.8] (see also [61, Chs. 1, 3, App.]), the case $m=2$ is due to Rellich [81, Sect. II.7]. The power-weighted extension of (1.3) is then represented by the first line of (1.1) (*i.e.*, by deleting the second line in (1.1) which contains additional logarithmic refinements).

Even though a detailed history of the power-weighted Birman–Hardy–Rellich inequalities was provided in the companion paper [41], we will now repeat the highlights of this history for matters of completeness.

We start with the observation that the inequalities (1.3) and their power weighted generalizations, that is, the first line in (1.1), are known to be strict, that is, equality holds in (1.3), resp., in the first line in (1.1) (in fact, for the entire inequality (1.1)) if and only if $f=0$ on $(0,\rho)$. Moreover, these inequalities are optimal, meaning, the constants $[(2m-1)!!]^2/2^{2m}$ in (1.3), respectively, the constants $A(m,\alpha)$ in (1.1) are sharp, although, this must be qualified and will be revisited below as different authors frequently prove sharpness for different function spaces. In the present one-dimensional context at hand, sharpness of (1.3) (and typically, its power weighted version, the first line in (1.1)), are often proved in an integral form (rather than the currently presented differential form) where $f^{(m)}$ on the left-hand side is replaced by F and f on the right-hand side by m repeated integrals over F. For pertinent one-dimensional sources, we refer, for instance, to [14, pp. 3–5], [22], [24, pp. 104–105], [42, 49, 51], [52, pp. 240–243], [61, Ch. 3], [62, pp. 5–11], [64, 72, 80]. We also note that higher-order Hardy inequalities, including various weight functions, are discussed in [60, Sect. 5], [61, Chs. 2–5], [62, Chs. 1–4], [63], and [79, Sect. 10] (however, Birman's sequence of inequalities (1.3) is not mentioned in these sources). In addition, there are numerous sources which treat multi-dimensional versions of these inequalities on various domains $\Omega \subseteq \mathbb{R}^n$, which, when specialized to radially symmetric functions (*e.g.*, when Ω represents a ball), imply one-dimensional Birman–Hardy–Rellich-type inequalities with power weights under various restrictions on these weights. However, none of the results obtained in this manner imply (1.1), under optimal hypotheses on α and γ. We also mention that a large number of these references treat the L^p-setting, and in some references $x \in (a,b)$ is replaced by $d(x)$, the distance of x to the boundary of (a,b), respectively, Ω, but this represents quite a different situation (especially in the multi-dimensional context) and hence is not further discussed in this paper.

To put the logarithmic refinements in (1.1) (*i.e.*, the second line in (1.1)) into some perspective and to compare with existing results in the literature, we offer the following comments: originally, logarithmic refinements of Hardy's inequality started with oscillation theoretic considerations going back to Hartman [53] (see also [54, pp. 324–325]) and have been used in connection with Hardy's inequality in [38, 43], and more recently, in [39, 40]. Since then there has been enormous activity in this context and we mention, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], [14, Chs. 3, 5], [16, 17, 18, 21, 23, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 37, 39, 44, 46, 47], [48, Chs. 2, 6, 7], [56, 57, 65, 66, 67, 68, 70, 71, 74, 76, 77], [81, Sect. 2.7], [82, 83, 84, 88, 89, 90, 91]. The vast majority of these references deals with analogous multi-dimensional settings (relevant to our setting in particular in the case of radially symmetric functions), several also with the L^p-context. For $m \geqslant 2$ the inequalities (1.1) and (1.2) proven in [41] were new in the following sense: the weight parameter $\alpha \in \mathbb{R}$ is unrestricted (as opposed to prior results) and at the same time the conditions on the logarithmic parameters γ and Γ are sharp.

The issue of sharpness of the constants $A(m,\alpha)$ and $B(m,\alpha)$ appearing in (1.1) is a rather delicate one and hence we offer the following remarks, the gist of which can be found in [41, Appendix A].

We start by noting that the smaller the underlying function space, the larger the efforts needed to prove optimality. Many of the results cited in the remainder of this remark, under particular restrictions on the weight parameter α, establish sharpness for larger classes of functions f which do not automatically continue to hold in the $C_0^\infty((0,\rho))$-context. It is this simple observation that adds considerable complexity to sharpness proofs for the space $C_0^\infty((0,\rho))$. (The issue of dependence of optimal constants on the underlying function space is nicely illustrated in [30].) By the same token, optimality proofs obtained for C_0^∞ function spaces automatically hold for larger function spaces as long as the inequalities have already been established for the larger function spaces with the same constants $A(m,\alpha), B(m,\alpha)$. This comment applies, in particular, to many papers that prove sharpness results in multi-dimensional situations for larger function spaces such as[1] $C_0^\infty(B(0;\rho))$ or (homogeneous, weighted) Sobolev spaces rather than $C_0^\infty(B(0;\rho)\backslash\{0\})$. Unless $C_0^\infty(B(0;\rho)\backslash\{0\})$ is dense in the appropriate norm, one cannot *a priori* assume that the optimal constants $A(m,\widetilde{\alpha})$ and $B(m,\widetilde{\alpha})$ (with $\widetilde{\alpha}$ appropriately depending on n, *e.g.*, $\widetilde{\alpha}=\alpha+n-1$) remain the same for $C_0^\infty(B(0;\rho))$ and $C_0^\infty(B(0;\rho)\backslash\{0\})$, say. At least in principle, they could actually increase for the space $C_0^\infty(B(0;\rho)\backslash\{0\})$.

Turning to a review of the existing literature, sharpness of the constant $A(m,0)$, $m\in\mathbb{N}$ (*i.e.*, in the unweighted case, $\alpha=0$), corresponding to the space $C_0^\infty((0,\infty))$ has been shown by Yafaev [91]. In fact, he also established this result for fractional m (in this context we also refer to appropriate norm bounds in $L^p(\mathbb{R}^n;d^nx)$ of operators of the form $|x|^{-\beta}\,|-i\nabla|^{-\beta}$, $1<p<n/\beta$, see [13, Sect. 1.7],[14, 55, 58, 59, 78, 86], [87, Sects. 1.7, 4.2]). Sharpness of $A(2,0)$ (*i.e.*, in the unweighted Rellich case) was shown by Rellich [81, pp. 91–101] in connection with the space $C_0^\infty((0,\infty))$; his multi-dimensional results also yield sharpness of $A(2,n-1)$ for $n\in\mathbb{N}$, $n\geqslant 3$, again for $C_0^\infty((0,\infty))$; in this context see also [14, Corollary 6.3.5]. An exhaustive study of optimality of $A(2,\widetilde{\alpha})$ (*i.e.*, Rellich inequalities with power weights) for the space $C_0^\infty(\Omega\backslash\{0\})$ for cones $\Omega\subseteq\mathbb{R}^n$, $n\geqslant 2$, appeared in Caldiroli and Musina [21]. The authors, in particular, describe situations where $A(2,\widetilde{\alpha})$ has to be replaced by other constants and also treat the special case of radially symmetric functions in detail. Additional results for power weighted Rellich inequalities appeared in [74, 75]; further extensions of power weighted Rellich inequalities with sharp constants on $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ were obtained in [69]; for optimal power weighted Hardy, Rellich, and higher-order inequalities on homogeneous groups, see [82, 83]. Many of these references also discuss sharp (power weighted) Hardy inequalities, implying optimality for $A(1,\widetilde{\alpha})$. Moreover, replacing $f(x)$ by $F(x)=\int_0^x dt\,f(t)$ $\big($or $F(x)=\int_x^\infty dt\,f(t)\big)$, optimality of the Hardy constant $A(1,0)$ for larger, L^p-based function spaces, can already be found in [52, Sect. 9.8] (see also [14, Theorem 1.2.1], [61, Ch. 3], [62, pp. 5–11], [64, 72, 80], in connection with $A(1,\alpha)$). We mention that Theorems 4.1 and 4.7, which assert optimality of $A(m,\alpha)$ in (1.1) and (1.2), were already proved in [41, Theorem A.1] using a different method.

Sharpness results for $A(m,\alpha)$ and $B(m,\alpha)$ together are much less frequently discussed in the literature, even under suitable restrictions on m and α. The results we found primarily follow upon specializing multi-dimensional results for function spaces such as $C_0^\infty(\Omega\backslash\{0\})$, or $C_0^\infty(\Omega)$, $\Omega\subseteq\mathbb{R}^n$

[1]Here $B(0;\rho)\subseteq\mathbb{R}^n$ denotes the open ball in $\mathbb{R}^n$, $n\geqslant 2$, with center at the origin $x=0$ and radius $\rho>0$.

open, and appropriate restrictions on m, α, and $n \geqslant 2$, for radially symmetric functions to the one-dimensional case at hand (*cf.* the previous paragraph). In this context we mention that the Hardy case $m = 1$, without a weight function, is studied in [1, 2, 5, 9, 20, 23, 26, 36, 50, 57, 65, 85, 89] (all for $N = 1$), and in [10, 28, 46] (all for $N \in \mathbb{N}$); the case with power weight functions is discussed in [17, 47], [48, Ch. 6] (for $N \in \mathbb{N}$); see also [66]. The Rellich case $m = 2$ with a general power weight on $C_0^\infty(\Omega \backslash \{0\})$ is discussed in [21] (for $N = 1$); the Rellich case $m = 2$, without weight function on $C_0^\infty(\Omega)$, is studied in [26, 27, 29] (all for $N = 1$), the case $N \in \mathbb{N}$ is studied in [4]; the case of additional power weights is treated in [47], [48, Ch. 6], [71]. The general case $m \in \mathbb{N}$ is discussed in [6] (for $N = 1$) and in [15, 47], [48, Ch. 6], [90] (all for $N \in \mathbb{N}$ and including power weights, but with additional restrictions). Employing oscillation theory, sharpness of the unweighted Hardy case $A(1,0) = B(1,0) = 1/4$, with $N \in \mathbb{N}$, was proved in [43].

As will become clear in the course of this paper, the special results available on sharpness of the N constants $B(m,\alpha)$ are all saddled with considerable complexity, especially, for larger values of $N \in \mathbb{N}$. For this reason only sharpness of the constants $A(m,\alpha)$ was derived in [41, Appendix A] and sharpness of $A(m,\alpha)$ and $B(m,\alpha)$ was postponed to this paper which therefore should be viewed as a companion of [41].

In Section 2 (a very massive one) we establish all the preliminary results, culminating in Lemmas 2.13 and 2.14, required in the remainder of this paper. The methods used in this section are adaptations of those in [15, Sect. 3]. The basic approximation procedure is introduced in Section 3, with Corollaries 3.12 and 3.13 summarizing the principal results. Our final Section 4 then proves optimality of the $N+1$ constants $A(m,\alpha)$ and $B(m,\alpha)$ for the interval $(0,\rho)$ in Theorems 4.1 and 4.2 and for the interval (r,∞) in Theorems 4.7 and 4.8 based on Lemmas 2.13 and 2.14 and Corollaries 3.12 and 3.13. We also mention that Theorems 4.2 and 4.8 still hold if the repeated log-terms $\ln_p(\,\cdot\,)$ (see (1.5) below) are replaced by the type of repeated log-terms used, for example, in [15, 16, 17, 90].[2]

We conclude this introduction by establishing the principal notation used in this paper: for $j \in \mathbb{N}_0$ (with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) we define e_j by

$$e_0 = 0, \quad e_1 = 1, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}. \tag{1.4}$$

For $N \in \mathbb{N}$, $\gamma, \rho \in (0,\infty)$, with $\gamma \geqslant \rho e_N$, and $1 \leqslant j \leqslant N$, we define $\ln_j(\gamma/x)$, for $0 < x < \rho$, by

$$\ln_1(\gamma/x) = \ln(\gamma/x), \quad \ln_{j+1}(\gamma/x) = \ln(\ln_j(\gamma/x)), \quad 1 \leqslant j \leqslant N-1. \tag{1.5}$$

For the rest of this paper we shall assume that $N \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\gamma, \rho \in (0,\infty)$, with

[2]Detailed proofs of Theorems 4.2 and 4.8 for the type of log-terms used in [15, 16, 17, 90] are available from the authors upon request.

$\gamma \geqslant \rho e_{N+1}$. We shall write

$$A(m,\alpha) = 4^{-m} \prod_{j=1}^{m} (2j-1-\alpha)^2, \tag{1.6}$$

$$B(m,\alpha) = 4^{-m} \sum_{k=1}^{m} \prod_{j=1, j\neq k}^{m} (2j-1-\alpha)^2. \tag{1.7}$$

Note that if $\alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant m}$, one has

$$B(m,\alpha) = A(m,\alpha) \sum_{j=1}^{m} (2j-1-\alpha)^{-2}. \tag{1.8}$$

We assume $\psi \in C^\infty(\mathbb{R})$ satisfies the following properties:

$$(i) \quad \psi \text{ is non-increasing}, \tag{1.9}$$

$$(ii) \quad \psi(x) = \begin{cases} 1, & x \leqslant 8\rho/10, \\ 0, & x \geqslant 9\rho/10. \end{cases} \tag{1.10}$$

For $g \in C^\infty((0,\rho))$ we shall write

$$\begin{aligned} J_N[g] = & \int_0^\rho dx\, x^\alpha \left|g^{(m)}(x)\right|^2 - A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} |g(x)|^2 \\ & - B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} |g(x)|^2 \prod_{j=1}^{k} [\ln_j(\gamma/x)]^{-2}, \end{aligned} \tag{1.11}$$

provided that

$$\int_0^\rho dx\, x^\alpha \left|g^{(m)}(x)\right|^2 < \infty, \quad \int_0^\rho dx\, x^{\alpha-2m} |g(x)|^2 < \infty. \tag{1.12}$$

For $j = 0, 1, \ldots, N$ and $\beta \in \mathbb{R}$ we introduce

$$\begin{aligned} &\sigma_0(\beta) = (2m-1-\alpha+\beta)/2, \\ &\sigma_j(\beta) = -(1-\beta)/2, \quad j = 1, \ldots, N. \end{aligned} \tag{1.13}$$

For $0 \leqslant j \leqslant k \leqslant N$ and $\underline{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$, where $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N > 0$, we shall write

$$\begin{aligned} \Gamma_{j,k}(\underline{\varepsilon}) &= \Gamma_{j,k}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N), \\ &= \int_0^\rho dx\, x^{-1+\varepsilon_0} [\ln_1(\gamma/x)]^{-1-\varepsilon_1} \cdots [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \\ &\quad \times [\ln_{j+1}(\gamma/x)]^{-\varepsilon_{j+1}} \cdots [\ln_k(\gamma/x)]^{-\varepsilon_k} \\ &\quad \times [\ln_{k+1}(\gamma/x)]^{1-\varepsilon_{k+1}} \cdots [\ln_N(\gamma/x)]^{1-\varepsilon_N} [\psi(x)]^2. \end{aligned} \tag{1.14}$$

In particular, if $N \in \mathbb{N}$,

$$\begin{aligned}
\Gamma_{0,0}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{k=1}^N [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2, \\
\Gamma_{0,k}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{\ell=1}^k [\ln_\ell(\gamma/x)]^{-\varepsilon_\ell} \prod_{p=k+1}^N [\ln_p(\gamma/x)]^{1-\varepsilon_p} [\psi(x)]^2, \quad k = 1, \dots, N, \\
\Gamma_{k,k}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{\ell=1}^k [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \prod_{p=k+1}^N [\ln_p(\gamma/x)]^{1-\varepsilon_p} [\psi(x)]^2, \quad k = 1, \dots, N, \\
\Gamma_{N,N}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{\ell=1}^N [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} [\psi(x)]^2.
\end{aligned} \tag{1.15}$$

For $k \in \mathbb{N}$ we shall write P_k for the polynomial

$$P_k(\sigma) = \sigma(\sigma - 1) \cdots (\sigma - k + 1), \quad \sigma \in \mathbb{R}. \tag{1.16}$$

For $\underline{\beta} = (\beta_0, \beta_1, \dots, \beta_N)$, where $\beta_0, \beta_1, \dots, \beta_N \in \mathbb{R}$, we introduce

$$v_{\underline{\beta}}(x) = v_{\beta_0,\beta_1,\dots,\beta_N}(x) = \begin{cases} x^{\sigma_0(\beta_0)}, & 0 < x < \rho,\ N = 0, \\ x^{\sigma_0(\beta_0)} \prod_{\ell=1}^N [\ln_\ell(\gamma/x)]^{-\sigma_\ell(\beta_\ell)}, & 0 < x < \rho,\ N \in \mathbb{N}, \end{cases} \tag{1.17}$$

and

$$f_{\underline{\beta}}(x) = f_{\beta_0,\beta_1,\dots,\beta_N}(x) = v_{\underline{\beta}}(x)\psi(x), \quad 0 < x < \rho. \tag{1.18}$$

If $N \in \mathbb{N}$ and $\underline{\varepsilon}_1 = (\varepsilon_1, \dots, \varepsilon_N)$, where $\varepsilon_1, \dots, \varepsilon_N > 0$, we define $h_{\ell,\underline{\varepsilon}_1} : (0, \rho) \to \mathbb{R}$, $\ell \in \mathbb{N}$, iteratively by

$$\begin{aligned}
h_{1,\underline{\varepsilon}_1}(x) &= h_{1,\varepsilon_1,\dots,\varepsilon_N}(x) = \sum_{k=1}^N \sigma_k(\varepsilon_k) \prod_{j=1}^k [\ln_j(\gamma/x)]^{-1}, \\
h_{\ell+1,\underline{\varepsilon}_1}(x) &= x h'_{\ell,\underline{\varepsilon}_1}(x), \quad \ell \in \mathbb{N}.
\end{aligned} \tag{1.19}$$

Note that, since $\gamma/x > \gamma/\rho \geqslant e_{N+1}$, one infers that

$$[\ln_j(\gamma/x)]^{-1} \leqslant 1, \quad 0 < x < \rho,\ j = 1, \dots, N. \tag{1.20}$$

For $0 \leqslant j \leqslant k \leqslant N$ and $\beta_0, \beta_1, \dots, \beta_N \in \mathbb{R}$, we define $a_{j,k}(\underline{\beta}) = a_{j,k}(\beta_0, \beta_1, \dots, \beta_N)$ by

$$\begin{aligned}
a_{0,0}(\underline{\beta}) &= \big[P_m(\sigma_0(\beta_0))\big]^2 - A(m, \alpha), \\
a_{N,N}(\underline{\beta}) &= \sigma_N(\beta_N)\Big\{P_m(\sigma_0(\beta_0))P''_m(\sigma_0(\beta_0))[\sigma_N(\beta_N) + 1] + \big[P'_m(\sigma_0(\beta_0))\big]^2 \sigma_N(\beta_N)\Big\}, \\
a_{j,j}(\underline{\beta}) &= \sigma_j(\beta_j)\Big\{P_m(\sigma_0(\beta_0))P''_m(\sigma_0(\beta_0))[\sigma_j(\beta_j) + 1] + \big[P'_m(\sigma_0(\beta_0))\big]^2 \sigma_j(\beta_j)\Big\} \\
&\quad - B(m, \alpha), \qquad 1 \leqslant j \leqslant N - 1, \\
a_{0,j}(\underline{\beta}) &= 2\sigma_j(\beta_j)P_m(\sigma_0(\beta_0))P'_m(\sigma_0(\beta_0)), \qquad 1 \leqslant j \leqslant N, \\
a_{j,k}(\underline{\beta}) &= \sigma_k(\beta_k)\Big\{P_m(\sigma_0(\beta_0))P''_m(\sigma_0(\beta_0))[2\sigma_j(\beta_j) + 1] + 2\big[P'_m(\sigma_0(\beta_0))\big]^2 \sigma_j(\beta_j)\Big\}, \\
&\qquad 1 \leqslant j < k \leqslant N.
\end{aligned} \tag{1.21}$$

If $N \in \mathbb{N}$, $\beta_0, \beta_1, \dots, \beta_N \in \mathbb{R}$, and $1 \leqslant j \leqslant k \leqslant N$, then we define $b_{j,k}(\underline{\beta}) = b_{j,k}(\beta_0, \beta_1, \dots, \beta_N)$ by

$$\begin{aligned} b_{j,j}(\underline{\beta}) &= \frac{1}{4}\Big[P_m(\sigma_0(\beta_0))P_m''(\sigma_0(\beta_0)) + \big[P_m'(\sigma_0(\beta_0))\big]^2\Big](\beta_j - \beta_j^2) + a_{j,j}(\underline{\beta}), \\ &\qquad\qquad 1 \leqslant j \leqslant N, \\ b_{j,k}(\underline{\beta}) &= a_{j,k}(\underline{\beta}) - \frac{1}{4}\Big[P_m(\sigma_0(\beta_0))P_m''(\sigma_0(\beta_0)) + \big[P_m'(\sigma_0(\beta_0))\big]^2\Big](1 - 2\beta_j)(1 - \beta_k), \\ &\qquad\qquad 1 \leqslant j < k \leqslant N. \end{aligned} \tag{1.22}$$

For the rest of this paper we shall assume that $M \in (0, \infty)$ is fixed and that $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, constants denoted by $c_j, j \in \mathbb{N}$, will depend on $N \in \mathbb{N} \cup \{0\}$, $\gamma, \rho \in (0, \infty)$ with $\gamma \geqslant \rho e_{N+1}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $M \in (0, \infty)$, and $\psi \in C^\infty(\mathbb{R})$, but will be independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$.

2 Preliminary results

We mention again that the methods used in this section are adapted from [15, Sect. 3].

Lemma 2.1. *Let $j \in \{1, \dots, N+1\}$ and $\beta \in \mathbb{R}$. Then, for all $0 < x < \rho$,*

$$\frac{d}{dx}[\ln_j(\gamma/x)]^{-\beta} = \beta x^{-1}[\ln_1(\gamma/x)]^{-1} \cdots [\ln_{j-1}(\gamma/x)]^{-1}[\ln_j(\gamma/x)]^{-1-\beta}. \tag{2.1}$$

Proof. For $j = 1$, (2.1) clearly holds. Suppose that (2.1) holds for $j \in \{1, \dots, N\}$. Then

$$\begin{aligned} \frac{d}{dx}[\ln_{j+1}(\gamma/x)]^{-\beta} &= \frac{d}{dx}[\ln(\ln_j(\gamma/x))]^{-\beta} \\ &= -\beta[\ln_{j+1}(\gamma/x)]^{-1-\beta}[\ln_j(\gamma/x)]^{-1}\frac{d}{dx}[\ln_j(\gamma/x)]^{-(-1)} \\ &= -\beta[\ln_{j+1}(\gamma/x)]^{-1-\beta}[\ln_j(\gamma/x)]^{-1}(-1)x^{-1}\prod_{k=1}^{j-1}[\ln_k(\gamma/x)]^{-1} \\ &= \beta x^{-1}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-1}[\ln_{j+1}(\gamma/x)]^{-1-\beta}. \end{aligned} \tag{2.2}$$

The result now follows by induction. $\square$

Lemma 2.2.

(i) $\big[P_m(\sigma_0(0))\big]^2 = A(m, \alpha)$.

(ii) $\dfrac{1}{4}\Big\{\big[P_m'(\sigma_0(0))\big]^2 - P_m(\sigma_0(0))P_m''(\sigma_0(0))\Big\} = B(m, \alpha)$.

Proof. Since *(i)* is clear, we only need to prove *(ii)*. Since both sides of *(ii)* are continuous in α, we may assume that $\alpha \in \mathbb{R}\backslash\{1, 3, \dots, 2m-1\}$. For $\sigma \in \mathbb{R}\backslash\{0, 1, \dots, m-1\}$ one gets

$$\begin{aligned} P_m'(\sigma) &= (\sigma - 1)(\sigma - 2)\cdots(\sigma - m + 1) \\ &\quad + \sigma(\sigma - 2)\cdots(\sigma - m + 1) + \cdots + \sigma(\sigma - 1)\cdots(\sigma - m + 2) \\ &= \sigma^{-1}P_m(\sigma) + (\sigma - 1)^{-1}P_m(\sigma) + \cdots + (\sigma - m + 1)^{-1}P_m(\sigma), \end{aligned} \tag{2.3}$$

hence

$$P_m'(\sigma)\big[P_m(\sigma)\big]^{-1} = \sum_{j=0}^{m-1}(\sigma - j)^{-1}, \tag{2.4}$$

thus, differentiating both sides,

$$P_m(\sigma)P_m''(\sigma) - [P_m'(\sigma)]^2 = -[P_m(\sigma)]^2\sum_{j=0}^{m-1}(\sigma - j)^{-2}. \tag{2.5}$$

Put $\sigma = (2m-1-\alpha)/2$. Then $\sigma \in \mathbb{R}\backslash\{0,1,\ldots,m-1\}$ if and only if $\alpha \in \mathbb{R}\backslash\{1,3,\ldots,2m-1\}$. So, by (2.5), part *(i)*, and (1.8), for $\alpha \in \mathbb{R}\backslash\{1,3,\ldots,2m-1\}$, one obtains

$$\begin{aligned}&[P_m'((2m-1-\alpha)/2)]^2 - P_m((2m-1-\alpha)/2)P_m''((2m-1-\alpha)/2)\\ &\quad = [P_m((2m-1-\alpha)/2)]^2\sum_{j=0}^{m-1}\left(\frac{2m-1-\alpha}{2} - j\right)^{-2},\end{aligned} \tag{2.6}$$

that is,

$$\begin{aligned}[P_m'(\sigma_0(0))]^2 - P_m(\sigma_0(0))P_m''(\sigma_0(0)) &= 4\big[P_m(\sigma_0(0))\big]^2\sum_{j=0}^{m-1}(2(m-j)-1-\alpha)^{-2}\\ &= 4A(m,\alpha)\sum_{j=1}^{m}(2j-1-\alpha)^{-2}\\ &= 4B(m,\alpha).\end{aligned} \tag{2.7}$$

$\square$

Remark 2.3. *Let $h_{\ell,\underline{\varepsilon}_1} : (0,\rho) \to \mathbb{R}$, $\ell \in \mathbb{N}$, be as in* (1.19)*. For all $\ell \in \mathbb{N}$ with $\ell \geqslant 3$, there exists $c_1(\ell) > 0$ such that for all $\varepsilon_1,\ldots,\varepsilon_N \in (0,M)$ one has*

$$|h_{\ell,\underline{\varepsilon}_1}(x)| \leqslant c_1(\ell)[\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho. \tag{2.8}$$

Lemma 2.4. *Suppose $N \in \mathbb{N}$. Let $v_{\underline{\varepsilon}} = v_{\varepsilon_0,\varepsilon_1,\ldots\varepsilon_N} : (0,\rho) \to (0,\infty)$ be defined as in* (1.17)*. Then, for $\tau \in \mathbb{N}$,*

$$\begin{aligned}v_{\underline{\varepsilon}}^{(\tau)}(x) = x^{\sigma_0(\varepsilon_0)-\tau}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j(\varepsilon_j)}\Big\{&P_\tau(\sigma_0(\varepsilon_0))\\ &+ P_\tau'(\sigma_0(\varepsilon_0))h_{1,\underline{\varepsilon}_1}(x) + (1/2)P_\tau''(\sigma_0(\varepsilon_0))[h_{1,\underline{\varepsilon}_1}(x)]^2 + (1/2)P_\tau'(\sigma_0(\varepsilon_0))h_{2,\underline{\varepsilon}_1}(x)\\ &+ E_{\tau,\underline{\varepsilon}}(x)\Big\}, \quad 0 < x < \rho,\end{aligned} \tag{2.9}$$

where $E_{\tau,\underline{\varepsilon}}(x)$ is of the form

$$\begin{aligned}E_{\tau,\underline{\varepsilon}}(x) &= E_{\tau,\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N}(x)\\ &= \sum_{j=1}^{Q(\tau)} p_{\tau,j}[h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,1}}\cdots[h_{\tau,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,\tau}}, \quad 0 < x < \rho,\end{aligned} \tag{2.10}$$

for some $Q(\tau) \in \mathbb{N}, w_{\tau,j,k} \in \mathbb{N} \cup \{0\}$ *for all* $j \in \{1, \dots, Q(\tau)\}$ *and* $k \in \{1, \dots, \tau\}$*,* $p_{\tau,j} \in \mathbb{R}$ *for all* $j \in \{1, \dots, Q(\tau)\}$*. Moreover, there exists* $c_2 = c_2(\tau) > 0$*, independent of* $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N$*, such that*

$$\left|p_{\tau,j}[h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,1}} \cdots [h_{\tau,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,\tau}}\right| \leqslant c_2[\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho, \tag{2.11}$$

for all $j \in \{1, \dots, Q(\tau)\}$*. Hence*

$$\left|E_{\tau,\underline{\varepsilon}}(x)\right| \leqslant c_2 Q(\tau)[\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho. \tag{2.12}$$

Proof. We prove this result by induction on $\tau \in \mathbb{N}$. For brevity we shall write $\sigma_j = \sigma_j(\varepsilon_j), j = 0, 1, \dots, N$, in this proof. For $\tau = 1$ we have, by Lemma 2.1,

$$v'_{\underline{\varepsilon}}(x) = x^{\sigma_0 - 1} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \left(\sigma_0 + h_{1,\underline{\varepsilon}_1}(x)\right), \quad 0 < x < \rho. \tag{2.13}$$

For $\tau = 2$ we have

$$\begin{aligned} v''_{\underline{\varepsilon}}(x) &= x^{\sigma_0 - 2} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \left(\sigma_0 - 1 + h_{1,\underline{\varepsilon}_1}(x)\right)\left(\sigma_0 + h_{1,\underline{\varepsilon}_1}(x)\right) \\ &\quad + x^{\sigma_0 - 1} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \left(x^{-1} h_{2,\underline{\varepsilon}_1}(x)\right) \\ &= x^{\sigma_0 - 2} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \left\{\sigma_0(\sigma_0 - 1) + (2\sigma_0 - 1) h_{1,\underline{\varepsilon}_1}(x) + [h_{1,\underline{\varepsilon}_1}(x)]^2 + h_{2,\underline{\varepsilon}_1}(x)\right\}. \end{aligned} \tag{2.14}$$

For $\tau = 3$ we have

$$\begin{aligned} v'''_{\underline{\varepsilon}}(x) &= x^{\sigma_0 - 3} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \left(\sigma_0 - 2 + h_{1,\underline{\varepsilon}_1}(x)\right)\left\{\sigma_0(\sigma_0 - 1)\right. \\ &\quad \left. + (2\sigma_0 - 1) h_{1,\underline{\varepsilon}_1}(x) + [h_{1,\underline{\varepsilon}_1}(x)]^2 + h_{2,\underline{\varepsilon}_1}(x)\right\} \\ &\quad + x^{\sigma_0 - 3} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \left\{(2\sigma_0 - 1) h_{2,\underline{\varepsilon}_1}(x) + 2 h_{1,\underline{\varepsilon}_1}(x) h_{2,\underline{\varepsilon}_1}(x) + h_{3,\underline{\varepsilon}_1}(x)\right\} \\ &= x^{\sigma_0 - 3} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \Big\{P_3(\sigma_0) + P'_3(\sigma_0) h_{1,\underline{\varepsilon}_1}(x) \\ &\quad + (1/2) P''_3(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2 + (1/2) P''_3(\sigma_0) h_{2,\underline{\varepsilon}_1}(x) + E_{3,\underline{\varepsilon}}(x)\Big\}, \end{aligned} \tag{2.15}$$

where

$$E_{3,\underline{\varepsilon}}(x) = [h_{1,\underline{\varepsilon}_1}(x)]^3 + 3 h_{1,\underline{\varepsilon}_1}(x) h_{2,\underline{\varepsilon}_1}(x) + h_{3,\underline{\varepsilon}_1}(x), \tag{2.16}$$

hence the result holds for $\tau = 3$ by Remark 2.3 and (1.20). Next, we assume that the lemma holds for $\tau \in \mathbb{N}$. Differentiating (2.9) yields

$$\begin{aligned}
v_{\underline{\varepsilon}}^{(\tau+1)}(x) &= x^{\sigma_0-\tau-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\big(\sigma_0-\tau+h_{1,\underline{\varepsilon}_1}(x)\big)\Big\{P_\tau(\sigma_0)\\
&\quad + P_\tau'(\sigma_0)h_{1,\underline{\varepsilon}_1}(x) + (1/2)P_\tau''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2 + (1/2)P_\tau''(\sigma_0)h_{2,\underline{\varepsilon}_1}(x) + E_{\tau,\underline{\varepsilon}}(x)\Big\}\\
&\quad + x^{\sigma_0-\tau-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\Big\{P_\tau'(\sigma_0)h_{2,\underline{\varepsilon}_1}(x)\\
&\quad + P_\tau''(\sigma_0)h_{1,\underline{\varepsilon}_1}(x)h_{2,\underline{\varepsilon}_1}(x) + (1/2)P_\tau''(\sigma_0)h_{3,\underline{\varepsilon}_1}(x) + xE_{\tau,\underline{\varepsilon}}'(x)\Big\}\\
&= x^{\sigma_0-(\tau+1)}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\Big\{P_\tau(\sigma_0)(\sigma_0-\tau) + \big[P_\tau(\sigma_0)\\
&\quad + P_\tau'(\sigma_0)(\sigma_0-\tau)\big]h_{1,\underline{\varepsilon}_1}(x) + \big[(1/2)P_\tau''(\sigma_0)(\sigma_0-\tau) + P_\tau'(\sigma_0)\big][h_{1,\underline{\varepsilon}_1}(x)]^2\\
&\quad + \big[(1/2)P_\tau''(\sigma_0)(\sigma_0-\tau) + P_\tau'(\sigma_0)\big]h_{2,\underline{\varepsilon}_1}(x) + E_{\tau+1,\underline{\varepsilon}}(x)\Big\}\\
&= x^{\sigma_0-(\tau+1)}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\Big\{P_{\tau+1}(\sigma_0) + P_{\tau+1}'(\sigma_0)h_{1,\underline{\varepsilon}_1}(x)\\
&\quad + (1/2)P_{\tau+1}''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2 + (1/2)P_{\tau+1}''(\sigma_0)h_{2,\underline{\varepsilon}_1}(x) + E_{\tau+1,\underline{\varepsilon}}(x)\Big\},
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
E_{\tau+1,\underline{\varepsilon}}(x) &= (1/2)P_\tau''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^3 + (3/2)P_\tau''(\sigma_0)h_{1,\underline{\varepsilon}_1}(x)h_{2,\underline{\varepsilon}_1}(x)\\
&\quad + (\sigma_0-\tau)E_{\tau,\underline{\varepsilon}}(x) + h_{1,\underline{\varepsilon}_1}(x)E_{\tau,\underline{\varepsilon}}(x) + (1/2)P_\tau''(\sigma_0)h_{3,\underline{\varepsilon}_1}(x) + xE_{\tau,\underline{\varepsilon}}'(x).
\end{aligned} \tag{2.18}$$

Thus, by (1.19), $E_{\tau+1,\underline{\varepsilon}}(x)$ can be written in the form

$$E_{\tau+1,\underline{\varepsilon}}(x) = \sum_{j=1}^{Q(\tau+1)} p_{\tau+1,j}[h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,1}}\cdots[h_{\tau+1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,\tau+1}} \tag{2.19}$$

for some $Q(\tau+1) \in \mathbb{N}, w_{\tau+1,j,k} \in \mathbb{N}\cup\{0\}$ for $j \in \{1,\dots,Q(\tau+1)\}$ and $k \in \{1,\dots,\tau+1\}$, $p_{\tau+1,j} \in \mathbb{R}$ for $j \in \{1,\dots,Q(\tau+1)\}$. By (2.18), (1.19), (1.20), and Remark 2.3, there exists $\widetilde{c}_2 > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0,M)$, such that, for all $0 < x < \rho$,

$$\big|p_{\tau+1,j}[h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,1}}\cdots[h_{\tau+1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,\tau+1}}\big| \leqslant \widetilde{c}_2[\ln(\gamma/x)]^{-3}. \tag{2.20}$$

Hence the lemma holds for $\tau+1$. $\square$

Lemma 2.5. *Suppose $N \in \mathbb{N}$. Let $v_{\underline{\varepsilon}} = v_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N} : (0,\rho) \to (0,\infty)$ be defined as in (1.17), $f_{\underline{\varepsilon}} = f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N} : (0,\rho) \to [0,\infty)$ be defined as in (1.18), and, for $0 \leqslant j \leqslant k \leqslant N$, $a_{j,k}(\underline{\varepsilon}) = a_{j,k}(\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N)$ be defined as in (1.21). Let $G_{1,\underline{\varepsilon}} = G_1(\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N) \in \mathbb{R}$ be defined by*[3]

$$\int_0^\rho dx\, x^\alpha\big|f_{\underline{\varepsilon}}^{(m)}(x)\big|^2 = \int_0^\rho dx\, x^\alpha\big|v_{\underline{\varepsilon}}^{(m)}(x)\big|^2[\psi(x)]^2 + G_{1,\underline{\varepsilon}}. \tag{2.21}$$

[3] One notes that, since $\varepsilon_0 > 0$, (1.10) and Lemma 2.4 imply that the integrals in (2.21) are finite and hence $G_{1,\underline{\varepsilon}}$ is well-defined.

Then there exists $c_3 > 0$, independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$, such that

$$\left|G_{1,\underline{\varepsilon}}\right| \leqslant c_3, \tag{2.22}$$

and

$$\begin{aligned} J_{N-1}[f_{\underline{\varepsilon}}] &= G_{1,\underline{\varepsilon}} + \sum_{0 \leqslant j \leqslant k \leqslant N} a_{j,k}(\underline{\varepsilon}) \Gamma_{j,k}(\underline{\varepsilon}) \\ &\quad + \int_0^\rho dx\, x^{2(\sigma_0(\varepsilon_0)-m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j(\varepsilon_j)} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2, \end{aligned} \tag{2.23}$$

where $G_{2,\underline{\varepsilon}} = G_{2,\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N} : (0,\rho) \to \mathbb{R}$ satisfies

$$\left|G_{2,\underline{\varepsilon}}(x)\right| \leqslant c_3 [\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho. \tag{2.24}$$

Proof. We shall write $\sigma_j = \sigma_j(\varepsilon_j)$, $j = 0, 1, \ldots, N$, in this proof. By Lemma 2.4 we have

$$\begin{aligned} \left|v_{\underline{\varepsilon}}^{(m)}(x)\right|^2 [\psi(x)]^2 &= x^{2(\sigma_0 - m)} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} \Big[P_m(\sigma_0) \\ &\quad + P_m'(\sigma_0) h_{1,\underline{\varepsilon}_1}(x) + \tfrac{1}{2} P_m''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2 + \tfrac{1}{2} P_m''(\sigma_0) h_{2,\underline{\varepsilon}_1}(x) + E_{m,\underline{\varepsilon}}(x) \Big]^2 [\psi(x)]^2 \\ &= x^{2(\sigma_0 - m)} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} \Big\{ \big[P_m(\sigma_0)\big]^2 + 2 P_m(\sigma_0) P_m'(\sigma_0) h_{1,\underline{\varepsilon}_1}(x) \\ &\quad + \Big[P_m(\sigma_0) P_m''(\sigma_0) + \big[P_m'(\sigma_0)\big]^2 \Big] [h_{1,\underline{\varepsilon}_1}(x)]^2 + P_m(\sigma_0) P_m''(\sigma_0) h_{2,\underline{\varepsilon}_1}(x) \\ &\quad + G_{2,\underline{\varepsilon}}(x) \Big\} [\psi(x)]^2, \end{aligned} \tag{2.25}$$

where, by Lemma 2.4, $G_{2,\underline{\varepsilon}} = G_{2,\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N} : (0,\rho) \to \mathbb{R}$ satisfies

$$\left|G_{2,\underline{\varepsilon}}(x)\right| \leqslant c_4 [\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho, \tag{2.26}$$

for some $c_4 > 0$ independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M)$. Direct computation shows

$$\int_0^\rho dx\, x^{2(\sigma_0 - m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} h_{1,\underline{\varepsilon}_1}(x)[\psi(x)]^2 = \sum_{j=1}^N \sigma_j \Gamma_{0,j}(\underline{\varepsilon}), \tag{2.27}$$

$$\begin{aligned} \int_0^\rho dx\, x^{2(\sigma_0 - m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} [h_{1,\underline{\varepsilon}_1}(x)]^2 [\psi(x)]^2 &= \sum_{j=1}^N \sigma_j^2 \Gamma_{j,j}(\underline{\varepsilon}) \\ &\quad + 2 \sum_{1 \leqslant j < k \leqslant N} \sigma_j \sigma_k \Gamma_{j,k}(\underline{\varepsilon}), \end{aligned} \tag{2.28}$$

$$\begin{aligned} \int_0^\rho dx\, x^{2(\sigma_0 - m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} h_{2,\underline{\varepsilon}_1}(x)[\psi(x)]^2 &= \sum_{j=1}^N \sigma_j \Gamma_{j,j}(\underline{\varepsilon}) \\ &\quad + \sum_{1 \leqslant j < k \leqslant N} \sigma_k \Gamma_{j,k}(\underline{\varepsilon}). \end{aligned} \tag{2.29}$$

Combining (2.25) and (2.27)-(2.29) yields

$$\begin{aligned}
&\int_0^\rho dx\, x^\alpha \big|v_{\underline{\varepsilon}}^{(m)}(x)\big|^2[\psi(x)]^2 = \big[P_m(\sigma_0)\big]^2\Gamma_{0,0}(\underline{\varepsilon}) \\
&\quad + \sum_{j=1}^N 2P_m(\sigma_0)P_m'(\sigma_0)\sigma_j\Gamma_{0,j}(\underline{\varepsilon}) \\
&\quad + \sum_{j=1}^N \Big\{\Big[P_m(\sigma_0)P_m''(\sigma_0) + \big[P_m'(\sigma_0)\big]^2\Big]\sigma_j^2 + P_m(\sigma_0)P_m''(\sigma_0)\sigma_j\Big\}\Gamma_{j,j}(\underline{\varepsilon}) \\
&\quad + \sum_{1\leqslant j<k\leqslant N} \Big\{2\Big[P_m(\sigma_0)P_m''(\sigma_0) + \big[P_m'(\sigma_0)\big]^2\Big]\sigma_j\sigma_k + P_m(\sigma_0)P_m''(\sigma_0)\sigma_k\Big\}\Gamma_{j,k}(\underline{\varepsilon}) \\
&\quad + \int_0^\rho dx\, x^{2(\sigma_0-m)+\alpha}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-2\sigma_j}G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2.
\end{aligned} \tag{2.30}$$

Equation (2.23) now follows from (1.11), (2.21), and (2.30). Since

$$f_{\underline{\varepsilon}}^{(m)}(x) = \sum_{j=0}^m \binom{m}{j} v_{\underline{\varepsilon}}^{(m-j)}(x)\psi^{(j)}(x), \tag{2.31}$$

we have, by (1.10),

$$\begin{aligned}
\big|G_{1,\underline{\varepsilon}}\big| &= \bigg|\int_0^\rho dx\, x^\alpha\big|f_{\underline{\varepsilon}}^{(m)}(x)\big|^2 - \int_0^\rho dx\, x^\alpha\big|v_{\underline{\varepsilon}}^{(m)}(x)\big|^2[\psi(x)]^2\bigg| \\
&= \bigg|\int_0^\rho dx\, x^\alpha\bigg\{2v_{\underline{\varepsilon}}^{(m)}(x)\psi(x)\sum_{j=1}^m\binom{m}{j}v_{\underline{\varepsilon}}^{(m-j)}(x)\psi^{(j)}(x) + \bigg(\sum_{j=1}^m\binom{m}{j}v_{\underline{\varepsilon}}^{(m-j)}(x)\psi^{(j)}(x)\bigg)^2\bigg\}\bigg| \\
&\leqslant 2\sum_{j=1}^m\binom{m}{j}\int_{(0.8)\rho}^{(0.9)\rho} dx\, x^\alpha\big|v_{\underline{\varepsilon}}^{(m)}(x)v_{\underline{\varepsilon}}^{(m-j)}(x)\big|\big|\psi(x)\big|\big|\psi^{(j)}(x)\big| \\
&\quad + \sum_{j,k=1}^m\binom{m}{j}\binom{m}{k}\int_{(0.8)\rho}^{(0.9)\rho} dx x^\alpha\,\big|v_{\underline{\varepsilon}}^{(m-j)}(x)v_{\underline{\varepsilon}}^{(m-k)}(x)\big|\big|\psi^{(j)}(x)\psi^{(k)}(x)\big|.
\end{aligned} \tag{2.32}$$

Hence Lemma 2.4 implies that there exists $c_5 > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that $|G_{1,\underline{\varepsilon}}| \leqslant c_5$. Thus Lemma 2.5 is proved upon putting $c_3 = \max\{c_4, c_5\}$. $\square$

Lemma 2.6. *Let $k \in \{0, 1, \dots, N\}$ and $\beta_0, \beta_1, \dots, \beta_k \geqslant 0$. Then*

$$\int_0^\rho dx\, x^{-1+\beta_0}[\ln_1(\gamma/x)]^{-1-\beta_1}\cdots[\ln_k(\gamma/x)]^{-1-\beta_k} < \infty \tag{2.33}$$

if and only if

$$\begin{cases} \quad\beta_0 > 0, \\ or\ \beta_0 = 0\ and\ \beta_1 > 0, \\ or\ \beta_0 = \beta_1 = 0\ and\ \beta_2 > 0, \\ \qquad\qquad\vdots \\ or\ \beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0\ and\ \beta_k > 0. \end{cases} \tag{2.34}$$

Proof. This follows from Lemma 2.1 and (1.20). ☐

Lemma 2.7. *Let $\beta \in (-\infty, 1)$. Then there exists $c_6 = c_6(\beta) > 0$, independent of $\varepsilon_0 \in (0, M)$, such that*

$$\int_0^\rho d\,x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{-\beta}[\psi(x)]^2 \leqslant c_6\varepsilon_0^{-1+\beta}. \tag{2.35}$$

Proof. Writing $\tau = \varepsilon_0^{-1}[\ln(\gamma/\rho)]^{-1} > 0$, and using the change of variables

$$\begin{aligned} s &= \varepsilon_0^{-1}[\ln(\gamma/x)]^{-1} \quad \Big(i.e.,\ x = \gamma e^{\frac{-1}{\varepsilon_0 s}}\Big), \\ ds &= \varepsilon_0^{-1}x^{-1}[\ln(\gamma/x)]^{-2}dx \quad \Big(i.e.,\ dx = \gamma\varepsilon_0^{-1}s^{-2}e^{\frac{-1}{\varepsilon_0 s}}ds\Big), \end{aligned} \tag{2.36}$$

one obtains

$$\begin{aligned} \int_0^\rho dx\,x^{-1+\varepsilon_0}[\ln(\gamma/x)]^{-\beta}[\psi(x)]^2 &\leqslant \int_0^\rho dx\,x^{-1+\varepsilon_0}[\ln(\gamma/x)]^{-\beta} = \gamma^{\varepsilon_0}\varepsilon_0^{-1+\beta}\int_0^\tau ds\,s^{-2+\beta}e^{\frac{-1}{s}} \\ &\leqslant \left(\gamma^{\varepsilon_0}\int_0^\infty ds\,s^{-2+\beta}e^{\frac{-1}{s}}\right)\varepsilon_0^{-1+\beta}. \end{aligned} \tag{2.37}$$

☐

Lemma 2.8. *Suppose $N \geqslant 2$. Let $\beta \in (-\infty, 1)$ and $1 \leqslant j \leqslant N-1$. Then there exists $c_7 = c_7(\beta) > 0$, independent of $\varepsilon_j \in (0, M)$, such that*

$$\int_0^\rho dx\,x^{-1}\prod_{k=1}^{j-1}[\ln_k(\gamma/x)]^{-1}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\ln_{j+1}(\gamma/x)]^{-\beta}[\psi(x)]^2 \leqslant c_7\varepsilon_j^{-1+\beta}. \tag{2.38}$$

Proof. Writing $\tau = \varepsilon_j^{-1}[\ln_{j+1}(\gamma/\rho)]^{-1} > 0$, and using the change of variables

$$s = \varepsilon_j^{-1}[\ln_{j+1}(\gamma/x)]^{-1}, \tag{2.39}$$

so that, by Lemma 2.1,

$$ds = \varepsilon_j^{-1}x^{-1}[\ln_1(\gamma/x)]^{-1}\cdots[\ln_j(\gamma/x)]^{-1}[\ln_{j+1}(\gamma/x)]^{-2}dx, \tag{2.40}$$

one gets

$$\begin{aligned} &\int_0^\rho dx\,x^{-1}\prod_{k=1}^{j-1}[\ln_k(\gamma/x)]^{-1}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\ln_{j+1}(\gamma/x)]^{-\beta}[\psi(x)]^2 \\ &\quad\leqslant \varepsilon_j\int_0^\tau ds\,[\ln_j(\gamma/x)]^{-\varepsilon_j}[\ln_{j+1}(\gamma/x)]^{2-\beta}. \end{aligned} \tag{2.41}$$

By (2.39) one has

$$(\varepsilon_j s)^{-1} = \ln(\ln_j(\gamma/x)) \quad \Big(i.e.,\ \ln_j(\gamma/x) = e^{\frac{1}{\varepsilon_j s}}\Big). \tag{2.42}$$

Hence

$$\begin{aligned} &\int_0^\rho dx\,x^{-1}\prod_{k=1}^{j-1}[\ln_k(\gamma/x)]^{-1}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\ln_{j+1}(\gamma/x)]^{-\beta}[\psi(x)]^2 \\ &\quad\leqslant \int_0^\tau ds\,\varepsilon_j e^{\frac{-1}{s}}(\varepsilon_j s)^{-2+\beta} \leqslant \left(\int_0^\infty ds\,e^{\frac{-1}{s}}s^{-2+\beta}\right)\varepsilon_j^{-1+\beta}. \end{aligned} \tag{2.43}$$

☐

Next, we need to introduce some more notation: For $\tau \in \{0,1,\dots,N-1\}$ and $\tau < j \leqslant k \leqslant N$ we write

$$\begin{aligned}\big(\Gamma_\tau(\underline{\varepsilon})\big)_{j,k} = \int_0^\rho dx \bigg\{ & x^{-1} \prod_{\ell=1}^{\tau} [\ln_\ell(\gamma/x)]^{-1} \prod_{\ell=\tau+1}^{j} [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \prod_{\ell=j+1}^{k} [\ln_\ell(\gamma/x)]^{-\varepsilon_\ell} \\ & \times \prod_{\ell=k+1}^{N} [\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell} [\psi(x)]^2 \bigg\}.\end{aligned} \tag{2.44}$$

By Lemma 2.6, $\big(\Gamma_\tau(\underline{\varepsilon})\big)_{j,k}$ is well-defined for $\tau \in \{0,1,\dots,N-1\}$ and $\tau < j \leqslant k \leqslant N$ as the integral on the right-hand side of (2.44) is finite.

Lemma 2.9.

(i) There exists $c_8 > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0,M)$, such that

$$\varepsilon_0 \Gamma_{0,0}(\underline{\varepsilon}) = \sum_{j=1}^{N} (1-\varepsilon_j) \Gamma_{0,j}(\underline{\varepsilon}) + G_{3,\underline{\varepsilon}}, \tag{2.45}$$

and for $j = 1, \dots, N$,

$$\varepsilon_0 \Gamma_{0,j}(\underline{\varepsilon}) = -\sum_{k=1}^{j} \varepsilon_k \Gamma_{k,j}(\underline{\varepsilon}) + \sum_{k=j+1}^{N} (1-\varepsilon_k) \Gamma_{j,k}(\underline{\varepsilon}) + G_{4,j,\underline{\varepsilon}}, \tag{2.46}$$

where

$$\big|G_{3,\underline{\varepsilon}}\big| \leqslant c_8, \quad \big|G_{4,j,\underline{\varepsilon}}\big| \leqslant c_8. \tag{2.47}$$

(ii) Suppose $N \geqslant 2$. Let $1 \leqslant j \leqslant N-1$. Then there exists $c_9 = c_9(j) > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0,M)$, such that

$$\varepsilon_j \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,j} = \sum_{k=j+1}^{N} (1-\varepsilon_k) \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,k} + G_{5,j,\underline{\varepsilon}_j}, \tag{2.48}$$

where $\underline{\varepsilon}_j = (\varepsilon_j, \dots, \varepsilon_N)$, and, for $j+1 \leqslant k \leqslant N$,

$$\varepsilon_j \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,k} = -\sum_{\ell=j+1}^{k} \varepsilon_\ell \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{\ell,k} + \sum_{\ell=k+1}^{N} (1-\varepsilon_\ell) \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{k,\ell} + G_{6,j,k,\underline{\varepsilon}_j}, \tag{2.49}$$

and where

$$\big|G_{5,j,\underline{\varepsilon}_j}\big| \leqslant c_9, \quad \big|G_{6,j,k,\underline{\varepsilon}_j}\big| \leqslant c_9. \tag{2.50}$$

(iii) There exists $c_{10} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0,M)$, such that

$$\begin{aligned}&\varepsilon_0^2 \Gamma_{0,0}(\underline{\varepsilon}) - 2\varepsilon_0 \sum_{j=1}^{N} (1-\varepsilon_j) \Gamma_{0,j}(\underline{\varepsilon}) \\ &\quad = \sum_{j=1}^{N} (\varepsilon_j - \varepsilon_j^2) \Gamma_{j,j}(\underline{\varepsilon}) - \sum_{1 \leqslant j < k \leqslant N} (1-2\varepsilon_j)(1-\varepsilon_k) \Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}},\end{aligned} \tag{2.51}$$

where

$$\left|G_{7,\underline{\varepsilon}}\right| \leqslant c_{10}. \tag{2.52}$$

Proof.

(i) We observe

$$\begin{aligned}
&\frac{d}{dx}\Big(x^{\varepsilon_0}[\ln_1(\gamma/x)]^{1-\varepsilon_1}\cdots[\ln_N(\gamma/x)]^{1-\varepsilon_N}[\psi(x)]^2\Big)\\
&\qquad - 2x^{\varepsilon_0}[\ln_1(\gamma/x)]^{1-\varepsilon_1}\cdots[\ln_N(\gamma/x)]^{1-\varepsilon_N}\psi(x)\psi'(x)\\
&\quad = \varepsilon_0 x^{-1+\varepsilon_0}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{1-\varepsilon_j}[\psi(x)]^2 - (1-\varepsilon_1)x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{-\varepsilon_1}\prod_{j=2}^{N}[\ln_j(\gamma/x)]^{1-\varepsilon_j}[\psi(x)]^2\\
&\qquad\qquad\vdots\\
&\qquad - (1-\varepsilon_N)x^{-1+\varepsilon_0}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\varepsilon_j}[\psi(x)]^2,
\end{aligned} \tag{2.53}$$

integrating both sides yields

$$G_{3,\underline{\varepsilon}} = \varepsilon_0\Gamma_{0,0}(\underline{\varepsilon}) - \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}). \tag{2.54}$$

Similarly, for $j \in \{1,\ldots,N\}$,

$$\begin{aligned}
&\frac{d}{dx}\Bigg(x^{\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2\Bigg)\\
&\qquad - 2x^{\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}\psi(x)\psi'(x)\\
&\quad = \varepsilon_0 x^{-1+\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2\\
&\qquad + \varepsilon_1 x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{-1-\varepsilon_1}\prod_{k=2}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2\\
&\qquad\qquad\vdots\\
&\qquad + \varepsilon_j x^{-1+\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-1-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2\\
&\qquad - (1-\varepsilon_{j+1})x^{-1+\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-1-\varepsilon_k}[\ln_{j+1}(\gamma/x)]^{-\varepsilon_{j+1}}\prod_{k=j+2}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2\\
&\qquad\qquad\vdots\\
&\qquad - (1-\varepsilon_N)x^{-1+\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-1-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{-\varepsilon_k}[\psi(x)]^2,
\end{aligned} \tag{2.55}$$

integrating both sides yields

$$G_{4,j,\underline{\varepsilon}} = \varepsilon_0 \Gamma_{0,j}(\underline{\varepsilon}) + \sum_{k=1}^{j} \varepsilon_k \Gamma_{k,j}(\underline{\varepsilon}) - \sum_{k=j+1}^{N} (1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}). \tag{2.56}$$

By (1.10), there exists $c_8 > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\left|G_{3,\underline{\varepsilon}}\right| \leqslant c_8, \quad \left|G_{4,j,\underline{\varepsilon}}\right| \leqslant c_8. \tag{2.57}$$

(ii) One has

$$\begin{aligned}
&\frac{d}{dx}\bigg([\ln_j(\gamma/x)]^{-\varepsilon_j} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2\bigg) - 2[\ln_j(\gamma/x)]^{-\varepsilon_j} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} \psi(x)\psi'(x) \\
&= \varepsilon_j x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2 \\
&- (1-\varepsilon_{j+1}) x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\ln_{j+1}(\gamma/x)]^{-\varepsilon_{j+1}} \prod_{k=j+2}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2 \\
&\qquad \vdots \\
&- (1-\varepsilon_N) x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{-\varepsilon_k} [\psi(x)]^2,
\end{aligned} \tag{2.58}$$

integrating both sides in (2.58) yields

$$G_{5,j,\underline{\varepsilon}_j} = \varepsilon_j \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,j} - \sum_{k=j+1}^{N} (1-\varepsilon_k) \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,k}. \tag{2.59}$$

Similarly one obtains, for $j+1 \leqslant k \leqslant N$,

$$\begin{aligned}
&\frac{d}{dx}\Big([\ln_j(\gamma/x)]^{-\varepsilon_j} \cdots [\ln_k(\gamma/x)]^{-\varepsilon_k} [\ln_{k+1}(\gamma/x)]^{1-\varepsilon_{k+1}} \cdots [\ln_N(\gamma/x)]^{1-\varepsilon_N} [\psi(x)]^2\Big) \\
&\quad - 2[\ln_j(\gamma/x)]^{-\varepsilon_j} \cdots [\ln_k(\gamma/x)]^{-\varepsilon_k} [\ln_{k+1}(\gamma/x)]^{1-\varepsilon_{k+1}} \cdots [\ln_N(\gamma/x)]^{1-\varepsilon_N} \psi(x)\psi'(x) \\
&= \varepsilon_j x^{-1} \prod_{\ell=1}^{j-1} [\ln_\ell(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \prod_{\ell=j+1}^{k} [\ln_\ell(\gamma/x)]^{-\varepsilon_\ell} \prod_{\ell=k+1}^{N} [\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell} [\psi(x)]^2 + \\
&\qquad \vdots \\
&+ \varepsilon_k x^{-1} \prod_{\ell=1}^{j-1} [\ln_\ell(\gamma/x)]^{-1} \prod_{\ell=j}^{k} [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \prod_{\ell=k+1}^{N} [\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell} [\psi(x)]^2 - (1-\varepsilon_{k+1}) x^{-1} \\
&\times \prod_{\ell=1}^{j-1} [\ln_\ell(\gamma/x)]^{-1} \prod_{\ell=j}^{k} [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} [\ln_{k+1}(\gamma/x)]^{-\varepsilon_{k+1}} \prod_{\ell=k+2}^{N} [\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell} [\psi(x)]^2 - \\
&\qquad \vdots \\
&- (1-\varepsilon_N) x^{-1} \prod_{\ell=1}^{j-1} [\ln_\ell(\gamma/x)]^{-1} \prod_{\ell=j}^{k} [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \prod_{\ell=k+1}^{N} [\ln_\ell(\gamma/x)]^{-\varepsilon_\ell} [\psi(x)]^2,
\end{aligned} \tag{2.60}$$

integrating both sides in (2.60) yields

$$G_{6,j,k,\underline{\varepsilon}_j} = \sum_{\ell=j}^{k} \varepsilon_\ell \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{\ell,k} - \sum_{\ell=k+1}^{N} (1-\varepsilon_\ell)\big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{k,\ell}. \tag{2.61}$$

By (1.10), there exists $c_9 > 0$, independent of $\varepsilon_j, \dots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{5,j,\underline{\varepsilon}_j}\big| \leqslant c_9, \quad \big|G_{6,j,k,\underline{\varepsilon}_j}\big| \leqslant c_9, \tag{2.62}$$

for $1 \leqslant j \leqslant N-1$ and $j+1 \leqslant k \leqslant N$.

(iii) By *(i)* we have

$$\begin{aligned}
&\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon}) - 2\varepsilon_0 \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}) = -\varepsilon_0 \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}) + \varepsilon_0 G_{3,\underline{\varepsilon}} \\
&= -\sum_{j=1}^{N}(1-\varepsilon_j)\bigg\{-\sum_{k=1}^{j}\varepsilon_k\Gamma_{k,j}(\underline{\varepsilon}) + \sum_{k=j+1}^{N}(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{4,j,\underline{\varepsilon}}\bigg\} + \varepsilon_0 G_{3,\underline{\varepsilon}} \\
&= \sum_{j=1}^{N}\sum_{k=1}^{j}(1-\varepsilon_j)\varepsilon_k\Gamma_{k,j}(\underline{\varepsilon}) - \sum_{j=1}^{N}\sum_{k=j+1}^{N}(1-\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) \ + G_{7,\underline{\varepsilon}},
\end{aligned} \tag{2.63}$$

where there exists $c_{10} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{7,\underline{\varepsilon}}\big| \leqslant c_{10}. \tag{2.64}$$

Thus

$$\begin{aligned}
&\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon}) - 2\varepsilon_0 \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}) = \sum_{j=1}^{N}(\varepsilon_j-\varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) + \sum_{1\leqslant j<k\leqslant N}(1-\varepsilon_k)\varepsilon_j\Gamma_{j,k}(\underline{\varepsilon}) \\
&+ \sum_{1\leqslant j<k\leqslant N}\varepsilon_j(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) - \sum_{1\leqslant j<k\leqslant N}(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}} \\
&= \sum_{j=1}^{N}(\varepsilon_j-\varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) - \sum_{1\leqslant j<k\leqslant N}(1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}}.
\end{aligned} \tag{2.65}$$

□

Lemma 2.10. *Suppose $N \in \mathbb{N}$. Then there exists a constant $c_{11} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, with the following property: Given any fixed $\varepsilon_1, \dots, \varepsilon_N \in (0, M)$, there exists a decreasing sequence $\{\varepsilon_{0,\ell}\}_{\ell=1}^{\infty} \subseteq (0, M)$ and $L_0 \in \mathbb{R}$ such that $\varepsilon_{0,\ell} \downarrow 0$ as $\ell \uparrow \infty$, $|L_0| \leqslant c_{11}$, and, writing $f_{\underline{\varepsilon}} = f_{\varepsilon_{0,\ell},\varepsilon_1,\dots,\varepsilon_N}$ as defined in* (1.18),

$$\lim_{\ell\uparrow\infty} J_{N-1}[f_{\underline{\varepsilon}}] = \sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k} + L_0. \tag{2.66}$$

Proof. We first note that by Lemma 2.7, there exists $c_{12} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that for all $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$ we have

$$\begin{aligned}\Gamma_{0,0}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{1-\varepsilon_1} \cdots [\ln_N(\gamma/x)]^{1-\varepsilon_N}[\psi(x)]^2 \\ &\leqslant \int_0^\rho dx\, x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{3/2}\left\{[\ln_1(\gamma/x)]^{-1/2}\prod_{k=2}^N[\ln_k(\gamma/x)]\right\}[\psi(x)]^2 \\ &\leqslant c_{12}\varepsilon_0^{-5/2}. \end{aligned} \tag{2.67}$$

For $j = 1, \dots, N$, by Lemma 2.7, there exists $c_{13} = c_{13}(j) > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that for all $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$ we have

$$\begin{aligned}\Gamma_{0,j}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0}\prod_{k=1}^j[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^N[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2 \\ &\leqslant \int_0^\rho dx\, x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{1/2}\left\{[\ln_1(\gamma/x)]^{-1/2}\prod_{k=j+1}^N[\ln_k(\gamma/x)]\right\}[\psi(x)]^2 \\ &\leqslant c_{13}\varepsilon_0^{-3/2}. \end{aligned} \tag{2.68}$$

Since we are fixing $\varepsilon_1, \dots, \varepsilon_n \in (0, M)$, for $0 \leqslant j \leqslant k \leqslant N$, we shall consider $a_{j,k}(\underline{\varepsilon}) = a_{j,k}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N)$ as functions of $\varepsilon_0 \in (0, M)$ only. Then

$$\begin{aligned} a_{0,0}(\varepsilon_0) &= \big[P_m(\sigma_0(\varepsilon_0))\big]^2 - A(m,\alpha), \\ a'_{0,0}(\varepsilon_0) &= P_m(\sigma_0(\varepsilon_0))P'_m(\sigma_0(\varepsilon_0)), \\ a''_{0,0}(\varepsilon_0) &= \frac{1}{2}\left\{P_m(\sigma_0(\varepsilon_0))P''_m(\sigma_0(\varepsilon_0)) + \big[P'_m(\sigma_0(\varepsilon_0))\big]^2\right\}, \\ a^{(k)}_{0,0}(\varepsilon_0) &= 2^{-k}\left\{\frac{d^k}{d\sigma^k}\Big(\big[P_m(\sigma)\big]^2\Big)\bigg|_{\sigma=\sigma_0(\varepsilon_0)}\right\}, \quad k = 3, \dots, 2m. \end{aligned} \tag{2.69}$$

Similarly one has, for $j = 1, \dots, N$, and $k = 2, \dots, 2m-1$,

$$\begin{aligned} a_{0,j}(\varepsilon_0) &= 2\sigma_j(\varepsilon_j)P_m(\sigma_0(\varepsilon_0))P'_m(\sigma_0(\varepsilon_0)), \\ a'_{0,j}(\varepsilon_0) &= \sigma_j(\varepsilon_j)\left\{\big[P'_m(\sigma_0(\varepsilon_0))\big]^2 + P_m(\sigma_0(\varepsilon_0))P''_m(\sigma_0(\varepsilon_0))\right\}, \\ a^{(k)}_{0,j}(\varepsilon_0) &= 2^{-(k-1)}\sigma_j(\varepsilon_j)\left\{\frac{d^k}{d\sigma^k}\Big(P_m(\sigma)P'_m(\sigma)\Big)\bigg|_{\sigma=\sigma_0(\varepsilon_0)}\right\}. \end{aligned} \tag{2.70}$$

Thus, by Lemma 2.2,

$$\begin{aligned} a_{0,0}(\varepsilon_0) &= a_{0,0}(0) + a'_{0,0}(0)\varepsilon_0 + \frac{1}{2}a''_{0,0}(0)\varepsilon_0^2 + \varepsilon_0^3\left(\sum_{k=3}^{2m}(k!)^{-1}a^{(k)}_{0,0}(0)\varepsilon_0^{k-3}\right) \\ &= P_m(\sigma_0(0))P'_m(\sigma_0(0))\varepsilon_0 + \frac{1}{4}\left\{P_m(\sigma_0(0))P''_m(\sigma_0(0)) + \big[P'_m(\sigma_0(0))\big]^2\right\}\varepsilon_0^2 \\ &\quad + \left(\sum_{k=3}^{2m}(k!)^{-1}2^{-k}\left\{\frac{d^k}{d\sigma^k}\Big(\big[P_m(\sigma)\big]^2\Big)\bigg|_{\sigma=\sigma_0(0)}\right\}\varepsilon_0^{k-3}\right)\varepsilon_0^3. \end{aligned} \tag{2.71}$$

Put

$$G_8(\varepsilon_0)=\sum_{k=3}^{2m}(k!)^{-1}2^{-k}\left\{\frac{d^k}{d\sigma^k}\Big(\big[P_m(\sigma)\big]^2\Big)\Big|_{\sigma=\sigma_0(0)}\right\}\varepsilon_0^{k-3}, \tag{2.72}$$

then there exists $c_{14}>0$, independent of $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N\in(0,M)$, such that

$$\big|G_8(\varepsilon_0)\big|\leqslant c_{14},\quad \varepsilon_0\in(0,M). \tag{2.73}$$

Similarly, for $j=1,\dots,N$,

$$\begin{aligned}a_{0,j}(\varepsilon_0)&=a_{0,j}(0)+a_{0,j}'(0)\varepsilon_0+\sum_{k=2}^{2m-1}(k!)^{-1}a_{0,j}^{(k)}(0)\varepsilon_0^k\\&=2\sigma_j(\varepsilon_j)P_m(\sigma_0(0))P_m'(\sigma_0(0))+\sigma_j(\varepsilon_j)\Big\{\big[P_m'(\sigma_0(0))\big]^2+P_m(\sigma_0(0))P_m''(\sigma_0(0))\Big\}\varepsilon_0\\&\quad+\left(\sum_{k=2}^{2m-1}(k!)^{-1}2^{-(k-1)}\sigma_j(\varepsilon_j)\left\{\frac{d^k}{d\sigma^k}\Big(P_m(\sigma)P_m'(\sigma)\Big)\Big|_{\sigma=\sigma_0(0)}\right\}\varepsilon_0^{k-2}\right)\varepsilon_0^2.\end{aligned} \tag{2.74}$$

For $j=1,\dots,N$, put

$$G_{9,j}(\varepsilon_0,\varepsilon_j)=\sum_{k=2}^{2m-1}(k!)^{-1}2^{-(k-1)}\sigma_j(\varepsilon_j)\left\{\frac{d^k}{d\sigma^k}\Big(P_m(\sigma)P_m'(\sigma)\Big)\Big|_{\sigma=\sigma_0(0)}\right\}\varepsilon_0^{k-2}, \tag{2.75}$$

then there exists $c_{15}=c_{15}(j)>0$, independent of $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N\in(0,M)$, such that

$$\big|G_{9,j}(\varepsilon_0,\varepsilon_j)\big|\leqslant c_{15},\quad j=1,\dots,N,\ \varepsilon_0,\varepsilon_j\in(0,M). \tag{2.76}$$

Hence, applying Lemma 2.9,

$$\begin{aligned}&a_{0,0}(\underline{\varepsilon})\Gamma_{0,0}(\underline{\varepsilon})+\sum_{j=1}^{N}a_{0,j}(\underline{\varepsilon})\Gamma_{0,j}(\underline{\varepsilon})\\&\quad=P_m(\sigma_0(0))P_m'(\sigma_0(0))\varepsilon_0\Gamma_{0,0}(\underline{\varepsilon})+\frac14\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0))+\big[P_m'(\sigma_0(0))\big]^2\Big\}\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon})\\&\qquad+G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon})+\sum_{j=1}^{N}\Big\{2\sigma_j(\varepsilon_j)P_m(\sigma_0(0))P_m'(\sigma_0(0))\Gamma_{0,j}(\underline{\varepsilon})\\&\qquad+\sigma_j(\varepsilon_j)\Big(\big[P_m'(\sigma_0(0))\big]^2+P_m(\sigma_0(0))P_m''(\sigma_0(0))\Big)\varepsilon_0\Gamma_{0,j}(\underline{\varepsilon})+G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon})\Big\}\\&\quad=P_m(\sigma_0(0))P_m'(\sigma_0(0))\Big\{\varepsilon_0\Gamma_{0,0}(\underline{\varepsilon})-\sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon})\Big\}\\&\qquad+\frac14\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0))+\big[P_m'(\sigma_0(0))\big]^2\Big\}\Big\{\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon})-2\varepsilon_0\sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon})\Big\}\\&\qquad+G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon})+\sum_{j=1}^{N}G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon})\end{aligned}$$

$$
\begin{aligned}
&= P_m(\sigma_0(0))P_m'(\sigma_0(0))G_{3,\underline{\varepsilon}} + \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\} \\
&\quad \times \left\{\sum_{j=1}^{N}(\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) - \sum_{1\leqslant j<k\leqslant N}(1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}}\right\} \\
&\quad + G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^{N} G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon}).
\end{aligned} \tag{2.77}
$$

Put

$$
\begin{aligned}
G_{10,\underline{\varepsilon}} &= P_m(\sigma_0(0))P_m'(\sigma_0(0))G_{3,\underline{\varepsilon}} + \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}G_{7,\underline{\varepsilon}} \\
&\quad + G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^{N} G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon}).
\end{aligned} \tag{2.78}
$$

Then by Lemma 2.9, (2.67), (2.68), (2.73), and (2.76), there exists $c_{16} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$
\big|G_{10,\underline{\varepsilon}}\big| \leqslant c_{16}, \quad \varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M). \tag{2.79}
$$

Let $\{\varepsilon_{0,\ell}\}_{\ell=1}^{\infty}$ be any decreasing sequence in $(0, M)$ with $\lim_{\ell\uparrow\infty}\varepsilon_{0,\ell} = 0$. Applying Lemma 2.5, (2.77), and (2.78), we have, with $\varepsilon_0 = \varepsilon_{0,\ell}$,

$$
\begin{aligned}
J_{N-1}[f_{\underline{\varepsilon}}] &= G_{1,\underline{\varepsilon}} + \int_0^{\rho} dx\, x^{-1+\varepsilon_{0,\ell}} \prod_{j=1}^{N}[\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2 \\
&\quad + a_{0,0}(\underline{\varepsilon})\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^{N} a_{0,j}(\underline{\varepsilon})\Gamma_{0,j}(\underline{\varepsilon}) + \sum_{1\leqslant j\leqslant k\leqslant N} a_{j,k}(\underline{\varepsilon})\Gamma_{j,k}(\underline{\varepsilon}) \\
&= G_{1,\underline{\varepsilon}} + \int_0^{\rho} dx\, x^{-1+\varepsilon_{0,\ell}} \prod_{j=1}^{N}[\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2 \\
&\quad + G_{10,\underline{\varepsilon}} + \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\bigg\{\sum_{j=1}^{N}(\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) \\
&\quad - \sum_{1\leqslant j<k\leqslant N}(1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon})\bigg\} + \sum_{1\leqslant j\leqslant k\leqslant N} a_{j,k}(\underline{\varepsilon})\Gamma_{j,k}(\underline{\varepsilon}) \\
&= \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\bigg\{\sum_{j=1}^{N}(\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) \\
&\quad - \sum_{1\leqslant j<k\leqslant N}(1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon})\bigg\} + G_{11,\underline{\varepsilon}} + \sum_{1\leqslant j\leqslant k\leqslant N} a_{j,k}(\underline{\varepsilon})\Gamma_{j,k}(\underline{\varepsilon}),
\end{aligned} \tag{2.80}
$$

where

$$
\begin{aligned}
G_{11,\underline{\varepsilon}} &= G_1(\varepsilon_{0,\ell}, \varepsilon_1, \dots, \varepsilon_N) + \int_0^{\rho} dx\, x^{-1+\varepsilon_{0,\ell}} \prod_{j=1}^{N}[\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2 \\
&\quad + G_{10}(\varepsilon_{0,\ell}, \varepsilon_1, \dots, \varepsilon_N).
\end{aligned} \tag{2.81}
$$

By (2.24) and Lemma 2.1 there exist $c_{17}, c_{18} > 0$, independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M)$, such that

$$\begin{aligned}
&\bigg| \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{j=1}^N [\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2 \bigg| \\
&\quad \leqslant \int_0^\rho dx\, c_3 x^{-1}[\ln_1(\gamma/x)]^{-3/2}\bigg\{[\ln_1(\gamma/x)]^{-1/2}\prod_{j=2}^N[\ln_j(\gamma/x)]\bigg\}[\psi(x)]^2 \\
&\quad \leqslant c_{17}\int_0^\rho dx\, x^{-1}[\ln_1(\gamma/x)]^{-3/2}[\psi(x)]^2 = c_{18} < \infty.
\end{aligned} \tag{2.82}$$

This, together with (2.22) and (2.79), implies that there exists $c_{11} > 0$, independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{11,\underline{\varepsilon}}\big| \leqslant c_{11}, \quad \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M). \tag{2.83}$$

By compactness of $[-c_{11}, c_{11}]$, there exist a subsequence $\{\varepsilon_{0,\ell_p}\}_{p=1}^\infty$ and $L_0 \in [-c_{11}, c_{11}]$, such that

$$\lim_{p\uparrow\infty} G_{11}(\varepsilon_{0,\ell_p}, \varepsilon_1, \ldots, \varepsilon_N) = L_0. \tag{2.84}$$

We shall regard this subsequence as $\{\varepsilon_{0,\ell}\}_{\ell=1}^\infty$. For $1 \leqslant j \leqslant k \leqslant N$ we have, by monotone convergence,

$$\lim_{\ell\uparrow\infty} \Gamma_{j,k}(\varepsilon_{0,\ell}, \varepsilon_1, \ldots, \varepsilon_N) = \big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}(\varepsilon_1, \ldots, \varepsilon_N). \tag{2.85}$$

The lemma now follows from taking the limit $\ell \uparrow \infty$ in (2.80) and using (2.81) and (2.83)–(2.85). $\square$

Lemma 2.11. *Suppose $N \geqslant 2$. Then there exists a constant $c_{19} > 0$, independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M)$, with the following property: Let $p \in \{1, \ldots, N-1\}$ and let $\varepsilon_{p+1}, \ldots, \varepsilon_N \in (0, M)$ be fixed. Then there exist $L_p \in \mathbb{R}$, with $|L_p| \leqslant c_{19}$, and a decreasing sequence $\{\varepsilon_{p,\ell}\}_{\ell=1}^\infty \subseteq (0, M)$ with $\varepsilon_{p,\ell} \downarrow 0$ as $\ell \uparrow \infty$, such that*

$$\begin{aligned}
&\lim_{\ell\uparrow\infty} \sum_{p\leqslant j\leqslant k\leqslant N} b_{j,k}(0, \ldots, 0, \varepsilon_{p,\ell}, \varepsilon_{p+1}, \ldots, \varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{j,k} \\
&\quad = \sum_{p+1\leqslant j\leqslant k\leqslant N} b_{j,k}(0, \ldots, 0, \varepsilon_{p+1}, \ldots, \varepsilon_N)\big(\Gamma_p(\underline{\varepsilon})\big)_{j,k} + L_p.
\end{aligned} \tag{2.86}$$

Proof. By Lemma 2.2 one obtains

$$\begin{aligned}
b_{p,p}(0, \ldots, 0, \varepsilon_p, \varepsilon_{p+1}, \ldots, \varepsilon_N) &= \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\big(\varepsilon_p - \varepsilon_p^2\big) \\
&\quad - \frac{1}{2}(1-\varepsilon_p)\bigg\{P_m(\sigma_0(0))P_m''(\sigma_0(0))\frac{1}{2}(1+\varepsilon_p) - \big[P_m'(\sigma_0(0))\big]^2\frac{1}{2}(1-\varepsilon_p)\bigg\} - B(m,\alpha) \\
&= \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) - \big[P_m'(\sigma_0(0))\big]^2\Big\}\varepsilon_p = -B(m,\alpha)\varepsilon_p,
\end{aligned} \tag{2.87}$$

and, for $j = p+1, \dots, N$, one gets

$$\begin{aligned} b_{p,j}(0,\dots,0,\varepsilon_p,\varepsilon_{p+1},\dots,\varepsilon_N) &= \sigma_j(\varepsilon_j)\Big\{P_m(\sigma_0(0))P''_m(\sigma_0(0))\varepsilon_p - \big[P'_m(\sigma_0(0))\big]^2(1-\varepsilon_p)\Big\} \\ &\quad + \frac{1}{2}\sigma_j(\varepsilon_j)\Big\{P_m(\sigma_0(0))P''_m(\sigma_0(0)) + \big[P'_m(\sigma_0(0))\big]^2\Big\}(1-2\varepsilon_p) \\ &= \frac{1}{2}\sigma_j(\varepsilon_j)\Big\{P_m(\sigma_0(0))P''_m(\sigma_0(0)) - \big[P'_m(\sigma_0(0))\big]^2\Big\} \\ &= B(m,\alpha)(1-\varepsilon_j). \end{aligned} \tag{2.88}$$

Thus, by Lemma 2.9,

$$\begin{aligned} &\bigg| b_{p,p}(0,\dots,0,\varepsilon_p,\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,p} + \sum_{j=p+1}^{N} b_{p,j}(0,\dots,0,\varepsilon_p,\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,j}\bigg| \\ &\quad = \bigg| - B(m,\alpha)\Big\{\varepsilon_p\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,p} - \sum_{j=p+1}^{N}(1-\varepsilon_j)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,j}\Big\}\bigg| \leqslant c_{19}, \end{aligned} \tag{2.89}$$

where $c_{19} = B(m,\alpha)\max\{c_9(1),\dots,c_9(N-1)\} > 0$ is once again independent of $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N \in (0,M)$. Hence by compactness of $[-c_{19},c_{19}]$ there exist a decreasing subsequence $\{\varepsilon_{p,\ell}\}_{\ell=1}^{\infty}$ of $\{\frac{1}{\ell}\}_{\ell=1}^{\infty}$ and $L_p \in [-c_{19},c_{19}]$ such that

$$\begin{aligned} L_p = \lim_{\ell\uparrow\infty} & b_{p,p}(0,\dots,0,\varepsilon_{p,\ell},\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,p} \\ &+ \sum_{j=p+1}^{N} b_{p,j}(0,\dots,0,\varepsilon_{p,\ell},\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,j}. \end{aligned} \tag{2.90}$$

By monotone convergence

$$\begin{aligned} &\lim_{\ell\uparrow\infty} \sum_{p+1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{p,\ell},\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{j,k} \\ &\quad = \sum_{p+1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p}(\underline{\varepsilon})\big)_{j,k}. \end{aligned} \tag{2.91}$$

The lemma now follows from (2.90), (2.91). □

Lemma 2.12. *We have*

$$\lim_{\varepsilon_N\downarrow 0} b_{N,N}(0,\dots,0,\varepsilon_N) = B(m,\alpha). \tag{2.92}$$

Proof. We have, by Lemma 2.2

$$\begin{aligned} \lim_{\varepsilon_N\downarrow 0} b_{N,N}(0,\dots,0,\varepsilon_N) &= \lim_{\varepsilon_N\downarrow 0} a_{N,N}(0,\dots,0,\varepsilon_N) \\ &= \lim_{\varepsilon_N\downarrow 0} -\frac{1}{4}(1-\varepsilon_N)\Big\{P_m(\sigma_0(0))P''_m(\sigma_0(0))(1+\varepsilon_N) - \big[P'_m(\sigma_0(0))\big]^2(1-\varepsilon_N)\Big\} \\ &= \frac{1}{4}\Big\{\big[P'_m(\sigma_0(0))\big]^2 - P_m(\sigma_0(0))P''_m(\sigma_0(0))\Big\} = B(m,\alpha). \end{aligned} \tag{2.93}$$

□

Lemma 2.13. *Suppose $N \in \mathbb{N}$. Then given any $\eta > 0$, there exist $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M)$ such that if $f_{\underline{\varepsilon}} = f_{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N}$ is as defined in* (1.18)*, one has*

$$\left| J_{N-1}[f_{\underline{\varepsilon}}] \left[\int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-2} \big|f_{\underline{\varepsilon}}(x)\big|^2 \right]^{-1} - B(m,\alpha) \right| \leqslant \eta. \tag{2.94}$$

Proof. Let $c_{20} = \max\{c_{11}, c_{19}\} > 0$, independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M)$, where c_{11} and c_{19} are as in Lemmas 2.10 and 2.11. By Lemma 2.6 and monotone convergence one infers

$$\lim_{\varepsilon_N \downarrow 0} \int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 = \infty. \tag{2.95}$$

Thus, we can choose $\varepsilon_N \in (0, M)$ sufficiently small such that

$$\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 > 1, \tag{2.96}$$

and

$$c_{20} \left[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \right]^{-1} < \eta, \tag{2.97}$$

and, by Lemma 2.12,

$$\big| b_{N,N}(0, \ldots, 0, \varepsilon_N) - B(m,\alpha) \big| < \eta. \tag{2.98}$$

Thus, for any $R_{N-1} \in [-c_{20}, c_{20}]$, one has

$$\begin{aligned}
&\bigg| \Big\{ b_{N,N}(0, \ldots, 0, \varepsilon_N) \big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + R_{N-1} \Big\} \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} \\
&\qquad \times [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \bigg]^{-1} - B(m,\alpha) \bigg| \\
&\leqslant \big| b_{N,N}(0, \ldots, 0, \varepsilon_N) - B(m,\alpha) \big| \\
&\qquad + c_{20} \left| \left[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \right]^{-1} \right| \\
&< 2\eta.
\end{aligned} \tag{2.99}$$

Suppose first that $N \geqslant 2$. Then, by Lemma 2.11, there exist $L_{N-1} \in [-c_{19}, c_{19}]$ and a decreasing sequence $\{\varepsilon_{N-1,\ell}\}_{\ell=1}^\infty \subseteq (0, M)$, with $\lim_{\ell\uparrow\infty} \varepsilon_{N-1,\ell} = 0$, such that

$$\begin{aligned}
&\lim_{\ell\uparrow\infty} \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0, \ldots, 0, \varepsilon_{N-1,\ell}, \varepsilon_N) \big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} \\
&\quad = b_{N,N}(0, \ldots, 0, \varepsilon_N) \big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + L_{N-1}.
\end{aligned} \tag{2.100}$$

By (2.96) and monotone convergence, and replacing $\{\varepsilon_{N-1,\ell}\}_{\ell=1}^{\infty}$ by a subsequence if necessary, one can assume that

$$\int_0^\rho dx \left\{x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1,\ell}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)[^2 \right\} > 1, \ \ell \in \mathbb{N}. \tag{2.101}$$

Combining (2.97), (2.100), (2.101), and (2.99) with $R_{N-1} = L_{N-1}$, and using monotone convergence, there exists $\varepsilon_{N-1} \in (0, M)$ satisfying

$$\begin{aligned}
&\Bigg|\Bigg\{ \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N) \big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} \Bigg\} \\
&\quad \times \left(\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \right)^{-1} - B(m,\alpha) \Bigg| \\
&\leqslant \Bigg|\Bigg\{ \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N) \big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} \\
&\quad - \Big[b_{N,N}(0,\dots,0,\varepsilon_N) \big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + L_{N-1} \Big] \Bigg\} \Bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \\
&\quad \times [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \Bigg]^{-1} \Bigg| \\
&\quad + \Bigg| \Big[b_{N,N}(0,\dots,0,\varepsilon_N) \big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + L_{N-1} \Big] \Bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \\
&\quad \times [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \Bigg]^{-1} - B(m,\alpha) \Bigg| \\
&< \eta + 2\eta = 3\eta,
\end{aligned} \tag{2.102}$$

and

$$c_{20} \left[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \right]^{-1} < \eta, \tag{2.103}$$

as well as

$$\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 > 1. \tag{2.104}$$

One notes that by (2.102), (2.103), for all $R_{N-2} \in [-c_{20}, c_{20}]$,

$$\begin{aligned}
&\Bigg|\Bigg\{ \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N) \big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} + R_{N-2} \Bigg\} \\
&\quad \times \left[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \right]^{-1} - B(m,\alpha) \Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \Bigg|\Bigg\{\sum_{N-1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k}\Bigg\}\\
&\quad\times\Bigg[\int_0^\rho dx\,x^{-1}\prod_{j=1}^{N-2}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}-B(m,\alpha)\Bigg|\\
&\quad+c_{20}\Bigg(\int_0^\rho dx\,x^{-1}\prod_{j=1}^{N-2}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg)^{-1}\\
&< 3\eta+\eta=4\eta. \qquad (2.105)
\end{aligned}$$

So we have chosen $\varepsilon_{N-1},\varepsilon_N\in(0,M)$. If $N-1\geqslant 2$, then, by Lemma 2.11, there exist $L_{N-2}\in[-c_{19},c_{19}]$ and a decreasing sequence $\{\varepsilon_{N-2,\ell}\}_{\ell=1}^\infty\subseteq(0,M)$ with $\lim_{\ell\uparrow\infty}\varepsilon_{N-2,\ell}=0$ such that

$$\begin{aligned}
&\lim_{\ell\uparrow\infty}\sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2,\ell},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k}\\
&\quad=\sum_{N-1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k}+L_{N-2}. \qquad (2.106)
\end{aligned}$$

By (2.104) and monotone convergence, and replacing $\{\varepsilon_{N-2,\ell}\}_{\ell=1}^\infty$ by a subsequence, if necessary, one can assume that

$$\begin{aligned}
&\int_0^\rho dx\,x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}[\ln_{N-2}(\gamma/x)]^{-1-\varepsilon_{N-2,\ell}}\\
&\quad\times\prod_{j=N-1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2>1,\quad \ell\in\mathbb{N}. \qquad (2.107)
\end{aligned}$$

Combining (2.103), (2.106), (2.107), and (2.105) with $R_{N-2}=L_{N-2}$, and monotone convergence, there exists $\varepsilon_{N-2}\in(0,M)$ satisfying

$$\begin{aligned}
&\Bigg|\Bigg\{\sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k}\Bigg\}\\
&\quad\times\Bigg[\int_0^\rho dx\,x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}-B(m,\alpha)\Bigg|\\
&\leqslant\Bigg|\Bigg\{\sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k}\\
&\quad-\Bigg[\sum_{N-1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k}+L_{N-2}\Bigg]\Bigg\}\\
&\quad\times\Bigg[\int_0^\rho dx\,x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}\Bigg|\\
&\quad+\Bigg|\Bigg[\sum_{N-1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k}+L_{N-2}\Bigg]\\
&\quad\times\Bigg[\int_0^\rho dx\,x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}-B(m,\alpha)\Bigg|\\
&<\eta+4\eta=5\eta, \qquad (2.108)
\end{aligned}$$

and

$$c_{20}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}<\eta, \tag{2.109}$$

as well as

$$\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2>1, \tag{2.110}$$

such that for all $R_{N-3}\in[-c_{20},c_{20}]$ one infers

$$\begin{aligned}
&\bigg|\bigg\{\sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k}+R_{N-3}\bigg\}\\
&\quad\times\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}-B(m,\alpha)\bigg|\\
&\leqslant\bigg|\bigg\{\sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k}\bigg\}\\
&\quad\times\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}-B(m,\alpha)\bigg|\\
&\quad+c_{20}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}\\
&<5\eta+\eta=6\eta.
\end{aligned}\tag{2.111}$$

Repeating the argument above $N-1$ times (or if $N=1$) one arrives at the following fact: there exist $\varepsilon_1,\dots,\varepsilon_N\in(0,M)$ such that

$$\begin{aligned}
&\bigg|\bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}\bigg\}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}\\
&\quad-B(m,\alpha)\bigg|\leqslant(2N-1)\eta,
\end{aligned}\tag{2.112}$$

and

$$c_{20}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}<\eta, \tag{2.113}$$

as well as

$$\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2>1, \tag{2.114}$$

so that for all $R_0 \in [-c_{20}, c_{20}]$ one obtains

$$\begin{aligned}
&\Bigg|\Bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}+R_0\Bigg\} \\
&\quad\times\Bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}-B(m,\alpha)\Bigg| \\
&\leqslant\Bigg|\Bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}\Bigg\}\Bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-1-\varepsilon_j} \\
&\quad\times[\psi(x)]^2\Bigg]^{-1}-B(m,\alpha)\Bigg|+c_{20}\Bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1} \\
&<(2N-1)\eta+\eta=2N\eta.
\end{aligned} \tag{2.115}$$

Then, by Lemma 2.10, there exist $L_0 \in [-c_{20}, c_{20}]$ and a decreasing sequence $\{\varepsilon_{0,\ell}\}_{\ell=1}^\infty \subseteq (0,M)$ with $\lim_{\ell\uparrow\infty}\varepsilon_{0,\ell}=0$ such that

$$\lim_{\ell\uparrow\infty} J_{N-1}[f_{\varepsilon_{0,\ell},\varepsilon_1,\dots,\varepsilon_N}] = \sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}+L_0. \tag{2.116}$$

By (2.114) and monotone convergence, and replacing $\{\varepsilon_{0,\ell}\}_{\ell=1}^\infty$ by a subsequence if necessary, we can assume that

$$\int_0^\rho dx\, x^{-1+\varepsilon_{0,\ell}}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2>1, \quad \ell\in\mathbb{N}. \tag{2.117}$$

Combining (2.112), (2.113), (2.115) with $R_0 = L_0$, (2.116), (2.117), and monotone convergence, there exists $\varepsilon_0 \in (0,M)$ satisfying

$$\begin{aligned}
&\Bigg|J_{N-1}[f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}]\Bigg[\int_0^\rho dx\, x^{-1+\varepsilon_0}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}-B(m,\alpha)\Bigg| \\
&\leqslant\Bigg|\Bigg[J_{N-1}[f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}]-\Bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_n)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}+L_0\Bigg\}\Bigg] \\
&\quad\times\Bigg[\int_0^\rho dx\, x^{-1+\varepsilon_0}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}\Bigg| \\
&\quad+\Bigg|\Bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_n)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}+L_0\Bigg\} \\
&\quad\times\Bigg[\int_0^\rho dx\, x^{-1+\varepsilon_0}\prod_{j=1}^N[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\Bigg]^{-1}-B(m,\alpha)\Bigg| \\
&<\eta+2N\eta=(2N+1)\eta.
\end{aligned} \tag{2.118}$$

□

Lemma 2.14. *Suppose $N = 0$ and let f_{ε_0} be as defined on* (1.18). *Then*

$$\lim_{\varepsilon_0\downarrow 0}\int_0^\rho dx\, x^\alpha\big|f_{\varepsilon_0}^{(m)}(x)\big|^2\Bigg[\int_0^\rho dx\, x^{\alpha-2m}|f_{\varepsilon_0}(x)|^2\Bigg]^{-1}=A(m,\alpha). \tag{2.119}$$

Proof. By (1.10) we have

$$\lim_{\varepsilon_0 \downarrow 0} \int_0^\rho dx\, x^{\alpha-2m} |f_{\varepsilon_0}(x)|^2 \geqslant \lim_{\varepsilon_0 \downarrow 0} \int_0^{(0.8)\rho} dx\, x^{-1+\varepsilon_0} = \infty. \tag{2.120}$$

In addition, one has

$$f_{\varepsilon_0}^{(m)}(x) = \sum_{j=0}^{m} \binom{m}{j} P_j(\sigma_0(\varepsilon_0)) x^{\sigma_0(\varepsilon_0)-j} \psi^{(m-j)}(x), \quad 0 < x < \rho. \tag{2.121}$$

Thus, for all $0 < x < \rho$,

$$\begin{aligned} x^\alpha \big| f_{\varepsilon_0}^{(m)}(x) \big|^2 &= \sum_{j,k=0}^{m} \binom{m}{j}\binom{m}{k} P_j(\sigma_0(\varepsilon_0)) P_k(\sigma_0(\varepsilon_0)) x^{\alpha+2\sigma_0(\varepsilon_0)-j-k} \psi^{(m-j)}(x) \psi^{(m-k)}(x) \\ &= \big[P_m(\sigma_0(\varepsilon_0))\big]^2 x^{-1+\varepsilon_0} [\psi(x)]^2 + G_{12}(\varepsilon_0, x) \\ &= A(m, \alpha - \varepsilon_0) x^{-1+\varepsilon_0} [\psi(x)]^2 + G_{12}(\varepsilon_0, x), \end{aligned} \tag{2.122}$$

where, again by (1.10),

$$\big| G_{12}(\varepsilon_0, x) \big| \leqslant c_{21}, \quad \varepsilon_0 \in (0, M),\ 0 < x < \rho, \tag{2.123}$$

for some $c_{21} > 0$, independent of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, M)$. Hence,

$$\int_0^\rho dx\, x^\alpha \big| f_{\varepsilon_0}^{(m)}(x) \big|^2 = A(m, \alpha - \varepsilon_0) \int_0^\rho dx\, x^{-1+\varepsilon_0} [\psi(x)]^2 + \int_0^\rho dx\, G_{12}(\varepsilon_0, x), \tag{2.124}$$

and the lemma follows by dividing both sides of (2.124) by

$$\int_0^\rho dx\, x^{\alpha-2m} |f_{\varepsilon_0}(x)|^2 = \int_0^\rho dx\, x^{-1+\varepsilon_0} [\psi(x)]^2 \tag{2.125}$$

and applying (2.120), (2.123). □

3 The approximation procedure

We start with some more notation. For the remainder of this paper we shall assume $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N \in (0, \rho/20)$, that is, we shall assume $M = \rho/20$. Let $f_{\underline{\varepsilon}} = f_{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N}$ be as defined in (1.18). Then for $\delta \in (0, \rho/20)$, we shall write, recalling $\underline{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$,

$$f_{(\delta),\underline{\varepsilon}}(x) = \begin{cases} 0, & x < \delta \text{ or } \rho \leqslant x, \\ f_{\underline{\varepsilon}}(x), & \delta \leqslant x < \rho. \end{cases} \tag{3.1}$$

We shall let $h \in C^\infty(\mathbb{R})$ satisfy the following properties:

(i) h is even on $\mathbb{R}$, (3.2)

(ii) $h(x) \geqslant 0, \quad x \in \mathbb{R}$, (3.3)

(iii) $\operatorname{supp}(h) \subseteq (-1, 1)$, (3.4)

(iv) $\int_{-1}^{1} dx\, h(x) = 1$, (3.5)

(v) h is non-increasing on $[0, \infty)$. (3.6)

For $\varepsilon > 0$ we write

$$h_\varepsilon(x) = \varepsilon^{-1} h(x/\varepsilon), \quad x \in \mathbb{R}. \tag{3.7}$$

For $\delta \in (0, \rho/20)$ and $\varepsilon \in (0, \delta/4]$, we write

$$f_{(\delta,\varepsilon),\underline{\varepsilon}} = f_{(\delta),\underline{\varepsilon}} * h_\varepsilon. \tag{3.8}$$

Remark 3.1.

(i) Since h is even, we have

$$\begin{aligned} f_{(\delta,\varepsilon),\underline{\varepsilon}}(x) &= \int_{-\infty}^{\infty} dt\, \varepsilon^{-1} h(t/\varepsilon) f_{(\delta),\underline{\varepsilon}}(x-t) = \int_{-\infty}^{\infty} dt\, \varepsilon^{-1} h(-t/\varepsilon) f_{(\delta),\underline{\varepsilon}}(x-t) \\ &= \int_{-\infty}^{\infty} du\, \varepsilon^{-1} h(u/\varepsilon) f_{(\delta),\underline{\varepsilon}}(x+u) = \int_{-\infty}^{\infty} dr\, \varepsilon^{-1} h((r-x)/\varepsilon) f_{(\delta),\underline{\varepsilon}}(r) \\ &= \int_{x-\varepsilon}^{x+\varepsilon} dr\, \varepsilon^{-1} h((r-x)/\varepsilon) f_{(\delta),\underline{\varepsilon}}(r), \quad x \in \mathbb{R}. \end{aligned} \tag{3.9}$$

(ii) Since $\varepsilon \in (0, \delta/4]$, $\operatorname{supp}(f_{(\delta,\varepsilon),\underline{\varepsilon}}) \subseteq [3\delta/4, 73\rho/80]$. Hence,

$$f_{(\delta,\varepsilon),\underline{\varepsilon}} \in C_0^\infty((0,\rho)). \tag{3.10}$$

(iii) Let $g \in L^\infty(\mathbb{R}), x \in \mathbb{R}, \tau \in \mathbb{R}\backslash\{0\}$. For $0 < \varepsilon \leqslant \delta/4 < \rho/80$, let $g_\varepsilon = h_\varepsilon * g$. By the sequence of change of variables in (3.9)*, we have*

$$\begin{aligned} \tau^{-1}[g_\varepsilon(x+\tau) - g_\varepsilon(x)] &= \int_{-\infty}^{\infty} dr\, (\tau\varepsilon)^{-1} \{h((r-x-\tau)/\varepsilon) - h((r-x)/\varepsilon)\} g(r) \\ &= -\int_{-\infty}^{\infty} dr\, (\tau\varepsilon)^{-1} h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon)(\tau/\varepsilon) g(r) \\ &= -\varepsilon^{-2} \int_{-\infty}^{\infty} dr\, h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon) g(r), \end{aligned} \tag{3.11}$$

where

$$0 \leqslant \lambda(x,r,\tau) \leqslant 1, \quad x, r \in \mathbb{R}. \tag{3.12}$$

Since $h', g \in L^\infty(\mathbb{R})$ and, for $-1 \leqslant \tau \leqslant 1$,

$$\operatorname{supp} h'([\,\cdot\, - x - \lambda(x, \cdot\,, \tau)\tau]/\varepsilon) \subseteq [x - \varepsilon - 1, x + \varepsilon + 1], \tag{3.13}$$

applying the dominated convergence theorem we get

$$\begin{aligned} g_\varepsilon'(x) &= \lim_{\tau\to 0} \tau^{-1}[g_\varepsilon(x+\tau) - g_\varepsilon(x)] \\ &= -\lim_{\tau\to 0} \varepsilon^{-2} \int_{-\infty}^{\infty} dr\, h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon) g(r) \\ &= -\varepsilon^{-2} \lim_{\tau\to 0} \int_{x-\varepsilon-1}^{x+\varepsilon+1} dr\, h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon) g(r) \\ &= -\varepsilon^{-2} \int_{x-\varepsilon-1}^{x+\varepsilon+1} dr\, h'((r-x)/\varepsilon) g(r) = -\varepsilon^{-2} \int_{x-\varepsilon}^{x+\varepsilon} dr\, h'((r-x)/\varepsilon) g(r). \end{aligned} \tag{3.14}$$

Let $\delta \in (0, \rho/20)$. For technical convenience, so that we can use the general theory of convolution, we shall write $\widetilde{f}_{(\delta),\underline{\varepsilon}}$ for a function in $C_0^\infty(\mathbb{R})$ satisfying:

$$\begin{aligned}
&(i)\ \widetilde{f}_{(\delta),\underline{\varepsilon}}(x) = f_{(\delta),\underline{\varepsilon}}, \quad x \geqslant \delta,\\
&(ii)\ \widetilde{f}_{(\delta),\underline{\varepsilon}}(x) \geqslant 0, \quad -\infty < x < \infty.
\end{aligned} \tag{3.15}$$

Constants denoted by $\nu_j, j \in \mathbb{N}$, will depend on $N \in \mathbb{N} \cup \{0\}$, $\gamma, \rho \in (0,\infty)$ with $\gamma \geqslant \rho e_{N+1}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $h, \psi \in C^\infty(\mathbb{R})$, and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, \rho/20)$, but are independent of $\delta \in (0, \rho/20)$ and $\varepsilon \in (0, \delta/4)$. $\diamond$

Lemma 3.2. *For all $k \in \mathbb{N} \cup \{0\}$ there exists $\nu_1 = \nu_1(k) > 0$ such that*

$$\big|f_{\underline{\varepsilon}}^{(k)}(x)\big| \leqslant \nu_1 x^{[2(m-k)-1-\alpha+(\varepsilon_0/2)]/2}, \quad 0 < x < \rho. \tag{3.16}$$

Proof. This lemma follows from Lemma 2.4, the product rule

$$f_{\underline{\varepsilon}}^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} v_{\underline{\varepsilon}}^{(k-j)}(x)\psi^{(j)}(x), \quad 0 < x < \rho, \tag{3.17}$$

and that, for all $\beta > 0$, the function $t \mapsto t^{-\beta}\ln(t)$ is bounded on $(1,\infty)$. $\square$

Lemma 3.3. *For $j = 1, \dots, m$, and $x \in [3\delta/4, 5\delta/4]$, we have, writing $\theta = \delta/4$,*

$$f_{(\delta,\theta),\underline{\varepsilon}}^{(j)}(x) = \sum_{k=1}^{j} (-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j-k)}(\delta) + \theta^{-1}\int_\delta^{x+\theta} dr\, h((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j)}(r). \tag{3.18}$$

Proof. For $3\delta/4 \leqslant x \leqslant 5\delta/4$ we have, by (3.14)

$$\begin{aligned}
f'_{(\delta,\theta),\underline{\varepsilon}}(x) &= -\theta^{-2}\int_{x-\theta}^{x+\theta} dr\, h'((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}(r)\\
&= -\theta^{-2}\int_{\delta}^{x+\theta} dr\, h'((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}(r)\\
&= -\theta^{-1}\int_{\delta}^{x+\theta} dr\, \frac{d}{dr}[h((r-x)/\theta)] f_{(\delta),\underline{\varepsilon}}(r)\\
&= -\theta^{-1}\bigg\{h((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}(r)\bigg|_\delta^{x+\theta} - \int_\delta^{x+\theta} dr\, h((r-x)/\theta) f'_{(\delta),\underline{\varepsilon}}(r)\bigg\}\\
&= -\theta^{-1}\bigg\{-h((\delta-x)/\theta) f_{(\delta),\underline{\varepsilon}}(\delta) - \int_\delta^{x+\theta} dr\, h((r-x)/\theta) f'_{(\delta),\underline{\varepsilon}}(r)\bigg\}\\
&= \theta^{-1}h((\delta-x)/\theta) f_{(\delta),\underline{\varepsilon}}(\delta) + \theta^{-1}\int_\delta^{x+\theta} dr\, h((r-x)/\theta) f'_{(\delta),\underline{\varepsilon}}(r).
\end{aligned} \tag{3.19}$$

Suppose $j \in \{1, \dots, m-1\}$ and that for all $x \in [3\delta/4, 5\delta/4]$ one has

$$f_{(\delta,\theta),\underline{\varepsilon}}^{(j)}(x) = \sum_{k=1}^{j} (-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j-k)}(\delta) + \theta^{-1}\int_\delta^{x+\theta} dr\, h((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j)}(r), \tag{3.20}$$

then, by (3.14), one concludes

$$\begin{aligned}
f^{(j+1)}_{(\delta,\theta),\underline{\varepsilon}}(x) &= \sum_{k=1}^{j}(-1)^{k+1}\theta^{-k}(-1/\theta)h^{(k)}((\delta-x)/\theta)f^{(j-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad + \frac{d}{dx}\left(\theta^{-1}\int_{x-\theta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(r)\right) \\
&= \sum_{k=1}^{j}(-1)^{k}\theta^{-(k+1)}h^{(k)}((\delta-x)/\theta)f^{(j-k)}_{(\delta),\underline{\varepsilon}}(\delta) - \frac{1}{\theta^2}\int_{x-\theta}^{x+\theta} dr\, h'((r-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(r) \\
&= \sum_{k=2}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) - \frac{1}{\theta}\int_{\delta}^{x+\theta} dr\left(\frac{d}{dr}[h((r-x)/\theta)]\right)f^{(j)}_{(\delta),\underline{\varepsilon}}(r) \\
&= \sum_{k=2}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) - \frac{1}{\theta}\bigg\{h((r-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(r)\bigg|_{\delta}^{x+\theta} \\
&\quad - \int_{\delta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j+1)}_{(\delta),\underline{\varepsilon}}(r)\bigg\} \\
&= \sum_{k=2}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) + \frac{1}{\theta}h((\delta-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad + \frac{1}{\theta}\int_{\delta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j+1)}_{(\delta),\underline{\varepsilon}}(r) \\
&= \sum_{k=1}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad + \frac{1}{\theta}\int_{\delta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j+1)}_{(\delta),\underline{\varepsilon}}(r). \qquad (3.21)
\end{aligned}$$

Hence, Lemma 3.3 follows by induction. □

Corollary 3.4. *There exists $\nu_2 > 0$ such that for all $\delta \in (0, \rho/20)$,*

$$\left|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\right| \leqslant \nu_2 x^{[-1-\alpha+(\varepsilon_0/2)]/2}, \quad 3\delta/4 \leqslant x \leqslant 5\delta/4. \qquad (3.22)$$

Proof. Let

$$K_m = \sup\left\{\left|h^{(k)}(t)\right| \,\middle|\, -1 \leqslant t \leqslant 1,\ k = 0,1,\ldots,m\right\}. \qquad (3.23)$$

By Lemmas 3.2 and 3.3 we have for $x \in [3\delta/4, 5\delta/4]$,

$$\begin{aligned}
\left|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\right| &\leqslant \sum_{k=1}^{m} 4^k\delta^{-k}K_m\nu_1(m-k)\delta^{[2k-1-\alpha+(\varepsilon_0/2)]/2} \\
&\quad + 4\delta^{-1}\left(\tfrac{6\delta}{4}-\delta\right)K_m \sup\left\{\left|f^{(m)}_{(\delta),\underline{\varepsilon}}(r)\right| \,\middle|\, \delta \leqslant r \leqslant 6\delta/4\right\} \\
&= \sum_{k=1}^{m} 4^k K_m\nu_1(m-k)\delta^{[-1-\alpha+(\varepsilon_0/2)]/2} \\
&\quad + 2K_m\nu_1(m)\sup\left\{r^{[-1-\alpha+(\varepsilon_0/2)]/2} \,\middle|\, \delta \leqslant r \leqslant 6\delta/4\right\}
\end{aligned}$$

$$\begin{aligned}
&\leqslant K_m\bigg(\sum_{k=1}^{m} 4^k \nu_1(m-k)\bigg)\big\{(4/3)^{[-1-\alpha+(\varepsilon_0/2)]/2} + (4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2}\big\} \\
&\quad \times x^{[-1-\alpha+(\varepsilon_0/2)]/2} \\
&\quad + 2K_m\nu_1(m)\big\{(4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2} + 2^{[-1-\alpha+(\varepsilon_0/2)]/2}\big\} x^{[-1-\alpha+(\varepsilon_0/2)]/2} \\
&= \nu_2 x^{[-1-\alpha+(\varepsilon_0/2)]/2},
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
\nu_2 = K_m&\bigg(\sum_{k=1}^{m} 4^k \nu_1(m-k)\bigg)\big\{(4/3)^{[-1-\alpha+(\varepsilon_0/2)]/2} + (4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2}\big\} \\
&+ 2K_m\nu_1(m)\big\{(4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2} + 2^{[-1-\alpha+(\varepsilon_0/2)]/2}\big\}.
\end{aligned} \tag{3.25}$$

□

Lemma 3.5. *There exists $\nu_3 > 0$ such that for all $\delta \in (0, \rho/20)$ we have*

$$\big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big| \leqslant \nu_3 x^{[-1-\alpha+(\varepsilon_0/2)]/2}, \quad 5\delta/4 \leqslant x \leqslant \rho. \tag{3.26}$$

Proof. We first note that, for $5\delta/4 \leqslant x \leqslant 73\rho/80$,

$$\begin{aligned}
f_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \int_{x-\delta/4}^{x+\delta/4} dr\,(4/\delta) h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r) \\
&= \int_{x-\delta/4}^{x+\delta/4} dr\,(4/\delta) h(4(r-x)/\delta) \widetilde{f}_{(\delta),\underline{\varepsilon}}(r) \\
&= \big(h_{\delta/4} * \widetilde{f}_{(\delta),\underline{\varepsilon}}\big)(x),
\end{aligned} \tag{3.27}$$

hence

$$\begin{aligned}
f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \Big(h_{\delta/4} * \widetilde{f}^{(m)}_{(\delta),\underline{\varepsilon}}\Big)(x) \\
&= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) \widetilde{f}^{(m)}_{(\delta),\underline{\varepsilon}}(r) \\
&= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f^{(m)}_{(\delta),\underline{\varepsilon}}(r),
\end{aligned} \tag{3.28}$$

therefore, by Lemma 3.2,

$$\begin{aligned}
\big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big| &\leqslant \sup\big\{\big|f^{(m)}_{(\delta),\underline{\varepsilon}}(r)\big| \,\big|\, x-(\delta/4) \leqslant r \leqslant x+(\delta/4)\big\} \\
&\leqslant \nu_1(m)\sup\big\{r^{[-1-\alpha+(\varepsilon_0/2)]/2} \,\big|\, x-(\delta/4) \leqslant r \leqslant x+(\delta/4)\big\} \\
&\leqslant \nu_1(m)\sup\big\{r^{[-1-\alpha+(\varepsilon_0/2)]/2} \,\big|\, 3x/4 \leqslant r \leqslant 5x/4\big\} \\
&\leqslant \nu_1(m)\big\{(3/4)^{[-1-\alpha+(\varepsilon_0/2)]/2} + (5/4)^{[-1-\alpha+(\varepsilon_0/2)]/2}\big\} x^{[-1-\alpha+(\varepsilon_0/2)]/2}.
\end{aligned} \tag{3.29}$$

By Remark 3.1 (ii), $\mathrm{supp}(f_{(\delta,\delta/4),\underline{\varepsilon}}) \subseteq [3\delta/4, 73\rho/80]$. So (3.26) holds for $x \in [73\rho/80, \rho]$, completing the proof. □

Lemma 3.6. *On any compact interval* $[a,b] \subseteq (0,\rho]$, $f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}$ *converges to* $f^{(m)}_{\underline{\varepsilon}}$ *uniformly as* $\delta \downarrow 0$.

Proof. Choose $\delta_0 \in (0,\rho/20)$ such that $0 < 5\delta_0/4 < a$. Then for all $0 < \delta < \delta_0$ and $x \in [a,b]$,

$$\begin{aligned} f_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r) \\ &= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta_0),\underline{\varepsilon}}(r) \\ &= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) \widetilde{f}_{(\delta_0),\underline{\varepsilon}}(r) \\ &= \big(h_{\delta/4} * \widetilde{f}_{(\delta_0),\underline{\varepsilon}}\big)(x). \end{aligned} \tag{3.30}$$

Since $\widetilde{f}_{(\delta_0),\underline{\varepsilon}} \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \Big(h_{\delta/4} * \widetilde{f}^{(m)}_{(\delta_0),\underline{\varepsilon}}\Big)(x), \quad x \in [a,b], \\ &\underset{\delta\downarrow 0}{\longrightarrow} \widetilde{f}^{(m)}_{(\delta_0),\underline{\varepsilon}}(x) \quad \text{uniformly for } x \in [a,b], \\ &= f^{(m)}_{\underline{\varepsilon}}(x). \end{aligned} \tag{3.31}$$

□

Corollary 3.7. *We have*

$$\lim_{\delta\downarrow 0}\int_0^\rho dx\, x^\alpha \big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 = \int_0^\rho dx\, x^\alpha \big|f^{(m)}_{\underline{\varepsilon}}(x)\big|^2. \tag{3.32}$$

Proof. Let $\nu_4 = \max\{\nu_2,\nu_3\} > 0$. Then by Corollary 3.4 and Lemma 3.5, we have, for all $\delta \in (0,\rho/20)$,

$$x^\alpha \big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 \leqslant \nu_4^2 x^{-1+(\varepsilon_0/2)}, \quad 0 < x < \rho. \tag{3.33}$$

By Lemma 3.6 we have

$$\lim_{\delta\downarrow 0} x^\alpha \big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 = x^\alpha \big|f^{(m)}_{\underline{\varepsilon}}(x)\big|^2, \quad 0 < x < \rho. \tag{3.34}$$

Since $x \mapsto \nu_4 x^{-1+(\varepsilon_0/2)}$ is integrable on $(0,\rho)$, the corollary now follows by dominated convergence. □

Lemma 3.8. *There exists* $\nu_5 > 0$ *such that for all* $\delta \in (0,\rho/20)$ *we have*

$$|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| \leqslant \nu_5 x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}, \quad 3\delta/4 \leqslant x \leqslant 5\delta/4. \tag{3.35}$$

Proof. For $3\delta/4 \leqslant x \leqslant 5\delta/4$ we have

$$\begin{aligned}|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| &= \left|4\delta^{-1}\int_{\delta}^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r)\right| \\ &\leqslant \sup\{|f_{(\delta),\underline{\varepsilon}}(r)| \,|\, \delta \leqslant r \leqslant 6\delta/4\} = \sup\{|f_{\underline{\varepsilon}}(r)| \,|\, \delta \leqslant r \leqslant 3\delta/2\} \\ &\leqslant \nu_1(0) \sup\left\{r^{[2m-1-\alpha+(\varepsilon_0/2)]/2} \,\middle|\, \delta \leqslant r \leqslant 3\delta/2\right\} \\ &\leqslant \nu_1(0)\big\{(4/5)^{[2m-1-\alpha+(\varepsilon_0/2)]/2} + 2^{[2m-1-\alpha+(\varepsilon_0/2)]/2}\big\} x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}. \end{aligned} \tag{3.36}$$

□

Lemma 3.9. *There exists $\nu_6 > 0$ such that for all $\delta \in (0, \rho/20)$ we have*

$$|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| \leqslant \nu_6 x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}, \quad 5\delta/4 \leqslant x < \rho. \tag{3.37}$$

Proof. For $x \in [5\delta/4, \rho)$ we have

$$\begin{aligned}|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| &= \left|4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r)\right| \\ &\leqslant \sup\{|f_{(\delta),\underline{\varepsilon}}(r)| \,|\, x-\delta/4 \leqslant r \leqslant x+\delta/4\} \\ &\leqslant \nu_1(0) \sup\left\{r^{[2m-1-\alpha+(\varepsilon_0/2)]/2} \,\middle|\, 3x/4 \leqslant r \leqslant 5x/4\right\} \\ &\leqslant \nu_1(0)\big\{(3/4)^{[2m-1-\alpha+(\varepsilon_0/2)]/2} + (5/4)^{[2m-1-\alpha+(\varepsilon_0/2)]/2}\big\} x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}. \end{aligned} \tag{3.38}$$

□

Lemma 3.10. *On any compact interval $[a, b] \subseteq (0, \rho]$, $f_{(\delta,\delta/4),\underline{\varepsilon}}$ converges to $f_{\underline{\varepsilon}}$ uniformly as $\delta \downarrow 0$.*

Proof. Choose $\delta_0 \in (0, \rho/20)$ with $0 < 5\delta_0/4 < a$. By (3.30), for all $0 < \delta < \delta_0$, we have

$$f_{(\delta,\delta/4),\underline{\varepsilon}}(x) = \big(h_{\delta/4} * \widetilde{f}_{(\delta_0),\underline{\varepsilon}}\big)(x), \quad a \leqslant x \leqslant b. \tag{3.39}$$

Since $\widetilde{f}_{(\delta_0),\underline{\varepsilon}} \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} f_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \big(h_{\delta/4} * \widetilde{f}_{(\delta_0),\underline{\varepsilon}}\big)(x) \\ &\underset{\delta\downarrow 0}{\longrightarrow} \widetilde{f}_{(\delta_0),\underline{\varepsilon}}(x) \quad \text{uniformly for } x \in [a, b] \\ &= f_{\underline{\varepsilon}}(x). \end{aligned} \tag{3.40}$$

□

Corollary 3.11. *For $k \in \{0, 1, \dots, N\}$ we have*

$$\lim_{\delta\downarrow 0} \int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2} \big|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 = \int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2} \big|f_{\underline{\varepsilon}}(x)\big|^2. \tag{3.41}$$

Proof. Let $\nu_7 = \max\{\nu_5, \nu_6\} > 0$. By Lemmas 3.8 and 3.9 we have, for all $\delta \in (0, \rho/20)$ and $x \in (0, \rho)$,

$$x^{\alpha-2m} \prod_{j=1}^{k} [\ln_j(\gamma/x)]^{-2} \big| f_{(\delta,\delta/4),\underline{\varepsilon}}(x) \big|^2 \leqslant \nu_7^2 x^{-1+(\varepsilon_0/2)} \prod_{j=1}^{k} [\ln_j(\gamma/x)]^{-2}. \tag{3.42}$$

By Lemma 3.10 we have for $x \in (0, \rho)$,

$$\lim_{\delta \downarrow 0} x^{\alpha-2m} \prod_{j=1}^{k} [\ln_j(\gamma/x)]^{-2} \big| f_{(\delta,\delta/4),\underline{\varepsilon}}(x) \big|^2 = x^{\alpha-2m} \prod_{j=1}^{k} [\ln_j(\gamma/x)]^{-2} \big| f_{\underline{\varepsilon}}(x) \big|^2. \tag{3.43}$$

Since $x \mapsto x^{-1+(\varepsilon_0/2)} \Big(\prod_{j=1}^{k} [\ln_j(\gamma/x)]^{-2} \Big)$ is integrable on $(0, \rho)$, the corollary now follows by dominated convergence. □

Corollary 3.12. *Suppose $N \in \mathbb{N}$. Then there exists a family $\{g_{\delta,\underline{\varepsilon}}\}_{\delta \in (0,(0.05)\rho)} \subseteq C_0^\infty((0, \rho))$ such that*

$$\begin{aligned} &\lim_{\delta \downarrow 0} J_{N-1}[g_{\delta,\underline{\varepsilon}}] \bigg(\int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-2} \big| g_{\delta,\underline{\varepsilon}}(x) \big|^2 \bigg)^{-1} \\ &\quad = J_{N-1}[f_{\underline{\varepsilon}}] \bigg(\int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-2} \big| f_{\underline{\varepsilon}}(x) \big|^2 \bigg)^{-1}. \end{aligned} \tag{3.44}$$

Proof. For $\delta \in (0, \rho/20)$ put $g_{\delta,\underline{\varepsilon}} = f_{(\delta,\delta/4),\underline{\varepsilon}}$. Then $g_{\delta,\underline{\varepsilon}} \in C_0^\infty((0, \rho))$ by Remark 3.1 *(ii)*. The result now follows from Corollaries 3.7 and 3.11. □

Corollary 3.13. *Suppose $N = 0$. Then there exists a family $\{g_{\delta,\underline{\varepsilon}}\}_{\delta \in (0,(0.05)\rho)} \subseteq C_0^\infty((0, \rho))$ such that*

$$\begin{aligned} &\lim_{\delta \downarrow 0} \int_0^\rho dx\, x^\alpha \big| g_{\delta,\underline{\varepsilon}}^{(m)}(x) \big|^2 \bigg(\int_0^\rho dx\, x^{\alpha-2m} \big| g_{\delta,\underline{\varepsilon}}(x) \big|^2 \bigg)^{-1} \\ &\quad = \int_0^\rho dx\, x^\alpha \big| f_{\underline{\varepsilon}}^{(m)}(x) \big|^2 \bigg(\int_0^\rho dx\, x^{\alpha-2m} \big| f_{\underline{\varepsilon}}(x) \big|^2 \bigg)^{-1}. \end{aligned} \tag{3.45}$$

Proof. The proof of this corollary is the same as that of Corollary 3.12. □

4 Principal results on optimal constants

In our final section we now prove optimality of the constants $A(m, \alpha)$ and $B(m, \alpha)$.

Starting with the interval $(0, \rho)$, we first establish optimality of $A(m, \alpha)$ in (1.1).

Theorem 4.1. *Suppose that $N = 0$. Then, given any $\eta > 0$, there exists $g \in C_0^\infty((0, \rho))$ such that*

$$\bigg| \int_0^\rho dx\, x^\alpha \big| g^{(m)}(x) \big|^2 \bigg[\int_0^\rho dx\, x^{\alpha-2m} \big| g(x) \big|^2 \bigg]^{-1} - A(m, \alpha) \bigg| \leqslant \eta. \tag{4.1}$$

In particular, the constant $A(m, \alpha)$ in (1.1) is sharp.

Proof. Given any $\eta > 0$ there exists $\varepsilon_0 \in (0, \rho/20)$ such that

$$\left| \int_0^\rho dx\, x^\alpha \big|f_{\varepsilon_0}^{(m)}(x)\big|^2 \left[\int_0^\rho dx\, x^{\alpha-2m} \big|f_{\varepsilon_0}(x)\big|^2 \right]^{-1} - A(m,\alpha) \right| \leqslant \eta/2, \tag{4.2}$$

by Lemma 2.14. With this value of $\varepsilon_0 \in (0, \rho/20)$, Corollary 3.13 implies that there exists $g \in C_0^\infty((0,\rho))$ such that

$$\begin{aligned} &\bigg| \int_0^\rho dx\, x^\alpha \big|g^{(m)}(x)\big|^2 \left[\int_0^\rho dx\, x^{\alpha-2m} \big|g(x)\big|^2 \right]^{-1} \\ &\quad - \int_0^\rho dx\, x^\alpha \big|f_{\varepsilon_0}^{(m)}(x)\big|^2 \left[\int_0^\rho dx\, x^{\alpha-2m} \big|f_{\varepsilon_0}(x)\big|^2 \right]^{-1} \bigg| \leqslant \eta/2. \end{aligned} \tag{4.3}$$

Theorem 4.1 now follows from (4.2), (4.3). □

Next, we prove optimality of the N constants $B(m,\alpha)$ in (1.1):

Theorem 4.2. *Suppose that $N \in \mathbb{N}$. Then for any $\eta > 0$, there exists $g \in C_0^\infty((0,\rho))$ such that*

$$\begin{aligned} &\bigg| \bigg[\int_0^\rho dx\, x^\alpha \big|g^{(m)}(x)\big|^2 - A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} \big|g(x)\big|^2 \\ &\quad - B(m,\alpha) \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2m} \big|g(x)\big|^2 \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} \bigg] \\ &\quad \times \left[\int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-2} \big|g(x)\big|^2 \right]^{-1} - B(m,\alpha) \bigg| \leqslant \eta. \end{aligned} \tag{4.4}$$

In particular, successively increasing N through $1, 2, 3, \dots,$ demonstrates that the N constants $B(m,\alpha)$ in (1.1) are sharp. Together with Theorem 4.1, this theorem asserts that the $N+1$ constants, $A(m,\alpha)$ and the N constants $B(m,\alpha)$, in (1.1) are sharp.

Proof. Given any $\eta > 0$ there exist $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, \rho/20)$ such that, writing $f_{\underline{\varepsilon}} = f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}$,

$$\left| J_{N-1}[f_{\underline{\varepsilon}}] \left[\int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-2} \big|f_{\underline{\varepsilon}}(x)\big|^2 \right]^{-1} - B(m,\alpha) \right| \leqslant \eta/2, \tag{4.5}$$

by Lemma 2.13. With these values of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, \rho/20)$, Corollary 3.12 implies that there exists $g \in C_0^\infty((0,\rho))$ such that

$$\begin{aligned} &\bigg| J_{N-1}[g] \left[\int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-2} \big|g(x)\big|^2 \right]^{-1} \\ &\quad - J_{N-1}[f_{\underline{\varepsilon}}] \left[\int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-2} \big|f_{\underline{\varepsilon}}(x)\big|^2 \right]^{-1} \bigg| \leqslant \eta/2. \end{aligned} \tag{4.6}$$

Theorem 4.2 now follows from (4.5), (4.6). □

Next we turn to analogous results for the half line (r,∞). We start with some preparations.

Writing

$$\begin{aligned} Q_{m,\alpha}(\lambda) &= \left(\lambda^2 - \frac{(1-\alpha)^2}{4}\right)\left(\lambda^2 - \frac{(3-\alpha)^2}{4}\right)\cdots\left(\lambda^2 - \frac{(2m-1-\alpha)^2}{4}\right) \\ &= \prod_{j=1}^{m}\left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right) = \sum_{\ell=0}^{2m} k_\ell(m,\alpha)\lambda^\ell, \end{aligned} \tag{4.7}$$

one infers that

$$(i)\ k_{2j-1}(m,\alpha) = 0, \quad j = 1,\dots,m, \tag{4.8}$$

$$(ii)\ k_{2j}(m,\alpha) = (-1)^{m-j}|k_{2j}(m,\alpha)|, \quad j = 0,1,\dots,m, \tag{4.9}$$

and thus,

$$Q_{m,\alpha}(\lambda) = \sum_{j=0}^{m}(-1)^{m-j}|k_{2j}(m,\alpha)|\lambda^{2j}. \tag{4.10}$$

Lemma 4.3 ([41, Sect. 2 and proof of Theorem 3.1 (i)])**.** *Suppose $\hat\rho > e_{N+1}$ and $\alpha \in \mathbb{R}\backslash\{1,\dots,2m-1\}$. For $g \in C_0^\infty((\hat\rho,\infty))$ let $w = w_g \in C_0^\infty((\ln(\hat\rho),\infty))$ be defined by*

$$g(e^t) = e^{[(2m-1-\alpha)/2]t}w(t), \quad t \in (\ln(\hat\rho),\infty). \tag{4.11}$$

Then for all $g \in C_0^\infty((\hat\rho,\infty))$,

$$\begin{aligned} \int_{\hat\rho}^\infty dy\, y^\alpha\big|g^{(m)}(y)\big|^2 &= \int_{\ln(\hat\rho)}^\infty dt\, \sum_{j=0}^{m}|k_{2j}(m,\alpha)|\big|w^{(j)}(t)\big|^2, \\ \int_{\hat\rho}^\infty dy\, y^{\alpha-2m}|g(y)|^2 &= \int_{\ln(\hat\rho)}^\infty dt\, |w(t)|^2, \end{aligned} \tag{4.12}$$

and, if $N \in \mathbb{N}$, one also has, for $k = 1,\dots,N$,

$$\left(e^t\right)^{\alpha-2m}\big|g(e^t)\big|^2\prod_{p=1}^{k}[\ln_p(e^t)]^{-2} = e^{-t}|w(t)|^2t^{-2}\prod_{p=1}^{k-1}[\ln_p(t)]^{-2}, \quad t \in (\ln(\hat\rho),\infty). \tag{4.13}$$

Hence, if $N \in \mathbb{N}$,

$$\begin{aligned} &\bigg[\int_{\hat\rho}^\infty dy\, y^\alpha\big|g^{(m)}(y)\big|^2 - A(m,\alpha)\int_{\hat\rho}^\infty dy\, y^{\alpha-2m}|g(y)|^2 \\ &\quad - B(m,\alpha)\int_{\hat\rho}^\infty dy\, y^{\alpha-2m}|g(y)|^2\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(y)]^{-2}\bigg] \\ &\quad \times\left[\int_{\hat\rho}^\infty dy\, y^{\alpha-2m}|g(y)|^2\prod_{p=1}^{N}[\ln_p(y)]^{-2}\right]^{-1} \end{aligned}$$

$$\begin{aligned}
&= \bigg[\int_{\ln(\hat\rho)}^{\infty} dt \sum_{j=0}^{m} |k_{2j}(m,\alpha)| \big|w^{(j)}(t)\big|^2 - A(m,\alpha) \int_{\ln(\hat\rho)}^{\infty} dt\, |w(t)|^2 \\
&\quad - B(m,\alpha) \int_{\ln(\hat\rho)}^{\infty} dt\, |w(t)|^2 t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2} \bigg] \\
&\quad \times \bigg[\int_{\ln(\hat\rho)}^{\infty} dt\, |w(t)|^2 t^{-2} \prod_{p=1}^{N-1} [\ln_p(t)]^{-2} \bigg]^{-1}, \quad g \in C_0^\infty((\hat\rho,\infty)). \qquad (4.14)
\end{aligned}$$

Corollary 4.4. *Lemma* 4.3 *holds for all* $\alpha \in \mathbb{R}$*, that is, it holds without the restriction* $\alpha \in \mathbb{R}\backslash\{1,\dots,2m-1\}$.

Proof. We first note that by (4.7), for $\ell = 0,1,\dots,2m$, $k_\ell(m,\alpha)$ is a polynomial in α and so it is continuous in α. For $g \in C_0^\infty((\hat\rho,\infty))$, to emphasize that the definition of $w = w_g \in C_0^\infty((\ln(\hat\rho),\infty))$ in (4.11) depends also on α, we shall write, for all $\alpha \in \mathbb{R}$,

$$w_\alpha(t) = e^{-[(2m-1-\alpha)/2]t} g(e^t), \quad t \in (\ln(\hat\rho),\infty). \qquad (4.15)$$

Then, for $j = 0,1,\dots,m$, one gets

$$w_\alpha^{(j)}(t) = \sum_{k=0}^{j} S(j,k,\alpha,t) g^{(k)}(e^t), \quad t \in (\ln(\hat\rho),\infty), \qquad (4.16)$$

where, for $j \in \{0,1,\dots,m\}$, $k \in \{0,1,\dots,j\}$, and $t \in (\ln(\hat\rho),\infty)$, $\alpha \mapsto S(j,k,\alpha,t)$ is continuous in α. We also note that, for $g \in C_0^\infty((\hat\rho,\infty))$,

$$\operatorname{supp}(w_\alpha) = \big\{ t \in (\ln(\hat\rho),\infty) \,\big|\, e^t \in \operatorname{supp}(g) \big\} \qquad (4.17)$$

is independent of $\alpha \in \mathbb{R}$. Now let $\alpha \in \{1,\dots,2m-1\}$. Then, by dominated convergence, for $g \in C_0^\infty((\hat\rho,\infty))$,

$$\begin{aligned}
&\lim_{\beta\to\alpha} \int_{\hat\rho}^{\infty} dy\, y^\beta \big|g^{(m)}(y)\big|^2 = \int_{\hat\rho}^{\infty} dy\, y^\alpha \big|g^{(m)}(y)\big|^2, \\
&\lim_{\beta\to\alpha} \int_{\hat\rho}^{\infty} dy\, y^{\beta-2m} |g(y)|^2 = \int_{\hat\rho}^{\infty} dy\, y^{\alpha-2m} |g(y)|^2,
\end{aligned} \qquad (4.18)$$

and, if $N \in \mathbb{N}$, one obtains

$$\begin{aligned}
&\lim_{\beta\to\alpha} \int_{\hat\rho}^{\infty} dy\, y^{\beta-2m} |g(y)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(y)]^{-2} = \int_{\hat\rho}^{\infty} dy\, y^{\alpha-2m} |g(y)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(y)]^{-2}, \\
&\lim_{\beta\to\alpha} \int_{\hat\rho}^{\infty} dy\, y^{\beta-2m} |g(y)|^2 \prod_{p=1}^{N} [\ln_p(y)]^{-2} = \int_{\hat\rho}^{\infty} dy\, y^{\alpha-2m} |g(y)|^2 \prod_{p=1}^{N} [\ln_p(y)]^{-2}.
\end{aligned} \qquad (4.19)$$

Similarly, for $g \in C_0^\infty((\hat\rho,\infty))$,

$$\begin{aligned}
&\lim_{\beta\to\alpha} \int_{\ln(\hat\rho)}^{\infty} dt \sum_{j=0}^{m} |k_{2j}(m,\beta)| \big|w_\beta^{(j)}(t)\big|^2 = \int_{\ln(\hat\rho)}^{\infty} dt \sum_{j=0}^{m} |k_{2j}(m,\alpha)| \big|w_\alpha^{(j)}(t)\big|^2, \\
&\lim_{\beta\to\alpha} A(m,\beta) \int_{\ln(\hat\rho)}^{\infty} dy\, |w_\beta(t)|^2 = A(m,\alpha) \int_{\ln(\hat\rho)}^{\infty} dy\, |w_\alpha(t)|^2,
\end{aligned} \qquad (4.20)$$

and, if $N \in \mathbb{N}$, one has

$$\begin{aligned}
&\lim_{\beta\to\alpha} B(m,\beta)\int_{\ln(\hat{\rho})}^{\infty} dt\,|w_\beta(t)|^2 t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k-1}[\ln_p(t)]^{-2}\\
&\quad = B(m,\alpha)\int_{\ln(\hat{\rho})}^{\infty} dt\,|w_\alpha(t)|^2 t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k-1}[\ln_p(t)]^{-2}, \qquad (4.21)\\
&\lim_{\beta\to\alpha}\int_{\ln(\hat{\rho})}^{\infty} dt\,|w_\beta(t)|^2 t^{-2}\prod_{p=1}^{N-1}[\ln_p(t)]^{-2} = \int_{\ln(\hat{\rho})}^{\infty} dt\,|w_\alpha(t)|^2 t^{-2}\prod_{p=1}^{N-1}[\ln_p(t)]^{-2}. \qquad (4.22)
\end{aligned}$$

The corollary now follows from (4.18)–(4.22) and Lemma 4.3. □

Lemma 4.5 ([41, Sect. 2 and proof of Theorem 3.1 (*iii*)]). *Suppose* $1/\widetilde{\rho} > e_{N+1}$ *and* $\alpha \in \mathbb{R}\backslash\{1,\dots,2m-1\}$. *For* $g \in C_0^\infty((0,\widetilde{\rho}))$ *let* $u = u_g \in C_0^\infty((\ln(1/\widetilde{\rho}),\infty))$ *be defined by*

$$g(e^{-t}) = e^{-[(2m-1-\alpha)/2]t}u(t), \quad t \in (\ln(1/\widetilde{\rho}),\infty). \qquad (4.23)$$

Then, for all $g \in C_0^\infty((0,\widetilde{\rho}))$,

$$\begin{aligned}
&\int_0^{\widetilde{\rho}} dy\, y^\alpha \big|g^{(m)}(y)\big|^2 = \int_{\ln(1/\widetilde{\rho})}^{\infty} dt\sum_{j=0}^{m}|k_{2j}(m,\alpha)|\big|u^{(j)}(t)\big|^2,\\
&\int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m}|g(y)|^2 = \int_{\ln(1/\widetilde{\rho})}^{\infty} dt\,|u(t)|^2,
\end{aligned} \qquad (4.24)$$

and, if $N \in \mathbb{N}$, *we also have, for* $k = 1,\dots,N$,

$$\big(e^{-t}\big)^{\alpha-2m}\big|g(e^{-t})\big|^2\prod_{p=1}^{k}[\ln_p(e^t)]^{-2} = e^t|u(t)|^2t^{-2}\prod_{p=1}^{k-1}[\ln_p(t)]^{-2}, \quad t \in (\ln(1/\widetilde{\rho}),\infty). \qquad (4.25)$$

Hence, if $N \in \mathbb{N}$,

$$\begin{aligned}
&\bigg[\int_0^{\widetilde{\rho}} dy\, y^\alpha\big|g^{(m)}(y)\big|^2 - A(m,\alpha)\int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m}|g(y)|^2\\
&\quad - B(m,\alpha)\int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m}|g(y)|^2\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(1/y)]^{-2}\bigg]\\
&\quad\times\bigg[\int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m}|g(y)|^2\prod_{p=1}^{N}[\ln_p(1/y)]^{-2}\bigg]^{-1}\\
&=\bigg[\int_{\ln(1/\widetilde{\rho})}^{\infty} dt\sum_{j=0}^{m}|k_{2j}(m,\alpha)|\big|u^{(j)}(t)\big|^2 - A(m,\alpha)\int_{\ln(1/\widetilde{\rho})}^{\infty} dt\,|u(t)|^2\\
&\quad - B(m,\alpha)\int_{\ln(1/\widetilde{\rho})}^{\infty} dt\,|u(t)|^2t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k-1}[\ln_p(t)]^{-2}\bigg]\\
&\quad\times\bigg[\int_{\ln(1/\widetilde{\rho})}^{\infty} dt\,|u(t)|^2t^{-2}\prod_{p=1}^{N-1}[\ln_p(t)]^{-2}\bigg]^{-1}, \quad g \in C_0^\infty((0,\widetilde{\rho})). \qquad (4.26)
\end{aligned}$$

Corollary 4.6. *Lemma* 4.5 *holds for all* $\alpha \in \mathbb{R}$*, that is, it holds without the restriction* $\alpha \in \mathbb{R}\backslash\{1, \dots, 2m-1\}$.

As the proof of this corollary is very similar to that of Corollary 4.4 we shall omit it.

At this point we are ready to establish optimality of $A(m, \alpha)$ on the interval (r, ∞) in (1.2).

Theorem 4.7. *Suppose that* $N = 0$*. Let* $r \in (1, \infty)$*. Then, for any* $\eta > 0$*, there exists* $\varphi \in C_0^\infty((r, \infty))$ *such that*

$$\left| \int_r^\infty dx\, x^\alpha \big|\varphi^{(m)}(x)\big|^2 \left[\int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \right]^{-1} - A(m, \alpha) \right| \leqslant \eta. \tag{4.27}$$

In particular, the constant $A(m, \alpha)$ *in* (1.2) *is sharp.*

Proof. Put $\rho = 1/r$ so that $1 > \rho$. Applying Theorem 4.1, there exists $g \in C_0^\infty((0, \rho))$ such that

$$\left| \int_0^\rho dy\, y^\alpha \big|g^{(m)}(y)\big|^2 \left[\int_0^\rho dy\, y^{\alpha-2m} |g(y)|^2 \right]^{-1} - A(m, \alpha) \right| \leqslant \eta. \tag{4.28}$$

By Corollary 4.6, writing

$$u(t) = e^{[(2m-1-\alpha)/2]t} g(e^{-t}), \quad t \in (\ln(1/\rho), \infty), \tag{4.29}$$

one obtains

$$\left| \int_{\ln(1/\rho)}^\infty dt \sum_{j=0}^m |k_{2j}(m, \alpha)| \big|u^{(j)}(t)\big|^2 \left[\int_{\ln(1/\rho)}^\infty dt\, |u(t)|^2 \right]^{-1} - A(m, \alpha) \right| \leqslant \eta. \tag{4.30}$$

Introducing

$$\varphi(x) = x^{(2m-1-\alpha)/2} u(\ln(x)), \quad x \in (1/\rho, \infty) = (r, \infty), \tag{4.31}$$

Corollary 4.4 implies

$$\left| \int_r^\infty dx\, x^\alpha \big|\varphi^{(m)}(x)\big|^2 \left[\int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \right]^{-1} - A(m, \alpha) \right| \leqslant \eta, \tag{4.32}$$

concluding the proof since $\varphi \in C_0^\infty((r, \infty))$. ☐

Next, we prove optimality of the N constants $B(m, \alpha)$ in (1.2):

Theorem 4.8. *Suppose that* $N \in \mathbb{N}$*. Let* $r, \Gamma \in (0, \infty)$ *satisfy* $r > \Gamma e_{N+1}$*. Then, for any* $\eta > 0$*, there exists* $\varphi \in C_0^\infty((r, \infty))$ *such that*

$$\begin{aligned} \Bigg| \Bigg[& \int_r^\infty dx\, x^\alpha \big|\varphi^{(m)}(x)\big|^2 - A(m, \alpha) \int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \\ & - B(m, \alpha) \sum_{k=1}^{N-1} \int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \prod_{p=1}^k [\ln_p(x/\Gamma)]^{-2} \Bigg] \\ & \times \left[\int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \prod_{p=1}^N [\ln_p(x/\Gamma)]^{-2} \right]^{-1} - B(m, \alpha) \Bigg| \leqslant \eta. \end{aligned} \tag{4.33}$$

In particular, successively increasing N through $1, 2, 3\ldots,$ demonstrates that the N constants $B(m,\alpha)$ in (1.2) are sharp. Together with Theorem 4.7, this theorem asserts that the $N+1$ constants, $A(m,\alpha)$ and the N constants $B(m,\alpha)$, in (1.2) are sharp.

Proof. Put $\rho = \Gamma/r$ so that $1 > \rho e_{N+1}$. Applying Theorem 4.2 with $\gamma = 1$, there exists $g \in C_0^\infty((0,\rho))$ such that

$$\begin{aligned}\bigg|\bigg[\int_0^\rho dy\, y^\alpha \big|g^{(m)}(y)\big|^2 - A(m,\alpha)\int_0^\rho dy\, y^{\alpha-2m}|g(y)|^2 \\ - B(m,\alpha)\int_0^\rho dy\, y^{\alpha-2m}|g(y)|^2 \sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(1/y)]^{-2}\bigg] \\ \times\bigg[\int_0^\rho dy\, y^{\alpha-2m}|g(y)|^2\prod_{p=1}^{N}[\ln_p(1/y)]^{-2}\bigg]^{-1} - B(m,\alpha)\bigg| \leqslant \eta. \end{aligned} \tag{4.34}$$

By Corollary 4.6, writing

$$u(t) = e^{[(2m-1-\alpha)/2]t} g(e^{-t}), \quad t \in (\ln(1/\rho),\infty), \tag{4.35}$$

one has

$$\begin{aligned}\bigg|\bigg[\int_{\ln(1/\rho)}^\infty dt \sum_{j=0}^{m}|k_{2j}(m,\alpha)|\big|u^{(j)}(t)\big|^2 - A(m,\alpha)\int_{\ln(1/\rho)}^\infty dt\,|u(t)|^2 \\ - B(m,\alpha)\int_{\ln(1/\rho)}^\infty dt\,|u(t)|^2 t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k-1}[\ln_p(t)]^{-2}\bigg] \\ \times\bigg[\int_{\ln(1/\rho)}^\infty dt\,|u(t)|^2 t^{-2}\prod_{p=1}^{N-1}[\ln_p(t)]^{-2}\bigg]^{-1} - B(m,\alpha)\bigg| \leqslant \eta. \end{aligned} \tag{4.36}$$

Introducing

$$\widetilde{\varphi}(\xi) = \xi^{(2m-1-\alpha)/2} u(\ln(\xi)), \quad \xi \in (1/\rho,\infty), \tag{4.37}$$

Corollary 4.4 implies

$$\begin{aligned}\bigg|\bigg[\int_{1/\rho}^\infty d\xi\, \xi^\alpha \big|\widetilde{\varphi}^{(m)}(\xi)\big|^2 - A(m,\alpha)\int_{1/\rho}^\infty d\xi\, \xi^{\alpha-2m}|\widetilde{\varphi}(\xi)|^2 \\ - B(m,\alpha)\int_{1/\rho}^\infty d\xi\, \xi^{\alpha-2m}|\widetilde{\varphi}(\xi)|^2\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(\xi)]^{-2}\bigg] \\ \times\bigg[\int_{1/\rho}^\infty d\xi\, \xi^{\alpha-2m}|\widetilde{\varphi}(\xi)|^2\prod_{p=1}^{N}[\ln_p(\xi)]^{-2}\bigg]^{-1} - B(m,\alpha)\bigg| \leqslant \eta. \end{aligned} \tag{4.38}$$

Putting

$$\varphi(x) = \widetilde{\varphi}(x/\Gamma), \quad x \in (\Gamma/\rho,\infty) = (r,\infty), \tag{4.39}$$

one infers

$$\begin{aligned}
&\Bigg|\Bigg[\Gamma^{2m-\alpha-1}\bigg\{\int_r^\infty dx\, x^\alpha \big|\varphi^{(m)}(x)\big|^2 - A(m,\alpha)\int_r^\infty dx\, x^{\alpha-2m}|\varphi(x)|^2 \\
&\quad - B(m,\alpha)\int_r^\infty dx\, x^{\alpha-2m}|\varphi(x)|^2 \sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(x/\Gamma)]^{-2}\bigg\}\Bigg] \\
&\quad \times \Bigg[\Gamma^{2m-\alpha-1}\int_r^\infty dx\, x^{\alpha-2m}|\varphi(x)|^2 \prod_{p=1}^{N}[\ln_p(x/\Gamma)]^{-2}\Bigg]^{-1} - B(m,\alpha)\Bigg| \leqslant \eta, \qquad (4.40)
\end{aligned}$$

finishing the proof since $\varphi \in C_0^\infty((r,\infty))$. □

Remark 4.9.

(i) *Theorem* 4.1 *(resp., Theorem* 4.7*) extends to* $\rho = \infty$ *(resp.,* $r = 0$*) upon disregarding all logarithmic terms (i.e., upon putting* $B(m,\alpha) = 0$*), we omit the details.*

(ii) *The sequence of logarithmically refined power-weighted Birman–Hardy–Rellich inequalities underlying Theorems* 4.1*,* 4.2*,* 4.7*, and* 4.8*, extend from* C_0^∞*−functions to functions in appropriately weighted (homogeneous) Sobolev spaces as shown in detail in [41, Sect. 3]. In the course of this extension, the constants* $A(m,\alpha)$ *and the* N *constants* $B(m,\alpha)$ *remain the same and hence optimal.*

(iii) *We note once more that Theorems* 4.1 *and* 4.7 *were proved in [41, Theorem A.1] using a different method.*

(iv) *Both Theorems* 4.2 *and* 4.8 *still hold if the repeated log-terms* $ln_p(\,\cdot\,)$ *are replaced by the type of repeated log-terms used in [15, 16, 17, 90]. Detailed proofs of Theorems* 4.2 *and* 4.8 *for the type of repeated log-terms used in [15, 16, 17, 90] are available upon request from the authors.*

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Uniqueness of entire functions whose difference polynomials share a polynomial with finite weight

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ABSTRACT

In this paper, we use the concept of weighted sharing of values to investigate the uniqueness results when two difference polynomials of entire functions share a nonzero polynomial with finite weight. Our result improves and extends some recent results due to Sahoo-Karmakar [J. Cont. Math. Anal. 52(2) (2017), 102–110] and that of Li *et al.* [Bull. Malays. Math. Sci. Soc., 39 (2016), 499–515]. Some examples have been exhibited which are relevant to the content of the paper.

RESUMEN

En este artículo, usamos el concepto de intercambio pesado de valores para investigar los resultados de unicidad cuando dos polinomios de diferencia de funciones enteras comparten un polinomio no cero con peso finito. Nuestro resultado mejora y extiende algunos resultados de Sahoo-Karmakar [J. Cont. Math. Anal. 52(2) (2017), 102–110] y los de Li *et al.* [Bull. Malays. Math. Sci. Soc., 39 (2016), 499–515]. Se exhiben algunos ejemplos que son relevantes para el contenido del artículo.

Keywords and Phrases: Entire function, difference polynomial, shift and difference operator, weighted sharing.

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1 Introduction

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, the zero of $f-a$ and $g-a$ have the same locations as well as same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities into account, then f and g are said to share the value a IM (ignoring multiplicities)(see [37]). We adopt the standard notations of the Nevanlinna theory of meromorphic functions (see [14, 22, 41]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic function of f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r \to \infty$ outside of an exceptional set of finite linear measure.

We define shift and difference operators of $f(z)$ by $f(z+c)$ and $\Delta_c f(z) = f(z+c) - f(z)$, respectively. Note that $\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z))$, where c is a nonzero complex number and $n \geq 2$ is a positive integer.

For further generalization of $\Delta_c f$, we now define the linear difference operator of an entire (meromorphic) function f as $L_c(f) = f(z+c) + c_0 f(z)$, where c_0 is a finite complex constant. Clearly, for the particular choice of the constant $c_0 = -1$, we get $L_c(f) = \Delta_c f$.

In 1959, Hayman [13] proved the following result.

Theorem A ([13]). *Let f be a transcendental entire function and let n be an integer such that $n \geq 1$. Then $f^n f' = 1$ has infinitely many solutions.*

A number of authors have shown their interest to find the uniqueness of entire and meromorphic functions whose differential polynomials share certain values or fixed points, and obtained some remarkable results (see [3, 9, 10, 26, 33, 34, 36, 37, 39, 42]).

In recent years, the difference variant of the Nevanlinna theory has been established in [8, 11, 12]. Using these theories, some mathematicians in the world began to study the uniqueness questions of meromorphic functions sharing values with their shifts, and study the value distribution of the nonlinear difference polynomials, and produced many fine works, for example, see [1, 5, 6, 7, 11, 15, 16, 23, 27, 29, 30, 31, 40, 44]. We recall the following result from Laine-Yang [23].

Theorem B ([23]). *Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Later on, Liu-Yang [28] extended Theorem B, and proved the following result:

Theorem C ([28]). *Let f be a transcendental entire function of finite order, and let η be a nonzero complex constant. Then for $n \geq 2$ the function $f(z)^n f(z+\eta) - P_0(z)$ has infinitely many zeros, where P_0 is any given polynomial such that $P_0 \not\equiv 0$.*

Regarding uniqueness corresponding to Theorem C, Li *et al.* [24] obtained the following result.

Theorem D ([24]). *Let f and g be two distinct transcendental entire functions of finite order, and let $P_0 \not\equiv 0$ be a polynomial. Let η is a nonzero complex constant and $n \geq 4$ is an integer such that $2\deg(P_0) < n+1$. Also, suppose that $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ share 0 CM. Then one of the following assertions holds.*

(I) If $n \geq 4$ and $f(z)^n f(z+\eta)/P_0(z)$ is a Mobius transformation of $g(z)^n g(z+\eta)/P_0(z)$, then either

(i) $f = tg$, where t is a constant satisfying $t^{n+1} = 1$

(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a non-constant polynomial.

(II) If $n \geq 6$, then (I)(i) or (I)(ii) holds.

In 2016, Li-Li [25] obtained the IM analogues of the above Theorem D as follows.

Theorem E ([25]). *Let f and g be two distinct transcendental entire functions of finite order, and let $P_0 \not\equiv 0$ be a polynomial. Let η is a nonzero complex constant and $n \geq 4$ is an integer such that $2\deg(P_0) < n+1$. Also, suppose that $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ share 0 IM. Then one of the following assertions holds.*

(I) If $n \geq 4$ and $f(z)^n f(z+\eta)/P_0(z)$ is a Mobius transformation of $g(z)^n g(z+\eta)/P_0(z)$, then either

(i) $f = tg$, where t is a constant satisfying $t^{n+1} = 1$,

(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a non-constant polynomial.

(II) If $n \geq 12$, then (I)(i) or (I)(ii) holds.

In 2001, the notion of weighted sharing was originally defined in the literature ([18, 19]), which is the gradual change of shared values from CM to IM. Below we recall the definition.

Definition 1.1 ([18, 19]). *Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

Clearly, if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Using the notion of weighted sharing, Sahoo-Karmakar [35] further improved Theorem D as follows.

Theorem F ([35]). *Let f, g, P_0 and n be defined as in Theorem D. Suppose that $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ share $(0, 2)$.*

(I) If $n \geq 4$ and $f(z)^n f(z+\eta)/P_0(z)$ is a Mobius transformation of $g(z)^n g(z+\eta)/P_0(z)$, then either

(i) $f = tg$, where t is a constant satisfying $t^{n+1} = 1$

(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a non-constant polynomial.

(II) If $n \geq 6$, then (I)(i) or (I)(ii) holds.

Observing the above results, it is natural to ask the following questions.

Question 1.2. *What can be said about the relationship of two finite order non-constant meromorphic functions f and g if their more general nonlinear difference polynomials $f(z)^n L_c(f)$ and $g(z)^n L_c(g)$ share a polynomial $P(z) \not\equiv 0$, where $L_c(f) = f(z+c) + c_0 f(z)$ with c and c_0 being finite nonzero complex constants, and $n \geq 2$ being a positive integer?*

Question 1.3. *Is it possible to further reduce the nature of sharing from* $(0, 2)$ *to* $(0, 1)$ *in Theorem* F*?*

Question 1.4. *Can the lower bound of n be further reduced in Theorems* E *and* F*?*

Question 1.5. *What can be said about the uniqueness of f and g if we consider the difference polynomial of the form $f(z)^n \Delta_c f$ and $g(z)^n \Delta_c g$ in Theorems* E *and* F*?*

The purpose of this paper is to answer all the questions raised above. In fact we have been successfully able to reduce the nature of sharing of $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ in Theorem F. We have also reduced the lower bound of n in Theorems E and F successfully.

2 Main results

Now we state our main result.

Theorem 2.1. *Let f and g be two transcendental entire functions of finite order, $P \not\equiv 0$ be a polynomial. Let c be a non-zero complex constant, and n be a positive integer such that $2\deg(P) < n+1$. Let l be a non-negative integer such that $f(z)^n L_c(f) - P(z)$ and $g(z)^n L_c(g) - P(z)$ share $(0, l)$ and $g(z)$, $g(z+c)$ share 0 CM. If $n \geq 4$ and $f(z)^n L_c(f)/P(z)$ is a Mobius transformation of $g(z)^n L_c(g)/P(z)$, or one of the following conditions holds:*

(i) $l \geq 2$ and $n \geq 5$;

(ii) $l = 1$ *and* $n \geq 6$*;*

(iii) $l = 0$ *and* $n \geq 11$*, then one of the following conclusions can be realized:*

(a) $f = tg$*, where* t *is a constant satisfying* $t^{n+1} = 1$*;*

(b) *When* $c_0 = 0$*,* $f = e^U$ *and* $g = te^{-U}$*, where* $P(z)$ *reduces to a nonzero constant* d*,* t *is a constant such that* $t^{n+1} = d^2$ *and* U *is a non-constant polynomial;*

(c) *When* $c_0 \neq 0$*,* $f = c_1 e^{az}$*,* $g(z) = c_2 e^{-az}$*, where* a*,* c_1*,* c_2 *and* d *are non-zero constants satisfying* $(c_1c_2)^{n+1}(e^{ac} + c_0)(e^{-ac} + c_0) = d^2$*.*

If $L_c(f) = \Delta_c f$, then one can easily get the following corollary from Theorem 2.1 which answers Question 1.5.

Corollary 2.2. *Let* f *and* g *be two transcendental entire functions of finite order,* $P \not\equiv 0$ *be a polynomial. Let* c *be a non-zero complex constant, and* n *be a positive integer such that* $2\deg(P) < n + 1$*. Let* l *be a non-negative integer such that* $f(z)^n \Delta_c f - P(z)$ *and* $g(z)^n \Delta_c g - P(z)$ *share* $(0, l)$ *and* $g(z)$*,* $g(z + c)$ *share* 0 *CM. If* $n \geq 4$ *and* $f(z)^n \Delta_c(f)/P(z)$ *is a Mobius transformation of* $g(z)^n \Delta_c(g)/P(z)$*, or one of the following conditions holds:*

(i) $l \geq 2$ *and* $n \geq 5$*;*

(ii) $l = 1$ *and* $n \geq 6$*;*

(iii) $l = 0$ *and* $n \geq 11$*, then one of the following conclusions can be realized:*

(a) $f = tg$*, where* t *is a constant satisfying* $t^{n+1} = 1$*;*

(b) $f = c_1 e^{az}$*,* $g(z) = c_2 e^{-az}$*, where* a*,* c_1*,* c_2 *and* d *are non-zero constants satisfying* $(c_1c_2)^{n+1}(e^{ac} + c_0)(e^{-ac} + c_0) = d^2$*.*

The following examples show that both the conclusions of Theorem 2.1 actually holds.

Example 2.3. *Let* $f(z) = e^z$ *and* $g = tf$*, where* t *is a constant such that* $t^{n+1} = 1$*, and* η *be any non-zero complex constant. Then for any given polynomial* p *such that* $p \not\equiv 0$ *with* $2\deg(p) < n+1$*,* $f(z)^n f(z+\eta) - p(z)$ *and* $g(z)^n g(z+\eta) - p(z)$ *share* $(0, \infty)$*. Also* $f(z)^n(f(z+\eta) - f(z)) - p(z)$ *and* $g(z)^n(g(z+\eta) - g(z)) - p(z)$*share* $(0, \infty)$*. Here* f *and* g *satisfy the conclusion* (a) *of Theorem 2.1.*

Example 2.4. *Let* $f(z) = e^{2\pi iz/\eta}$ *and* $g(z) = te^{-2\pi iz/\eta}$*, where* t *is a constant such that* $t^{n+1} = 1$*,* η *is a non-zero complex constant. Then* $f(z)^n f(z+\eta)$ *and* $g(z)^n g(z+\eta)$ *share* $(1, \infty)$*. Here* f *and* g *satisfy the conclusion* (b) *of Theorem 2.1.*

Example 2.5. *Let* $f(z) = e^z$*,* $g(z) = e^{-z}$*,* $\eta = -\log(-1)$ *and* $P(z) = 2$*. Then one can easily verify that* $f(z)^n(f(z+\eta) - f(z))$ *and* $g(z)^n(g(z+\eta) - g(z))$ *share* $(2, \infty)$*. Here* f *and* g *satisfy the conclusion* (b) *of Theorem 2.1.*

The following example shows that Theorem 2.1 is not true for infinite order entire functions.

Example 2.6. *Let* $f(z) = \dfrac{e^{2\pi iz/\eta}}{e^{e^{2\pi iz/\eta}}}$ *and* $g(z) = \dfrac{1}{e^{e^{2\pi iz/\eta}}}$, *where* η *is a non-zero constant. Then it is easy to verify that* $f(z)^n f(z+\eta)$ *and* $g(z)^n g(z+\eta)$ *share* $(1,\infty)$. *But there does not exist a non-zero constant* t *such that* $f = tg$ *or* $fg = t$, *where* $t^{n+1} = 1$.

3 Auxiliary definitions

Throughout the paper we have used the following definitions and notations.

Definition 3.1 ([17]). *Let* $a \in \mathbb{C} \cup \{\infty\}$. *We denote by* $N(r,a;f \mid= 1)$ *the counting function of simple* a *points of* f. *For* $p \in \mathbb{N}$ *we denote by* $N(r,a;f \mid\leq p)$ *the counting function of those* a*-points of* f *(counted with multiplicities) whose multiplicities are not greater than* p. *By* $\overline{N}(r,a;f \mid\leq p)$ *we denote the corresponding reduced counting function. In a similar manner we can define* $N(r,a;f \mid\geq p)$ *and* $\overline{N}(r,a;f \mid\geq p)$.

Definition 3.2 ([19]). *Let* $p \in \mathbb{N} \cup \{\infty\}$. *We denote by* $N_p(r,a;f)$ *the counting function of* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$. *Then* $N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \cdots + \overline{N}(r,a;f \mid\geq p)$. *Clearly* $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 3.3 ([43]). *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* $(a,0)$. *Let* z_0 *be an* a*-point of* f *with multiplicity* p, *an* a*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r,a;f)$ *the reduced counting function of those* a*-points of* f *and* g *where* $p > q$, *by* $N_E^{1)}(r,a;f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q = 1$, *by* $\overline{N}_E^{(2}(r,a;f)$ *the reduced counting function of those* a*-points of* f *and* g *where* $p = q \geq 2$. *In the same way we can define* $\overline{N}_L(r,a;g)$, $\overline{N}_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$. *In a similar manner we can define* $\overline{N}_L(r,a;f)$ *and* $\overline{N}_L(r,a;g)$ *for* $a \in \mathbb{C} \cup \{\infty\}$.

When f and g share (a,m), $m \geq 1$, then $N_E^{1)}(r,a;f) = N(r,a;f \mid= 1)$.

Definition 3.4 ([19]). *Let* f, g *share a value* $(a,0)$. *We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g. *Clearly* $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

4 Some lemmas

We now prove several lemmas which will play key roles in proving the main results of the paper. Let F and G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{4.1}$$

Lemma 4.1 ([8]). *Let $f(z)$ be a meromorphic function of finite order ρ, and let c be a fixed non-zero complex constant. Then for each $\epsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\epsilon}) + O\{\log r\}.$$

Lemma 4.2 ([8]). *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a non-zero complex number. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 4.3 ([32]). *Let f be a non-constant meromorphic function and let $\mathcal{R}(f) = \sum\limits_{i=0}^{n} a_i f^i / \sum\limits_{j=0}^{m} b_j f^j$ be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then*

$$T(r, \mathcal{R}(f)) = d\, T(r, f) + S(r, f), \text{ where } d = \max\{n, m\}.$$

Lemma 4.4 ([25]). *Let f and g be two transcendental entire functions of finite order, $c \neq 0$ be a complex constant, $\alpha(z)$ be a small function of f and g, $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a nonzero polynomial, where $a_0, a_1, \ldots, a_n (\neq 0)$ are complex constants, and let $n > \Gamma_1$ be an integer. If $P(f)f(z+c)$ and $P(g)g(z+c)$ share $\alpha(z)$ IM, then $\rho(f) = \rho(g)$.*

Lemma 4.5. *Let f be a transcendental entire function of finite order, and $L_c(f) = f(z+c) + c_0 f(z)$, where $c,\ c_0 \in \mathbb{C} - \{0\}$. Then for $n \in \mathbb{N}$,*

$$nT(r, f) + S(r, f) \leq T(r, f(z)^n L_c(f)) \leq (n+1)T(r, f) + S(r, f).$$

Proof. This lemma can be proved in a similar manner as done in the proof of Lemma 2.4 and Remark 2.1 of [30]. □

Remark 4.6. *If $c_0 = 0$, then $L_c(f) = f(z+c)$ and therefore by Lemma 2.3 of [30], we can get*

$$T(r, f(z)^n L_c(f)) = (n+1)T(r, f) + S(r, f). \tag{4.2}$$

Remark 4.7. *If $c_0 \neq 1$, then the following example shows that one can not get equality just like* (4.2).

Example 4.8 ([30]). *If $f(z) = e^z$, $e^c = 2$, $c_0 = -1$, then $T(r, f(z)^n L_c(f)) = T(r, e^{(n+1)z}) = (n+1)T(r, f) + S(r, f)$. If $f(z) = e^z + z$, $c = 2\pi i$, $c_0 = -1$, then $T(r, f(z)^n L_c(f)) = T(r, 2\pi i(e^z + z)^n) = nT(r, f) + S(r, f)$.*

Remark 4.9. *From the above example, it can be easily seen that $f(z)$ and $f(z+c)$ share 0 CM for the first one, but for the second one $f(z)$ and $f(z+c)$ do not share 0 CM. Regarding this one may ask, in order to get equality just like* (4.2), *is it sufficient to assume that $f(z)$ and $f(z+c)$ share 0 CM? In this direction, we prove the following lemma.*

Lemma 4.10. *Let $F = f(z)^n L_c(f)$, where $f(z)$ is an entire function of finite order, and $f(z)$, $f(z+c)$ share 0 CM. Then*

$$T(r, F) = (n+1)T(r, f) + S(r, f).$$

Proof. Keeping in view of Lemmas 4.1 and 4.3, we have

$$\begin{aligned} T(r, F) &= T(r, f(z)^n L_c(f)) = m(r, f^n L_c(f)) \\ &\leq m(r, f(z)^n) + m(r, L_c(f)) + S(r, f) \\ &\leq T(f(z)^n) + m\left(r, \frac{L_c(f)}{f(z)}\right) + m(r, f(z)) + S(r, f) \\ &\leq (n+1)T(r, f) + S(r, f). \end{aligned}$$

Since $f(z)$ and $f(z+c)$ share 0 CM, we must have $N\left(r, \infty; \frac{L_c(f)}{f(z)}\right) = S(r, f)$.

So, keeping in view of Lemmas 4.2 and 4.3, we obtain

$$\begin{aligned} (n+1)T(r, f) = T(r, f(z)^{n+1}) &= m(r, f(z)^{n+1} P(f(z))) \\ &= m\left(r, F\frac{f(z)}{L_c(f)}\right) \leq m(r, F) + m\left(r, \frac{f(z)}{L_c(f)}\right) + S(r, f) \\ &\leq T(r, F) + T\left(r, \frac{L_c(f)}{f(z)}\right) + S(r, f) = T(r, F) + N\left(r, \infty; \frac{L_c(f)}{f(z)}\right) \\ &\quad + m\left(r, \frac{L_c(f)}{f(z)}\right) + S(r, f) = T(r, F) + S(r, f). \end{aligned}$$

From the above two inequalities, we must have

$$T(r, F) = (n+1)T(r, f) + S(r, f). \quad \square$$

Lemma 4.11 ([37]). *Let F and G be non-constant meromorphic functions such that G is a Mobius transformation of F. Suppose that there exists a subset $I \subset \mathbb{R}^+$ with linear measure $mesI = +\infty$ such that for $r \in I$ and $r \longrightarrow \infty$*

$$\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) < (\lambda + o(1))T(r, G),$$

where $\lambda < 1$. If there exists a point $z_0 \in \mathbb{C}$ satisfying $F(z_0) = G(z_0) = 1$, then either $F = G$ or $FG = 1$.

Lemma 4.12 ([38]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then*

$$N\left(r, \infty; \frac{f}{g}\right) - N\left(r, \infty; \frac{g}{f}\right) = N(r, \infty; f) + N(r, 0; g) - N(r, \infty; g) - N(r, 0; f).$$

Lemma 4.13. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C}-\{0\}$ be finite complex constant and $n \in \mathbb{N}$. Let $F(z) = f(z)^n L_c(f)$, where $L_c(f) \not\equiv 0$. Then*

$$nT(r, f) \leq T(r, F) - N(r, 0; L_c(f)) + S(r, f).$$

Proof. Using Lemmas 4.2 and 4.12, and the first fundamental theorem of Nevanlinna, we obtain

$$\begin{aligned} m(r, f(z)^{n+1}) &= m\left(r, \frac{f(z)F}{L_c(f)}\right) \leq m(r,F) + m\left(r, \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\leq m(r,F) + T\left(r, \frac{f(z)}{L_c(f)}\right) - N\left(r, \infty; \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\leq m(r,F) + T\left(r, \frac{L_c(f)}{f(z)}\right) - N\left(r, \infty; \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\leq m(r,F) + N\left(r, \infty; \frac{L_c(f)}{f(z)}\right) + m\left(r, \frac{L_c(f)}{f(z)}\right) - N\left(r, \infty; \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\leq m(r,F) + N(r,0;f) - N(r,0;L_c(f)) + S(r,f). \end{aligned}$$

i.e.,

$$m(f(z)^{n+1}) \leq T(r,F) + T(r,f) - N(r,0;L_c(f)) + S(r,f).$$

By Lemma 4.3, we obtain

$$(n+1)T(r,f) = m(r,f^{n+1}) \leq T(r,F) + T(r,f) - N(r,0;L_c(f)) + S(r,f),$$

i.e.,

$$nT(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f). \quad \square$$

Lemma 4.14 ([2])**.** *If f, g be two non-constant meromorphic functions sharing $(1,1)$, then*

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 4.15 ([4])**.** *If f, g be two non-constant meromorphic functions sharing $(1,1)$, then*

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f),$$

where $N_0(r,0;f')$ is the counting function of those zeros of f' which are not the zeros of $f(f-1)$.

Lemma 4.16 ([43])**.** *If f, g be two non-constant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$, then*

$$N_E^{1)}(r,1;f) \leq N(r,0;H) + S(r,f) \leq N(r,\infty;H) + S(r,f) + S(r,g).$$

Lemma 4.17 ([4])**.** *If f, g be two non-constant meromorphic functions such that they share $(1,0)$, then*

$$\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \leq N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 4.18 ([4])**.** *If f, g be share $(1,0)$, then*

(i) $\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$.

(ii) $\overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f)$.

(iii) $\overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + S(r,g)$.

Lemma 4.19 ([20])**.** *If f, g be be two non-constant meromorphic functions that share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$, then*

$$\begin{aligned} N(r,\infty;H) \quad &\leq \quad \overline{N}(r,0;f\mid\geq 2) + \overline{N}(r,0;g\mid\geq 2) + \overline{N}_*(r,1;f,g) + \overline{N}_*(r,\infty;f,g) \\ &\quad + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') + S(r,f) + S(r,g), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Lemma 4.20 ([21])**.** *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N\left(r,0;f^{(k)}\mid f\neq 0\right) \leq k\overline{N}(r,\infty;f) + N\left(r,0;f\mid< k\right) + k\overline{N}\left(r,0f\mid\geq k\right) + S(r,f).$$

5 Proofs of the theorems

Proof of Theorem 2.1. Let $F = f(z)^n L_c(f)/P(z)$ and $G = g(z)^n L_c(g)/P(z)$. Then F and G are two transcendental meromorphic functions that share $(1,l)$ except the zeros and poles of $P(z)$. Since $g(z)$ and $g(z+c)$ share 0 CM, from Lemma 4.10, we obtain

$$T(r,G) = (n+1)T(r,g) + O\{r^{\rho(f)-1+\epsilon}\} + O\{\log r\}. \tag{5.1}$$

Since f and g are of finite order, it follows from Lemma (4.5) and (5.1) that F and G are also of finite order. Moreover, from Lemma 4.4 we deduce that $\rho(f)=\rho(g)=\rho(F)=\rho(G)$.

We consider the following two cases separately.

Case 1: Suppose that F is a Mobius transformation of G, *i.e.*,

$$F = \frac{AG+B}{CG+D}, \tag{5.2}$$

where $A,\ B,\ C,\ D$ are complex constants satisfying $AD-BC\neq 0$. Let z_0 be a 1-point such that F. Since $F,\ G$ share $(1,2)$, z_0 is also a 1-point of G. Therefore, from (5.2), we obtain $A+B=C+D$, and hence (5.2) can be written as

$$F-1 = \frac{G-1}{\alpha G+\beta},$$

where $\alpha = C/(A-C)$ and $\beta = D/(A-C)$. From this we can say that F, G share $(1,\infty)$.

Now using the standard Valiron-Mohon'ko Lemma 4.3, we obtain from (5.2) that

$$T(r,F) = T(r,G) + O(\log r).$$

Then using Lemmas 4.5 and 4.10 and the fact that f and g are transcendental entire functions of finite order, we deduce

$$T(r,f) \le \frac{n+1}{n} T(r,g) + S(r,f) + S(r,g) \quad \text{and} \quad \frac{T(r,G)}{T(r,g)} \longrightarrow n+1 \tag{5.3}$$

as $r \longrightarrow \infty$, $r \in I$.

Now keeping in view of (5.3), Lemma 4.2 and the condition that f and g are transcendental entire functions, we obtain

$$\begin{aligned}
\overline{N}(r,0;F) + \overline{N}(r,\infty;F) &= \overline{N}(r,0;f(z)^n L_c(f)) + O(\log r)\\
&\le \overline{N}(r,0;f(z)) + \overline{N}(r,0;L_c(f)) + O(\log r)\\
&\le \overline{N}(r,0;f(z)) + T(r,L_c(f)) + O(\log r)\\
&\le \overline{N}(r,0;f(z)) + m(r,L_c(f)) + O(\log r)\\
&\le \overline{N}(r,0;f(z)) + m\left(r,\frac{L_c(f)}{f(z)}\right) + m(r,f(z)) + O(\log r)\\
&\le 2T(r,f) + S(r,f) \le \frac{2n+2}{n} T(r,g) + S(r,g).
\end{aligned}$$

Similarly, we obtain $\overline{N}(r,0;G) + \overline{N}(r,\infty;G) \le 2T(r,g) + S(r,g)$. Thus using (5.3), we obtain

$$\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) \le \frac{2(2n+1)}{n(n+1)} T(r,G) + S(r,g). \tag{5.4}$$

Since, $g(z)$ and $g(z+c)$ share 0 CM, we get that $N(r,0;L_c(g)/g(z)) = S(r,g)$. Thus, keeping in view of this, Lemmas 4.2, 4.10 and applying the second fundamental theorem of Nevanlinna on G, we obtain

$$\begin{aligned}
(n+1)T(r,g) = T(r,G) &\le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) + S(r,g)\\
&\le \overline{N}(r,0;g) + \overline{N}(r,0;L_c(g)) + \overline{N}(r,1;G) + S(r,g)\\
&\le \overline{N}(r,0;g) + T(r,L_c(g)) + \overline{N}(r,1;G) + S(r,g)\\
&\le \overline{N}(r,0;g) + T\left(r,\frac{L_c(g)}{g(z)}\right) + T(r,g) + S(r,g)\\
&\le 2T(r,g) + \overline{N}(r,1;G) + S(r,g),
\end{aligned}$$

i.e.,

$$(n-1)T(r,g) \le 2T(r,g) + \overline{N}(r,1;G) + S(r,g).$$

From this and the fact that F and G share $(1,2)$, we conclude that there exists a point $z_0 \in \mathbb{C}$ such that $F(z_0) = G(z_0) = 1$. Hence from (5.4), Lemma 4.11 and the condition $n \ge 4$, we conclude that either $FG = 1$ or $F = G$. Now we consider the following sub-cases.

Subcase 1.1: $F \equiv G$. Then we get

$$f(z)^n(f(z+c)+c_0f(z)) \equiv g(z)^n(g(z+c)+c_0g(z)).$$

Let $h(z)=f(z)/g(z)$. Then we deduce that

$$(h(z)^nh(z+c)-1)g(z+c)=-c_0(h^{n+1}(z)-1)g(z). \tag{5.5}$$

Suppose h is not constant. Then from (5.5), we obtain

$$\frac{g(z)}{g(z+c)}=\frac{h(z)^nh(z+c)-1}{c_0(h(z)^{n+1}-1)}.$$

As $g(z)$ and $g(z+c)$ share 0 CM, from the above equation we can say that $h(z)^{n+1}$ and $h(z)^nh(z+c)$ share $(1,\infty)$. Let z_0 be a zero of $h^{n+1}-1$. Then we must have $h(z)_0^{n+1}=1$ and $h(z_0)^nh(z_0+c)=1$. Hence $h(z_0+c)=h(z_0)$, and therefore by Lemma 4.1, we obtain

$$\overline{N}(r,1;h^{n+1}) \leq \overline{N}(r,0;h(z+c)-h(z)) \leq 2T(r,h)+S(r,h).$$

Keeping in mind the above inequality and Lemma 4.3 and applying the second fundamental theorem of Nevanlinna to h^{n+1}, we obtain

$$\begin{aligned}(n+1)T(r,h) &= T(r,h^{n+1}) \leq \overline{N}(r,\infty;h^{n+1})+\overline{N}(r,0;h^{n+1})+\overline{N}(r,1;h^{n+1})+S(r,h)\\ &\leq 4T(r,h)+S(r,h),\end{aligned}$$

i.e.,

$$(n-3)T(r,h) \leq S(r,h),$$

which is not possible since $n \geq 4$. Hence h is constant. Then (5.5) reduces to $(h^{n+1}-1)L_c(g)=0$. As $L_c(g) \not\equiv 0$, we must have $h^{n+1}=1$ and thus $f=tg$, for a constant t such that $t^{n+1}=1$, which is the conclusion (a).

Subcase 1.2: Suppose $FG \equiv 1$. Then we have

$$f(z)^nL_c(f)g(z)^nL_c(g)=P_0(z)^2. \tag{5.6}$$

From (5.6) and the condition that f and g are transcendental entire functions, one can immediately say that both f and g have at most finitely many zeros. So, we may write

$$f(z)=P_1(z)e^{Q_1(z)},\ g(z)=P_1(z)e^{Q_2(z)}, \tag{5.7}$$

where P_1, P_2, Q_1, Q_2 are polynomials, and Q_1, Q_2 are non-constants. Substituting (5.7) in (5.6), we obtain

$$\begin{aligned}&(P_1P_2)^ne^{n(Q_1+Q_2)}[P_1(z+c)P_2(z+c)e^{Q_1(z+c)+Q_2(z+c)}+c_0^2P_1P_2e^{Q_1+Q_2}\\ &+c_0P_1P_2(z+c)e^{Q_1+Q_2(z+c)}+c_0P_1(z+c)P_2e^{Q_1(z+c)+Q_2}]=P(z)^2.\end{aligned} \tag{5.8}$$

Keeping in view of (5.7), we must have

$$n(Q_1(z)+Q_2(z))+Q_1(z+c)+Q_2(z+c)=A_1, \tag{5.9}$$

$$n(Q_1(z)+Q_2(z))+Q_1(z)+Q_2(z+c)=A_2, \tag{5.10}$$

$$n(Q_1(z)+Q_2(z))+Q_1(z+c)+Q_2(z)=A_3, \tag{5.11}$$

$$(n+1)(Q_1(z)+Q_2(z))=A_4, \tag{5.12}$$

where A_1, A_2, A_3, A_4 are constants. Let $Q_1(z)+Q_2(z)=W(z)$. Then (5.9) can be written as

$$nW(z)+W(z+c)=A_1, \tag{5.13}$$

for all $z\in\mathbb{C}$. Therefore, from (5.13), we must have $W=B$, where B is a constant, and therefore, we have

$$Q_2=B-Q_1. \tag{5.14}$$

Keeping in view of (5.14), (5.7) can be written as

$$f(z)=P_1e^{Q_1(z)},\ g(z)=P_2e^Be^{-Q_1(z)}. \tag{5.15}$$

Now (5.8) can be written as

$$\begin{aligned}&(P_1P_2)^n[P_1(z+c)P_2(z+c)e^{A_4}+c_0P_1(z+c)P_2e^{A_3}+c_0P_1P_2(z+c)e^{A_2}\\&+c_0^2P_1P_2e^{A_4}]=P(z)^2.\end{aligned} \tag{5.16}$$

If P_1P_2 is not a constant, then the degree of the left side of (5.16) is at least $n+1$. But the condition $2\deg(P)<n+1$ implies that the degree of the right side of (5.16) is less than $n+1$, which is a contradiction. Thus P_1P_2 and P reduce to non-zero constants.

Since P_1, P_2 are both polynomials and their product is constant, each of them must be constant. Therefore, (5.15) can be written as

$$f(z)=e^U,\ g(z)=e^Be^{-U}, \tag{5.17}$$

where U is a non-constant polynomial. Using the above forms of f and g and keeping in mind that P is a constant, say d, (5.6) reduces to

$$e^{(n+1)B}(e^{U(z+c)-U(z)}+c_0)(e^{-(U(z+c)-U(z))}+c_0)=d^2. \tag{5.18}$$

If $c_0=0$, (5.18) reduces to $e^{(n+1)B}=d^2$. Set $e^B=t$. Then (5.17) can be written as

$$f(z)=e^U,\ g(z)=te^{-U},\ \text{where } t \text{ is a constant such that }\ t^{n+1}=1,$$

which is the conclusion (b).

If $c_0 \neq 0$, then from (5.18), one can say that $e^{U(z+c)-U(z)} + c_0$ has no zeros. Then $\phi(z) = e^{U(z+c)-U(z)} \neq 0, -c_0, \infty$. By Picard's theorem, ϕ is constant and so $\deg(U(z)) = 1$. Therefore, from (5.17), one may obtain

$$f(z) = c_1 e^{az},\ g(z) = c_2 e^{-az},$$

where $a,\ c_1$ and c_2 are non-zero constants. Using these in (5.6), we obtain

$$(c_1c_2)^{n+1}(e^{ac} + c_0)(e^{-ac} + c_0) = d^2,$$

which is the conclusion (c).

Case 2: Suppose $n \geq 5$.

Since $f(z)^n L_c(f) - P(z)$ and $g(z)^n L_c(g) - P(z)$ share $(0, l)$, it follows that F and G share $(1, l)$. Let $H \not\equiv 0$. First suppose $l \geq 2$.

Using Lemmas 4.16 and 4.19, we obtain

$$\begin{aligned}
\overline{N}(r,1;F) &= N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\geq 2) \leq N(r,\infty;H) + \overline{N}(r,1;F \mid\geq 2) \\
&\leq \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid\geq 2) \\
&\quad + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \qquad (5.19)
\end{aligned}$$

Keping in view of the above observation and Lemma 4.20, we see that

$$\begin{aligned}
&\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_*(r,1;F,G) \\
\leq\ &\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}(r,1;F \mid\geq 3) + S(r,F) \\
\leq\ &\overline{N}_0(r,0;G') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3) + S(r,F) + S(r,G) \\
\leq\ &\overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) + S(r,F) + S(r,G) \\
\leq\ &N(r,0;G' \mid G \neq 0) \leq \overline{N}(r,0;G) + S(r,G). \qquad (5.20)
\end{aligned}$$

Since $g(z)$ and $g(z+c)$ share 0 CM, we must have $N(r,\infty, L_c(g)/g(z)) = 0$.

Hence using (5.19), (5.20), Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to F, we obtain

$$\begin{aligned}
&nT(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f) \\
\leq\ &\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F') - N(r,0;L_c(f)) + S(r,f) \\
\leq\ &N_2(r,0;F) + N_2(r,0;G) - N(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\leq\ &N_2(r,0;f^n L_c(f)) + N_2\left(r,0;g^{n+1}\frac{L_c(g)}{g(z)}\right) - N(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\leq\ &2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N\left(r,0;\frac{L_c(g)}{g(z)}\right) + S(r,f) + S(r,g)
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad 2(T(r,f)+T(r,g))+T\left(r,\frac{L_c(g)}{g(z)}\right)+S(r,f)+S(r,g)\\
&\leq \quad 2(T(r,f)+T(r,g))+N\left(r,\infty;\frac{L_c(g)}{g(z)}\right)+m\left(r,\frac{L_c(g)}{g(z)}\right)+S(r,f)+S(r,g)\\
&\leq \quad 2(T(r,f)+T(r,g))+S(r,f)+S(r,g). \qquad (5.21)
\end{aligned}
$$

Similarly, using Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to G, we obtain

$$
\begin{aligned}
& nT(r,g)\leq T(r,G)-N(r,0;L_c(g))+S(r,g)\\
\leq \quad & \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,1;G)-\overline{N}(r,0;G')-N(r,0;L_c(g))+S(r,g)\\
\leq \quad & N_2(r,0;F)+N_2(r,0;G)-N(r,0;L_c(g))+S(r,f)+S(r,g)\\
\leq \quad & N_2(r,0;f(z)^nL_c(f))+N_2\left(r,0;g^nL_c(g)\right)-N(r,0;L_c(g))+S(r,f)+S(r,g)\\
\leq \quad & 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+N\left(r,0;L_c(f)\right)+S(r,f)+S(r,g)\\
\leq \quad & 2(T(r,f)+T(r,g))+T\left(r,L_c(f)\right)+S(r,f)+S(r,g)\\
\leq \quad & 2(T(r,f)+T(r,g))+m\left(r,\frac{L_c(f)}{f(z)}\right)+m(r,f(z))+S(r,f)+S(r,g)\\
\leq \quad & 2(T(r,f)+T(r,g))+T(r,f)+S(r,f)+S(r,g). \qquad (5.22)
\end{aligned}
$$

Combining (5.21) and (5.22), we get

$$(n-5)T(r,f)+(n-4)T(r,g)\leq S(r,f)+S(r,g),$$

which contradicts with $n\geq 5$.

When $l=1$, Keeping in view of Lemmas 4.14, 4.15, 4.16, 4.19 and 4.20, we obtain

$$
\begin{aligned}
\overline{N}(r,1;F) \quad = \quad & N(r,1;F\mid=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
\leq \quad & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)\\
& +\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\leq \quad & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+2\overline{N}_L(r,1;F)+\overline{2}N_L(r,1;G)\\
& +\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\leq \quad & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_{F>2}(r,1;G)+N(r,1;G)\\
& -\overline{N}(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\leq \quad & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+N(r,0;G'\mid G\neq 0)\\
& +\frac{1}{2}\overline{N}(r,0;F)+\overline{N}_0(r,0;F')+S(r,F)+S(r,G)\\
\leq \quad & \overline{N}(r,0;F\mid\geq 2)+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;G)+\overline{N}_0(r,0;F')\\
& +S(r,F)+S(r,G). \qquad (5.23)
\end{aligned}
$$

Since $g(z)$, $g(z+c)$ share 0 CM, $N(r,\infty;g(z+c)/g(z))=0$, and therefore, using Lemma 4.2, we obtain $T(r,g(z+c)/g(z))=0$.

Hence using (5.23), Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to F, we obtain

$$\begin{aligned}
& nT(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f) \\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F') - N(r,0;L_c(f)) + S(r,f) \\
\leq\ & N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) - N(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\leq\ & 2\overline{N}(r,0;f) + N_2\left(r,0;g^{n+1}\frac{L_c(g)}{g}\right) + \frac{1}{2}\overline{N}(r,0;F) + S(r,f) + S(r,g) \\
\leq\ & 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + \frac{1}{2}(\overline{N}(r,0;f) + \overline{N}(r,0,L_c(f))) + N\left(r,0;\frac{L_c(g)}{g}\right) \\
& + S(r,f) + S(r,g) \\
\leq\ & \frac{5}{2}T(r,f) + 2T(r,g) + \frac{1}{2}T(r,L_c(f)) + T\left(r,\frac{L_c(g)}{g}\right) + S(r,f) + S(r,g) \\
\leq\ & \frac{5}{2}T(r,f) + 2T(r,g) + \frac{1}{2}m\left(r,\frac{L_c(f)}{f}\right) + \frac{1}{2}m(r,f(z)) + S(r,f) + S(r,g) \\
\leq\ & 3T(r,f) + 2T(r,g) + S(r,f) + S(r,g).
\end{aligned} \tag{5.24}$$

In a similar manner, we may obtain

$$nT(r,g) \leq 3T(r,f) + \frac{5}{2}T(r,g) + S(r,f) + S(r,g). \tag{5.25}$$

Combining (5.24) and (5.25), we obtain

$$(n-6)T(r,f) + \left(n - \frac{5}{2}\right)T(r,g) \leq S(r,f) + S(r,g),$$

which is a contradiction since $n \geq 6$.

When $l = 0$, using Lemmas 4.16, 4.17, 4.18, 4.19 and 4.20, we obtain

$$\begin{aligned}
\overline{N}(r,1;F) =\ & N(r,1;F\mid=1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_L(r,1;F) \\
& + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + 2\overline{N}_L(r,1;F) + \overline{2}N_L(r,1;G) \\
& + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + \overline{N}_L(r,1;F) + \overline{N}_{F>1}(r,1;G) \\
& + \overline{N}_{G>1}(r,1;F) + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;F') \\
& + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + N(r,0;G'\mid G\neq 0) \\
& + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_0(r,0;F') + S(r,F) + S(r,G) \\
\leq\ & N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G) + \overline{N}(r,0;G) + \overline{N}_0(r,0;F') \\
& + S(r,F) + S(r,G).
\end{aligned} \tag{5.26}$$

Hence using (5.26), Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to F, we obtain

$$
\begin{aligned}
& nT(r,f) \le T(r,F) - N(r,0;L_c(f)) + S(r,f) \\
\le\ & \overline{N}(r,0;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F') - N(r,0;L_c(f)) + S(r,f) \\
\le\ & N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) - N(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\le\ & 2\overline{N}(r,0;f) + N_2\left(r,0;g^{n+1}(z)\frac{L_c(g)}{g(z)}\right) + \overline{N}\left(r,0;g^{n+1}(z)\frac{L_c(g)}{g(z)}\right) \\
& + 2\overline{N}(r,0;f^n(z)L_c(f)) + S(r,f) + S(r,g) \\
\le\ & 4\overline{N}(r,0;f) + 3\overline{N}(r,0;g) + \overline{N}\left(r,0;\frac{L_c(g)}{g(z)}\right) + N\left(r,0;\frac{L_c(g)}{g(z)}\right) \\
& + 2\overline{N}(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\le\ & 4T(r,f) + 3T(r,g) + 2T\left(r,\frac{L_c(g)}{g(z)}\right) + 2T(r,L_c(f)) + S(r,f) + S(r,g) \\
\le\ & 4T(r,f) + 3T(r,g) + 2m(r,L_c(f)) + S(r,f) + S(r,g) \\
\le\ & 4T(r,f) + 3T(r,g) + 2m\left(r,\frac{L_c(f)}{f(z)}\right) + 2m(r,f(z)) + S(r,f) + S(r,g) \\
\le\ & 6T(r,f) + 3T(r,g) + S(r,f) + S(r,g).
\end{aligned}
\tag{5.27}
$$

In a similar manner, we obtain

$$nT(r,g) \le 5T(r,f) + 6T(r,g) + S(r,f) + S(r,g). \tag{5.28}$$

Combining (5.27) and (5.28), we get

$$(n-11)T(r,f) + (n-9)T(r,g) \le S(r,f) + S(r,g),$$

which is a contradiction since $n \ge 11$.

Thus $H \equiv 0$. Then by integration we obtain (5.2). Therefore, the results follows from Case 1. This completes the proof of the theorem. □

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