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SUMMARY

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Dual digraphs of finite semidistributive lattices

Andrew Craig[1, ✉]
Miroslav Haviar[1, 2]
José São João[3, 4]

[1] Department of Mathematics and Applied Mathematics, University of Johannesburg, PO Box 524, Auckland Park, 2006, South Africa.
acraig@uj.ac.za ✉

[2] Department of Mathematics, Faculty of Natural Sciences, M. Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia.
miroslav.haviar@umb.sk

[3] Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden.
zealvro98@outlook.com

[4] Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden.

ABSTRACT

Dual digraphs of finite join-semidistributive lattices, meet-semidistributive lattices and semidistributive lattices are characterised. The vertices of the dual digraphs are maximal disjoint filter-ideal pairs of the lattice. The approach used here combines representations of arbitrary lattices due to Urquhart (1978) and Ploščica (1995). The duals of finite lattices are mainly viewed as TiRS digraphs as they were presented and studied in Craig–Gouveia–Haviar (2015 and 2022). When appropriate, Urquhart's two quasi-orders on the vertices of the dual digraph are also employed. Transitive vertices are introduced and their role in the domination theory of the digraphs is studied. In particular, finite lattices with the property that in their dual TiRS digraphs the transitive vertices form a dominating set (respectively, an in-dominating set) are characterised. A characterisation of both finite meet- and join-semidistributive lattices is provided via minimal closure systems on the set of vertices of their dual digraphs.

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RESUMEN

Se caracterizan los digrafos duales de reticulados finitos unión-semidistributivos, encuentro-semidistributivos y semidistributivos. Los vértices de los digrafos duales son pares filtro-ideales disjuntos maximales del reticulado. El enfoque usado combina las representaciones de reticulados arbitrarios de Urquhart (1978) and Ploščica (1995). Los duales de reticulados finitos son vistos principalmente como digrafos TiRS como fueron presentados y estudiados en Craig–Gouveia–Haviar (2015 y 2022). Cuando sea apropiado, también se emplean los dos cuasi-órdenes de Urquhart en los vértices del digrafo dual. Se introducen los vértices transitivos y se estudia su rol en la teoría de dominación de digrafos. En particular, se caracterizan los reticulados finitos con la propiedad que en sus digrafos TiRS duales los vértices transitivos forman un conjunto dominante (respectivamente un conjunto dominante interior). Se entrega una caracterización de reticulados encuentro- y unión-semidistributivos a través de sistemas de clausura mínima en el conjunto de vértices de sus digrafos duales.

Keywords and Phrases: semidistributive lattice, TiRS digraph, join-semidistributive lattice, meet-semidistributive lattice, dual digraph, domination.

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1 Introduction

Semidistributivity was first described by Jónsson [16] while he was studying sublattices of a free lattice. He proved [16, Lemma 2.6] that every free lattice is semidistributive.

A lattice is *join-semidistributive* if it satisfies the following quasi-equation for all $x, y, z \in L$:

$(\mathrm{SD}_\vee)$ $x \vee y = x \vee z \quad \Longrightarrow \quad x \vee y = x \vee (y \wedge z)$.

Dually, L is *meet-semidistributive* if it satisfies:

$(\mathrm{SD}_\wedge)$ $x \wedge y = x \wedge z \quad \Longrightarrow \quad x \wedge y = x \wedge (y \vee z)$.

A lattice is *semidistributive* if it satisfies both $(\mathrm{SD}_\vee)$ and $(\mathrm{SD}_\wedge)$.

For background on semidistributive lattices we refer to the papers by Adaricheva *et al.* [1] and [2], the chapter by Adaricheva and Nation [3], and the paper by Davey *et al.* [10].

The aim of our paper is to investigate dual digraphs of finite semidistributive lattices. Theorem 3.6 provides a representation of finite semidistributive lattices via a certain class of TiRS digraphs (see Definition 2.4). This theorem is a generalisation of Birkhoff's representation of finite distributive lattices via finite ordered sets [6] (see comments in the next paragraph regarding the distributive case). In addition, we study transitive vertices in the dual digraphs and their role in the domination theory of the digraphs, and also explore closure systems on the set of vertices of the dual digraphs.

We employ representations for finite lattices due to Urquhart [20] and Ploščica [17]. In Urquhart's representation the elements of the dual space are maximal disjoint filter-ideal pairs of the lattice. Urquhart considered two quasi-orders $\leqslant_1$ and $\leqslant_2$ on them and studied the dual of the lattice as a certain doubly (quasi-) ordered space. In Ploščica's representation, the dual space of a lattice L is formed by maximal partial homomorphisms from L into the two-element lattice, which correspond to Urquhart's maximal disjoint filter-ideal pairs of L. When L is a distributive lattice, these maximal partial homomorphisms become total homomorphisms from L into the two-element lattice, which form the Priestley dual of L [18]. The close relationship between Ploščica's representation of general lattices and Priestley's representation of distributive lattices lies in the single binary relation E, which Ploščica considered on his dual space. When L is distributive, E becomes exactly Priestley's order on the dual space. Ploščica's dual space of a finite lattice L is therefore a finite digraph where the vertices are the maximal partial homomorphisms from L into the two-element lattice and the binary relation E, which mimics Priestley's order, forms the edge set of the digraph. These dual digraphs of lattices were presented and studied as TiRS digraphs in two papers by Craig, Gouveia and Haviar [7, 8].

In our approach we combine Urquhart's and Ploščica's representations of finite lattices: the vertices of our dual digraphs are maximal disjoint filter-ideal pairs of the lattice in the Urquhart style, but we mainly study them as TiRS digraphs using the Ploščica binary relation E on the vertices. Only in a small part of our investigation do we swap Ploščica's relation E for Urquhart's two quasi-orders on the vertices to present our results in a different yet rather satisfactory way (the end of Section 3).

In Section 2 we give preliminary results that will prove useful in the subsequent three sections of the paper. In Section 3 we provide several characterisations of the dual digraphs of finite meet-semidistributive, finite join-semidistributive, and finite semidistributive lattices. In Section 4 we introduce transitive vertices in the dual digraphs and we study their role in the domination theory of the digraphs. In particular, we are able to characterise finite lattices having the properties that in their dual TiRS digraphs the transitive vertices form a dominating set, respectively an in-dominating set. In Section 5 we characterise both finite meet-semidistributive and finite join-semidistributive lattices via minimal closure systems on the set of vertices of their dual digraphs.

In Section 6 we make some concluding remarks and observations. In particular, we note connections to other representations of finite semidistributive lattices, and we propose several directions for future research in this area.

2 Preliminaries

Ploščica's representation of arbitrary bounded lattices [17] uses the set of maximal partial homomorphisms (MPHs) from a bounded lattice L to the two-element bounded lattice $(\{0,1\},\wedge,\vee,0,1)$ as the underlying set of the dual space of L. We recall that a *partial homomorphism* from a bounded lattice $(L,\wedge,\vee,0,1)$ into the two-element bounded lattice $(\{0,1\},\wedge,\vee,0,1)$ is a partial map $f : L \to \{0,1\}$ such that $\operatorname{dom} f$ is a bounded sublattice of L and the restriction $f{\restriction}_{\operatorname{dom} f}$ is a bounded lattice homomorphism. A *maximal partial homomorphism* is a partial homomorphism with no proper extension. The set of MPHs is then equipped with a binary relation and a topology.

Definition 2.1 ([20, Section 3])**.** Let L be a lattice. Then $\langle F, I\rangle$ is a *disjoint filter-ideal pair* of L if F is a filter of L and I is an ideal of L such that $F \cap I = \varnothing$. We say that a disjoint filter-ideal pair $\langle F, I\rangle$ is maximal if there is no disjoint filter-ideal pair $\langle G, J\rangle \neq \langle F, I\rangle$ such that $F \subseteq G$ and $I \subseteq J$. A maximal disjoint filter-ideal pair $\langle F, I\rangle$ of L is *total in* L if $F \cup I = L$.

There is a one-to-one correspondence between the set of MPHs from L to $2 = (\{0,1\},\wedge,\vee,0,1)$ and the maximal disjoint filter-ideal pairs (MDFIPs) of L. The latter were used in the dual representation of Urquhart [20]. We will use a combination of the two approaches: for a lattice L, the elements of our dual set X_L will be MDFIPs, but we will equip the set with the binary relation due to Ploščica, and hence will obtain a digraph. (Later, when desirable, we will also equip the set X_L of all MDFIPs of L with Urquhart's two quasi-orders $\leqslant_1$ and $\leqslant_2$.) We do not require the topologies used by Ploščica and Urquhart because we are only working with finite lattices.

Ploščica's binary relation on the set of MPHs is defined as follows for MPHs f and g from L to 2:

$$\text{(E1)}\ \ fEg \quad\Longleftrightarrow\quad (\forall x \in \operatorname{dom} f \cap \operatorname{dom} g)(f(x) \leqslant g(x)).$$

The digraph dual to a finite bounded lattice L in Ploščica's representation is $G_L = (V_L, E)$ where the set of vertices V_L is formed by all MPHs from L to 2 and the relation E is defined by (E1) above. We will now present this dual digraph as $G_L = (X_L, E)$ where the set of vertices will be X_L, *i.e.* is formed by all MDFIPs of L, and the corresponding Ploščica relation E will be defined below by (E2).

For two MDFIPs $\langle F, I\rangle$ and $\langle G, J\rangle$, Ploščica's relation E is determined as follows:

$$\text{(E2)}\ \ \langle F, I\rangle E\langle G, J\rangle \quad\Longleftrightarrow\quad F \cap J = \emptyset.$$

For finite lattices every filter is the up-set of a unique element and every ideal is the down-set of a unique element, so we can represent every disjoint filter-ideal pair $\langle F, I\rangle$ by an ordered pair $\langle{\uparrow}x, {\downarrow}y\rangle$ where $x = \bigwedge F$ and $y = \bigvee I$. Hence for finite lattices we have $\langle{\uparrow}x, {\downarrow}y\rangle E\langle{\uparrow}a, {\downarrow}b\rangle$ if and only if $x \nleqslant b$.

In Figure 1 we present a number of examples of finite (non-distributive) lattices and their dual digraphs. To make the labelling more compact, we denote by xy the MDFIP $\langle{\uparrow}x, {\downarrow}y\rangle$. Also, to keep the display simpler, we have not included the loop on each vertex. Notice that the directed edge set is not a transitive relation.

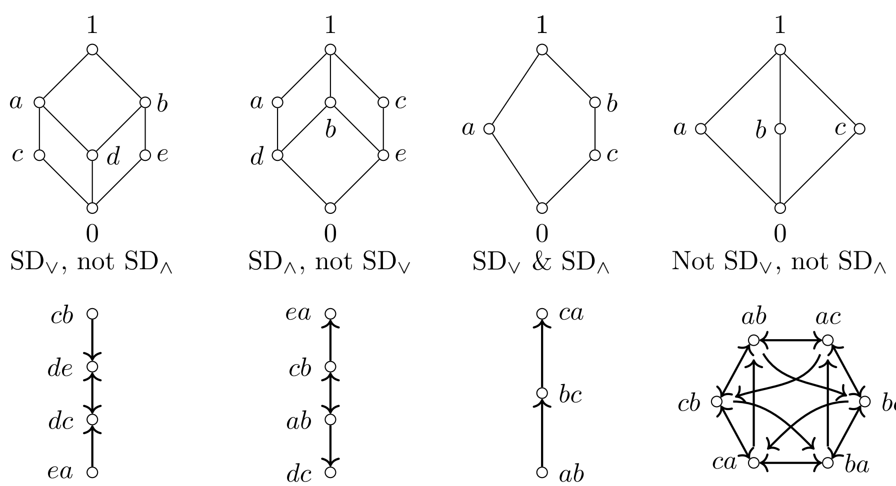

Figure 1: Some finite lattices and their dual digraphs.

The fact below was noted by Urquhart and will be useful later.

Proposition 2.2 ([20, p. 52])**.** *Let L be a finite lattice. If $\langle F, I\rangle$ is a maximal disjoint ideal-filter pair of L then $\bigwedge F$ is join-irreducible and $\bigvee I$ is meet-irreducible.*

Some of what appears in the proposition below can be found in the paper by Gaskill and Nation [13, p. 353]. We will make frequent use of this result and its proof reveals some important features of MDFIPs.

Proposition 2.3. *Let L be a finite lattice and $\langle F, I\rangle$ be a maximal disjoint filter-ideal pair of L. Then the following are equivalent:*

- (i) $\bigwedge F$ *is join-prime;*
- (ii) $\bigvee I$ *is meet-prime;*
- (iii) $F \cup I = L$*;*
- (iv) F *is a prime filter;*
- (v) I *is a prime ideal.*

The equivalences $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ hold even when L is not finite.

Proof. Let L be a finite lattice and let $\langle F, I\rangle$ be a maximal disjoint filter-ideal pair of L. Let $\bigwedge F = x$ and $\bigvee I = y$.

First we show that $(iii) \Rightarrow (i)$. Assume that $F \cup I = L$. Let $a, b \in L$ such that $x \leqslant a \vee b$. We claim that $a \in F$ or $b \in F$. Suppose for a contradiction that $a \notin F$ and $b \notin F$. Then $a, b \in L \setminus F = I$. That implies $a \vee b \in I$, whence $x \in I$, a contradiction.

Now we show that $(i) \Rightarrow (iii)$. Assume that x is join-prime. Let $a \in L$ such that $a \notin F \cup I$. We will consider three cases for the element $a \vee y$ and derive a contradiction for each case.

Case 1: If $a \vee y \in I$ then $a \leqslant a \vee y = y$, thus $a \in I$, a contradiction.

Case 2: If $a \vee y \in F$ then $x \leqslant a \vee y$. Since x is join-prime, $x \leqslant a$ or $x \leqslant y$. If $x \leqslant a$ then $a \in F$, contradicting $a \notin F \cup I$. If $x \leqslant y$ then $x \in I$, contradicting $F \cap I = \varnothing$.

Case 3: Suppose $a \vee y \notin F \cup I$. Since $a \vee y \notin {\uparrow}x$, ${\downarrow}(a \vee y) \cap {\uparrow}x = \varnothing$. From $a \vee y \notin {\downarrow}y$ it follows that ${\downarrow}y \subset {\downarrow}(a \vee y)$. Hence $\langle {\uparrow}x, {\downarrow}(a \vee y)\rangle$ is a disjoint filter-ideal pair properly containing $\langle F, I\rangle$, which contradicts the maximality of $\langle F, I\rangle$.

The equivalence of (ii) and (iii) can be shown analogously.

Now we drop the assumption that L is finite and show that $(iii) \Rightarrow (iv)$. Let $a \vee b \in F$. If $a \notin F$ and $b \notin F$ then we have $a, b \in L \backslash F = I$. Since I is an ideal we would get $a \vee b \in I$, a contradiction. Therefore $a \in F$ or $b \in F$.

To show $(iv) \Rightarrow (iii)$, and the equivalence of (iv) and (v), one uses the fact that a filter $F \subseteq L$ is prime if and only if $L \backslash F$ is a prime ideal. $\square$

The properties of the digraphs dual to bounded lattices were described by Craig, Gouveia and Haviar [7]. There they were called *TiRS graphs*; in this paper we will use the terminology *TiRS digraphs*. We recall the necessary facts. (We note that in the definition below $xE = \{ y \in V \mid (x, y) \in E \}$ and $Ex = \{ y \in V \mid (y, x) \in E \}$.)

Definition 2.4 ([7, Definition 2.2])**.** A TiRS digraph $G = (V, E)$ is a set V and a reflexive relation $E \subseteq V \times V$ such that:

(S) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$.

(R) For all $x, y \in V$, if $xE \subset yE$ then $(x, y) \notin E$, and if $Ey \subset Ex$ then $(x, y) \notin E$.

(Ti) For all $x, y \in V$, if xEy then there exists $z \in V$ such that $zE \subseteq xE$ and $Ez \subseteq Ey$.

We recall that the vertices of the dual digraph G_L of a bounded lattice L are formed by the set X_L of MDFIPs of L and Ploščica's relation E is determined by (E2). Using these facts, the following result can be stated.

Proposition 2.5 ([7, Proposition 2.3])**.** *For any bounded lattice L, its dual digraph $G_L = (X_L, E)$ is a TiRS digraph.*

We recall from [17] a fact concerning general digraphs $G = (X, E)$. Let $\underset{\sim}{2} = (\{0, 1\}, \leqslant)$ denote the two-element digraph. A partial map $\varphi \colon X \to \underset{\sim}{2}$ is said to preserve the relation E if $\varphi(x) \leqslant \varphi(y)$ whenever $x, y \in \operatorname{dom} \varphi$ and $(x, y) \in E$. The lattice of maximal partial E-preserving maps from G to $\underset{\sim}{2}$ is denoted by $\mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$.

Lemma 2.6 ([17, Lemma 1.3])**.** *Let $G = (X, E)$ be a digraph and let us consider $\varphi \in \mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$. Then*

(i) $\varphi^{-1}(0) = \{ x \in X \mid$ *there is no* $y \in \varphi^{-1}(1)$ *with* $(y, x) \in E \}$*;*

(ii) $\varphi^{-1}(1) = \{ x \in X \mid$ *there is no* $y \in \varphi^{-1}(0)$ *with* $(x, y) \in E \}$*.*

The above lemma allows us to observe that for a digraph $G = (X, E)$ and $\varphi, \psi \in \mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$ we have

$$\varphi^{-1}(1) \subseteq \psi^{-1}(1) \iff \psi^{-1}(0) \subseteq \varphi^{-1}(0).$$

This implies that the reflexive and transitive binary relation $\leqslant$ defined on $\mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$ by

$$\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

is a partial order. For a digraph $G = (X, E)$ we let $\mathbb{C}(G) = (\mathcal{G}^{\mathrm{mp}}(G, \underset{\sim}{2}), \leqslant)$.

Theorem 2.7 ([7, Theorem 1.7 and p. 87]). *For any finite bounded lattice L we have that L is isomorphic to $\mathbb{C}(G_L)$ and for any finite TiRS digraph $G = (V, E)$ we have that G is isomorphic to $G_{\mathbb{C}(G)}$.*

In later sections, we will frequently make use of Theorem 2.7 in the following way: given any finite TiRS digraph $G = (V, E)$, we can consider G to be the dual digraph $G_L = (X_L, E)$ for some finite lattice L.

There are a number of different constructions that yield complete lattices isomorphic to the complete lattice $\mathbb{C}(G)$ described above, which is assigned to a digraph $G = (X, E)$ (see [9]). In particular, later we will use the lattice obtained via the polarity $\mathbb{K}(G) = (X, X, E^{\mathsf{C}})$, which will be described in Section 5.

At the end of this preliminary section we recall from [20] how the set X_L of all MDFIPs of a finite bounded lattice L can be equipped with two quasi-orders $\leqslant_1$ and $\leqslant_2$. Urquhart in [20, p. 47] defined two binary relations $\leqslant_1$ and $\leqslant_2$ on the set set X_L of all MDFIPs of an arbitrary lattice L as follows: for two MDFIPs $\langle F, I\rangle$ and $\langle G, J\rangle$,

($\leqslant_1$) $\langle F, I\rangle \leqslant_1 \langle G, J\rangle \iff F \subseteq G$;

($\leqslant_2$) $\langle F, I\rangle \leqslant_2 \langle G, J\rangle \iff I \subseteq J$.

It is clear that the binary relations $\leqslant_1$ and $\leqslant_2$ are reflexive and transitive on the set X_L, and hence are quasi-orders.

3 Characterisation of dual digraphs

The theorem below will play a crucial role in the proof of our first result. Our presentation is slightly different to [3]; we have re-stated their items to suit our purposes. We use $\mathsf{J}(L)$, respectively $\mathsf{M}(L)$, to denote the join-irreducible, respectively meet-irreducible, elements of L.

Theorem 3.1 ([3, Theorem 3-1.4]). *Let L be a finite lattice. Then the following are equivalent:*

(*i*) *L satisfies $SD_\vee$;*

(*ii*) *For each $x \in \mathsf{M}(L)$, there exists a unique minimal element of the set*

$$S(x) = \{k \in L \mid k \nleqslant x \;\&\; k \leqslant x^*\},$$

where x^* is the unique upper cover of x, and moreover, this minimal element of $S(x)$ is in $\mathsf{J}(L)$.

(*iii*) *There exists a map* $\kappa\colon \mathsf{M}(L) \to \mathsf{J}(L)$ *such that for each* $x \in \mathsf{M}(L)$, $\kappa(x)$ *is the minimal element of the set* $S(x)$.

Using the previous result, in the next theorem we characterise finite join-semidistributive and meet-semidistributive lattices via their MDFIPs.

Theorem 3.2. *Let* L *be a finite lattice.*

(*i*) L *is not join-semidistributive if and only if there exist distinct maximal disjoint filter-ideal pairs of the form* $\langle{\uparrow}y, {\downarrow}x\rangle$ *and* $\langle{\uparrow}z, {\downarrow}x\rangle$ *for some* $x, y, z \in L$.

(*ii*) L *is not meet-semidistributive if and only if there exist distinct maximal disjoint filter-ideal pairs of the form* $\langle{\uparrow}x, {\downarrow}y\rangle$ *and* $\langle{\uparrow}x, {\downarrow}z\rangle$ *for some* $x, y, z \in L$.

Proof. For the necessity, assume L is not join-semidistributive, whence by Theorem 3.1, for some $x \in \mathsf{M}(L)$ there exist two minimal elements y and z of the set $S(x)$. Then ${\uparrow}y \cap {\downarrow}x = \varnothing$ and ${\uparrow}z \cap {\downarrow}x = \varnothing$ so $\langle{\uparrow}y, {\downarrow}x\rangle$ and $\langle{\uparrow}z, {\downarrow}x\rangle$ are disjoint filter-ideal pairs. We claim that $\langle{\uparrow}y, {\downarrow}x\rangle$ and $\langle{\uparrow}z, {\downarrow}x\rangle$ are maximal. Suppose on the contrary that there is a disjoint filter-ideal pair $\langle{\uparrow}a, {\downarrow}b\rangle$ of L such that ${\uparrow}y \subseteq {\uparrow}a$ and ${\downarrow}x \subseteq {\downarrow}b$ but $\langle{\uparrow}a, {\downarrow}b\rangle \neq \langle{\uparrow}y, {\downarrow}x\rangle$. This gives us two possible cases:

Case 1: If $a \neq y$ then since y is minimal in the set $S(x)$ and $a \leqslant y \leqslant x^*$ we have that $a \leqslant x$. But $x \leqslant b$, which implies that $a \leqslant b$, contradicting ${\uparrow}a \cap {\downarrow}b = \varnothing$.

Case 2: If $b \neq x$ then $x^* \leqslant b$ since x^* is the unique upper cover of x. But $a \leqslant y \leqslant x^*$, which implies that $a \leqslant b$, contradicting again ${\uparrow}a \cap {\downarrow}b = \varnothing$.

Thus $\langle{\uparrow}y, {\downarrow}x\rangle$ is maximal and we can use a similar argument to prove that $\langle{\uparrow}z, {\downarrow}x\rangle$ is maximal.

For the sufficiency, assume that there exist distinct maximal disjoint filter-ideal pairs of the form $\langle{\uparrow}y, {\downarrow}x\rangle$ and $\langle{\uparrow}z, {\downarrow}x\rangle$ for some $x, y, z \in L$. We will prove that y and z are both minimal elements of the set $S(x)$. If follows from ${\uparrow}y \cap {\downarrow}x = \varnothing$ and ${\uparrow}z \cap {\downarrow}x = \varnothing$ that $y \nleqslant x$ and $z \nleqslant x$. We will argue $y \leqslant x^*$ by contradiction. Suppose $y \nleqslant x^*$, then ${\uparrow}y \cap {\downarrow}x^* = \varnothing$. Since $x < x^*$ implies that ${\downarrow}x \subset {\downarrow}x^*$, we get that $\langle{\uparrow}y, {\downarrow}x\rangle$ is properly contained in $\langle{\uparrow}y, {\downarrow}x^*\rangle$, which is a contradiction. Therefore $y \leqslant x^*$ and $y \in S(x)$. Using a similar argument, $z \in S(x)$. Now if $a \in S(x)$ and $a < y$, then ${\uparrow}y \subset {\uparrow}a$. Since $a \nleqslant x$, we have ${\uparrow}a \cap {\downarrow}x = \varnothing$. Therefore $\langle{\uparrow}a, {\downarrow}x\rangle$ is a disjoint filter-ideal pair with ${\uparrow}y \subset {\uparrow}a$, contradicting the maximality of $\langle{\uparrow}y, {\downarrow}x\rangle$. Similarly, if $b \in S(x)$ such that $b < z$, then $\langle{\uparrow}b, {\downarrow}x\rangle$ is a disjoint filter-ideal pair properly containing $\langle{\uparrow}z, {\downarrow}x\rangle$, which is a contradiction. Therefore y and z are both minimal elements of $S(x)$.

The proof of (*ii*) follows by an order-dual argument. $\square$

Corollary 3.3. *Let $G = (V, E)$ be a finite TiRS digraph which is the dual digraph of a finite lattice L. If the relation E is antisymmetric, then L is semidistributive.*

Proof. In accordance with our remarks after Theorem 2.7, we can consider G to be G_L and so its vertex set V will be X_L.

Suppose for a contradiction that L is not semidistributive. Then L is not join-semidistributive or L is not meet-semidistributive. If L is not join-semidistributive then by Theorem 3.2 (i) there are maximal disjoint filter-ideal pairs of the form $\langle \uparrow y, \downarrow x\rangle$ and $\langle \uparrow z, \downarrow x\rangle$ for some $x, y, z \in L$. Since G is the dual digraph of L, we have $\langle \uparrow y, \downarrow x\rangle, \langle \uparrow z, \downarrow x\rangle \in V$. Clearly $\langle \uparrow y, \downarrow x\rangle E \langle \uparrow z, \downarrow x\rangle$ and $\langle \uparrow z, \downarrow x\rangle E \langle \uparrow y, \downarrow x\rangle$. This contradicts the antisymmetry of the relation E.

If L is not meet-semidistributive, then the argument is analogous. $\square$

Remark 3.4. The converse to Corollary 3.3 does not hold. We can see it on the lattice in Figure 2.

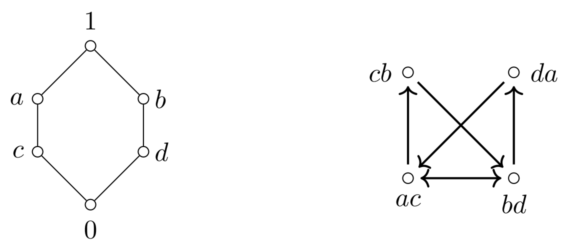

Figure 2: A finite semidistributive lattice and its dual digraph.

The lattice is semidistributive but we see on its dual digraph, which contains a "double arrow" between the elements ac and bd, that the relation E of the digraph is not antisymmetric.

Hence the condition in Corollary 3.3 is sufficient but not necessary for a finite lattice to be semidistributive. An interesting task that is left open is to possibly weaken the given sufficient condition to some form of "weak antisymmetry" of the relation E so that the resulting condition on E is necessary and sufficient for a finite lattice to be semidistributive.

In the statement and the proof of the following result we again use the fact that, by Theorem 2.7, $G = (V, E)$ is isomorphic to the dual digraph $G_L = (X_L, E_L)$ of the lattice L, whose vertex set X_L is the set of all MDFIPs of L.

Lemma 3.5. *Let $G = (V, E)$ be a finite TiRS digraph with dual lattice L. Let $u, v \in V$ be distinct. Then:*

- (i) *$Eu = Ev$ if and only if u and v are the isomorphic images of $\langle \uparrow x, \downarrow y\rangle$ and $\langle \uparrow z, \downarrow y\rangle$ in X_L for some $x, y, z \in L$;*

(ii) *$uE = vE$ if and only if u and v are the isomorphic images of $\langle \uparrow x, \downarrow y\rangle$ and $\langle \uparrow x, \downarrow z\rangle$ in X_L for some $x, y, z \in L$.*

Proof. Let $u, v \in V$. To show the sufficiency of the condition in (i), let u and v be the isomorphic images of the vertices $\langle \uparrow x, \downarrow y\rangle$ and $\langle \uparrow z, \downarrow y\rangle$ in G_L for some $x, y, z \in L$. Since G is isomorphic to G_L, we only need to show that $E_L\langle \uparrow x, \downarrow y\rangle = E_L\langle \uparrow z, \downarrow y\rangle$. To this end, let $\langle F, I\rangle \in E_L\langle \uparrow x, \downarrow y\rangle$, then $F \cap \downarrow y = \varnothing$. Thus $\langle F, I\rangle \in E_L\langle \uparrow z, \downarrow y\rangle$. Similarly, if $\langle F, I\rangle \in E_L\langle \uparrow z, \downarrow y\rangle$, then $F \cap \downarrow y = \varnothing$ and $\langle F, I\rangle \in E_L\langle \uparrow x, \downarrow y\rangle$. Therefore $E_L\langle \uparrow x, \downarrow y\rangle = E_L\langle \uparrow z, \downarrow y\rangle$ and $Eu = Ev$.

For the necessity of the condition in (i), let $\langle \uparrow x, \downarrow y\rangle$ and $\langle \uparrow z, \downarrow w\rangle$ be isomorphic images of u and v in X_L and let $Eu = Ev$. We will show $\downarrow y = \downarrow w$. Let $a \in \downarrow y$. For all $\langle F, I\rangle \in E_L\langle \uparrow z, \downarrow w\rangle$ we have $F \cap \downarrow y = \varnothing$ since $E_L\langle \uparrow x, \downarrow y\rangle = E_L\langle \uparrow z, \downarrow w\rangle$. For $S = \bigcup\{F \mid \langle F, I\rangle \in E_L\langle \uparrow z, \downarrow w\rangle\}$ now $a \notin S$ as $a \in \downarrow y$. We claim that $a \in \downarrow w$. Suppose on the contrary that $a \notin \downarrow w$. Then $a \nleq w$ and $\uparrow a \cap \downarrow w = \varnothing$. This shows $\langle \uparrow a, \downarrow w\rangle$ is a disjoint filter-ideal pair. Hence there is an MDFIP $\langle H, J\rangle$ such that $\uparrow a \subseteq H$ and $\downarrow w \subseteq J$. But $\downarrow w \subseteq J$ and $H \cap J = \varnothing$ implies that $H \cap \downarrow w = \varnothing$. Then $\langle H, J\rangle \in E_L\langle \uparrow z, \downarrow w\rangle$, so $H \subseteq S$, which means $a \in S$, a contradiction. Thus $a \in \downarrow w$. The reverse inclusion can be shown analogously. Therefore $\downarrow y = \downarrow w$ and the proof of (i) is complete. Part (ii) can be proven analogously. □

Theorem 3.6. *Let $G = (V, E)$ be a finite TiRS digraph with $u, v \in V$. Then*

(i) *G is the dual digraph of a join-semidistributive lattice if and only if whenever $u \neq v$ then $Eu \neq Ev$.*

(ii) *G is the dual digraph of a meet-semidistributive lattice if and only if whenever $u \neq v$ then $uE \neq vE$.*

(iii) *G is the dual digraph of a semidistributive lattice if and only if whenever $u \neq v$ then $Eu \neq Ev$ and $uE \neq vE$.*

Proof. Let G be a finite TiRS digraph with dual lattice L. To show the necessity in (i), assume there exist distinct $u, v \in V$ such that $Eu = Ev$. Then by Lemma 3.5 there exist distinct MDFIPs $\langle \uparrow x, \downarrow y\rangle$ and $\langle \uparrow z, \downarrow y\rangle$ in L. It then follows from Theorem 3.2(i) that L is not join-semidistributive.

To show the sufficiency in (i), assume that L is not join-semidistributive. Then by Theorem 3.2(i) there exist distinct MDFIPs $\langle \uparrow x, \downarrow y\rangle$ and $\langle \uparrow z, \downarrow y\rangle$. By Lemma 3.5 there exist distinct vertices $u, v \in V$ such that $Eu = Ev$.

Part (ii) can be shown analogously, and part (iii) follows directly from (i) and (ii). □

We recall that the "separation property" (S) in the definition of TiRS digraphs is defined as follows:

(S) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$.

Hence it should be emphasized that the condition (iii) in the theorem above characterising the semidistributivity is clearly strengthening the separation condition (S) by replacing in it the logical connective "or" with "and". Thus it can be considered as a certain "strong separation property":

(sS) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ and $Ex \neq Ey$.

It is interesting to realise that finite semidistributive lattices are exactly those finite lattices whose dual digraphs have the "separation property" (S) strengthened to the "strong separation property" (sS).

A remark of Urquhart [20, Section 7] says that a finite lattice L is meet-semidistributive if and only if the quasi-order $\leqslant_1$ is an order. We state that result (and its dual) below and prove it using the results from earlier in the section.

Theorem 3.7. *Let L be a finite lattice.*

(i) *L is join-semidistributive if and only if the quasi-order $\leqslant_2$ on the vertices of the dual digraph is an order.*

(ii) *L is meet-semidistributive if and only if the quasi-order $\leqslant_1$ on the vertices of the dual digraph is an order.*

Proof. Assume firstly that the quasi-order $\leqslant_2$ on the vertices of the dual digraph is not an order, that is, the relation $\leqslant_2$ is not antisymmetric. Then there exist distinct vertices x and y such that $x \leqslant_2 y$ and $y \leqslant_2 x$. If we consider the vertices x and y as the MDFIPs $x = \langle F, I \rangle$ and $y = \langle G, J \rangle$, then by definition of $\leqslant_2$ we have $I \subseteq J$ and $J \subseteq I$, hence the MDFIPs x and y have the same ideal part. By Theorem 3.2 it follows that L is not join-semidistributive.

Conversely, if L is not join-semidistributive, then by Theorem 3.2 there exist distinct MDFIPs x and y with the same ideal part, whence $x \leqslant_2 y$ and $y \leqslant_2 x$. It follows that the relation $\leqslant_2$ is not antisymmetric, hence the quasi-order $\leqslant_2$ is not an order. ☐

Now we can rephrase Lemma 3.5 in terms of quasi-orders $\leqslant_1$ and $\leqslant_2$:

Corollary 3.8. *Let $G = (V, E)$ be a finite TiRS digraph with dual lattice L. Let $u, v \in V$ be distinct. Then:*

(i) *$Eu = Ev$ if and only if $u \leqslant_2 v$ and $v \leqslant_2 u$;*

(ii) *$uE = vE$ if and only if $u \leqslant_1 v$ and $v \leqslant_1 u$.*

We can finally summarise the previous results in the following characterisations of join-semidistributivity, meet-semidistributivity and semidistributivity of finite lattices via the properties of their dual digraphs:

Corollary 3.9. *Let $G = (V, E)$ be a finite TiRS digraph.*

(1) The following are equivalent:

(i) *G is the dual digraph of a join-semidistributive lattice;*

(ii) *for all $u, v \in V$, if $u \neq v$ then $Eu \neq Ev$;*

(iii) *the quasi-order $\leqslant_2$ on V is an order.*

(2) The following are equivalent:

(i) *G is the dual digraph of a meet-semidistributive lattice;*

(ii) *for all $u, v \in V$, if $u \neq v$ then $uE \neq vE$;*

(iii) *the quasi-order $\leqslant_1$ on V is an order.*

(3) The following are equivalent:

(i) *G is the dual digraph of a semidistributive lattice;*

(ii) *for all $u, v \in V$, if $u \neq v$ then $Eu \neq Ev$ and $uE \neq vE$;*

(iii) *both the quasi-orders $\leqslant_1$ and $\leqslant_2$ on V are orders.*

4 Domination in dual digraphs

In the dual digraph of a lattice L, there are certain vertices that play an important role. It turns out that these vertices correspond to MDFIPs where $F \cup I = L$.

Definition 4.1. A vertex v of a digraph $G = (V, E)$ is said to be *transitive* in G if uEv and vEw imply uEw for all $u, w \in V$.

With respect to the illustration of the following result, the reader is reminded to return to Figure 1 for examples.

Theorem 4.2. *Let L be a lattice with dual digraph $G_L = (X_L, E)$. A maximal disjoint filter-ideal pair $\langle F, I\rangle$ is total in L if and only if it is transitive in G_L.*

Proof. Let $\langle F, I\rangle$ be total in L. Assume that $\langle G, J\rangle$ and $\langle H, K\rangle$ are maximal disjoint filter-ideal pairs such that $\langle G, J\rangle E\langle F, I\rangle$ and $\langle F, I\rangle E\langle H, K\rangle$. By the definition of E we have that $G \cap I = \varnothing$ and $F \cap K = \varnothing$. We claim that $G \cap K = \varnothing$. Notice that since $F \cap K = \varnothing$ and $\langle F, I\rangle$ is total, it follows that $K \subseteq L \backslash F = I$. But $G \cap I = \varnothing$ and hence $G \cap K = \varnothing$. By the definition of E we get $\langle G, J\rangle E\langle H, K\rangle$ and therefore $\langle F, I\rangle$ is transitive.

For the converse, assume that $\langle F, I\rangle$ is not total in L. Take $x \in L\backslash(F \cup I)$ and consider the disjoint filter-ideal pairs $\langle\uparrow x, I\rangle$ and $\langle F, \downarrow x\rangle$. These can be extended to maximal disjoint filter-ideal pairs $\langle G, J\rangle$ (where $\uparrow x \subseteq G$ and $I \subseteq J$) and $\langle H, K\rangle$ (with $F \subseteq H$ and $\downarrow x \subseteq K$). Since $I \subseteq J$, we have $G \cap I = \varnothing$ and hence $\langle G, J\rangle E\langle F, I\rangle$. Since $F \subseteq H$ we get $F \cap K = \varnothing$ and hence $\langle F, I\rangle E\langle H, K\rangle$. But, since $x \in G \cap K$ we do not have $\langle G, J\rangle E\langle H, K\rangle$ and so $\langle F, I\rangle$ is not transitive. □

The following result was first shown in a more restricted context by Gaskill and Nation [13]. This more general statement is folklore.

Proposition 4.3 ([13, Lemma 1]). *Let L be a join-semidistributive lattice with greatest element 1. Then L has a prime ideal. Dually, if L is a meet-semidistributive lattice with least element 0, then L has a prime filter.*

Proof. Let I be an ideal that is maximal with respect to not containing 1. Suppose that $y, z \notin I$. Then there is an element $x \in I$ such that $x \vee y = x \vee z = 1$. Since L satisfies $SD_\vee$ we get $x \vee (y \wedge z) = 1$ and hence $y \wedge z \notin I$. □

Corollary 4.4. *Let L be a bounded lattice. If the dual digraph $G_L = (X_L, E)$ does not have a transitive vertex then L satisfies neither $SD_\vee$ nor $SD_\wedge$.*

Proof. Assume that G_L does not have a transitive element. Then every MDFIP of L is such that $F \cup I \neq L$. By Proposition 2.3 we have that no filter $F \subseteq L$ can be prime. Since L has both a greatest and least element, by Proposition 4.3, L cannot be join-semidistributive and it cannot be meet-semidistributive. □

Notice that the converse of Corollary 4.4 does not hold. The lattice L_3 from [10] satisfies neither $SD_\vee$ nor $SD_\wedge$ but there exists a maximal disjoint filter-ideal pair $\langle F, I\rangle$ with $F \cup I = L$ (or, a total homomorphism from L_3 to 2).

As stated earlier, the transitive elements in a finite TiRS digraph can play a special role. Notice that when a TiRS digraph G is a poset (*i.e.* it is the dual digraph of a finite distributive lattice) then *every* element of G is transitive.

The next lemma captures two familiar facts about finite join-semidistributive and meet-semidistributive lattices.

Lemma 4.5 ([13, Lemma 1]). (*i*) *The co-atoms of a finite join-semidistributive lattice are meet-prime.*

(*ii*) *The atoms of a finite meet-semidistributive lattice are join-prime.*

Proof. We prove only (*i*) as the proof of (*ii*) will follow using a dual argument.

Let L be a finite join-semidistributive lattice and let $x \in L$ be a co-atom such that $x \geqslant a \wedge b$ for some $a, b \in L$. Suppose that $x \ngeqslant a$ and $x \ngeqslant b$. We then have $x \vee a > x$ and $x \vee b > x$. Since x is a co-atom, we get $x \vee a = 1 = x \vee b$. However, since L is join-semidistributive, we get $x = x \vee (a \wedge b) = x \vee a = 1$, a contradiction. Thus $x \geqslant a$ or $x \geqslant b$. □

In the definition below we note that the original source uses 'arc' instead of 'edge'.

Definition 4.6 ([15, Definition 2])**.** Given a digraph $D = (V, E)$, with vertex set V and edge set E, a set $S \subseteq V$ is a *dominating set* if for every vertex $v \in V \backslash S$, there is a vertex $u \in S$ such that uEv.

Proposition 4.7. *Let $G = (V, E)$ be a finite TiRS digraph. If G is dual to a finite join-semidistributive lattice L, then the transitive vertices of G form a dominating set.*

Proof. Assume that $G = G_L = (X_L, E)$ for some finite join-semidistributive lattice L. If x is a vertex of G then $x = \langle \uparrow a, \downarrow b \rangle$ for some $a, b \in L$. Since $b \neq 1$ we have that $b \leqslant c$ for some co-atom c. By Lemma 4.5 we have that c is meet-prime and so by Proposition 2.3 we know that $\downarrow c$ is a prime ideal and that there exists $d \in L$ such that $\uparrow d$ is a prime filter with $\uparrow d \cap \downarrow c = \varnothing$ and $\uparrow d \cup \downarrow c = L$. By Theorem 4.2, $y = \langle \uparrow d, \downarrow c \rangle$ is a transitive vertex of G_L. Since $\downarrow b \subseteq \downarrow c$ we have $\uparrow d \cap \downarrow b = \varnothing$ and hence yEx. □

The converse of the above proposition does not hold. Let L' be the diamond M_3 with a new top element t. Then its dual digraph G is the same as the dual digraph of M_3 (see Figure 1) except it has an extra vertex $v = \langle \uparrow t, \downarrow 1 \rangle$, which is transitive since it is total. In G the edges obviously go from the vertex v to every other vertex. Hence the set $\{v\}$ of transitive vertices of G is the dominating set, yet the lattice L' is not join-semidistributive as it contains a sublattice isomorphic to M_3 (*cf.* [10]).

Since transitive elements are connected to join- and meet-prime elements, the previous result is partly related to how the join-primes or meet-primes sit inside the lattice. The next result characterises finite TiRS digraphs G dual to finite lattices, in which the transitive vertices of G form a dominating set.

Theorem 4.8. *Let $G = (V, E)$ be a finite TiRS digraph. Then G is dual to a finite lattice L in which every co-atom is meet-prime if and only if the transitive vertices of G form a dominating set.*

Proof. Let $G = (V, E)$ be the dual digraph G_L for some finite lattice L in which every co-atom is meet-prime. If $x \in V$ then $x = \langle \uparrow a, \downarrow b \rangle$ for some $a, b \in L$. Since $b \neq 1$ we have that $b \leqslant c$ for some co-atom c. By Proposition 2.3 we know that $\downarrow c$ is a prime ideal and that there exists $d \in L$

such that $\uparrow d$ is a prime filter with $\uparrow d \cap \downarrow c = \varnothing$ and $\uparrow d \cup \downarrow c = L$. By Theorem 4.2, $y = \langle \uparrow d, \downarrow c \rangle$ is a transitive vertex of $G_L = G$. Since $\downarrow b \subseteq \downarrow c$ we have $\uparrow d \cap \downarrow b = \varnothing$ and hence yEx.

Next, assume that the transitive vertices of G form a dominating set and let c be a co-atom of L. The pair $\langle \uparrow 1, \downarrow c \rangle$ is a disjoint filter-ideal pair that can be extended to a maximal disjoint filter-ideal pair $\langle \uparrow b, \downarrow c \rangle$. Since the transitive vertices form a dominating set, there exists a transitive vertex $\langle \uparrow x, \downarrow y \rangle$ such that $\langle \uparrow x, \downarrow y \rangle E \langle \uparrow b, \downarrow c \rangle$, *i.e.* $\uparrow x \cap \downarrow c = \emptyset$. Since $\langle \uparrow x, \downarrow y \rangle$ is transitive, we have by Proposition 2.3 and Theorem 4.2 that x is join-prime. Now, we have that $\langle \uparrow x, \downarrow c \rangle$ is a disjoint filter-ideal pair which can be extended to a maximal disjoint filter-ideal pair $\langle \uparrow a, \downarrow c \rangle$ where $a \leqslant x$. Since $a \nleqslant c$ we have $c < a \vee c = 1$. Clearly now $x \leqslant a \vee c$ and hence $x \leqslant a$ or $x \leqslant c$. The latter cannot happen as $\uparrow x \cap \downarrow c = \varnothing$ so $x \leqslant a$ and hence $x = a$. Now $\langle \uparrow x, \downarrow c \rangle$ is a maximal disjoint filter-ideal pair with x join-prime, and hence c is meet-prime. $\square$

Remark 4.9. It is well-known (*cf.* [11, Theorem 2.24]; see also [3, Theorem 3-1.4]) that a finite lattice L satisfies $SD_\vee$ if and only if each element in L has a so-called canonical join representation. Using [13, Lemma 1(ii)] we are able to show that the equivalent conditions of Theorem 4.8 hold for the TiRS digraph G dual to a finite lattice L if and only if the top element 1 of L has a canonical join representation. Since canonical join representations are not the focus of this paper, we have decided to present the proof in a separate paper where this will be explored with the proper context and in more depth.

Definition 4.10 ([15, Definition 3])**.** Given a digraph $D = (V, E)$, with vertex set V and edge set E, a set $S \subseteq V$ is an *in-dominating set* if for every vertex $v \in V \backslash S$, there is a vertex $u \in S$ such that vEu.

Theorem 4.11. *Let $G = (V, E)$ be a finite TiRS digraph. Then G is dual to a finite lattice L in which every atom is join-prime if and only if the transitive vertices of G form an in-dominating set.*

Proof. Let $G_L = (X_L, E)$ be the dual digraph of some finite lattice L in which every atom is join-prime. If $x \in V$ then $x = \langle \uparrow a, \downarrow b \rangle$ for some $a, b \in L$. Assume that x is not transitive. Since $a \neq 0$ we have that $c \leqslant a$ for some atom $c \in L$. By Proposition 2.3 we know that $\uparrow c$ is a prime filter and that there exists $d \in L$ such that $\downarrow d$ is a prime ideal with $\uparrow c \cap \downarrow d = \varnothing$ and $\uparrow c \cup \downarrow d = L$. By Theorem 4.2, $y = \langle \uparrow c, \downarrow d \rangle$ is a transitive vertex of G_L. Since $\uparrow a \subseteq \uparrow c$ we have $\uparrow c \cap \downarrow b = \varnothing$ and hence xEy.

Next, assume that the transitive vertices of $G = (V, E)$ form an in-dominating set and let c be an atom of L. The pair $\langle \uparrow c, \downarrow 0 \rangle$ is a disjoint filter-ideal pair that can be extended to an MDFIP $\langle \uparrow c, \downarrow b \rangle$. Since the transitive vertices form an in-dominating set, there exists a transitive vertex $\langle \uparrow x, \downarrow y \rangle$ such that $\langle \uparrow c, \downarrow b \rangle E \langle \uparrow x, \downarrow y \rangle$, *i.e.* $\uparrow c \cap \downarrow y = \emptyset$. Since $\langle \uparrow x, \downarrow y \rangle$ is transitive, we have by Proposition 2.3 and Theorem 4.2 that y is meet-prime.

Now, we have that $\langle \uparrow c, \downarrow y\rangle$ is a disjoint filter-ideal pair which can be extended to a maximal disjoint filter-ideal pair $\langle \uparrow c, \downarrow a\rangle$ where $y \leqslant a$. Since $c \nleqslant a$ we have $0 = a \wedge c < c$. Clearly now $a \wedge c < y$ and hence $a \leqslant y$ or $c \leqslant y$. The latter cannot happen as $\uparrow c \cap \downarrow y = \varnothing$ so $a \leqslant y$ and hence $y = a$. Now $\langle \uparrow c, \downarrow y\rangle$ is an MDFIP with y is meet-prime, and hence c is join-prime. □

Corollary 4.12. *Let $G = (V, E)$ be a finite TiRS digraph. If G is dual to a finite meet-semidistributive lattice L, then the transitive vertices of G form an in-dominating set.*

Proof. Let $G = (V, E)$ be a finite TiRS digraph. Assume G is dual to a finite meet-semidistributive lattice L. Then by Lemma 4.5 the atoms of L are join-prime. It then follows from Theorem 4.11 that the transitive elements of L form an in-dominating set. □

We think it is an interesting problem to try and characterise the dual digraphs of finite join-semidistributive lattices within the class of finite TiRS digraphs whose transitive vertices form a dominating set (and dually). We attempted to do so but were unable to identify the required condition.

5 Minimal closure systems from dual digraphs

Closure systems appear in many different areas of mathematics. They were investigated in relation to join-semidistributive lattices by Adaricheva *et al.* [1]. A comprehensive account of the theory can be found in the book chapters by Adaricheva and Nation [4, 5]. The definitions below all follow the notational conventions used in Adaricheva and Nation [4, Section 4-2] although in some cases the reference is to another source.

Definition 5.1 ([14, Definition 30])**.** Let X be a set and $\phi : \wp(X) \to \wp(X)$. Then ϕ is a *closure operator* on X if for all $Y, Z \in \wp(X)$,

- *(i)* $Y \subseteq \phi(Y)$,
- *(ii)* $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$,
- *(iii)* $\phi(\phi(Y)) = \phi(Y)$.

If X is a set and ϕ a closure operator on X then the pair $\langle X, \phi\rangle$ is called a *closure system.*

For $Y \subseteq X$ we say that Y is *closed* if $\phi(Y) = Y$. The closed sets of a closure operator ϕ on X form a complete lattice, denoted by $\mathrm{Cld}(X, \phi)$.

Example 5.2. Let L be a finite lattice. If $a \in L$ let $J_a = \{x \in \mathsf{J}(L) \mid x \leqslant a\}$ and define $\tau\colon \wp(\mathsf{J}(L)) \to \wp(\mathsf{J}(L))$ by $\tau(A) = \bigcap\{J_a \mid a \in L \text{ and } A \subseteq J_a\}$. Then $\langle \mathsf{J}(L), \tau\rangle$ is a closure system. Notice that every finite lattice L is isomorphic to $\mathrm{Cld}(\mathsf{J}(L), \tau)$ via the isomorphism $a \mapsto J_a$.

From any digraph $G = (X, E)$ we get the closure system $\langle X, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd} \rangle$ (see [9, Theorem 3.3]). Here we recall necessary facts from [9, Section 3].

For a digraph $G = (X, E)$ one can consider the triple (called a *context*) $\mathbb{K}(G) := (X, X, E^{\complement})$, where the relation $E^{\complement} \subseteq X \times X$ is the complement of the digraph relation E: $E^{\complement} = (X \times X) \backslash E$. One can then define a Galois connection via so-called *polars* as follows. The maps

$$E^{\complement}_{\rhd} : (\wp(X), \subseteq) \to (\wp(X), \supseteq) \quad \text{and} \quad E^{\complement}_{\lhd} : (\wp(X), \supseteq) \to (\wp(X), \subseteq)$$

are given by

$$E^{\complement}_{\rhd}(Y) = \{ x \in X \mid (\forall y \in Y)(y, x) \notin E \},$$
$$E^{\complement}_{\lhd}(Y) = \{ z \in X \mid (\forall y \in Y)(z, y) \notin E \}.$$

The so-called *concept lattice* $\mathrm{CL}(\mathbb{K}(G))$ of the context $\mathbb{K}(G) = (X, X, E^{\complement})$, given by

$$\mathrm{CL}(\mathbb{K}(G)) = \{ Y \subseteq X \mid (E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd})(Y) = Y \},$$

is a complete lattice when ordered by inclusion.

The isomorphism in Proposition 5.3 below is different to the original source but is equivalent because of the one-to-one correspondence between the sets V_L and X_L. We recall that the definition of the lattice $\mathbb{C}(G_L)$ is given directly before Theorem 2.7.

Proposition 5.3 ([9, Proposition 3.1 and Corollary 3.2])**.** *If L is a finite lattice and $G_L = (X_L, E)$ is its dual digraph, we have*

$$L \cong \mathbb{C}(G_L) \cong \mathrm{CL}(\mathbb{K}(G_L)).$$

The map $a \mapsto \{ \langle F, I \rangle \in X_L \mid a \in F \}$ is the isomorphism from L to $\mathrm{CL}(\mathbb{K}(G_L))$.

The definition below is important in understanding the notion of a minimal closure system later on.

Definition 5.4 ([4, Definition 4-2.1])**.** Closure systems $\langle X, \phi \rangle$ and $\langle Y, \psi \rangle$ are called *equivalent* if $\mathrm{Cld}(X, \phi) \cong \mathrm{Cld}(Y, \psi)$. Two equivalent systems are called *isomorphic* if there exists a bijection $\rho : X \to Y$ such that $\rho(\phi(Z)) = \psi(\rho(Z))$ for all $Z \subseteq X$.

The left-most lattice in Figure 1 is referred to as L^{∂}_4 in [10]. We use this lattice to provide an illustration of Definition 5.4.

Example 5.5. Let $L = L^{\partial}_4$ and consider its dual digraph $G_L = (X_L, E) = (\{cb, de, dc, ea\}, E)$. From this digraph we get the closure system $\langle X_L, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd} \rangle$ with

$$\mathrm{Cld}(X_L, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd}) = \{\emptyset, \{cb\}, \{ea\}, \{de, dc\}, \{cb, de, dc\}, \{ea, de, dc\}, X_L\}.$$

If we let $Y = \{cb, de, ea\}$ and $\phi_Y(S) = Y \cap (E_\lhd^{\complement} \circ E_\rhd^{\complement})(S)$ then

$$\mathrm{Cld}(Y, \phi_Y) = \{\emptyset, \{cb\}, \{ea\}, \{de\}, \{cb, de\}, \{ea, de\}, Y\}.$$

It is easy to see that $\langle X_L, E_\lhd^{\complement} \circ E_\rhd^{\complement}\rangle$ and $\langle Y, \phi_Y\rangle$ are equivalent but not isomorphic.

Proposition 5.6. *Let $\langle X, \phi\rangle$ and $\langle Y, \psi\rangle$ be closure systems and let $f\colon X \to Y$ be a mapping. If $f(A)$ is closed in Y for all closed sets $A \subseteq X$ and $f^{-1}(B)$ is closed in X for all closed sets $B \subseteq Y$ then $f(\phi(A)) = \psi(f(A))$ for all $A \subseteq X$.*

Proof. Let f be such that $f(A)$ is closed in Y for all closed sets $A \subseteq X$ and $f^{-1}(B)$ is closed in X for all closed sets $B \subseteq Y$. Notice that for all $S \subseteq X$ we have that $\phi(S) = \bigcap\{A \subseteq X \mid S \subseteq A \text{ and } A \text{ is closed in } X\}$, and similarly for ψ. Let $S \subseteq X$. To show the inclusion $f(\phi(A)) \subseteq \psi(f(A))$, let $B \in \mathrm{Cld}(Y, \psi)$ such that $f(S) \subseteq B$. Then $S \subseteq f^{-1}(B)$. But $f^{-1}(B)$ is closed in X by our assumption. Hence $\phi(S) \subseteq f^{-1}(B) = \phi(f^{-1}(B))$. This implies that $f(\phi(S)) \subseteq B$. Since B was arbitrary, this is true for all closed sets containing $f(S)$. Therefore $f(\phi(S)) \subseteq \psi(f(S)) = \bigcap\{A \subseteq X \mid f(S) \subseteq A \text{ and } A \text{ is closed in } X\}$. For the reverse inclusion notice that since $A \subseteq \phi(A)$ we get that $f(A) \subseteq f(\phi(A))$. But $f(\phi(A))$ is closed by our assumption. Thus $\psi(f(A)) \subseteq f(\phi(A))$. ◻

Further, Adaricheva and Nation [4] posed the following problem: given a closure system $\langle X, \phi\rangle$, can we find a $\subseteq$-minimal subset Y of X and a closure operator ψ on Y such that $\langle Y, \psi\rangle$ is equivalent to $\langle X, \phi\rangle$? Such a closure system is then said to be *minimal* for $\langle X, \phi\rangle$.

Theorem 5.7 ([5, Lemma 4-2.13])**.** *A closure system $\langle X, \phi\rangle$ with lattice of closed sets L is minimal if and only if it is isomorphic to $\langle \mathsf{J}(L), \tau\rangle$.*

Proposition 5.8. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then the mapping $f\colon X \to \mathsf{J}(L)$ defined by $f(\langle F, I\rangle) = \bigwedge F$ is surjective and satisfies $f(E_\lhd^{\complement} \circ E_\rhd^{\complement}(S)) = \tau(f(S))$ for all $S \subseteq X$.*

Proof. We start by proving the surjectivity of f. Let $x \in \mathsf{J}(L)$ and let $T(x)$ denote the set $\{a \in L \mid x_* \leqslant a \text{ and } x \nleqslant a\}$ where x_* is the unique lower cover of x. We notice that the set $T(x)$ is non-empty since $x_* \in T(x)$. Let $y \in T(x)$ be a maximal element (which exists since $T(x)$ is a finite ordered set). Then we claim that $\langle {\uparrow}x, {\downarrow}y\rangle$ is an MDFIP. We have that ${\uparrow}x \cap {\downarrow}y = \varnothing$ since $x \nleqslant y$. Now let $\langle {\uparrow}a, {\downarrow}b\rangle$ be an MDFIP such that ${\uparrow}x \subseteq {\uparrow}a$ and ${\downarrow}y \subseteq {\downarrow}b$ and $\langle {\uparrow}a, {\downarrow}b\rangle \neq \langle {\uparrow}x, {\downarrow}y\rangle$. We get two cases from this.

Case 1: If ${\uparrow}x \neq {\uparrow}a$ then $a < x$ so $a \leqslant x_*$. Thus we get that $a \leqslant x_* \leqslant y \leqslant b$, which is a contradiction.

Case 2: If ${\downarrow}y \neq {\downarrow}b$ then $y < b$ and so $x_* \leqslant y < b$. But y is maximal in $T(x)$ so we have that $a \leqslant x \leqslant b$. Again, this is a contradiction.

Thus $\langle \uparrow x, \downarrow y \rangle$ is an MDFIP and $f(\langle \uparrow x, \downarrow y \rangle) = x$. Hence f is surjective.

To help us prove that f preserves closure, we define $B_a = \{\langle F, I \rangle \in X_L \mid a \in F\}$ and $J_a = \{x \in \mathsf{J}(L) \mid x \leqslant a\}$ for $a \in L$. Notice that the closed sets from $\langle X_L, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd} \rangle$ are exactly the sets B_a for all $a \in L$ and the closed sets from $\langle \mathsf{J}(L), \tau \rangle$ are exactly the sets J_a for all $a \in L$ (see Proposition 5.3 and Example 5.2). We claim that $f(B_a) = J_a$ and $f^{-1}(J_a) = B_a$ for all $a \in L$.

Let $a \in L$. We prove firstly that $f(B_a) = J_a$. To show the inclusion $f(B_a) \subseteq J_a$, let $x \in f(B_a)$. Then $x = \bigwedge F$ for some $\langle F, I \rangle \in B_a$. Since $\langle F, I \rangle \in B_a$ we have that $a \in F$. This implies that $x \leqslant a$. But $x \in \mathsf{J}(L)$ and thus $x \in J_a$. To show the reverse inclusion $f(B_a) \supseteq J_a$, let $x \in J_a$. Then by the surjectivity there is $y \in L$ such that $\langle \uparrow x, \downarrow y \rangle \in X_L$. Then since $x \in J_a$, we have that $x \leqslant a$. This implies that $a \in \uparrow x$ and that $\langle \uparrow x, \downarrow y \rangle \in B_a$. Since $\langle \uparrow x, \downarrow y \rangle \in B_a$, we get that $x \in f(B_a)$. Thus $f(B_a) = J_a$.

Now we prove that $f^{-1}(J_a) = B_a$ for all $a \in L$. To show $f^{-1}(J_a) \subseteq B_a$, let $\langle F, I \rangle \in f^{-1}(J_a)$. Then $f(\langle F, I \rangle) = x \in J_a$. Since $x \in J_a$, we have that $x \leqslant a$ and that $a \in \uparrow x = F$. Therefore $\langle F, I \rangle \in B_a$. To show $f^{-1}(J_a) \supseteq B_a$, let $\langle F, I \rangle \in B_a$. Then $a \in F$ and $f(\langle F, I \rangle) = \bigwedge F \leqslant a$. Therefore $f(\langle F, I \rangle) \in J_a$ and $\langle F, I \rangle \in f^{-1}(J_a)$. Thus $f^{-1}(J_a) = B_a$.

By Proposition 5.6 we get $f(E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd}(S)) = \tau(f(S))$ for all $S \subseteq X$. $\square$

The main result of this section is the theorem below. We again refer the reader to Figure 1 for basic illustrative examples, while Example 5.5 provides a demonstration of what can happen when L is not meet-semidistributive.

Theorem 5.9. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then $\langle X_L, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd} \rangle$ is a minimal closure system for itself if and only if L is meet-semidistributive.*

Proof. The necessity will be proved by contraposition. Assume L is not meet-semidistributive. By Proposition 5.8 we have that $|\mathsf{J}(L)| \leq |X_L|$ since f is surjective. But by Theorem 3.2 there exist distinct MDFIPs $\langle \uparrow x, \downarrow y \rangle$ and $\langle \uparrow x, \downarrow z \rangle$ where $x \in \mathsf{J}(L)$. This implies that f is not injective and hence $|\mathsf{J}(L)| < |X_L|$. Therefore by [5, Lemma 4-2.13], $\langle X, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd} \rangle$ is not minimal.

For the sufficiency, assume that L is meet-semidistributive. We will show that f defined in Proposition 5.8 is a bijection. We only need to show that f is injective. Let $\langle F, I \rangle, \langle G, J \rangle \in X$ be such that $f(\langle F, I \rangle) = f(\langle G, J \rangle) = x$. Then $F = G = \uparrow x$. By Theorem 3.2 we have that $I = J$. Therefore $\langle F, I \rangle = \langle G, J \rangle$ and hence f is injective. Thus it follows from Propositions 5.6 and 5.8 that f is an isomorphism of closure systems. By [5, Lemma 4-2.13] this implies that $\langle X, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd} \rangle$ is minimal. $\square$

Before stating the dual of Theorem 5.9, we need to make some observations. As observed earlier in the section, if L is a finite lattice, with $G_L = (X_L, E)$ its dual digraph, then $L \cong \mathrm{Cld}(X_L, E^{\complement}_{\lhd} \circ E^{\complement}_{\rhd}) \cong$

$\mathrm{Cld}(\mathsf{J}(L),\tau)$. If we reverse the order of the polar maps $E_{\triangleleft}^{\complement}$ and $E_{\triangleright}^{\complement}$, we again get a closure operator, but with $L^{\partial} \cong \mathrm{Cld}(X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement})$. For a finite lattice L, it is easy to show that $g : X_L \to X_{L^{\partial}}$, defined for $\langle \uparrow a, \downarrow b\rangle \in X_L$ by $g(\langle \uparrow a, \downarrow b\rangle) = \langle \uparrow b, \downarrow a\rangle$, is a bijection. From this we get that $\langle X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement}\rangle$ is isomorphic to $\langle X_{L^{\partial}}, E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement}\rangle$.

Theorem 5.10. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then $\langle X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement}\rangle$ is a minimal closure system for itself if and only if L is join-semidistributive.*

Proof. We know that L is join-semidistributive if and only if L^{∂} is meet-semidistributive. We can then apply Theorem 5.9 to the closure system $\langle X_{L^{\partial}}, E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement}\rangle$. □

Corollary 5.11. *Let L be a finite lattice and $G_L = (X_L, E)$ its dual digraph. Then $\langle X_L, E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement}\rangle$ and $\langle X_L, E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement}\rangle$ are minimal closure systems for themselves if and only if L is semidistributive.*

6 Conclusion and future research

In this paper we characterised dual digraphs of finite meet-semidistributive, join-semidistributive and semidistributive lattices. We combined Urquhart's and Ploščica's representations of finite lattices in the following sense: the vertices of our dual digraphs were maximal disjoint filter-ideal pairs of the lattice in the Urquhart style, but we mainly viewed the duals as TiRS digraphs using the Ploščica binary relation E on the vertices. We introduced transitive vertices in our digraphs and explored their role in the domination theory. In particular, we characterised the finite lattices with the property that in their dual digraphs the transitive vertices form a dominating set resp. an in-dominating set. Finally, we characterised finite meet-semidistributive and join-semidistributive lattices via minimal closure systems on the set of vertices of their dual digraphs.

We wish to take note of two other settings in which dual representations of finite semidistributive lattices have been developed. The older of these is that of Formal Concept Analysis, where a characterisation of both finite join-semidistributive and meet-semidistributive lattices is available [12, Section 6.3]. There is also a recent paper by Reading, Speyer and Thomas [19] where they give a representation of finite semidistributive lattices via *two-acyclic factorization systems*. They define a two-acyclic factorization system to be a 4-tuple $\langle W, \to, \twoheadrightarrow, \hookrightarrow\rangle$ with a set W and three binary relations $\to, \hookrightarrow, \twoheadrightarrow$ on W. The relations $\twoheadrightarrow$ and $\hookrightarrow$ are required to be partial orders. The representation then comes from defining a factorization system on the set of join-irreducible elements of a semidistributive lattice. The triple $(X, \hookrightarrow, \twoheadrightarrow)$ is isomorphic to Urquhart's dual of the lattice L. We note that, in our representation, join- and meet-semidistributive lattices can be considered separately, but in the setting of factorization systems this separation is not yet possible (see [19, Remark 5.14]).

Lastly, we wish to point to some promising directions for future research. These would build on the representation of finite join- and meet-semidistributive lattices obtained in Section 3. The first of these would be to attempt to study finite sublattices of free lattices via their dual digraphs. This would require first finding a dual description of the well-known Whitman's Condition. The second direction would be the study of finite convex geometries (see [1, 5]) via their dual digraphs. Finite convex geometries are closure systems that are often studied via their lattice of closed sets. These lattices of closed sets are join-semidistributive and lower semimodular. Work is already under way to find a dual characterisation of upper and lower semimodularity.

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References

[1] K. V. Adaricheva, V. A. Gorbunov and V. I. Tumanov, “Join-semidistributive lattices and convex geometries”, Adv. Math., vol. 173, no. 1, pp. 1–49, 2003.

[2] K. Adaricheva, M. Maróti, R. McKenzie, J. B. Nation and E. R. Zenk, “The Jónsson–Kiefer Property”, Studia Logica, vol. 83, no. 1–3, pp. 111–131, 2006.

[3] K. Adaricheva and J. B. Nation, “Classes of semidistributive lattices”, in *Lattice Theory: Special Topics and Applications*, vol. 2, G. Grätzer and F. Wehrung, Basel: Birkhäuser, 2016, pp. 59–101.

[4] K. Adaricheva and J. B. Nation, “Lattices of algebraic subsets and implicational classes”, in *Lattice Theory: Special Topics and Applications*, vol. 2, G. Grätzer and F. Wehrung, Basel: Birkhäuser, 2016, pp. 103–151.

[5] K. Adaricheva and J. B. Nation, “Convex geometries”, in *Lattice Theory: Special Topics and Applications*, vol. 2, G. Grätzer and F. Wehrung, Basel: Birkhäuser, 2016, pp. 153–179.

[6] G. Birkhoff, “On the combination of subalgebras”, Proc. Camb. Phil. Soc., vol. 29, no. 4, pp. 441–464, 1933.

[7] A. P. K. Craig, M. J. Gouveia and M. Haviar, “TiRS graphs and TiRS frames: a new setting for duals of canonical extensions”, Algebra Universalis, vol. 74, no. 1–2, pp. 123–138, 2015.

[8] A. P. K. Craig, M. J. Gouveia and M. Haviar, “Canonical extensions of lattices are more than perfect”, Algebra Universalis, vol. 83, no. 2, Paper No. 12, 17 pages, 2022.

[9] A. Craig and M. Haviar, “Reconciliation of approaches to the construction of canonical extensions of bounded lattices”, Math. Slovaca, vol. 64, no. 6, pp. 1335–1356, 2014.

[10] B. A. Davey, W. Poguntke and I. Rival, “A characterization of semi-distributivity”, Algebra Universalis, vol. 5, pp. 72–75, 1975.

[11] R. Freese, J. Ježek and J. B. Nation, *Free Lattices*, Mathematical Surveys and Monographs, vol. 42, Providence, R.I.: American Mathematical Society, 1995.

[12] B. Ganter and R. Wille, *Formal Concept Analysis: Mathematical Foundations*, Berlin: Springer, 1999.

[13] H. S. Gaskill and J. B. Nation, “Join-prime elements in semidistributive lattices”, Algebra Universalis, vol. 12, no. 3, pp. 352–369, 1981.

[14] G. Grätzer, *Lattice Theory: Foundation*, Basel: Birkhäuser, 2011.

[15] T. W. Haynes, S. T. Hedetniemi and M. A. Henning, "Domination in digraphs", in *Structures of Domination in Graphs*, vol. 66, Cham: Springer, 2021, pp. 387–428.

[16] B. Jónsson, "Sublattices of a free lattice", Canad. J. Math., vol. 13, pp. 256–264, 1961.

[17] M. Ploščica, "A natural representation of bounded lattices", Tatra Mountains Math. Publ., vol. 5, pp. 75–88, 1995.

[18] H. A. Priestley, "Representation of distributive lattices by means of ordered Stone spaces", Bull. Lond. Math. Soc., vol. 2, no. 2, pp. 186–190, 1970.

[19] N. Reading, D. E. Speyer and H. Thomas, "The fundamental theorem of finite semidistributive lattices", Selecta Math., vol. 27, no. 4, Paper No. 59, 53 pages, 2021.

[20] A. Urquhart, "A topological representation theory for lattices", Algebra Universalis, vol. 8, no. 1, pp. 45–58, 1978.

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Two nonnegative solutions for two-dimensional nonlinear wave equations

Svetlin Georgiev[1, ✉]
Mohamed Majdoub[2,3]

[1] *Department of Differential Equations, Faculty of Mathematics and Informatics, University of Sofia, Sofia, Bulgaria.*
`svetlingeorgiev1@gmail.com` ✉

[2] *Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia.*

[3] *Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, 31441, Dammam, Saudi Arabia.*
`mmajdoub@iau.edu.sa`

ABSTRACT

We study a class of initial value problems for two-dimensional nonlinear wave equations. A new topological approach is applied to prove the existence of at least two nonnegative classical solutions. The arguments are based upon a recent theoretical result.

RESUMEN

Estudiamos una clase de problemas de valor inicial para ecuaciones de onda no lineales en dimensión dos. Se aplica un nuevo enfoque topológico para probar la existencia de al menos dos soluciones clásicas no negativas. Los argumentos se basan en un resultado teórico reciente.

Keywords and Phrases: Hyperbolic Equations, positive solution, fixed point, cone, sum of operators.

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1 Introduction

Global existence for nonlinear wave equations is an important mathematical topic. Mathematicians, including F. John, S. Kleinerman, L. Hörmander, etc., have made investigations to this subject. The first non-trivial long-time existence result was established by F. John and S. Kleinerman in [19], where it is proved the almost global existence for a class $3D$ quasilinear scalar wave equations. Global existence for $3D$ quasilinear wave equations was established firstly by S. Kleinerman in [20] and by D. Christodoulou, independently by S. Kleinerman, in [5]. The problem in $2D$ case is quite delicate. Introducing the ghost weight, in [1] was proved the global well-posedness for a class $2D$ nonlinear wave equations. Using a class Hardy-type inequality depending on the compact support of the initial data, in [21] was proved almost global existence for $2D$ case. Here we propose a new approach for investigations for classical solutions of a class $2D$ nonlinear wave equations. We investigate for existence of at least two positive solutions for the following IVP

$$\begin{aligned} u_{tt} - \Delta u &= f(t, x, u, u_t, u_x), \quad t > 0, && x = (x_1, x_2) \in \mathbb{R}^2, \\ u(0, x) &= u_0(x), && x = (x_1, x_2) \in \mathbb{R}^2, \\ u_t(0, x) &= u_1(x), && x = (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \tag{1.1}$$

where $\Delta u = u_{x_1x_1} + u_{x_2x_2}$, $u_x = (u_{x_1}, u_{x_2})$.

The initial value problem (1.1) has attracted considerable attention in the mathematical community and the well-posedness theory in the Sobolev spaces for polynomial type nonlinearities has been extensively studied. The case of exponential nonlinearity was recently investigated (see [18] and references therein). In particular, if the nonlinearity f and the initial data u_0, u_1 are smooth then the Cauchy problem (1.1) has a classical local (in time) solution. This follows from Duhamel's formula via the usual fixed point argument in the space $H^s_{loc} \times H^{s-1}_{loc}$, $s > 2$. Such an s guarantee that $u, u_t, \nabla u$ are in L^∞. Note that $u \in H^s_{loc}$ means that the H^s norm over a ball centered at x_0 and with radius 1 is uniformly bounded by a constant independent of x_0. We refer the reader to [23] and references therein for more properties and information on nonlinear wave equations. In [17] is proved existence and uniqueness of generalized solutions of the first initial boundary value problem for strongly hyperbolic systems in bounded domains. In the case when

$$f(t, x, u, u_t, u_x) = f(u(x)), \quad t > 0, \quad x \in \mathbb{R}^2,$$

and

$$u_0(x) = u_1(x) = 0, \quad x \in \mathbb{R}^2,$$

the problem (1.1) is investigated in [14] where the authors prove existence of at least one nontrivial classical solution of the problem (1.1).

We make the following assumptions on the non-linearity and initial data trough the paper.

(H1) $u_0, u_1 \in \mathcal{C}^2(\mathbb{R}^2)$,

$$\begin{aligned} 0 &\leq u_0, |u_{0x_1}|, |u_{0x_1x_1}|, |u_{0x_2}|, |u_{0x_2x_2}| \leq r, \\ 0 &\leq u_1, |u_{1x_1}|, |u_{1x_1x_1}|, |u_{1x_2}|, |u_{1x_2x_2}| \leq r \quad \text{on} \quad \mathbb{R}^2, \end{aligned}$$

where $r > 0$ is a given constant.

(H2) $f \in \mathcal{C}([0, \infty) \times \mathbb{R}^6)$,

$$\begin{aligned} 0 &\leq f(t, x, w_1, w_2, w_3, w_4) \\ &\leq \sum_{j=1}^{l} \left(a_j(t, x)|w_1|^{p_j} + b_j(t, x)|w_2|^{p_j} + c_j(t, x)|w_3|^{p_j} + d_j(t, x)|w_4|^{p_j}\right), \end{aligned}$$

$(t, x) \in [0, \infty) \times \mathbb{R}^2$, where $a_j, b_j, c_j, d_j \in \mathcal{C}([0, \infty) \times \mathbb{R}^2)$,

$$0 \leq a_j, b_j, c_j, d_j \leq a, \quad p_j > 0, \quad j \in \{1, \ldots, l\},$$

where $a > 0$ and $l \in \mathbb{N}$ are given constants.

Our main result reads as follows.

Theorem 1.1. *Suppose* **(H1)** *and* **(H2)**. *Then the IVP* (1.1) *has at least two nonnegative classical solutions.*

To prove our main result we use a new topological approach. This approach can be used for investigations for existence of at least one and at least two classical solutions for initial value problems, boundary value problems and initial boundary value problems for some classes ordinary differential equations, partial differential equations and fractional differential equations (see [2, 3, 4, 7, 10, 12, 13, 15, 16] and references therein). So far, for the authors they are not known investigations for existence of multiple solutions for the IVP (1.1).

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3, we prove our main result. In Section 4, we give an example.

2 Auxiliary Results

Let X be a real Banach space.

Definition 2.1. *A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0, \infty)$ is defined by*

$$\alpha(Y) = \inf \left\{ \delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad diam(Y_j) \leq \delta, \quad j \in \{1, \ldots, m\} \right\},$$

where $diam(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \ldots, m\}$.

For the main properties of measure of noncompactness we refer the reader to [6].

Definition 2.3. *For a given number $k \geq 0$, a map $K : X \to X$ is said to be k-set contraction if it is continuous, bounded and*

$$\alpha(K(Y)) \leq k\alpha(Y)$$

for any bounded set $Y \subset X$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction.

Definition 2.4. *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Definition 2.5. *A closed, convex set $\mathcal{P}$ in X is said to be a cone if*

(1) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,

(2) $x, -x \in \mathcal{P}$ implies $x = 0$.

Let $\mathcal{P} \subset X$ be a cone and define

$$\mathcal{P}^* = \mathcal{P}\backslash\{0\},$$
$$\mathcal{P}_{r_1} = \Big\{u \in \mathcal{P} : \|u\| \le r_1\Big\},$$
$$\mathcal{P}_{r_1,r_2} = \Big\{u \in \mathcal{P} : r_1 \le \|u\| \le r_2\Big\}$$

for positive constants r_1, r_2 such that $0 < r_1 \le r_2$. The following result will be used to prove Theorem 1.1 . We refer the reader to [8] and [11] for more details.

Theorem 2.6. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \leqslant k < h-1$ and $S(\overline{U}_3) \subset (I-T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $Sx \neq (I-T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

(ii) *there exists* $\epsilon > 0$ *such that* $Sx \neq (I-T)(\lambda x)$, *for all* $\lambda \ge 1+\epsilon$, $x \in \partial U_2$ *and* $\lambda x \in \Omega$,

(iii) $Sx \neq (I-T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Note that (see [9]) the function

$$G(t,x,\tau,\xi) = -\frac{1}{2\pi}\frac{H(t-\tau-|x-\xi|)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}, \quad t,\tau > 0, \quad x,\xi \in \mathbb{R}^2,$$

where $|x-\xi| = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}$, is the Green function for the two-dimensional wave equation

$$u_{tt} - \Delta u = h(t,x), \qquad t > 0, \quad x = (x_1,x_2) \in \mathbb{R}^2,$$
$$u(0,x) = u_t(0,x) = 0, \quad x = (x_1,x_2) \in \mathbb{R}^2,$$

where $H(\cdot)$ denotes the Heaviside function. Observe that

$$G(t,x,\tau,\xi) \le 0, \quad t,\tau > 0, \quad x,\xi \in \mathbb{R}^2.$$

A key lemma in our proof is the following.

Lemma 2.7. *For $h_1, h_2, p > 0$, we have*

$$\left| \int_{\mathbb{R}^2} \int_0^\infty (h_1 + h_2\tau)^p G(t, x, \tau, \xi) d\tau d\xi \right| \leq (h_1 + h_2 t)^p I(t), \quad (t, x) \in (0, \infty) \times \mathbb{R}^2, \tag{2.1}$$

where $I(t) = t^3 + t^2(1 + |\log t|)$.

Proof. Let $h_1, h_2, p > 0$ and $t > 0$. One has

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \int_0^\infty (h_1 + h_2\tau)^p G(t, x, \tau, \xi) d\tau d\xi \right| \leq \frac{1}{2\pi} \int_{|x-\xi|\leq t} \int_0^{t-|x-\xi|} \frac{(h_1 + h_2\tau)^p}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\tau d\xi \\
&\leq \frac{(h_1 + h_2 t)^p}{2\pi} \int_{|x-\xi|\leq t} \left(\log(t + \sqrt{t^2 - |x-\xi|^2}) - \log|x-\xi| \right) d\xi \\
&= \frac{(h_1 + h_2 t)^p}{2\pi} \left(\int_{|x-\xi|\leq t} \log(t + \sqrt{t^2 - |x-\xi|^2}) d\xi - \int_{|x-\xi|\leq t} \log|x-\xi| d\xi \right) \\
&\leq \frac{(h_1 + h_2 t)^p}{2\pi} \left(\log(2t) \int_{|x-\xi|\leq t} d\xi - 2\pi \int_0^t r_1 \log r_1 dr_1 \right) \\
&= \frac{(h_1 + h_2 t)^p}{2\pi} \left(\pi t^2 \log(2t) - \pi \left(t^2 \log t - \frac{t^2}{2} \right) \right) \\
&\leq \frac{(h_1 + h_2 t)^p}{2} \left(t^2 \log(1 + 2t) + t^2 |\log t| + \frac{t^2}{2} \right) \\
&\leq \frac{(h_1 + h_2 t)^p}{2} \left(2t^3 + t^2 |\log t| + \frac{t^2}{2} \right) \\
&\leq (h_1 + h_2 t)^p \left(t^3 + t^2(1 + |\log t|) \right).
\end{aligned}$$

This gives (2.1) as desired. □

We make the change $u = v + u_0 + tu_1$. Then, we get the IVP

$$\begin{aligned}
v_{tt} - \Delta v &= f(t, x, v + u_0 + tu_1, v_t + u_1, v_x + u_{0x} + tu_{1x}) + \Delta u_0 + t\Delta u_1 \\
&= f_1(t, x, v, v_t, v_x), \quad t > 0, \quad x \in \mathbb{R}^2, \\
v(0, x) &= v_t(0, x) = 0, \quad x \in \mathbb{R}^2.
\end{aligned} \tag{2.2}$$

Lemma 2.8. *Suppose* **(H2)**. *If $w_k \in \mathbb{R}$, $|w_k| \leq b$, $k \in \{1, \ldots, 4\}$, for some positive b, then*

$$f(t, x, w_1, w_2, w_3, w_4) \leq 4a \sum_{j=1}^{l} b^{p_j}.$$

Proof. We have

$$\begin{aligned}
0 &\leq f(t,x,w_1,w_2,w_3,w_4) \\
&\leq \sum_{j=1}^{l}(a_j(t,x)|w_1|^{p_j} + b_j(t,x)|w_2|^{p_j} + c_j(t,x)|w_3|^{p_j} + d_j(t,x)|w_4|^{p_j}) \\
&\leq \sum_{j=1}^{l}(ab^{p_j} + ab^{p_j} + ab^{p_j} + ab^{p_j}) \\
&= 4a\sum_{j=1}^{l} b^{p_j}, \quad (t,x,w_1,w_2,w_3,w_4) \in [0,\infty)\times\mathbb{R}^6.
\end{aligned}$$

This completes the proof. □

Let $E = \mathcal{C}^2([0,\infty)\times\mathbb{R}^2)$ and for any $u \in E$, denote

$$\|u\| = \max\left\{\|u\|_\infty, \quad \|u_t\|_\infty, \quad \|u_{tt}\|_\infty\|u_{x_j}\|_\infty, \quad \|u_{x_jx_j}\|_\infty, \quad j\in\{1,2\}\right\},$$

provided that it is finite, where

$$\|v\|_\infty = \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^2}|v(t,x)|.$$

Lemma 2.9. *Suppose* **(H1)** *and* **(H2)***. Let $v \in E$, $\|v\| \leq b$, for some positive b. Then*

$$f(t,x,v+u_0+tu_1, v_t+u_1, v_x+u_{0x}+tu_{1x}) \leq 4a\sum_{j=1}^{l}(b+r(1+t))^{p_j}, \quad (t,x)\in[0,\infty)\times\mathbb{R}^2.$$

Proof. Let

$$\begin{aligned}
w_1 &= v+u_0+tu_1, \\
w_2 &= v_t+u_0+tu_1, \\
w_3 &= u_{x_1}+u_{0x_1}+tu_{1x_1}, \\
w_4 &= v_{x_2}+u_{0x_2}+tu_{1x_2}.
\end{aligned}$$

Then

$$|w_j| \leq b+r(1+t), \quad j\in\{1,\ldots,4\}, \quad t\geq 0.$$

Hence and Lemma 2.8, we get the desired result. This completes the proof. □

Lemma 2.10. *Suppose* **(H1)** *and* **(H2)**. *Let $v \in E$, $\|v\| \le b$, for some positive b. Then*

$$|f_1(t,x,v,v_t,v_x)| \le 4a\sum_{j=1}^{l}(b+r(1+t))^{p_j} + 2r(1+t), \quad (t,x) \in [0,\infty)\times\mathbb{R}^2.$$

Proof. By **(H1)**, we get

$$|\Delta u_0| \le 2r, \quad |\Delta u_1| \le 2r \quad \text{on} \quad \mathbb{R}^2.$$

Using Lemma 2.9, we obtain

$$\begin{aligned}|f_1(t,x,v,v_t,v_x)| &\le f(t,x,v+u_0+tu_1, v_t+u_1, v_x+u_{0x}+tu_{1x}) + |\Delta u_0| + t|\Delta u_1|\\ &\le 4a\sum_{j=1}^{l}(b+r(1+t))^{p_j} + 2r(1+t), \quad (t,x)\in[0,\infty)\times\mathbb{R}^2.\end{aligned}$$

This completes the proof. □

Now, applying Lemma 2.10 and (2.1), we obtain the following result.

Lemma 2.11. *Suppose* **(H1)** *and* **(H2)**. *Then*

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\int_0^\infty G(t,x,\tau,\xi)f_1(\tau,\xi,v(\tau,\xi),v_t(\tau,\xi),v_x(\tau,\xi))d\tau d\xi\right|\\ &\le \left(4a\sum_{j=1}^{l}(b+r(1+t))^{p_j} + 2r(1+t)\right)I(t)\\ &\le \left(4a\sum_{j=1}^{l}(2(b+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j}t^{p_j} + 2r(1+t)\right)I(t), \quad (t,x)\in[0,\infty)\times\mathbb{R}^2.\end{aligned}$$

Take a nonnegative function $g \in \mathcal{C}([0,\infty)\times\mathbb{R}^2)$. Suppose that $v \in E$ is a solution to the integral equation.

$$\begin{aligned}0 &= \frac{1}{8}\int_0^t\int_0^{x_1}\int_0^{x_2}(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)v(t_1,s_1,s_2)ds_2ds_1dt_1\\ &\quad -\frac{1}{16\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^\infty G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\\ &\quad \times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1ds_2ds_1dt_1, \qquad (2.3)\end{aligned}$$

$t \ge 0$, $(x_1,x_2)\in\mathbb{R}^2$. We differentiate three times in t, three times in x_1 and three times in x_2 the equation (2.3) and we obtain

$$0 = g(t,x)v(t,x) - \frac{1}{2\pi}g(t,x)\int_{\mathbb{R}^2}\int_0^\infty G(t,x,\tau,\xi)f_1(\tau,\xi,v(\tau,\xi),v_t(\tau,\xi),v_x(\tau,\xi))d\tau d\xi,$$

$t \geq 0$, $x \in \mathbb{R}^2$, whereupon

$$0 = v(t,x) - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_0^\infty G(t,x,\tau,\xi) f_1(\tau,\xi,v(\tau,\xi),v_t(\tau,\xi),v_x(\tau,\xi))d\tau d\xi,$$

$t \geq 0$, $x \in \mathbb{R}^2$. Hence, using the Green function, we conclude that v is a solution of the IVP (2.2). Thus, any solution $v \in E$ of the integral equation (2.3) is a solution to the IVP (2.2).

(H3) Let $m > 0$ be large enough and A, r_1, L_1, R_1 be positive constants that satisfy the following conditions

$$r_1 < L_1 < R_1, \quad r_1 < r, \quad R_1 > \left(\frac{2}{5m} + 1\right) L_1,$$

$$A\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) < \frac{L_1}{20}.$$

(H4) There exists a nonnegative function $g \in \mathcal{C}([0,\infty)\times\mathbb{R}^2)$ such that

$$\begin{aligned} q(t,x_1,x_2) = &\int_0^t\int_0^{x_1}\int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2) g(t_1,s_1,s_2) \\ &\times \left(1+|x_1-s_1|+(x_1-s_1)^2\right)\left(1+|x_2-s_2|+(x_2-s_2)^2\right) \\ &\times \left(1+(t-t_1)+(t-t_1)^2\right)\left(1+\left(1+t_1+\sum_{j=1}^{l} t_1^{p_j}\right)I(t_1)\right) ds_2 ds_1 dt_1 \\ &\leq A, \quad (t,x_1,x_2) \in [0,\infty)\times\mathbb{R}^2. \end{aligned}$$

In the last section we will give an example for the constants m, A, r, L_1, R_1 and R and for a function g that satisfy **(H3)** and **(H4)**. For $v \in E$, define the operator

$$\begin{aligned} Fv(t,x_1,x_2) = &\frac{1}{8}\int_0^t\int_0^{x_1}\int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2) \\ &\times v(t_1,s_1,s_2) ds_2 ds_1 dt_1 \\ &- \frac{1}{16\pi}\int_0^t\int_0^{x_1}\int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2) \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(t_1,s_1,s_2,t_2,\xi_1,\xi_2) \\ &\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2)) dt_2 d\xi_2 d\xi_1 ds_2 ds_1 dt_1, \end{aligned}$$

$(t,x_1,x_2) \in [0,\infty)\times\mathbb{R}^2$.

Lemma 2.12. *Suppose* **(H1)**–**(H3)**. *Then, for $v \in E$, $\|v\| \leq b$, for some positive b, we have*

$$\|Fv\| \leq A\left(b + 4a\sum_{j=1}^{l}(2(b+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right).$$

Proof. Using Lemma 2.11 and **(H3)**, we get

$$
\begin{aligned}
|Fv(t,x_1,x_2)| &\leq \frac{1}{8}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)\\
&\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1\\
&+\frac{1}{16\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)\\
&\times\Bigg|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\\
&\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\Bigg|ds_2ds_1dt_1\\
&\leq bA+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1\\
&+4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1\\
&+2r\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1\\
&\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),
\end{aligned}
$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$
\begin{aligned}
\left|\frac{\partial}{\partial t}Fv(t,x_1,x_2)\right| &\leq \frac{1}{4}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)g(t_1,s_1,s_2)\\
&\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1\\
&+\frac{1}{8\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)g(t_1,s_1,s_2)\\
&\times\Bigg|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\\
&\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\Bigg|ds_2ds_1dt_1\\
&\leq bA\\
&+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)\\
&\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1
\end{aligned}
$$

$$
\begin{aligned}
&+ 4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)\\
&\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1\\
&+ 2r\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)\\
&\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1\\
&\le A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),
\end{aligned}
$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial t^2}Fv(t,x_1,x_2)\right| &\le \frac{1}{4}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2g(t_1,s_1,s_2)\\
&\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1\\
&+\frac{1}{8\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2g(t_1,s_1,s_2)\\
&\times\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\right.\\
&\left.\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\right|ds_2ds_1dt_1\\
&\le bA+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2\\
&\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1\\
&+4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2\\
&\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1\\
&+2r\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2\\
&\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1\\
&\le A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),
\end{aligned}
$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$
\begin{aligned}
\left|\frac{\partial}{\partial x_1}Fv(t,x_1,x_2)\right| &\le \frac{1}{4}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)|x_1-s_1|(x_2-s_2)^2(t-t_1)^2g(t_1,s_1,s_2)\\
&\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1\\
&+\frac{1}{8\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)|x_1-s_1|(x_2-s_2)^2(t-t_1)^2g(t_1,s_1,s_2)
\end{aligned}
$$

$$\begin{aligned}
&\times \left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\right.\\
&\left.\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\right| ds_2ds_1dt_1\\
&\le bA\\
&+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)|x_1-s_1|(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1\\
&+4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)|x_1-s_1|(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1\\
&+2r\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)|x_1-s_1|(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1\\
&\le A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),
\end{aligned}$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x_1^2}Fv(t,x_1,x_2)\right| &\le \frac{1}{4}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_2-s_2)^2(t-t_1)^2g(t_1,s_1,s_2)\\
&\times |v(t_1,s_1,s_2)|ds_2ds_1dt_1\\
&+\frac{1}{8\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_2-s_2)^2(t-t_1)^2g(t_1,s_1,s_2)\\
&\times \left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\right.\\
&\left.\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\right| ds_2ds_1dt_1\\
&\le bA\\
&+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1\\
&+4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1\\
&+2r\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_2-s_2)^2(t-t_1)^2\\
&\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1
\end{aligned}$$

$$\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$. As above, one can obtain

$$\left|\frac{\partial}{\partial x_2}Fv(t,x_1,x_2)\right|\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$\left|\frac{\partial^2}{\partial x_2^2}Fv(t,x_1,x_2)\right|\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$. Consequently

$$\|Fv\|\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right).$$

This completes the proof. □

3 Proof of the Main Result

Let

$$\widetilde{\mathcal{P}}=\{u\in E: u\geq 0 \quad \text{on} \quad [0,\infty)\times\mathbb{R}^2\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{\mathcal{P}}$. Note that $Fv\geq 0$ for any $v\in\mathcal{P}$. Let $\epsilon>0$. For $v\in E$, define the operators

$$Tv(t,x)=(1+m\epsilon)v(t,x)-\epsilon\frac{L_1}{10},$$
$$Sv(t,x)=-\epsilon Fv(t,x)-m\epsilon v(t,x)-\epsilon\frac{L_1}{10},$$

$(t,x)\in[0,\infty)\times\mathbb{R}^2$. Note that any fixed point $v\in E$ of the operator $T+S$ is a solution to the IVP (2.2). Define

$$U_1=\mathcal{P}_{r_1}=\{v\in\mathcal{P}:\|v\|<r_1\},$$
$$U_2=\mathcal{P}_{L_1}=\{v\in\mathcal{P}:\|v\|<L_1\},$$
$$U_3=\mathcal{P}_{R_1}=\{v\in\mathcal{P}:\|v\|<R_1\},$$

$$R_2 = R_1 + \frac{A}{m}\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) + \frac{L_1}{5},$$
$$\Omega = \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \le R_2\}.$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|Tv_1 - Tv_2\| = (1+m\epsilon)\|v_1 - v_2\|,$$

whereupon $T : \Omega \to E$ is an expansive operator with a constant $1 + m\epsilon > 1$.

2. For $v \in \overline{\mathcal{P}_{R_1}}$, we get

$$\begin{aligned}\|Sv\| &\le \epsilon\|Fv\| + m\epsilon\|v\| + \frac{L_1}{10}\\ &\le \epsilon\left(A\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) + mR_1 + \frac{L_1}{10}\right).\end{aligned}$$

Therefore $S(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S : \overline{\mathcal{P}_{R_1}} \to E$ is continuous, we have that $S(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently $S : \overline{\mathcal{P}_{R_1}} \to E$ is a 0-set contraction.

3. Let $v_1 \in \overline{\mathcal{P}_{R_1}}$. Set

$$v_2 = v_1 + \frac{1}{m}Fv_1 + \frac{L_1}{5m}.$$

Note that by the second inequality of $(H3)$ and by Lemma 2.12, it follows that $\epsilon Fv + \epsilon\frac{L_1}{5} \ge 0$ on $[0,\infty)\times\mathbb{R}^2$. We have $v_2 \ge 0$ on $[0,\infty)\times\mathbb{R}^2$ and

$$\begin{aligned}\|v_2\| &\le \|v_1\| + \frac{1}{m}\|Fv_1\| + \frac{L_1}{5m}\\ &\le R_1 + \frac{A}{m}\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) + \frac{L_1}{5}\\ &= R_2.\end{aligned}$$

Therefore $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon F v_1 - \epsilon\frac{L_1}{10} - \epsilon\frac{L_1}{10}$$

or

$$(I-T)v_2 = -\epsilon m v_2 + \epsilon\frac{L_1}{10} = Sv_1.$$

Consequently $S(\overline{\mathcal{P}_{R_1}}) \subset (I-T)(\Omega)$.

4. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v - \lambda v_0) \in \mathcal{P}$, $v \in \partial\mathcal{P}_{r_1}$, $v \in \partial\mathcal{P}_{r_1} \bigcap(\Omega + \lambda u_0)$

and $Sv = v - \lambda v_0$ for some $\lambda \geq 0$. Then

$$\begin{aligned} r_1 &= \|v - \lambda v_0\| \\ &= \|Sv\| \\ &\geq -Sv(t,x) \\ &= \epsilon Fv(t,x) + \epsilon mv(t,x) + \epsilon\frac{L_1}{10} \\ &\geq \epsilon\frac{L_1}{20}, \quad (t,x) \in [0,\infty) \times \mathbb{R}^2, \end{aligned}$$

because by the second inequality of **(H3)** and by Lemma 2.12, it follows that $\epsilon Fv + \epsilon\frac{L_1}{20} \geq 0$ on $[0,\infty) \times \mathbb{R}^2$.

5. Suppose that for any $\epsilon_1 > 0$ small enough there exist a $u \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 u \in \overline{\mathcal{P}_{R_1}}$ and
$$Su = (I - T)(\lambda_1 u). \tag{3.1}$$
In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $u \in \partial\mathcal{P}_L$, $\lambda_1 u \in \overline{\mathcal{P}_{R_1}}$, $\lambda_1 \geq 1 + \epsilon_1$ and (3.1) holds. Since $u \in \partial\mathcal{P}_L$ and $\lambda_1 u \in \overline{\mathcal{P}_{R_1}}$, it follows that
$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1\|u\| \leq R_1.$$
Moreover,
$$-\epsilon Fu - m\epsilon u - \epsilon\frac{L}{10} = -\lambda_1 m\epsilon u + \epsilon\frac{L}{10},$$
or
$$Fu + \frac{L}{5} = (\lambda_1 - 1)mu.$$
From here,
$$2\frac{L}{5} \geq \left\|Fu + \frac{L}{5}\right\| = (\lambda_1 - 1)m\|u\| = (\lambda_1 - 1)mL$$
and
$$\frac{2}{5m} + 1 \geq \lambda_1,$$
which is a contradiction.

Therefore all conditions of Theorem 2.6 hold. Hence, the IVP (2.2) has at least two solutions v_1 and v_2 so that
$$r_1 < \|v_1\| < L_1 < \|v_2\| < R_1,$$
and
$$u = v_1 + u_0 + tu_1, \quad w = v_2 + u_0 + tu_1$$

are two different positive solutions of the IVP (1.1). This completes the proof.

4 An Example

Let

$$l = 1, \quad p_1 = \frac{3}{5}, \quad R_1 = r = 1, \quad a = 200, \quad L_1 = \frac{1}{2}, \quad r_1 = \frac{1}{100},$$
$$m = 10^{50}, \quad \epsilon = 50, \quad A = \frac{1}{10^{10}}, \quad R = 100.$$

Then

$$R_1 > \left(\frac{2}{5m} + 1q\right) L_1, \quad r_1 < L_1 < R_1, \quad r_1 < \frac{L_1}{20}.$$

Also,

$$\begin{aligned} A\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) &= \frac{1}{10^{10}}\left(1 + 800\cdot(4)^2 + 800\cdot 4 + 2\right) \\ &< \frac{1}{40} = \frac{L_1}{20}. \end{aligned}$$

Consequently **(H3)** holds. Now, we will construction the function g in **(H4)**. Let

$$h(x) = \log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}. \end{aligned}$$

Therefore

$$-\infty < \lim_{s\to\pm\infty}(1 + s + s^2)h(s) < \infty,$$
$$-\infty < \lim_{s\to\pm\infty}(1 + s + s^2)l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$(1 + s + s^2)\left(\frac{1}{44\sqrt{2}}\log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) \le C_1, \quad s \in \mathbb{R}.$$

Note that by [22, p. 707, Integral 79], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2(1+((1+s+s^2)I(s))^2)}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x_1,x_2) = Q(t)Q(x_1)Q(x_2), \quad t \in [0,\infty), \quad x_1,x_2 \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ so that

$$\begin{aligned} C_2 \geq & \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2) g_1(t_1,s_1,s_2) \\ & \times \left(1+|x_1-s_1|+(x_1-s_1)^2\right)\left(1+|x_2-s_2|+(x_2-s_2)^2\right) \\ & \times \left(1+(t-t_1)+(t-t_1)^2\right)\left(1+(1+t_1+t_1^2)I(t_1)\right) ds_2 ds_1 dt_1, \quad (t,x_1,x_2) \in [0,\infty)\times\mathbb{R}^2. \end{aligned}$$

Now, we take

$$g(t,x_1,x_2) = \frac{1}{10^{20}C_2} g_1(t,x_1,x_2), \quad (t,x_1,x_2) \in [0,\infty)\times\mathbb{R}^2.$$

Then

$$\begin{aligned} A &= \frac{1}{10^{10}} \\ &\geq \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2) g(t_1,s_1,s_2) \\ &\times \left(1+|x_1-s_1|+(x_1-s_1)^2\right)\left(1+|x_2-s_2|+(x_2-s_2)^2\right) \\ &\times \left(1+(t-t_1)+(t-t_1)^2\right)\left(1+(1+t_1+t_1^2)I(t_1)\right) ds_2 ds_1 dt_1, \quad (t,x_1,x_2) \in [0,\infty)\times\mathbb{R}^2. \end{aligned}$$

Now, consider the IVP

$$\begin{aligned} u_{tt} - u_{x_1x_1} - u_{x_2x_2} &= w(t)u^{\frac{3}{5}}, \quad (t,x_1,x_2) \in (0,\infty)\times\mathbb{R}^2, \\ u(0,x) &= u_t(0,x) = 0, \quad (x_1,x_2) \in \mathbb{R}^2, \end{aligned} \tag{4.1}$$

where

$$w(t) = \begin{cases} 10(9t^2-9t+2) & t \in [0,1] \\ \\ 20 & t > 1. \end{cases}$$

Here $l = 1$,

$$a_1(t, x_1, x_2) = |w(t)| \leq a = 200,$$
$$b_1(t, x_1, x_2) = c_1(t, x_1, x_2) = d_1(t, x_1, x_2) = 0,$$

$(t, x_1, x_2) \in [0, \infty) \times \mathbb{R}^2$, and

$$u_0(x) = u_1(x) = 0 \leq 1 = r, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We have that **(H1)** and **(H2)** hold. The IVP (4.1) has two nonnegative solutions $u_1(t, x) = 0$, $(t, x) \in [0, \infty) \times \mathbb{R}^2$, and

$$u_2(t, x) = \begin{cases} (t(1-t))^5 \quad (t, x) \in [0, 1] \times \mathbb{R}^2 \\ \\ 0 \quad (t, x) \in (1, \infty) \times \mathbb{R}^2. \end{cases}$$

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References

[1] S. Alinhac, "The null condition for quasilinear wave equations in two space dimensions I", Invent. Math., vol. 145, no. 3, pp. 597–618, 2001.

[2] L. Benzenati, S. G. Georgiev and K. Mebarki, "Existence of positive solutions for some kinds of BVPs in Banach spaces", submitted for publication.

[3] L. Benzenati and K. Mebarki, "Multiple positive fixed points for the sum of expansive mappings and k-set contractions", Math. Methods Appl. Sci., vol. 42, no. 13, pp. 4412–4426, 2019.

[4] L. Benzenati, K. Mebarki and R. Precup, "A vector version of the fixed point theorem of cone compression and expansion for a sum of two operators", Nonlinear Stud., vol. 27, no. 3, pp. 563–575, 2020.

[5] D. Christodoulou, "Global solutions of nonlinear hyperbolic equations for small data", Comm. Pure Appl. Math., vol. 39, no. 2, pp. 267–282, 1986.

[6] K. Deimling, *Nonlinear functional Analysis*, Heidelberg: Springer Berlin, 1985.

[7] S. Djebali and K. Mebarki, "Fixed Point Theory for Sums of Operators", J. Nonlinear Convex Anal., vol. 19, no. 6, pp. 1029–1040, 2018.

[8] S. Djebali and K. Mebarki, "Fixed point index for expansive perturbation of k-set contraction mappings", Topol. Methods Nonlinear Anal., vol. 54, no. 2, pp. 613–640, 2019.

[9] D. Duffy, *Green's function with Applications*, 1st edition, Boca Raton: Chapman & Hall/CRC Press, 2001.

[10] S. G. Georgiev and Z. Khaled, *Multiple fixed-point theorems and applications in the theory of ODEs, FDEs and PDEs*, Monographs and research notes in mathematics, Boca Raton: CRC Press, 2020.

[11] S. Georgiev, A. Kheloufi and K. Mebarki, "Classical solutions for the Korteweg-De Vries equation", New Trends in Nonlinear Analysis and Applications, to be published.

[12] S. G. Georgiev and K. Mebarki, "Existence of positive solutions for a class ODEs, FDEs and PDEs via fixed point index theory for the sum of operators", Comm. Appl. Nonlinear Anal., vol. 26, no. 4, pp. 16–40, 2019.

[13] S. G. Georgiev and K. Mebarki, "On fixed point index theory for the sum of operators and applications in a class ODEs and PDEs", submitted for publication.

[14] S. Georgiev and K. Mebarki, "Leggett-Williams fixed point theorem type for sums of two operators and application in PDEs", Differ. Equ. Appl., vol. 13, no. 3, pp. 321–344, 2021.

[15] S. G. Georgiev, K. Mebarki and Kh. Zennir, "Existence of solutions for a class of nonlinear hyperbolic equations", submitted for publication.

[16] S. G. Georgiev, K. Mebarki and Kh. Zennir, "Existence of solutions for a class IVP for nonlinear wave equations", submitted for publication.

[17] N. M. Hung, "Asymptotic behaviour of solutions of the first boundary-value problem for strongly hyperbolic systems near a conical point at the boundary of the domain", Sb. Math., vol. 190, no. 7, pp. 1035–1058, 1999.

[18] S. Ibrahim, M. Majdoub and N. Masmoudi, "Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity", Comm. Pure Appl. Math., vol. 59, no. 11, pp. 1639–1658, 2006.

[19] F. John and S. Klainerman, "Almost global existence to nonlinear wave equations in three space dimensions", Comm. Pure Appl. Math., vol. 37, no. 4, pp. 443–455, 1984.

[20] S. Klainerman, "The null condition and global existence to nonlinear wave equations", Lectures in Appl. Math., vol. 23, pp. 293–326, 1986.

[21] Z. Lei, T. C. Sideris and Y. Zhou, "Almost Global existence for two dimensional incompressible isotropic elastodynamics", Trans. Amer. Math. Soc., vol. 367, no. 11, pp. 8175–8197, 2015.

[22] A. Polyanin and A. Manzhirov, *Handbook of integral equations*, Boca Raton: CRC Press, 1998.

[23] C. D. Sogge, *Lectures on nonlinear wave equations*, 2nd Edition, Boston: International press, Inc., 2013.

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Existence of positive solutions for a nonlinear semipositone boundary value problems on a time scale

Saroj Panigrahi[1, ✉]
Sandip Rout[1]

[1] *School of Mathematics and Statistics, University of Hyderabad, Hyderabad, 500 046, India.*
panigrahi2008@gmail.com ✉
sandiprout7@gmail.com

ABSTRACT

In this paper, we are concerned with the existence of positive solution of the following semipositone boundary value problem on time scales:

$$(\psi(t)y^{\Delta}(t))^{\nabla}+\lambda_1 g(t,\, y(t))+\lambda_2 h(t,\, y(t)) = 0,\; t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}},$$

with mixed boundary conditions

$$\alpha y(\rho(c)) - \beta\psi(\rho(c))y^{\Delta}(\rho(c)) = 0,$$
$$\gamma y(\sigma(d)) + \delta\psi(d)y^{\Delta}(d) = 0,$$

where $\psi : C[\rho(c),\, \sigma(d)]_{\mathbb{T}}$, $\psi(t) > 0$ for all $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$; both g and $h : [\rho(c),\, \sigma(d)]_{\mathbb{T}} \times [0,\, \infty) \to \mathbb{R}$ are continuous and semipositone. We have established the existence of at least one positive solution or multiple positive solutions of the above boundary value problem by using fixed point theorem on a cone in a Banach space, when g and h are both superlinear or sublinear or one is superlinear and the other is sublinear for $\lambda_i > 0$; $i = 1, 2$ are sufficiently small.

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RESUMEN

En este artículo estudiamos la existencia de soluciones positivas del siguiente problema de valor de frontera semipositón en escalas de tiempo:

$$(\psi(t)y^{\Delta}(t))^{\nabla}+\lambda_1 g(t,\, y(t))+\lambda_2 h(t,\, y(t)) = 0,\; t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}},$$

con condiciones de frontera mixtas

$$\begin{aligned}\alpha y(\rho(c)) - \beta\psi(\rho(c))y^{\Delta}(\rho(c)) &= 0,\\ \gamma y(\sigma(d)) + \delta\psi(d)y^{\Delta}(d) &= 0,\end{aligned}$$

donde $\psi : C[\rho(c),\, \sigma(d)]_{\mathbb{T}}$, $\psi(t) > 0$ para todo $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$; ambas g y $h : [\rho(c),\, \sigma(d)]_{\mathbb{T}} \times [0,\, \infty) \to \mathbb{R}$ son continuas y semipositón. Hemos establecido la existencia de al menos una solución positiva o múltiples soluciones positivas del problema de valor en la frontera anterior usando un teorema de punto fijo en un cono en un espacio de Banach, cuando g y h son ambas superlineales o sublineales o una es superlineal y la otra es sublineal para $\lambda_i > 0$; $i = 1,\, 2$ suficientemente pequeños.

Keywords and Phrases: Positive solutions, boundary value problems, fixed point theorem, cone, time scales.

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1 Introduction

The study of dynamic equations on time scales goes to the seminal work of Stefan Hilger [11] and has received a lot of attention in recent years. Time scales were created to unify the study of continuous and discrete mathematics and particularly used in differential and difference equations. We are interested to prove the results for a dynamic equation where the domain of the unknown function is a time scale $\mathbb{T}$, which is a non-empty closed subset of real numbers $\mathbb{R}$.

We consider the second order semipositone boundary value problem on time scales:

$$(\psi(t)y^{\Delta}(t))^{\nabla} + \lambda_1 g(t,\, y(t)) + \lambda_2 h(t,\, y(t)) = 0,\; t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}, \tag{1.1}$$

with mixed boundary conditions

$$\begin{aligned} \alpha y(\rho(c)) - \beta\psi(\rho(c))y^{\Delta}(\rho(c)) &= 0, \\ \gamma y(\sigma(d)) + \delta\psi(d)y^{\Delta}(d) &= 0, \end{aligned} \tag{1.2}$$

where λ_1 and λ_2 are positive and

(H_1) $\psi : C[\rho(c),\, \sigma(d)]_{\mathbb{T}}$, $\psi(t) > 0$ for all $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$;

(H_2) $\alpha,\, \beta,\, \gamma,\, \delta, \geq 0$ and $\alpha\delta + \beta\gamma + \alpha\gamma > 0$;

(H_3) g and h : $[\rho(c),\, \sigma(d)]_{\mathbb{T}} \times [0,\, \infty) \to \mathbb{R}$ are continuous satisfying with both g and h are semipositone.

D. R. Anderson and P. Y. Wong [1], have established the existence result for the SL-BVP (1.1) and (1.2) where g is superlinear such that $g(t,\, y) \geq -M$ for some constant $M > 0$ and λ is in some interval of $\mathbb{R}$ with $h(t,\, y) = 0$. They did not establish any results concerning the existence of positive solutions for the boundary value problem (1.1) and (1.2), when g is sublinear. Many findings have also been obtained for the existence of positive solution of the boundary value problem (1.1) and (1.2), when $h(t,\, y) = 0$, but only a few results have been established for the existence of positive solutions when $h(t,\, y) \neq 0$. Motivated by the work of [1] and the references cited therein, we would like to establish the sufficient conditions for the existence of positive solution of the boundary value problem (1.1) and (1.2), when g and h are both superlinear or sublinear or one is superlinear and the other is sublinear for $\lambda_i > 0$; $i = 1,\, 2$ are sufficiently small.

It is worthy of mention that results of this paper not only apply to the set of real numbers or the set of integers but also to more general time scales such as $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$, etc. For basic notations and concepts on time scale calculus, we refer the readers to monographs [5, 6] and references cited therein. The study of nonlinear, semipositone boundary value problem has considerable importance even in differential equations. In recent years, several researchers studied semipositone boundary value problem on time scales [1, 2, 4, 7, 10, 16, 17]. Semipositone problems arise in many physical and chemical processes such as in chemical reactor theory, astrophysics, gas dynamics and fluidmechanics, relativistic mechanics, nuclear physics, design of suspension bridges, bulking of mechanical systems, combustion and management of natural resources (see [3, 9, 12, 15]). Let a and b such that $0 \leq \rho(a) \leq a < b \leq \sigma(b) < \infty$ and $(\rho(a),\ \sigma(b))_{\mathbb{T}}$ has at least two points.

The plan of the paper is as follows. In Section 2, we provided some preliminary results concerning the Green's function for the homogeneous boundary value problem and some important Lemmas. These results allow us in Section 3 to discuss the existence of at least one or multiple positive solutions. Finally, in Section 4, we illustrate few examples to justify the results obtained in the previous section.

2 Preliminaries

In this section, we have obtained some basic results related to Green's function for the homogeneous boundary value problem and some important Lemmas.

Now let us consider the homogenoeous dynamic boundary value problem

$$(\psi(t)y^{\Delta}(t))^{\nabla} = 0,\ t \in [\rho(c),\ \sigma(d)]_{\mathbb{T}}, \tag{2.1}$$

with boundary conditions (1.2). Green's function $\mathcal{G}(t,\ s)$ (see [7]) for the boundary value problem (2.1) and with the boundary conditions (1.2) is given by

$$\mathcal{G}(t,\ s) = \frac{1}{\varphi}\begin{cases} \left(\beta + \alpha\int_{\rho(c)}^{t}\frac{\nabla\tau}{\psi(\tau)}\right)\left(\delta + \gamma\int_{s}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)}\right),\ \rho(c) \leq t \leq s \leq \sigma(d), \\ \\ \left(\beta + \alpha\int_{\rho(c)}^{s}\frac{\nabla\tau}{\psi(\tau)}\right)\left(\delta + \gamma\int_{t}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)}\right),\ \rho(a) \leq s \leq t \leq \sigma(b), \end{cases} \tag{2.2}$$

where

$$\varphi = \alpha\delta + \beta\gamma + \alpha\gamma\int_{\rho(c)}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)} > 0.$$

Lemma 2.1 ([17])**.** *Assume (H_1) and (H_2) hold. Then the Green function $\mathcal{G}(t,\ s)$ satisfies*

$$(\psi(t)y^{\Delta}(t))^{\nabla} + Q(t) = 0, \quad t \in (\rho(c),\ \sigma(d))_{\mathbb{T}}, \tag{2.3}$$

with mixed boundary conditions (1.2), where $Q \in C_{rd}[\rho(c),\ \sigma(d)]_{\mathbb{T}}$, $Q(t) \geq 0$; then

$$y(t) \geq q(t)\|y\|,\ t \in [\rho(c),\ \sigma(d)]_{\mathbb{T}},\ s \in [a,\ b]_{\mathbb{T}}, \tag{2.4}$$

where $q(t)$ is given by

$$q(t)=\min\left\{\frac{\beta+\alpha\int_{\rho(c)}^{t}\frac{\nabla\tau}{\psi(\tau)}}{\beta+\alpha\int_{\rho(c)}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)}},\ \frac{\delta+\gamma\int_{t}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)}}{\delta+\gamma\int_{\rho(c)}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)}}\right\}.$$

Lemma 2.2 ([1])**.** *For all $t\in[\rho(c),\,\sigma(d)]_{\mathbb{T}}$ and $s\in[c,\,d]_{\mathbb{T}}$, then*

$$q(t)\mathcal{G}(s,\,s)\le\mathcal{G}(t,\,s)\le\mathcal{G}(s,\,s),\tag{2.5}$$

where $\mathcal{G}(t,\,s)$ is given in (2.2) and $q(t)$ is defined as in Lemma 2.1.

Lemma 2.3 ([1])**.** *Let (H_1) and (H_2) hold and let y_1 be the solution of*

$$(\psi(t)y^{\Delta}(t))^{\nabla}+1=0,\ t\in(\rho(c),\,\sigma(d))_{\mathbb{T}},\tag{2.6}$$

with mixed boundary conditions (1.2), then there exists a positive constant C such that

$$y_1(t)\le C\,q(t),\ t\in[\rho(c),\,\sigma(d)]_{\mathbb{T}},\tag{2.7}$$

where

$$C=\frac{1}{\varphi}(\sigma(d)-\rho(c))\left(\beta+\alpha\int_{\rho(c)}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)}\right)\left(\delta+\gamma\int_{\rho(c)}^{\sigma(d)}\frac{\nabla\tau}{\psi(\tau)}\right).$$

Lemma 2.4 ([8]). *Let $\lim\limits_{y\to\infty}\dfrac{g(t,\,y)}{y}=\infty$ and define $\mathbb{G}:[0,\,\infty)\to[0,\,\infty)$ by*

$$\mathbb{G}=\max_{\rho(c)\le t\le\sigma(d),\,0\le y\le r}g(t,\,y).\tag{2.8}$$

Then

- (I) $\mathbb{G}$ *is non-decreasing;*
- (II) $\lim\limits_{r\to\infty}\dfrac{\mathbb{G}(r)}{r}=\infty$*;*
- (III) *there exists $r^*>0$ such that $\mathbb{G}(r)>0$ for $r\ge r^*$.*

Lemma 2.5 ([8])**.** *Let $lim_{y\to\infty}\dfrac{g(t,\,y)}{y}=0$ holds. Then $\mathbb{G}$ defined by (2.8) is a nondecreasing function, such that*

$$\lim_{r\to\infty}\frac{\mathbb{G}(r)}{r}=0.$$

Define a function for $y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}$,

$$\overline{g}(t,\,y)=\begin{cases}g(t,\,y),\ y\ge0,\\ g(t,\,0),\ y<0.\end{cases}$$

and

$$\overline{h}(t,\, y) = \begin{cases} h(t,\, y),\, y \geq 0, \\ h(t,\, 0),\, y < 0. \end{cases}$$

Let us consider the nonlinear boundary value problem:

$$(\psi(t)y^{\Delta})^{\nabla} = -[\lambda_1\overline{g}(t,\, y-x) + \lambda_2\overline{h}(t,\, y-x) + M], \tag{2.9}$$

with boundary conditions (1.2).

Lemma 2.6. *Assume that $x(t) = My_1(t)$, where $y_1(t)$ is a unique solution of the boundary value problem (2.6) and (1.2). Then $y(t)$ is a solution of the boundary value problem (1.1) and (1.2) if and only if $\overline{y}(t) = y(t) + x(t)$ is a positive solution of the boundary value problem (2.9) and (1.2) with $\overline{y}(t) > x(t)$ for $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$.*

Proof. Let us assume that $\overline{y}(t)$ is a solution of the boundary value problem (2.9) and (1.2) such that $\overline{y}(t) \geq x(t)$ for any $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$. Let $y(t) = \overline{y}(t) - x(t) > 0$ on $[\rho(c),\, \sigma(d)]_{\mathbb{T}}$ as $\overline{y}(t) \geq x(t)$. Now, for any $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$, we have

$$(\psi(t)\overline{y}^{\Delta}(t))^{\nabla} + [\lambda_1\overline{g}(t,\, (\overline{y}(t) - x(t)) + \lambda_2\overline{h}(t,\, (\overline{y}(t) - x(t))) + M] = 0,$$

that is,

$$(\psi(t)y^{\Delta}(t))^{\nabla} + (\psi(t)x^{\Delta}(t))^{\nabla} + [\lambda_1\overline{g}(t,\, (\overline{y}(t) - x(t)) + \lambda_2\overline{h}(t,\, (\overline{y}(t) - x(t))) + M] = 0.$$

By using the definition of y together with the definition of x, we have

$$(\psi(t)y^{\Delta}(t))^{\nabla} + [\lambda_1\overline{g}(t,\, y(t)) + \lambda_2\overline{h}(t,\, y(t)) + M] + M(\psi(t)y_1^{\Delta}(t))^{\nabla}(t) = 0.$$

Thus,

$$(\psi(t)y^{\Delta}(t))^{\nabla} + \lambda_1\overline{g}(t,\, y(t)) + \lambda_2\overline{h}(t,\, y(t)) = 0.$$

On the other hand,

$$\begin{aligned} &\alpha y(\rho(c)) - \beta\psi(\rho(c))y(\rho(c)) \\ &= (\alpha\overline{y}(\rho(c)) - \beta\psi(\rho(c))\overline{y}^{\Delta}(\rho(c))) - (\alpha x(\rho(c)) - \beta\psi(\rho(c))x^{\Delta}(\rho(c))) \\ &= (\alpha\overline{y}(\rho(c)) - \beta\psi(\rho(c))\overline{y}^{\Delta}(\rho(c))) - M(\alpha y_1(\rho(c)) - \beta\psi(\rho(c))y_1^{\Delta}(\rho(c))) = 0, \end{aligned}$$

and

$$\gamma y(\sigma(d)) + \delta\psi(d)y^{\Delta}(d) = \gamma\overline{y}(\sigma(d)) + \delta\psi(d)\overline{y}^{\Delta}(d) - (\gamma x(\sigma(d)) + \delta\psi(d)x^{\Delta}(d))$$

$$= \gamma\overline{y}(\sigma(d)) + \delta\psi(d)\overline{y}^{\Delta}(d) - M(\gamma y_1(\sigma(d)) + \delta\psi(d)y_1^{\Delta}(d)) = 0.$$

Hence, $y(t)$ is a solution of the boundary value problem (1.1) and (1.2). Hence this completes the proof of the lemma. □

Let us define a Banach space

$$E = \{y : C[\rho(c),\, \sigma(d)]_{\mathbb{T}} \to \mathbb{R}\}$$

endowed with the norm

$$\|y\| = \max\{|y(t)|,\, t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}\}.$$

Define a cone K on E by

$$K = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : y(t) \geq q(t)\|y\|,\, t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}\},$$

where $q(t)$ is defined as in Lemma 2.1. Let us define an operator T_λ on K by

$$T_\lambda y(t) = \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s. \tag{2.10}$$

Lemma 2.7. *Assume that (H_1)–(H_3) hold. Then $T_\lambda(K) \subset K$ and $T_\lambda : K \to K$ is a completely continuous operator.*

Proof. First we show that $T_\lambda(K) \subset K$. Let $y \in K$ and $t \in [\rho(a),\ \sigma(b)]_{\mathbb{T}}$. Note that

$$(T_\lambda y)(t) = \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s,$$

that is,

$$(T_\lambda y)(t) \leq \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s.$$

Hence,

$$\|T_\lambda y\| \leq \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s.$$

By use of the Lemma (2.2), we obtain

$$(T_\lambda y)(t) \geq q(t)\int_{\rho(c)}^{d} \mathcal{G}(s,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s,$$

which implies

$$(T_\lambda y)(t) \geq q(t)\|T_\lambda y\|.$$

Thus, $T_\lambda(K) \subset K$. Since f and g are continuous, it shows that T_λ is continuous and by the

Arzelà-Ascoli Theorem [14], it is easy to verify that T_λ is a completely continuous operator. Hence this completes the proof of the lemma $\square$

Lemma 2.8 ([13]). *Let E be a real Banach space, and let $K \subset E$ be a cone. Let Ω_1, Ω_2 be two bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Assume that $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

$$\|Ty\| \leq \|y\| \ \text{ for all } \ y \in K \cap \partial\Omega_1 \ \text{ and and } \|Ty\| \geq \|y\| \ \text{ for all } \ y \in K \cap \partial\Omega_2,$$

or

$$\|Ty\| \geq \|y\| \ \text{ for all } \ y \in K \cap \partial\Omega_1 \ \text{ and } \ \|Ty\| \leq \|y\| \ \text{ for all } \ y \in K \cap \partial\Omega_2,$$

then T has at least one fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let us define the following:

(L_1) $\lim\limits_{y\to\infty} \dfrac{g(t, y)}{y} = \infty$;

(L_2) $\lim\limits_{y\to\infty} \dfrac{g(t, y)}{y} = 0$;

(L_3) $\lim\limits_{y\to 0} \dfrac{g(t, y)}{y} = 0$;

(L_4) $\lim\limits_{y\to 0} \dfrac{g(t, y)}{y} = \infty$;

(L_5) $\lim\limits_{y\to\infty} \dfrac{h(t, y)}{y} = \infty$;

(L_6) $\lim\limits_{y\to\infty} \dfrac{h(t, y)}{y} = 0$;

(L_7) $\lim\limits_{y\to 0} \dfrac{h(t, y)}{y} = 0$;

(L_8) $\lim\limits_{y\to 0} \dfrac{h(t, y)}{y} = \infty$.

Note that the limits $(L_i), i \in \mathbb{N}_1^8$, are assumed to be inform with respect t.

We would like to establish the existence of solutions for the boundary value problem (1.1) and (1.2) under the following cases:

(I) L_1 and L_5;

(II) L_1 and L_6;

(III) L_1 and L_7;

(IV) L_2 and L_5;

(V) L_2 and L_6;

(VI) L_2 and L_8;

(VII) L_3 and L_5;

$(VIII)$ L_3 and L_7;

(IX) L_3 and L_8;

(X) L_4 and L_6;

(XI) L_4 and L_7;

(XII) L_4 and L_8.

Remark 2.9. *We fails to apply the Lemma 2.8 for the pairs such as $(XIII)$ L_1 and L_8, (XIV) L_2 and L_7, (XV) L_3 and L_6 & (XVI) L_4 and L_5.*

3 Main Results

Theorem 3.1. *Let (H_1)–(H_3), (L_1) and (L_5) hold. Then the boundary value problem (1.1) and (1.2) has a positive solution for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. Let λ_1 and λ_2 satisfy

$$0 < \lambda_1 + \lambda_2 < \frac{1}{\max\limits_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_1}} g(t, y) + \max\limits_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_1}} h(t, y)}, \tag{3.1}$$

where $r_1 = \max\{(M+1)\|y_1\|, r^*, CM\}$, C and r^* are defined as in Lemma 2.3 and Lemma 2.4, respectively and y_1 be the solution of (1.2) and (2.6). Define $\Omega_{r_1} = \{y \in C[\rho(c), \sigma(d)]_{\mathbb{T}} : \|y\| < r_1\}$. For $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t, s)[\lambda_1 \overline{g}(s, y-x) + \lambda_2 \overline{h}(s, y-x) + M]\nabla s \\
&\leq (\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_1}} g(t, y) + \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_1}} h(t, y)\right)\int_{\rho(c)}^{d} \mathcal{G}(t, s)\nabla s + \int_{\rho(c)}^{d} \mathcal{G}(t, s) M\nabla s \\
&= \left((\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_1}} g(t, y) + \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_1}} h(t, y)\right) + M\right) y_1(t) \\
&\leq (1+M) y_1(t) \\
&\leq r_1 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \leq \|y\| \quad \text{for } y \in K \cap \partial\Omega_{r_1}. \tag{3.2}$$

Let us choose a constant $\overline{M} > 0$ such that

$$\frac{1}{2}\overline{M}(\lambda_1+\lambda_2)\mu\left(\min_{t_1\leq t\leq t_2}\int_{t_1}^{t_2} \mathcal{G}(t, s)\nabla s\right) \geq 1, \tag{3.3}$$

where

$$\mu = \min_{t_1\leq s\leq t_2} q(s). \tag{3.4}$$

From (L_1) and (L_5), we have for same $\overline{M} > 0$ there exists a constant $l > 0$ such that

$$\begin{aligned}
g(t, y) &\geq \overline{M} y \quad \text{for} \quad y \in [l, \infty), \\
h(t, y) &\geq \overline{M} y \quad \text{for} \quad y \in [l, \infty).
\end{aligned}$$

Now set $r_2 = \max\left\{2r_1, 2CM, \frac{2l_1}{\mu}\right\}$. Define $\Omega_{r_2} = \{y \in C[\rho(c), \sigma(d)]_{\mathbb{T}} : \|y\| < r_2\}$. For $y \in$

$K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned} y(s) - x(s) &= y(s) - My_1(s) \\ &\geq y(s) - MCq(s) \\ &\geq y(s) - \frac{CM}{\|y\|} y(s) \\ &\geq y(s) - \frac{CM}{r_2} y(s) \\ &\geq \frac{1}{2} y(s), \end{aligned}$$

and

$$\begin{aligned} \min_{t_1 \leq s \leq t_2} (y(s) - x(s)) &\geq \min_{t_1 \leq s \leq t_2} \frac{y(s)}{2} \\ &\geq \min_{t_1 \leq s \leq t_2} \frac{\|y\|}{2} q(s) \\ &= \frac{r_2 \mu}{2} \\ &\geq l. \end{aligned}$$

For $y \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned} \min_{t \in [t_1,\, t_2]} (T_\lambda y)(t) &= \min_{t \in [t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y - x) + \lambda_2 \overline{h}(s,\, y - x) + M] \nabla s \\ &\geq \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y - x) + \lambda_2 \overline{h}(s,\, y - x) + M] \nabla s \\ &\geq \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)(\lambda_1 + \lambda_2) \overline{M} (y(s) - x(s)) \nabla s \\ &\geq \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)(\lambda_1 + \lambda_2) \overline{M} \frac{y(s)}{2} \nabla s \\ &\geq \frac{1}{2} (\lambda_1 + \lambda_2) \overline{M} \mu \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s) \|y\| \nabla s \\ &\geq \|y\|. \end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_2}. \tag{3.5}$$

By Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1 \leq \|\overline{y}\| \leq r_2$. By use of the Lemma 2.3, it follows that

$$\begin{aligned}\overline{y}(t) &\geq r_1 q(t)\\ &\geq r_1 \frac{y_1(t)}{C}\\ &\geq M y_1(t)\\ &= x(t).\end{aligned}$$

Hence, $y = \overline{y} - x$ is a positive solution of the boundary value problem (1.1) and (1.2). This completes the proof of the theorem. □

Theorem 3.2. *Let (H_1)–(H_3), (L_4) and (L_8) hold. Then the boundary value problem (1.1) and (1.2) has a positive solution for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1, hence it is omitted. □

Theorem 3.3. *Assume that (H_1), (H_2), (L_2) and (L_6) hold. Let there exist two constant $D > 0$ and $\eta > 0$ such that*

$$\begin{aligned}g(t, y) \geq \eta \quad &\textit{for} \quad t \in [\rho(c), \sigma(d)],\, y \in [D, \infty),\\ h(t, y) \geq \eta \quad &\textit{for} \quad t \in [\rho(c), \sigma(d)],\, y \in [D, \infty),\end{aligned}$$

then the boundary value problem (1.1) and (1.2) has a positive solution for λ_i, $i = 1, 2$ are sufficiently small.

Proof. Set

$$r_1 = \max\left\{\frac{2D}{\mu},\ 2MC\right\}, \tag{3.6}$$

and

$$A = 2r_1 \left(\min_{t_1 \leq t \leq t_2} \int_{t_1}^{t_2} \mathcal{G}(t, s)(\lambda_1 + \lambda_2)\eta \nabla s\right)^{-1}, \tag{3.7}$$

where $\mu = \min\limits_{t_1 \leq s \leq t_2} q(s)$. Our claim is that for $\lambda_i \in [A, \infty)$, $i = 1, 2$, the boundary value problem (1.1) and (1.2) has a positive solution. Define $\Omega_{r_1} = \{y \in C[\rho(c), \sigma(d)]_{\mathbb{T}} : \|y\| < r_1\}$. For $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}y(s) - x(s) &= y(s) - M y_1(s)\\ &\geq y(s) - MCq(s)\\ &\geq y(s) - \frac{CM}{r_3} y(s)\\ &\geq \frac{1}{2} y(s),\end{aligned}$$

and

$$\begin{aligned}\min_{t_1\le s\le t_2}(y(s)-x(s)) &\ge \min_{t_1\le s\le t_2}\frac{y(s)}{2}\\ &\ge \min_{t_1\le s\le t_2}\frac{\|y\|}{2}q(s)\\ &= \frac{r_1\mu}{2}\\ &\ge D.\end{aligned}$$

For $y\in K\cap\partial\Omega_{r_1}$, we have

$$\begin{aligned}\min_{t\in[t_1,\,t_2]}(T_\lambda y)(t) &= \min_{t\in[t_1,\,t_2]}\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M]\nabla s\\ &\ge \min_{t\in[t_1,\,t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\,s)[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M]\nabla s\\ &\ge \min_{t\in[t_1,\,t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\,s)(\lambda_1+\lambda_2)\eta\nabla s\\ &= r_1=\|y\|.\end{aligned}$$

Thus,

$$\|T_\lambda y\|\ge\|y\| \quad \text{for} \quad y\in K\cap\partial\Omega_{r_1}. \tag{3.8}$$

From (L_2) and (L_6), we have

$$\begin{aligned}\overline{g}(t,\,y)\le\epsilon y \quad &\text{for} \quad t\in[\rho(c),\,\sigma(d)],\ y\ge l,\\ \overline{h}(t,\,y)\le\epsilon y \quad &\text{for} \quad t\in[\rho(c),\,\sigma(d)],\ y\ge l,\end{aligned}$$

On the other hand, by use of the Lemma 2.4, there exists a $R>0$ such that

$$R>\max\left\{2r_1,\ \max_{\rho(c)\le t\le\sigma(d)}\int_{\rho(c)}^{d}[\mathcal{G}(t,\,s)M+1]\nabla s\right\}.$$

and ϵ satisfies

$$\max_{\rho(c)\le t\le\sigma(d)}\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)[\epsilon\lambda_1 R+\epsilon\lambda_2 R+M]\nabla s\le R.$$

Let

$$\Omega_R=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<R\}.$$

For $y \in K \cap \partial\Omega_R$, we have

$$\begin{aligned} T_\lambda y(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M]\nabla s \\ &\le \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1 \epsilon R + \epsilon \lambda_2 R + M]\nabla s \\ &\le R = \|y\|. \end{aligned}$$

Thus,

$$\|T_\lambda y\| \le \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_R. \tag{3.9}$$

By Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1 \le \|\overline{y}\| \le R$. It follows that

$$\begin{aligned} \overline{y}(t) &\ge r_1 q(t) \\ &\ge r_1 \frac{y_1(t)}{C} \\ &\ge 2M y_1(t) \\ &\ge x(t). \end{aligned}$$

Hence, $y = \overline{y} - x$ is a positive solution of the boundary value problem (1.1) and (1.2). This completes the proof of the theorem. □

Theorem 3.4. *Assume that (H_1)–(H_3), (L_3) and (L_7) hold. Let there exist two constant $D > 0$ and $\eta > 0$ such that*

$$\begin{aligned} g(t,\, y) \ge \eta \quad &\textit{for} \quad t \in [\rho(c),\, \sigma(d)]\, y \in [D,\, l], \\ h(t,\, y) \ge \eta \quad &\textit{for} \quad t \in [\rho(c),\, \sigma(d)]\, y \in [D,\, l], \end{aligned}$$

then the boundary value problem (1.1) and (1.2) has a positive solution for λ_i, $i = 1, 2$ are sufficiently small.

Proof. The proof of the Theorem 3.4 is similar to that of Theorem 3.3, hence it is omitted. □

Theorem 3.5. *Let (H_1)–(H_3), (L_1) and (L_6) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $1 = 1, 2$ are sufficiently small.*

Proof. If (L_6) holds, then by the Lemma 2.5, there exists a constant $r_1 > 0$ such that

$$\mathbb{G}(r_1) \le N r_1.$$

Since λ_1 and λ_2 are sufficiently small, we have

$$\left[\lambda_1 \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\,y)+\lambda_2 \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\,y)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\le r_1.$$

Let $\Omega_{r_1}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_1\}$. For $y\in\partial\Omega_{r_1}$, we have

$$\begin{aligned}(T_\lambda y)(t)&=\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)\left[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\right]\nabla s\\ &\le\left(\lambda_1 \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\,y)+\lambda_2 \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\,y)\right)\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\\ &\quad+\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)M\nabla s\le r_1=\|y\|.\end{aligned}$$

Thus,

$$\|T_\lambda y\|\le\|y\|\quad\text{for all}\quad y\in K\cap\partial\Omega_{r_1}. \tag{3.10}$$

From (L_1), we have

$$g(t,\,y)>N_1 y\quad\text{for all}\quad y\le l.$$

Let $r_2=\max\left\{2CM,\frac{2l}{\mu},2r_1\right\}$ and $\Omega_{r_2}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_2\}$. For $y\in\partial K\cap\Omega_{r_2}$, we have

$$\begin{aligned}y(s)-x(s)&=y(s)-My_1(s)\\ &\ge y(s)-MCq(s)\\ &\ge y(s)-\frac{CM}{r_5}y(s)\\ &\ge\frac{1}{2}y(s),\end{aligned}$$

and

$$\begin{aligned}\min_{t_1\le s\le t_2}(y(s)-x(s))&\ge\min_{t_1\le s\le t_2}\frac{y(s)}{2}\\ &\ge\min_{t_1\le s\le t_2}\frac{\|y\|}{2}q(s)\\ &=\frac{r_2\mu}{2}\\ &\ge l.\end{aligned}$$

For $y \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
\min_{t\in[t_1,\, t_2]} (T_\lambda y)(t) &= \min_{t\in[t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M]\nabla s \\
&\geq \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M]\nabla s \\
&\geq \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\lambda_1 N_1(y(s)-x(s))\nabla s \\
&= r_2 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for all} \quad y \in K \cap \partial\Omega_{r_2}. \tag{3.11}$$

Let

$$R = \max\left\{ \left(\lambda_1 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} g(t,\, y) + \lambda_2 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} h(t,\, y) + M \right) \left(\int_{\rho(c)}^{d} \mathcal{G}(s,\, s)\nabla s \right), 2r_2 \right\},$$

then $r_1 < r_2 < R$. Let $\Omega_R = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < R\}$. For $y \in K \cap \Omega_R$, $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$, we have

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)\left[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right]\nabla s \\
&\leq \left(\lambda_1 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} g(t,\, y) + \lambda_2 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} h(t,\, y) + M \right) \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)\nabla s \\
&\leq R = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \leq \|y\| \quad \text{for all} \quad y \in K \cap \partial\Omega_R. \tag{3.12}$$

Thus by the Lemma 2.8, T_λ has at least two fixed points. Hence, the boundary value problem (1.1) and (1.2) has at least two positive solutions. □

Theorem 3.6. *Let (H_1)–(H_3), (L_2) and (L_5) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $i = 1,\, 2$ are sufficiently small.*

Proof. The proof of the Theorem 3.6 is similar to that of Theorem 3.5. □

Theorem 3.7. *Let (H_1)–(H_3), (L_4) and (L_7) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $i = 1,\, 2$ are sufficiently small.*

Proof. From (L_7), we have

$$\lim_{y\to 0}\frac{h(t,\,y)}{y}=0.$$

For $\epsilon>0$, there exists a $r_1>0$ such that

$$h(t,\,y)\le \epsilon y \quad \text{for} \quad y\in[0,\,r_1).$$

Since λ_1 and λ_2 are sufficiently small, we have

$$\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\,y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\le r_1.$$

Let $\Omega_{r_1}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_1\}$. For $y\in\partial\Omega_{r_1}$, we have

$$\begin{aligned}
(T_\lambda y)(t)&=\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)\left[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\right]\nabla s\\
&\le\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\left(\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)\right)\nabla s+\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)M\nabla s\\
&\le\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\,y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\\
&\le r_1=\|y\|
\end{aligned}$$

Thus,

$$\|T_\lambda y\|\le\|y\| \quad \text{for all} \quad y\in K\cap\partial\Omega_{r_1}. \tag{3.13}$$

From (L_4), we have

$$g(t,\,y)>N_1 y \quad \text{for all} \quad y\le l.$$

Let $r_2=\max\left\{2CM,\frac{2l}{\mu},2r_1\right\}$ and $\Omega_{r_2}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_2\}$. For $y\in K\cap\partial\Omega_{r_2}$, we have

$$\begin{aligned}
y(s)-x(s)&=y(s)-My_1(s)\\
&\ge y(s)-MCq(s)\\
&\ge y(s)-\frac{CM}{r_5}y(s)\\
&\ge\frac{1}{2}y(s),
\end{aligned}$$

and

$$\begin{aligned}
\min_{t_1 \le s \le t_2} (y(s) - x(s)) &\ge \min_{t_1 \le s \le t_2} \frac{y(s)}{2} \\
&\ge \min_{t_1 \le s \le t_2} \frac{\|y\|}{2} q(s) \\
&= \frac{r_2 \mu}{2} \\
&\ge l.
\end{aligned}$$

For $y \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
\min_{t\in[t_1,\, t_2]} (T_\lambda y)(t) &= \min_{t\in[t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s) \left[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\ge \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s) \left[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\ge \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s) \lambda_1 N_1 (y(s) - x(s)) \nabla s \\
&= r_2 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \ge \|y\| \quad \text{for all} \quad y \in K \cap \partial\Omega_{r_2}. \tag{3.14}$$

Let

$$R = \max\left\{ \left(\lambda_1 \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le R}} g(t,\, y) + \lambda_2 \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le R}} h(t,\, y) + M\right) \left(\int_{\rho(c)}^{d} \mathcal{G}(s,\, s) \nabla s\right), 2r_2 \right\},$$

then $r_1 < r_2 < R$. Let $\Omega_R = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < R\}$. For $y \in K \cap \Omega_R$, $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$, we have

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t,\, s) \Big[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\Big] \nabla s \\
&\le \left[(\lambda_1 + \lambda_2) \left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le R}} g(t,\, y) + \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le R}} h(t,\, y)\right) + M\right] \int_{\rho(c)}^{d} \mathcal{G}(s,\, s) \nabla s \\
&\le R = \|y\|
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \le \|y\| \quad \text{for all} \quad y \in K \cap \partial\Omega_R. \tag{3.15}$$

Thus by the Lemma 2.8, T_λ has at least two fixed points. Hence, the boundary value problem (1.1) and (1.2) has at least two positive solutions. □

Theorem 3.8. *Let (H_1)–(H_3), (L_3) and (L_8) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. The proof of the Theorem 3.8 is similar to that of Theorem 3.5. □

Theorem 3.9. *Let (H_1)–(H_3), (L_1) and (L_7) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. From (L_1), we have

$$\lim_{y\to\infty} \frac{g(t,\, y)}{y} = \infty.$$

For $k > 0$, there exists a $r_1 > 0$ such that

$$g(t,\, y) \geq ky \quad \text{for} \quad y > r_1.$$

Let $\Omega_{r_1} = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < r_1\}$ and let k satisfy

$$\frac{k\mu}{2}\lambda_1 \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\nabla s \geq 1.$$

For $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}
\min_{t\in[t_1,\, t_2]} (T_\lambda y)(t) &= \min_{t\in[t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s) \left[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\geq \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\lambda_1 k(y-x)\nabla s \\
&\geq \frac{k}{2}\lambda_1 \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\|y\|q(s)\nabla s \\
&\geq \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_1}. \tag{3.16}$$

From (L_7), we have

$$\lim_{y\to 0} \frac{h(t,\, y)}{y} = 0.$$

For $\epsilon > 0$, there exists a $r_2 > 0$ such that

$$h(t,\, y) \leq \epsilon y \quad \text{for} \quad y \in [0,\, \infty).$$

Since λ_1 and λ_2 are sufficiently small, let

$$\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} g(t,\, y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} h(t,\, y)+M\right)\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\, s)\nabla s\le r_2.$$

Let $\Omega_{r_2}=\{y\in C[\rho(c),\, \sigma(d)]_{\mathbb{T}}:\|y\|<r_2\}$. Now for any $y\in K\cap\partial\Omega_{r_2}$, we have

$$\begin{aligned}(T_\lambda y)(t) &= \int_{\rho(c)}^{d}\mathcal{G}(t,\, s)\left[\lambda_1\overline{g}(t,\, y-x)+\lambda_2\overline{h}(s,\, y-x)+M\right]\nabla s\\ &\le \int_{\rho(c)}^{d}\mathcal{G}(s,\, s)\left[\lambda_1\overline{g}(t,\, y-x)+\lambda_2\overline{h}(s,\, y-x)+M\right]\nabla s\\ &\le \left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} g(t,\, y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} h(t,\, y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\, s)\nabla s\\ &\le r_2=\|y\|.\end{aligned}$$

Thus,

$$\|T_\lambda y\|\le\|y\|\quad\text{for}\quad y\in K\cap\partial\Omega_{r_2}. \tag{3.17}$$

Hence, by the Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1<\|\overline{y}\|<r_2$. By the Lemma 2.6, the boundary value problem (1.1) and (1.2) has at least one positive solution. □

Theorem 3.10. *Let (H_1)–(H_3), (L_3) and (L_5) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $i=1,\, 2$ are sufficiently small.*

Proof. The proof of the Theorem 3.10 is similar to that of Theorem 3.9. □

Theorem 3.11. *Let (H_1)–(H_3), (L_2) and (L_8) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $i=1,\, 2$ are sufficiently small.*

Proof. From (L_2), we have

$$\lim_{y\to\infty}\frac{g(t,\, y)}{y}=0.$$

By Lemma 2.5, there exist $r_1>0$ and $k_1>0$ such that

$$\mathbb{G}(r_1)\le k_1 r_1.$$

Let $\Omega_{r_1}=\{y\in C[\rho(c),\, \sigma(d)]_{\mathbb{T}}:\|y\|<r_1\}$. Since λ_1 and λ_2 are sufficiently small, we have

$$\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\, y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\, y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\, s)\nabla s\le r_1.$$

For any $y \in K \cap \partial\Omega_{r_1}$, we obtain

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)\left[\lambda_1 \overline{g}(t,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\leq \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)\left[\lambda_1 \overline{g}(t,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\leq \left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_2}} g(t,\, y) + \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_2}} h(t,\, y)\right) + M\right] \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)\nabla s \\
&\leq r_1 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \leq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_1}. \tag{3.18}$$

From (L_8), we have

$$\lim_{y\to 0} \frac{h(t,\, y)}{y} = \infty.$$

For $k > 0$, there exists a $l > 0$ such that

$$h(t,\, y) \geq ky \quad \text{for} \quad y \in [0,\, l].$$

Let $r_2 = \left\{2cm, \frac{2l}{\mu}, 2r_1\right\}$ and $\Omega_{r_2} = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < r_2\}$. For any $y \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
\min_{t\in[t_1,\, t_2]} (T_\lambda y)(t) &= \min_{t\in[t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)\left[\lambda_1 \overline{g}(t,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\geq \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\lambda_2 k(y(s)-x(s))\nabla s \\
&\geq \frac{k}{2}\lambda_2\mu \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\|y\|\nabla s \\
&\geq r_2 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_2}. \tag{3.19}$$

Hence, by the Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1 < \|\overline{y}\| < r_2$. By the Lemma 2.6, the boundary value problem (1.1) and (1.2) has at least one positive solution. □

Theorem 3.12. *Let (H_1)–(H_3), (L_4) and (L_6) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $, i = 1,\, 2$ are sufficiently small.*

Proof. The proof of the Theorem 3.12 is similar to that of Theorem 3.11. □

4 Examples

We shall illustrate few examples in different time scales to justify the results obtained in the preceding section.

Example 4.1. *Let us consider the following boundary value problem on time scale* $\mathbb{T} = \mathbb{R}$*,*

$$((1+t^2)y')' + \frac{1}{2}\frac{1+y^2}{52} + \frac{1}{4}\frac{y^2\sin^2 y}{35} = 0, \quad t \in [0,\, 1], \tag{4.1}$$

with boundary conditions

$$\begin{aligned} y(0) - y'(0) &= 0, \\ y(1) + 2y'(1) &= 0, \end{aligned} \tag{4.2}$$

where $\psi(t) = 1+t^2$*,* $M = 1$*,* $\alpha, \beta, \gamma, \delta \geq 0$*,* $g(t,\, y) = \frac{1+y^2}{52}$ *and* $h(t,\, y) = \frac{y^2\sin^2 y}{35}$*. Green's function for the boundary value problem (4.1) and (4.2) is given by*

$$\mathcal{G}(t,\, s) = \frac{1}{2+\frac{\pi}{4}}\begin{cases} \left(1+\tan^{-1} t\right)\left(1+\frac{\pi}{4}-\tan^{-1} s\right), & t \leq s, \\ \left(1+\tan^{-1} s\right)\left(1+\frac{\pi}{4}-\tan^{-1} t\right), & s \leq t. \end{cases}$$

All the conditions (H_1)*–*(H_3)*,* (L_1) *and* (L_5) *are satishfied for* $(t,\, y) \in [0,\, 1]\times[0,\, 100]$*. By Theorem 3.1, boundary value problem (4.1) and (4.2) has at least one positive solution for* $\lambda_1 = \frac{1}{2}$ *and* $\lambda_2 = \frac{1}{4}$*.*

Example 4.2. *Let us consider the following boundary value problem on time scale* $\mathbb{T} = \mathbb{Z}$*,*

$$\nabla((1+t)^{-1}y^{\Delta}) + \lambda_1 \sin^2 y + \lambda_2\sqrt{y}\cos y = 0, \quad t \in [0,\, 3], \tag{4.3}$$

with boundary conditions

$$\begin{aligned} y(0) - \Delta y(0) &= 0, \\ y(3) + \frac{1}{3}\Delta y(2) &= 0, \end{aligned} \tag{4.4}$$

where $\psi(t) = (1+t)^{-1}$*,* $M = 1$*,* $\alpha, \beta, \gamma, \delta \geq 0$*,* $g(t,\, y) = \sin^2 y$ *and* $h(t,\, y) = \sqrt{y}\cos y$*. Green's function for the boundary value problem (4.3) and (4.4) is given by*

$$\mathcal{G}(t,\, s) = \frac{1}{11}\begin{cases} \left(1+\frac{t^2+3t}{2}\right)\left(1+\frac{(3-s)(s+6)}{2}\right), & t \leq s, \\ \left(1+\frac{s^2+3s}{2}\right)\left(1+\frac{(3-t)(t+6)}{2}\right), & s \leq t. \end{cases}$$

All the conditions (H_1)*–*(H_3)*,* (L_2) *and* (L_6) *are satishfied for* $(t,\, y) \in [0,\, 3]\times[0,\, 100]$*. Let* $D = 1$

and $\eta = \frac{1}{2}$ *such that* $g(t, y) \geq \frac{1}{2}$ *and* $h(t, y) \geq \frac{1}{2}$ *for* $t \in [0, 3]$, $y \in [1, \infty)$. *By Theorem 3.3, boundary value problem (4.3) and (4.4) has at least one positive solution for* λ_i; $i = 1, 2$ *are sufficiently small.*

Example 4.3. *Consider the boundary value problem on time scale* $\mathbb{T} = q^{\overline{\mathbb{Z}}} = \{2^k : k \in \mathbb{Z}\} \cup \{0\}$, *where* $q = 2 > 1$,

$$D^q\left((1+t)^{-1} D_q y(t)\right) + \lambda_1 \frac{y^2}{\sin\, y} + \lambda_2 \ln(y) = 0, \quad t \in [0, 2], \tag{4.5}$$

with boundary conditions

$$\begin{aligned} y(0) - D_q y(0) = 0, \\ y(2) + \frac{1}{2} D_q y(1) = 0, \end{aligned} \tag{4.6}$$

where $\psi(t) = (1+t)^{-1}$, $M = 1$, $\alpha, \beta, \gamma, \delta \geq 0$, $g(t, y) = \frac{y^2}{\sin\, y}$ *and* $h(t, y) = \ln(y)$. *Green's function for the boundary value problem (4.5) and (4.6) is given by*

$$\mathcal{G}(t, s) = \frac{3}{20} \begin{cases} \left(\frac{2t^2+3t+3}{3}\right)\left(\frac{17-3s-2s^2}{3}\right), \, t \leq s, \\ \left(\frac{2s^2+3s+3}{3}\right)\left(\frac{17-3t-2t^2}{3}\right), \, s \leq t. \end{cases}$$

The conditions (H_1)–(H_3), (L_1) *and* (L_6) *are satishfied for* $(t, y) \in [0, 2] \times [0, 100]$. *By Theorem 3.5, boundary value problem (4.5) and (4.6) has at least two positive solutions for* λ_i; $i = 1, 2$ *are sufficiently small.*

Example 4.4. *Let us consider the time scale* $\mathbb{T} = \mathbb{P}_{a,\,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a] = \mathbb{P}_{1,\,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$, *where* $a = b = 1$. *Consider the following boundary value problem:*

$$\begin{cases} y^{\Delta\nabla} + \lambda_1 \sqrt{y} + \lambda_2\, y\, \ln(1+y), \quad t \in (0, 2), \\ y(0) = 0, \quad y(2) = 0, \end{cases} \tag{4.7}$$

where $\psi(t) = 1$, $M = 1$, $\alpha, \beta, \gamma, \delta \geq 0$, $g(t, y) = \sqrt{y}$ *and* $h(t, y) = y \ln(y)$. *Green's function for the boundary value problem (4.7) is given by*

$$\mathcal{G}(t, s) = \frac{1}{2} \begin{cases} t(1-s), \, t \leq s, \\ (1+s)(2-t), \, s \leq t. \end{cases}$$

The conditions (H_1)–(H_3), (L_4) *and* (L_7) *are satishfied for* $(t, y) \in [0, 2] \times [0, 100]$. *By Theorem 3.7, boundary value problem (4.7) has at least two positive solutions for* λ_i; $i = 1, 2$ *are sufficiently small.*

Example 4.5. *Consider the boundary value problem on time scale $\mathbb{T} = \{\frac{n}{2} : t \in \mathbb{N}_0\}$:*

$$y^{\Delta\nabla}(t) + \lambda_1 y \ln(1+y) + \lambda_2 \frac{\sqrt{y}\sin y}{6} = 0, \quad t \in \left[0, \frac{3}{2}\right], \tag{4.8}$$

with boundary conditions

$$\begin{aligned} y(0) - y^{\Delta}(0) &= 0, \\ y\left(\frac{3}{2}\right) + y^{\Delta}(1) &= 0, \end{aligned} \tag{4.9}$$

where $\psi(t) = 1$, $M = 1$, $\alpha, \beta, \gamma, \delta \geq 0$, $g(t, y) = y\ln(1+y)$ and $h(t, y) = \frac{\sqrt{y}\sin y}{6}$. Green's function for the boundary value problem (4.8) and (4.9) is given by

$$\mathcal{G}(t, s) = \frac{2}{7}\begin{cases} (1+s)\left(\frac{5}{2} - t\right), \ t \leq s, \\ (1+t)\left(\frac{5}{2} - s\right), \ s \leq t. \end{cases}$$

The conditions (H_1)–(H_3), (L_1) and (L_7) are satisfied for $(t, y) \in [0, \frac{3}{2}] \times [0, 100]$. By Theorem 3.9, boundary value problem (4.8) and (4.9) has at least one positive solutions for λ_i; $i = 1, 2$ are sufficiently small.

Example 4.6. *Consider the following boundary value problem in time scale $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h = \frac{1}{2} > 0$,*

$$\left((1+t)^{-1} y^{\Delta}\right)^{\nabla} + \lambda_1 \sqrt{y}\sin y + \lambda_2 = 0 \quad \textit{for} \quad t \in [0, 2], \tag{4.10}$$

with boundary conditions

$$\begin{aligned} y(0) - y^{\Delta}(0) &= 0, \\ y(2) + \frac{2}{5} y^{\Delta}\left(\frac{3}{2}\right) &= 0, \end{aligned} \tag{4.11}$$

where $\psi(t) = (1+t)^{-1}$, $M = 1$, $\alpha, \beta, \gamma, \delta \geq 0$, $g(t, y) = \sqrt{y}\sin y$ and $h(t, y) = 1$. Green's function for the boundary value problem (4.10) and (4.11) is given by

$$\mathcal{G}(t, s) = \frac{2}{13}\begin{cases} \left(1 + \frac{s(2s+5)}{4}\right)\left(1 + \frac{(2-t)(2t+9)}{4}\right), \ s \leq t, \\ \left(1 + \frac{t(2t+5)}{4}\right)\left(1 + \frac{(2-s)(2s+9)}{4}\right), \ t \leq s. \end{cases}$$

The conditions (H_1)–(H_3), (L_2) and (L_8) are satishfied for $(t, y) \in [0, 2] \times [0, 100]$. By Theorem 3.11, boundary value problem (4.10) and (4.11) has at least one positive solutions for λ_i; $i = 1, 2$ are sufficiently small.

References

[1] D. R. Anderson and P. J. Y. Wong, "Positive solutions for second-order semipositone problems on time scales", Comput. Math. Appl., vol. 58, no. 2, pp. 281–291, 2009.

[2] D. R. Anderson and C. Zhai, "Positive solutions to semi-positone second-order three-point problems on time scale", Appl. Math. Comput., vol. 215, no. 10, pp. 3713–3720, 2010.

[3] R. Aris, *Introduction to the analysis of chemical reactors*, New Jersey: Prentice Hall, Englewood Cliffs, 1965.

[4] D. Bai, and Y. Xu, "Positive solutions for semipositone BVPs of second-order difference equations", Indian J. Pure Appl. Math., vol. 39, no.1, pp. 59–68, 2008.

[5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Boston: Birkhäuser, 2001.

[6] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Boston: Birkhäuser, 2003.

[7] R. Dahal, "Positive solutions for a second-order, singular semipositone dynamic boundary value problem", Int. J. Dyn. Syst. Differ. Equ., vol. 3, no. 1–2, pp. 178–188, 2011.

[8] L. Erbe and A. Peterson, "Positive solutions for a nonlinear differential equations on a measure chain", Math. Comput. Modelling, vol. 32, no. 5–6, pp. 571–585, 2000.

[9] C. Giorgi and E. Vuk, "Steady-state solutions for a suspension bridge with intermediate supports", Bound. Value Probl., Paper No. 204, 17 pages, 2013.

[10] C. S. Goodrich, "Existence of a positive solution to a nonlocal semipositone boundary value problem on a time scale", Comment. Math. Univ. Carolin., vol. 54, no. 4, pp. 509–525, 2013.

[11] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus", Results Math., vol. 18, no. 1–2, pp. 18–56, 1990.

[12] G. Infante, P. Pietramala and M. Tenuta, "Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory", Commun. Nonlinear Sci. Numer. Simul., vol. 19, no 7, pp. 2245–2251, 2014.

[13] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Groningen: P. Noordhoff, 1964.

[14] E. Kreyszig, *Introductory Functional Analysis With Applications*, New York: John Wiley & Sons, Inc., 1978.

[15] J. Selgrade, "Using stocking and harvesting to reverse period-doubling bifurcations in models in population biology", J. Differ. Equations Appl., vol. 4, no. 2, pp. 163–183, 1998.

[16] J. P. Sun and W. T. Li, "Existence of positive solutions to semipositone Dirichlet BVPs on time scales", Dynam. Systems Appl., vol. 16, no. 3, pp. 571–578, 2007.

[17] Y. Yang and F. Meng, "Positive solutions of the singular semipositone boundary value problem on time scales", Math. Comput. Modelling, vol. 52, no. 3–4, pp. 481–489,

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A class of nonlocal impulsive differential equations with conformable fractional derivative

Mohamed Bouaouid[1, ✉]
Ahmed Kajouni[1]
Khalid Hilal[1]
Said Melliani[1]

[1] *Sultan Moulay Slimane University, Faculty of Sciences and Technics, Department of Mathematics, BP 523, 23000, Béni Mellal, Morocco.*
bouaouidfst@gmail.com ✉
kajjouni@gmail.com
Khalid.hilal.usms@gmail.com
said.melliani@gmail.com

ABSTRACT

In this paper, we deal with the Duhamel formula, existence, uniqueness, and stability of mild solutions of a class of nonlocal impulsive differential equations with conformable fractional derivative. The main results are based on the semigroup theory combined with some fixed point theorems. We also give an example to illustrate the applicability of our abstract results

RESUMEN

En este artículo, tratamos la fórmula de Duhamel, la existencia, unicidad y estabilidad de soluciones mild de una clase de ecuaciones diferenciales no locales impulsivas con derivadas fraccionarias conformables. Los resultados principales se basan en teoría de semigrupos, combinada con algunos teoremas de punto fijo. También entregamos un ejemplo para ilustrar la aplicabilidad de nuestros resultados abstractos.

Keywords and Phrases: Functional-differential equations with fractional derivatives; Groups and semigroups of linear operators; Nonlocal conditions; Impulsive conditions; Conformable fractional derivatives.

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1 Introduction

Fractional calculus has attracted the attention of many researchers, due to its wide range of applications in modeling of various natural phenomena in different fields of sciences and engineering including: physics, engineering, biology, finance, chemistry [3, 26, 31, 35, 38, 41, 43, 44, 45, 46, 47, 48]. For better understanding these phenomena, several definitions of fractional derivatives have been introduced such as Riemann-Liouville and Caputo definitions, for more details we refer to the books [31, 41]. Unfortunately, these definitions are very complicated to handle in real models. However, in [30] a new definition of fractional derivative named conformable fractional derivative was initiated. This novel fractional derivative is very easy and satisfies all the properties of the classical derivative. The advantage of the conformable fractional derivative is very remarkable compared to other fractional derivatives in many comparisons. Indeed, for example, in the work [15] the authors gave the solution of conformable-fractional telegraph equations in terms of the classical exponential function, however for the Caputo-fractional telegraph equations considered in the very good papers [19, 20, 36], the fundamental solution cannot be given in terms of the exponential function as in the conformable-fractional case, and therefore the authors have been introduced the so-called Mittag-Leffler function. Another comparison, we notice that the constants of increases of the norms of the control bounded operators W and W^{-1} in the application of the work [27] are given directly in a simple way in terms of the exponential function, contrary, for the Caputo fractional derivative in the application of the nice work [51] these constants are given in terms of the so-called Mittag-Leffler function. For more details and conclusions concerning the uses and applications of conformable fractional calculus, we refer to the works [2, 4, 5, 7, 8, 10, 11, 12, 13, 14, 16, 17, 22, 23, 24, 25, 28, 29, 42, 49].

On the other hand, impulsive differential equations are crucial in description of dynamical processes with short-time perturbations [6, 32, 50]. Actually, the Cauchy problem of impulsive differential equations attracts the attention of many authors [1, 9, 33, 34, 37]. For example, Liang *et al.* [33] have proved the existence and uniqueness of mild solutions for the Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \in [0, \tau], \quad t \neq t_1, t_2, \ldots, t_n, \\ x(0) = x_0 + g(x), \\ x(t_i^+) = x(t_i^-) + h_i(x(t_i)), \quad i = 1, 2, \ldots, n, \end{cases} \qquad . \tag{1.1}$$

by using the following classical Duhamel formula:

$$x(t) = T(t)[x_0 + g(x)] + \sum_{0<t_i<t} T(t - t_i)h_i(x(t_i)) + \int_0^t T(t - s)f(s, x(s))ds, \tag{1.2}$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by the linear part A on a Banach space $(X, \| \, . \, \|)$ [40] and $x_0 \in X$. The expression $x(t_i^+) = x(t_i^-) + h_i(t_i)$ means the impulsive condition, with $x(t_i^+)$, $x(t_i^-)$ are the right and left limits of $x(.)$ at $t = t_i$, respectively. The condition $x(0) = x_0 + g(x)$

represents the nonlocal condition, which can be applied in physics with better effects than the classical initial condition [18, 21, 39]. The functions $f : [0,\tau] \times X \longrightarrow X$, $h_i : X \longrightarrow X$ and $g : \mathcal{C} \longrightarrow X$ satisfied some assumptions, with $\mathcal{C}$ is the space of functions $x(.)$ defined from $[0,\tau]$ into X such that $x(.)$ is continuous on each interval $]t_i, t_{i+1}]$ and $x(t_i^+)$, $x(t_i^-)$ exist.
The analogous of equation (1.1) in the frame of the Caputo fractional derivative have been considered by Mophou [37], when the author proved the existence and uniqueness of mild solutions for the following fractional Cauchy problem

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in [0,\tau], \quad t \neq t_1, t_2, \ldots, t_n, \quad 0 < \alpha < 1, \\ x(0) = x_0 + g(x), \\ x(t_i^+) = x(t_i^-) + h_i(x(t_i)), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.3}$$

by using the following fractional Duhamel formula:

$$\begin{aligned} x(t) = {} & T(t)[x_0 + g(x)] + \sum_{0<t_i<t} T(t-t_i)h_i(x(t_i)) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) f(s, x(s)) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} T(t-s) f(s, x(s)) ds, \end{aligned} \tag{1.4}$$

with Γ is the Gamma function and ${}^cD^\alpha x(t)$ presents the Caputo fractional derivative.

In the present work, we are interested in studying of equation (1.1) in the frame of the conformable fractional derivative. Precisely, we will be concerned with the study of the existence, uniqueness, and stability of mild solutions for the following conformable fractional Cauchy problem

$$\begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha} & = Ax(t) + f(t, x(t)), \quad t \in [0,\tau], \quad t \neq t_1, t_2, \ldots, t_n, \quad 0 < \alpha < 1, \\ x(0) & = x_0 + g(x), \\ x(t_i^+) & = x(t_i^-) + h_i(x(t_i)), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.5}$$

where $\dfrac{d^\alpha x(t)}{dt^\alpha}$ is the conformable fractional derivative.
The main novelty of this paper is to prove the analogous of Duhamel formulas (1.2) and (1.4) for the Cauchy problem (1.5) as follows:

$$\begin{aligned} x(t) = {} & T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) \\ & + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds. \end{aligned} \tag{1.6}$$

Then, based on this conformable fractional Duhamel formula, we discuss some results concerning the existence, uniqueness, and stability of the mild solution of the conformable fractional Cauchy problem (1.5).

This paper is organized as follows. In section 2, we briefly recall some tools related to the conformable fractional calculus. In section 3, we prove the main results. Section 4 is devoted to a concrete application of the main abstract results.

2 Preliminaries

Recalling some preliminary facts on the conformable fractional calculus.

Definition 2.1 ([30]). *Let $\alpha \in]0,1]$. The conformable fractional derivative of order α of a function $x(.): \ [0,+\infty[\longrightarrow \mathbb{R}$ is defined by*

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\varepsilon \longrightarrow 0} \frac{x(t+\varepsilon t^{1-\alpha}) - x(t)}{\varepsilon}, \ \ for \ \ t>0 \ \ and \ \ \frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t\longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha},$$

provided that the limits exist.

The fractional integral $I^\alpha(.)$ associated with the conformable fractional derivative is defined by

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s) ds.$$

Theorem 2.2 ([30]). *If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$\frac{d^\alpha (I^\alpha(x)(t))}{dt^\alpha} = x(t).$$

Definition 2.3 ([41]). *The Laplace transform of a function $x(.)$ is defined by*

$$\mathcal{L}(x(t))(\lambda) := \int_0^{+\infty} e^{-\lambda t} x(t) dt, \quad \lambda > 0.$$

It is remarkable that the above transform is not adequate to solve conformable fractional differential equations. For this reason, we consider the following definition, which appeared in [2].

Definition 2.4 ([2]). *The fractional Laplace transform of order α of a function $x(.)$ is defined by*

$$\mathcal{L}_\alpha(x(t))(\lambda) := \int_0^{+\infty} t^{\alpha-1} e^{-\lambda \frac{t^\alpha}{\alpha}} x(t) dt, \quad \lambda > 0.$$

The following proposition gives us the actions of the fractional integral and the fractional Laplace transform on the conformable fractional derivative, respectively.

Proposition 2.5 ([2]). *If $x(.)$ is a differentiable function, then we have the following results*

$$I^\alpha \left(\frac{d^\alpha x(.)}{dt^\alpha} \right)(t) = x(t) - x(0),$$
$$\mathcal{L}_\alpha \left(\frac{d^\alpha x(t)}{dt^\alpha} \right)(\lambda) = \lambda \mathcal{L}_\alpha(x(t))(\lambda) - x(0).$$

We end this preliminaries by the following remark.

Remark 2.6 ([14])**.** *For two arbitrary functions $x(.)$ and $y(.)$, we have*

$$\mathcal{L}_\alpha\left(x\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda) = \mathcal{L}(x(t))(\lambda),$$

$$\mathcal{L}_\alpha\left(\int_0^t s^{\alpha-1}x\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)y(s)ds\right)(\lambda) = \mathcal{L}(x(t))(\lambda)\mathcal{L}_\alpha(y(t))(\lambda).$$

3 Main results

We first prove the conformable fractional Duhamel formula (1.6). To do so, for $t \in [0, t_1]$, we apply the fractional Laplace transform in equation (1.5), we obtain

$$\mathcal{L}_\alpha(x(t))(\lambda) = (\lambda - A)^{-1}[x_0 + g(x)] + (\lambda - A)^{-1}\mathcal{L}_\alpha(f(t, x(t)))(\lambda).$$

According to the inverse fractional Laplace transform and Remark (2.6), we get

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)f(s, x(s))ds,$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by the linear part A on the Banach space X, that is, $(T(t))_{t\geq 0}$ is one parameter family of bounded linear operators on X satisfying the following properties

(1) $T(0) = I$,

(2) $T(s + t) = T(s)T(t)$ for all $t, s \in \mathbb{R}^+$,

(3) $\lim\limits_{t\downarrow 0} \| T(t)x - x \| = 0$ for each fixed $x \in X$,

(4) $\lim\limits_{t\downarrow 0}\dfrac{T(t)x - x}{t} = Ax$, for $x \in X$, provided that the limit exists.

As in [37], we assume that the solution of equation (1.5) is such that at the point of discontinuity t_k, we have $x(t_k^-) = x(t_k)$. Hence, one has

$$x(t_1^-) = T\left(\frac{t_1^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^{t_1} s^{\alpha-1}T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)f(s, x(s))ds.$$

For $t \in (t_1, t_2]$, using the fractional Laplace transform in equation (1.5), we obtain

$$\begin{aligned} x(t) &= T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)x(t_1^+) + \int_{t_1}^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)f(s, x(s))ds \\ &= T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)[x(t_1^-) + h_1(x(t_1))] + \int_{t_1}^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)f(s, x(s))ds. \end{aligned}$$

Replacing $x(t_1^-)$ by its expression in the above equation, we get

$$\begin{aligned} x(t) = {} & T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)\left[T\left(\frac{t_1^\alpha}{\alpha}\right)(x_0 + g(x)) + \int_0^{t_1} s^{\alpha-1}T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)f(s, x(s))ds + h_1(x(t_1))\right] \\ & + \int_{t_1}^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)f(s, x(s))ds. \end{aligned}$$

By using a computation, the above equation becomes

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

In particular, for $t = t_2^-$, one has

$$x(t_2^-) = T\left(\frac{t_2^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))] + \int_0^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

As the same, for $t \in (t_2, t_3]$, we obtain

$$\begin{aligned} x(t) &= T\left(\frac{t^\alpha - t_2^\alpha}{\alpha}\right) x(t_2^+) + \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds \\ &= T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)[x(t_2^-) + h_2(x(t_2))] + \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds. \end{aligned}$$

Hence, replacing $x(t_2^-)$ by its expression, we have

$$\begin{aligned} x(t) = T\left(\frac{t^\alpha - t_2^\alpha}{\alpha}\right) &\left[T\left(\frac{t_2^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))]\right. \\ &\left. + \int_0^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds + h_2(x(t_2))\right] \\ &+ \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds. \end{aligned}$$

Using a computation, we get

$$\begin{aligned} x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))] + T\left(\frac{t^\alpha - t_2^\alpha}{\alpha}\right)[h_2(x(t_2))] \\ + \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds. \end{aligned}$$

Repeating the same process, we obtain the following conformable fractional Duhamel formula

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

Definition 3.1. *A function $x \in \mathcal{C}$ is called a mild solution of conformable fractional Cauchy problem* (1.5) *if*

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

In the rest of this paper, we endow the space $\mathcal{C}$ with the norm $| x |_c := \sup_{t\in[0,\tau]} \| x(t) \|$. It is well known that the space $(\mathcal{C}, | \, . \, |_c)$ becomes a Banach space. We also denote by $|.|$ the norm in the space $\mathcal{L}(X)$ of bounded operators defined form X into itself.

To prove the main results, we need to use the following assumptions:

(H_1) The function $f(t, .) : X \longrightarrow X$ is continuous and for all $r > 0$ there exists a function $\mu_r \in L^\infty([0, \tau], \mathbb{R}^+)$ such that $\sup_{\|x\|\leq r} \| f(t, x) \| \leq \mu_r(t)$, for all $t \in [0, \tau]$.

(H_2) The function $f(.,x) : [0,\tau] \longrightarrow X$ is continuous, for all $x \in X$.

(H_3) There exists a constant $L_1 > 0$ such that $\| g(y) - g(x) \| \leq L_1 \mid y - x \mid_c$, for all $x, y \in \mathcal{C}$.

(H_4) There exist constants $C_i > 0$ such that $\| h_i(y(t_i)) - h_i(x(t_i)) \| \leq C_i \mid y - x \mid_c$, for all $x, y \in \mathcal{C}$.

Theorem 3.2. *If $(T(t))_{t>0}$ is compact and $(H_1) - (H_4)$ are satisfied, then the conformable fractional Cauchy problem* (1.5) *has at least one mild solution, provided that*

$$\left(L_1 + \sum_{i=1}^{n} C_i\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| < 1.$$

Proof. Let $B_r = \{x \in \mathcal{C}, \mid x \mid_c \leq r\}$, where

$$r \geq \frac{\sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left[\| x_0 \| + \| g(0) \| + \sum_{i=1}^{n} \| h_i(0) \| + \frac{\tau^\alpha}{\alpha} \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)}\right]}{1 - \left(L_1 + \sum_{i=1}^{n} C_i\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right|}.$$

In order to use the Krasnoselskii fixed-point theorem, we consider the following operators Γ_1 and Γ_2 defined by

$$\Gamma_1(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)), \quad x \in B_r,$$

$$\Gamma_2(x)(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds, \quad x \in B_r.$$

It is very easy to justify that the operator $\Gamma := \Gamma_1 + \Gamma_2$ is well defined, that is, $\Gamma(x) \in \mathcal{C}$ for all $x \in \mathcal{C}$. The rest of the proof will be given in four steps:

Step 1: Prove that $\Gamma_1(x) + \Gamma_2(y) \in B_r$, whenever $x,\ y \in B_r$.

Let $x,\ y \in B_r$, we have

$$\begin{aligned}\Gamma_1(x)(t) + \Gamma_2(y)(t) = {} & T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) \\ & + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y(s))ds.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\| \Gamma_1(x)(t) + \Gamma_2(y)(t) \| \leq {} & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \| x_0 \| + \| g(0) \| + \| g(x) - g(0) \| \,] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} [\, \| h_i(0) \| + \| h_i(x(t_i)) - h_i(0) \| \,] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \int_0^t s^{\alpha-1} \| f(s, y(s)) \| ds.\end{aligned}$$

By using assumptions (H_1), (H_3) and (H_4), we get

$$\begin{aligned}\parallel \Gamma_1(x)(t)+\Gamma_2(y)(t) \parallel \leq & \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \parallel x_0 \parallel + \parallel g(0) \parallel + L_1 \mid x \mid_c] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} [\, \parallel h_i(0) \parallel + C_i \mid x \mid_c] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \int_0^t s^{\alpha-1} ds.\end{aligned}$$

According to the fact that $x,\ y \in B_r$, we conclude that

$$\begin{aligned}\parallel \Gamma_1(x)(t)+\Gamma_2(y)(t) \parallel \leq & \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \parallel x_0 \parallel + \parallel g(0) \parallel + L_1 r] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} [\, \parallel h_i(0) \parallel + C_i r] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \int_0^t s^{\alpha-1} ds.\end{aligned}$$

Taking the supremum, we get

$$\begin{aligned}\mid \Gamma_1(x)+\Gamma_2(y) \mid_c \leq & \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| [\parallel x_0 \parallel + \parallel g(0) \parallel + L_1 r] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{i=1}^{n} [\, \parallel h_i(0) \parallel + C_i r] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \int_0^\tau s^{\alpha-1} ds \\ = & \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \parallel x_0 \parallel + \parallel g(0) \parallel + L_1 r] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{i=1}^{n} [\, \parallel h_i(0) \parallel + C_i r] \\ & + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \frac{\tau^\alpha}{\alpha} \\ \leq & \ r.\end{aligned}$$

Hence, the above inequality combined with the continuity of the function $\Gamma_1(x)(.)+\Gamma_2(y)(.)$ on $[0,\tau]$ show that $\Gamma_1(x)+\Gamma_2(y)\in B_r$, for all $x,\ y\in B_r$.

Step 2: Prove that Γ_1 is a contraction operator on B_r.

For $x, y \in \mathcal{C}$, we have

$$\Gamma(y)(t)-\Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[g(y)-g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right)[h_i(y(t_i))-h_i(x(t_i))].$$

Consequently, one has

$$\begin{aligned}\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \| g(y) - g(x) \| \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} \| h_i(y(t_i)) - h_i(x(t_i)) \| .\end{aligned}$$

Using assumptions (H_3) and (H_4), we get

$$\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq \left(L_1 + \sum_{i=1}^{n} C_i\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

Taking the supremum in above equation, we obtain

$$\mid \Gamma(y) - \Gamma(x) \mid_c \leq \left(L_1 + \sum_{i=1}^{n} C_i\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

This implies that Γ_1 is a contraction operator on B_r.

Step 3: Prove that Γ_2 is continuous.

Let $(x_n) \subset B_r$ such that $x_n \longrightarrow x$ in B_r. We have

$$\Gamma_2(x_n)(t) - \Gamma_2(x)(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s, x_n(s)) - f(s, x(s))]ds.$$

Then, by using a computation, we obtain

$$\mid \Gamma_2(x_n) - \Gamma_2(x) \mid_c \leq \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \int_0^\tau s^{\alpha-1} \| f(s, x_n(s)) - f(s, x(s)) \| ds.$$

Using assumption (H_1), we get $\| s^{\alpha-1}[f(s, x_n(s)) - f(s, x(s))] \| \leq 2\mu_r(s)s^{\alpha-1}$ and $f(s, x_n(s)) \longrightarrow f(s, x(s))$ as $n \longrightarrow +\infty$.

According to the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\longrightarrow+\infty} \mid \Gamma_2(x_n) - \Gamma_2(x) \mid_c = 0.$$

Thus, the operator Γ_2 is continuous.

Step 4: Prove that Γ_2 is compact by using the Arzelà-Ascoli theorem.

`Claim 1:` We prove that $\Gamma_2(B_r)$ is equicontinuous.

Let $t_1, t_2 \in [0, \tau]$ such that $t_1 < t_2$. Then, we have

$$\begin{aligned}\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) = & \int_0^{t_1} s^{\alpha-1} \left[T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)\right] f(s, x(s))ds \\ & + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds \\ = & \left[T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I)\right] \int_0^{t_1} s^{\alpha-1} T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds \\ & + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.\end{aligned}$$

By using a computation and assumption (H_1), we obtain

$$\| \Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) \| \leq \frac{\sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)}}{\alpha} \left[(t_2^\alpha - t_1^\alpha) + \tau^\alpha \left| T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I \right| \right].$$

According to [40], the compactness of $(T(t))_{t>0}$ assures that $\lim\limits_{t_2 \longrightarrow t_1} \left| T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I \right| = 0$. Hence, combining this fact with the above inequality, we conclude that $\Gamma_2(x)$, $x \in B_r$ are equicontinuous on $[0,\tau]$.

`Claim 2:` We prove that the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X.

For some fixed $t \in]0,\tau]$ let $\varepsilon \in]0,t[$, $x \in B_r$ and define the operator Γ_2^ε as follows

$$\Gamma_2^\varepsilon(x)(t) = T\left(\frac{\varepsilon^\alpha}{\alpha}\right) \int_0^{(t^\alpha - \varepsilon^\alpha)^{\frac{1}{\alpha}}} s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha - \varepsilon^\alpha}{\alpha}\right) f(s,x(s))ds.$$

Since $(T(t))_{t>0}$ is compact, then the set $\{\Gamma_2^\varepsilon(x)(t),\ x \in B_r\}$ is relatively compact in X. By using a computation combined with assumption (H_1), we get

$$\| \Gamma_2^\varepsilon(x)(t) - \Gamma_2(x)(t) \| \leq \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \frac{\varepsilon^\alpha}{\alpha}.$$

Therefore, we deduce that the $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X. For $t = 0$ the set $\{\Gamma_2(x)(0),\ x \in B_r\}$ is compact. Thus, the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X for all $t \in [0,\tau]$. By using the Arzelà-Ascoli theorem, we conclude that the operator Γ_2 is compact.

In conclusion, by the above steps combined with the Krasnoselskii fixed-point theorem, we conclude that $\Gamma_1 + \Gamma_2$ has at least one fixed point in $\mathcal{C}$, which is a mild solution of conformable fractional Cauchy problem (1.5). □

To obtain the uniqueness of the mild solution, we need the following assumption:

(H_5) There exists a constant $L_2 > 0$ such that $\| f(t,y) - f(t,x) \| \leq L_2 \| y - x \|$, for all $x,y \in X$ and $t \in [0,\tau]$.

Theorem 3.3. *Assume that $(H_2) - (H_5)$ hold, then the conformable fractional Cauchy problem* (1.5) *has an unique mild solution, provided that*

$$\left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| < 1.$$

Proof. Define the operator $\Gamma : \mathcal{C} \longrightarrow \mathcal{C}$ by:

$$\Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s,x(s))ds.$$

For $x, y \in \mathcal{C}$, we have

$$\Gamma(y)(t) - \Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right)[h_i(y(t_i)) - h_i(x(t_i))]$$
$$+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s, y(s)) - f(s, x(s))]ds.$$

Then, we obtain

$$\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \| g(y) - g(x) \|$$
$$+ \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} \| h_i(y(t_i)) - h_i(x(t_i)) \|$$
$$+ \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \int_0^t s^{\alpha-1} \| f(s, y(s)) - f(s, x(s)) \| ds.$$

According to assumptions (H_3), (H_4) and (H_5), we conclude that

$$\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq \left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

Taking the supremum, we obtain

$$\mid \Gamma(y) - \Gamma(x) \mid_c \leq \left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

Since $\left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| < 1$ then Γ is a contraction operator on the Banach space $(\mathcal{C}, \mid . \mid_c)$. Hence, by using the Banach contraction principle, we conclude that the operator Γ has an unique fixed point in $\mathcal{C}$, which is the mild solution of the conformable fractional Cauchy problem (1.5). $\square$

Now, we are in position to prove the continuous dependence of the mild solution to the initial condition. Precisely, we have the following result.

Theorem 3.4. *Assume that the conditions of Theorem* (3.3) *are satisfied. Let* x_0, $y_0 \in X$ *and denote by* x *and* y *the solutions associated with* x_0 *and* y_0*, respectively. Then, we have the following estimate*

$$\mid y - x \mid_c \leq \frac{\alpha \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right|}{\alpha - \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left(\alpha L_1 + \sum_{i=1}^n \alpha C_i + L_2 \tau^\alpha\right)} \| y_0 - x_0 \| .$$

Proof. For $t \in [0, \tau]$, we have

$$y(t) - x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[y_0 - x_0 + g(y) - g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right)[h_i(y(t_i)) - h_i(x(t_i))]$$
$$+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s, y(s)) - f(s, x(s))]ds.$$

Then, by using a computation combined with assumptions (H_1), (H_3) and (H_4), we obtain

$$\| y(t) - x(t) \| \leq \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left[\| y_0 - x_0 \| + \left(L_1 + \frac{L_2\tau^\alpha}{\alpha} + \sum_{i=1}^{n} C_i \right) | y - x |_c \right].$$

Taking the supremum, we get

$$| y - x |_c \leq \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left[\| y_0 - x_0 \| + \left(L_1 + \frac{L_2\tau^\alpha}{\alpha} + \sum_{i=1}^{n} C_i \right) | y - x |_c \right].$$

Thus, we deduce the desired estimate

$$| y - x |_c \leq \frac{\alpha \sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right|}{\alpha - \sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left(\alpha L_1 + \sum\limits_{i=1}^{n} \alpha C_i + L_2 \tau^\alpha \right)} \| y_0 - x_0 \| .$$

□

4 Application

We consider the nonlocal impulsive partial differential equation with conformable fractional derivative of the form

$$\begin{cases} \dfrac{\partial^{\frac{1}{2}} u(t,\xi)}{\partial t^{\frac{1}{2}}} = -\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + \displaystyle\int_0^t \frac{|\cos(u(t-s,\xi))|}{1+|\sin(u(t-s,\xi))|} ds, \quad (t,\xi) \in [0,1]\times]0,\pi[, \quad t \neq \dfrac{1}{2}, \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ u(0,\xi) = \dfrac{1}{n^2}[u(t_1,\xi) + 2u(t_2,\xi) + 3u(t_3,\xi) + \cdots + nu(t_n,\xi)], \quad \xi \in [0,\pi], \\ \lim\limits_{\varepsilon\longrightarrow 0^+} u\left(\dfrac{1}{2}+\varepsilon,\xi\right) = \lim\limits_{\varepsilon\longrightarrow 0^+} u\left(\dfrac{1}{2}-\varepsilon,\xi\right) + \dfrac{|u(\frac{1}{2},\xi)|}{n+|u(\frac{1}{2},\xi)|}, \quad \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $n \in \mathbb{N}$ such that $3 < n$ and $0 < t_1 < t_2 < t_3 < \cdots < t_n < 1$ are given real constants.

Let $X = L^2([0,\pi],\mathbb{R})$ and define the operator A as follows

$$A = -\frac{\partial^2(.)}{\partial \xi^2}, \quad D(A) = \{\varphi \in X : \ \varphi, \ \dot{\varphi} \text{ are absolutely continuous}, \ \ddot{\varphi} \in X \text{ and } \varphi(0) = \varphi(\pi) = 0\}.$$

It is well known that the operator A generates a compact semigroup $(T(t))_{t\geq 0}$ on X such that $\sup\limits_{t\geq 0}|T(t)| \leq 1$.

Next, we consider the change $x(t)(\xi) = u(t,\xi)$ and the following notations

$$f(t,x(t)) = \int_0^t \frac{|\cos(x(t-s))|}{1+|\sin(x(t-s))|} ds,$$

$$g(x) = \frac{1}{n^2}\sum_{i=1}^{n} ix(t_i),$$

$$h_1\left(x\left(\frac{1}{2}\right)\right) = \frac{|x(\frac{1}{2})|}{n+|x(\frac{1}{2})|}.$$

Then, equation (4.1) becomes as follows:

$$\begin{cases} \dfrac{d^{\frac{1}{2}}x(t)}{dt^{\frac{1}{2}}} & = Ax(t) + f(t, x(t)), t \in [0,1], \quad t \neq \frac{1}{2}, \\ x(0) & = g(x), \\ x(\frac{1}{2}^{+}) & = x(\frac{1}{2}^{-}) + h_1(x(\frac{1}{2})). \end{cases} \tag{4.2}$$

In this concrete application, we have $L_1 = \dfrac{1+2+3+\cdots+n}{n^2} = \dfrac{n+1}{2n}$, $C_1 = \dfrac{1}{n}$ and $\sup\limits_{t\in[0,1]} \left|T\left(\dfrac{t^\alpha}{\alpha}\right)\right| = \sup\limits_{t\in[0,1]} \left|T(2\sqrt{t})\right|$.

Now, returning back to Theorem (3.2), we obtain that

$$(L_1 + C_1) \sup_{t\in[0,1]} \left|T\left(\frac{t^\alpha}{\alpha}\right)\right| = \left(\frac{n+1}{2n} + \frac{1}{n}\right) \sup_{t\in[0,1]} |T(2\sqrt{t})| \leq \frac{n+1}{2n} + \frac{1}{n} = \frac{n+3}{2n} < 1.$$

Hence, we conclude that the above equation has at least one mild solution.

Conclusion

In this work, we have proved the Duhamel formula, existence, uniqueness, and stability of mild solutions of a class of nonlocal impulsive differential equations in the frame of the conformable fractional derivative. The main results are obtained by using the semigroup theory combined with some fixed point theorems. The ideas of this paper can be extended to other models in physics, biology, chemistry, economics and so forth.

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References

[1] N. Abada, M. Benchohra and H. Hammouche, “Existence results for semilinear differential evolution equations with impulses and delay”, Cubo, vol. 12, pp. 1–17, 2010.

[2] T. Abdeljawad, “On conformable fractional calculus”, J. Comput. Appl. Math., vol. 279, pp. 57–66, 2015.

[3] B. Ahmad, B. Alghamdi, R. P Agarwal and A. Alsaedi, “Riemann-Liouville fractional integro-differential equations with fractional non-local multi-point boundary conditions”, Fractals, vol. 30, no. 1, 11 pages, 2022.

[4] M. Al-Masaeed, E. Rabei and A. Al-Jamel, “Wkb approximation with conformable operator”, 2021, ArXiv:2111.01547.

[5] M. Atraoui and M. Bouaouid, “On the existence of mild solutions for nonlocal differential equations of the second order with conformable fractional derivative”, Adv. Difference Equ., Paper No. 447, 11 pages, 2021.

[6] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive differential equations and inclusions*, Comtemporary mathematics and its applications 2, New York: Hindawi Publishing Corporation, 2006.

[7] S. A. Bhanotar and M. K. Kaabar, “Analytical solutions for the nonlinear partial differential equations using the conformable triple Laplace transform decomposition method”, Int. J. Differ. Equ., 18 pages, 2021.

[8] T. T. Binh, N. H. Luc, D. O’Regan and N. H. Can, “On an initial inverse problem for a diffusion equation with a conformable derivative”, Adv. Difference Equ., Paper No. 481, 24 pages, 2019.

[9] G. Bonnano, R. Rodríguez-López and S. Tersian, “Existence of solutions to boundary value problem for impulsive fractional differential equations”, Fract. Calc. Appl. Anal., vol. 17, no. 3, pp. 717–744, 2014.

[10] M. Bouaouid, “Mild solutions of a class of conformable fractional differential equations with nonlocal conditions”, Acta Math. Appl. Sin. Engl. Ser., vol. 38, pp.1–13, 2022.

[11] M. Bouaouid, M. Atraoui, K. Hilal and S. Melliani, “Fractional differential equations with nonlocal-delay condition”, J. Adv. Math. Stud., vol. 11, no. 2, pp. 214–225, 2018.

[12] M. Bouaouid, M. Hannabou and K. Hilal, “Nonlocal conformable-fractional differential equations with a measure of noncompactness in Banach spaces”, J. Math., Art. ID 5615080, 6 pages, 2020.

[13] M. Bouaouid, K. Hilal and M. Hannabou, “Integral solutions of nondense impulsive conformable-fractional differential equations with nonlocal condition”, J. Appl. Anal., vol. 27, no. 2, pp. 187–197, 2021.

[14] M. Bouaouid, K. Hilal and S. Melliani, “Nonlocal conformable fractional Cauchy problem with sectorial operator”, Indian J. Pure Appl. Math., vol. 50, no. 4, pp. 999–1010, 2019.

[15] M. Bouaouid, K. Hilal and S. Melliani, “Nonlocal telegraph equation in frame of the conformable time-fractional derivative”, Adv. Math. Phys., 7 pages, 2019.

[16] M. Bouaouid, K. Hilal and S. Melliani, “Sequential evolution conformable differential equations of second order with nonlocal condition”, Adv. Difference Equ., Paper No. 21, 13 pages, 2019.

[17] M. Bouaouid, K. Hilal and S. Melliani, “Existence of mild solutions for conformable-fractional differential equations with non local conditions”, Rocky Mountain J. Math, vol. 50, no. 3, pp. 871–879, 2020.

[18] L. Byszewski, “Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem”, J. Math. Anal. Appl., vol. 162, no. 2, pp. 494–505, 1991.

[19] R. C. Cascaval, E. C. Eckstein, C. L. Frota and J. A. Goldstein, “Fractional telegraph equations”, J. Math. Anal. Appl., vol. 276, no. 1, pp. 145–159, 2002.

[20] J. Chen, F. Liu and V. Anh, “Analytical solution for the time-fractional telegraph equation by the method of separating variables”, J. Math. Anal. Appl., vol. 338, no. 2, pp. 1364–1377, 2008.

[21] K. Deng, “Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions”, J. Math. Anal. Appl., vol. 179, no. 2, pp. 630–637, 1993.

[22] A. El-Ajou, “A modification to the conformable fractional calculus with some applications”, Alexandria Eng. J., vol. 59, no. 4, pp. 2239–2249, 2020.

[23] H. Eltayeb, I. Bachar and M. Gad-Allah, “Solution of singular one-dimensional Boussinesq equation by using double conformable Laplace decomposition method”, Adv. Difference Equ., Paper No. 293, 19 pages, 2019.

[24] H. Eltayeb and S. Mesloub, “A note on conformable double Laplace transform and singular conformable pseudoparabolic equations”, J. Funct. Spaces, Art. ID 8106494, 12 pages, 2020.

[25] F. Gao and C. Chunmei, “Improvement on conformable fractional ferivative and its applications in fractional differential equations”, J. Funct. Spaces, Art. ID 5852414, 10 pages, 2020.

[26] Y. Giga, H. Mitake and S. Sato, "On the equivalence of viscosity solutions and distributional solutions for the time-fractional diffusion equation". J. Differential Equations, vol. 316, pp. 364–386, 2022.

[27] M. Hannabou, M. Bouaouid and K. Hilal, "Controllability of mild solution of nonlocal conformable fractional differential equations", Adv. Math. Phys., Art. ID 3671909, 8 pages, 2022.

[28] S. Injrou, R. Karroum and N. Deeb, "Various exact solutions for the conformable time-fractional generalized Fitzhugh-Nagumo equation with time-dependent coefficients", Int. J. Differ. Equ., Art. ID 8888989, 11 pages, 2021.

[29] M. Jneid and A. El Chakik, "Analytical solution for some systems of nonlinear conformable fractional differential equations", Far East J. Math. Sci. (FJMS), vol. 109, no. 2, pp. 243–259, 2018.

[30] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, "A new definition of fractional derivative", J. Comput. Appl. Math., vol. 264, pp. 65–70, 2014.

[31] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 204, Amsterdam: Elsevier, 2006.

[32] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of impulsive differential equations*, Series in modern applied mathematics 6, Singapore: World Scientific, 1989.

[33] J. Liang, J. H. Liu and T. J. Xiao, "Nonlocal impulsive problems for nonlinear differential equations in Banach spaces", Math. Comput. Modelling, vol. 49, no. 3–4, pp. 798–804, 2009.

[34] J. Liu and C. Zhang, "Existence and stability of almost periodic solutions to impulsive stochastic differential equations", Cubo, vol. 15, no. 1, pp. 77–96, 2013.

[35] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: John Wiley & Sons, Inc., 1993.

[36] S. Momani, "Analytic and approximate solutions of the space and time-fractional telegraph equations", Appl. Math. Comput., vol. 170, no. 2, pp. 1126–1134, 2005.

[37] G. M. Mophou, "Existence and uniqueness of mild solutions to impulsive fractional differential equations", Nonlinear Anal., vol. 72, no. 3–4, pp. 1604–1615, 2010.

[38] K. B. Oldham and J. Spanier, *The fractional calculus*, Mathematics in Science and Engineering 111, New York: Academic Press, Inc., 1974.

[39] W. E. Olmstead and C. A. Roberts, "The one-dimensional heat equation with a nonlocal initial condition", Appl. Math. Lett., vol. 10, no. 3, pp. 89–94, 1997.

[40] A. Pazy, *Semigropus of linear operators and applications to partial differential equations*, New York: Springer New York, 1983.

[41] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering 198, San Diego: Academic Press, Inc., 1999.

[42] E. Rabei, A. Al-Jamel and M. Al-Masaeed, "The solution of conformable Laguerre differential equation using conformable Laplace transform", 2021, ArXiv:2112.01322.

[43] S. G. Samko, A. A Kilbas and O. I Marichev, *Fractional integrals and derivatives: theory and applications*, Switzerland: Gordon & Breach Science Publishers, 1993.

[44] T. N. Thach, N. H. Can, V. V. Tri, "Identifying the initial state for a parabolic diffusion from their time averages with fractional derivative", Math. Meth. Appl. Sci, 16 pages, 2021.

[45] T. N. Thach, , D. Kumar, N. H. Luc and N. D. Phuong, "On a semilinear fractional reaction-diffusion equation with nonlocal conditions", Alexandria Eng. J., vol. 60, no. 6, pp. 5511–5520, 2021.

[46] T. N. Thach, D. Kumar, N. H. Luc and N. H. Tuan, "Existence and regularity results for stochastic fractional pseudo-parabolic equations driven by white noise", Discrete Contin. Dyn. Syst. Ser. S, vol. 15, no. 2, pp. 481–499, 2022.

[47] T. N. Thach and N. H. Tuan, "Stochastic pseudo-parabolic equations with fractional derivative and fractional Brownian motion", Stoch. Anal. Appl., vol. 40, no.2, pp. 328–351, 2022.

[48] N. H. Tuan, N. D. Phuong and T. N. Thach, "New well-posedness results for stochastic delay Rayleigh-Stokes equations", Discrete Contin. Dyn. Syst. Ser. B, vol. 28, no. 1, pp. 347–358, 2023.

[49] N. H. Tuan, T. N. Thach, N. H Can and D. O'Regan, "Regularization of a multidimensional diffusion equation with conformable time derivative and discrete data", Math. Methods Appl. Sci., vol. 44, no. 4, pp. 2879–2891, 2021.

[50] A. Zavalishchin, "Impulse dynamic systems and applications to mathematical economics", Dynam. Systems Appl., vol. 3, no. 3, pp. 443–449, 1994.

[51] Y. Zhou, V. Vijayakumar and R. Murugesu, "Controllability for fractional evolution inclusions without compactness", Evol. Equ. Control Theory, vol. 4, no. 4, pp. 507–524, 2015.

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On the minimum ergodic average and minimal systems

Manuel Saavedra[1]
Helmuth Villavicencio[2,✉]

[1] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.*
saavmath@pg.im.ufrj.br

[2] *Instituto de Matemática y Ciencias Afines, Universidad Nacional de Ingeniería, Lima, Peru.*
hvillavicencio@imca.edu.pe ✉

ABSTRACT

We prove some equivalences associated with the case when the average lower time is minimal. In addition, we characterize the minimal systems by means of the positivity of invariant measures on open sets and also the minimum ergodic averages. Finally, we show that a minimal system admits an open set whose measure is minimal with respect to a set of ergodic measures and its value can be chosen in $[0, 1]$.

RESUMEN

Demostramos algunas equivalencias asociadas con el caso cuando el tiempo inferior promedio es mínimo. Además, caracterizamos los sistemas minimales a través de la positividad de medidas invariantes en conjuntos abiertos y también los promedios ergódicos mínimos. Finalmente, mostramos que un sistema minimal admite un conjunto abierto cuya medida es mínima con respecto a un conjunto de medidas ergódicas y su valor puede ser elegido en $[0, 1]$.

Keywords and Phrases: Time average, minimum ergodic average, minimal systems.

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1 Introduction

The main motivation of this paper is the result obtained by Jenkinson in [3] which states that given an invariant measure there exists a continuous function that achieves the maximum ergodic average using such measure. This result has been used in several recent works [1, 6, 7]. A minimizing version of this result is possible to obtain in a straightforward way. In this sense, given the behavior of uniquely ergodic systems, it is natural to ask whether this version admits any relation to minimal systems and time averages. The present paper addresses both problems in the following way. Firstly, we prove some equivalences associated with the case when the average lower time is minimal. We also characterize the minimal systems by means of the positivity of invariant measures on open sets and also the minimum ergodic averages (this result was inspired by Theorem 6.17 in [9]). Finally, we show that given a finite set of ergodic measures in a minimal system it is possible to find an open set whose measure is minimal and its value can be chosen in $[0, 1]$. Let us state our results in a precise way.

Throughout this paper, the pair (X, d) denotes a compact metric space and $C(X)$ denotes the space of all continuous real-valued functions on X. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures of X, provided with the weak* topology. Let $T : X \to X$ be a continuous transformation. Given μ an element of $\mathcal{M}(X)$, we say that μ is T-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for every Borel subset A of X. We denote by $M_T(X)$ the set of T-invariant probability measures. A probability measure μ is called ergodic if $\mu(A) \in \{0, 1\}$ for each T-invariant set A. Denote by $\mathcal{E}_T(X)$ the set of ergodic measures. For $x \in X$, let δ_x be denote the Dirac point measure of x defined by $\delta_x(A) = 1$ when $x \in A$ and $\delta_x(A) = 0$ otherwise.

Let $f : X \to \mathbb{R}$ be a continuous function, we say that an invariant measure μ is f-minimizing if the minimum ergodic average [4] defined by

$$\alpha(f) = \min\left\{\int_X f dm : m \in M_T(X)\right\},$$

satisfies $\alpha(f) = \displaystyle\int_X f d\mu$. Given $x \in X$, recall that the lower time average is

$$\underline{\tau}(x, f) = \liminf_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} f \circ T^i(x).$$

We consider the number $E(x, f) = \underline{\tau}(x, f) - \alpha(f)$. This number quantifies the non-minimal time average. Note that $E(x, f) \geq 0$.

Next we state our first result that characterizes the cases where the non-minimal time average is equal to zero totally and partially uniform. For this purpose, we must recall that (X, T) is said to be uniquely ergodic if there is a unique invariant probability measure on X.

Theorem 1.1. *Let $T : X \to X$ be a continuous transformation of a compact metric space. For every $x \in X$ and $f \in C(X)$, we have the following equivalences*

(1) $E \equiv 0$ *if and only if* (X, T) *is uniquely ergodic.*

(2) $E(\cdot, f) = 0$ *if and only if* $\alpha(f) = \int_X f d\mu$, *for all* $\mu \in \mathcal{M}_T(X)$.

(3) $E(x, \cdot) = 0$ *if and only if every ergodic measure is a limit point of the sequence* $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \right\}$.

Our next result shows a characterization of the minimal systems through the open sets and minimum ergodic averages. Recall that a dynamical system (X, T) is called minimal if X does not contain any non-empty, proper, closed T-invariant subset.

Theorem 1.2. *Let $T : X \to X$ be a continuous transformation of a compact metric space. The following statements are equivalents:*

(1) (X, T) *is a minimal system.*

(2) *For each non empty open set* $A \subset X$ *and each* $\mu \in \mathcal{M}_T(X)$, *we have* $\mu(A) > 0$.

(3) *Every non-zero* $f \in C(X)$ *with* $f \geq 0$ *satisfies* $\alpha(f) > 0$.

Finally, in the case of non-discrete minimal systems, it is satisfied that the minimum ergodic average reaches all values of $[0, 1]$ for continuous functions of norm one. (see Lemma 2.8). Motivated by this, we found a condition on the ergodic measures [2] to obtain a version of this result through open sets.

Theorem 1.3. *Let $T : X \to X$ be a continuous transformation of a non-discrete compact metric space. If (X, T) is minimal and $\mathcal{F}$ is a finite subset of $\mathcal{E}_T(X)$, then for every $r \in [0, 1]$ there is an open set A such that r is the minimun value of $\mu(A)$ whenever $\mu \in \mathcal{F}$.*

The paper is organized as follows. In Section 2, we will prove several results necessary for the proof of the main theorems. Finally, in Section 3, we will prove Theorems 1.1, 1.2 and 1.3.

2 Preliminary lemmas

Let X be a compact metric space and $T : X \to X$ be a continuous transformation. We denote the applications

$$\begin{aligned} \alpha : C(X) &\longrightarrow \mathbb{R} \\ f &\longmapsto \min_{\mu \in \mathcal{M}_T(X)} \int_X f d\mu, \end{aligned}$$

and

$$\begin{aligned} E : X \times C(X) &\longrightarrow [0, +\infty) \\ (x, f) &\longmapsto \underline{\tau}(x, f) - \alpha(f). \end{aligned}$$

Below are some properties of these applications that are straightforward from the definition. Let Z be a convex set of a vector space V. A subset F of Z is called face of Z if whenever $x, y \in Z$ and $\lambda x + (1 - \lambda)y \in F$ with $0 < \lambda < 1$, then $\{x, y\} \subset F$.

Proposition 2.1. *We have the following properties*

(1) α *is continuous and* T*-invariant.*

(2) $\alpha(1) = 1$.

(3) $E(x, f) = E(x, f \circ T)$.

(4) $\alpha(f) \leq \alpha(g)$ *whenever* $f \leq g$.

(5) *The set* $\left\{\mu \in M_T(X) : \alpha(f) = \int_X f d\mu\right\}$ *is a non-empty closed face of* $M_T(X)$.

We will prove some additional properties of E

Lemma 2.2. *Let* $T : X \to X$ *be a continuous transformation of a compact metric space. It holds that* $E(x, f) = 0$ *for every* $x \in X$ *and* $f \in C(X)$ *if and only if the system* (X, T) *is uniquely ergodic.*

Proof. It is sufficient to prove that if $E \equiv 0$ then the system is uniquely ergodic. By Theorem 1 in [3] for every ergodic measure ν there exists an $f \in C(X)$ such that ν is the unique f-minimizing measure, that is, ν is the unique satisfying

$$\int_X f d\nu = \alpha(f).$$

Since $E(x, f) = 0$ for each $x \in X$, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = \int_X f d\nu. \tag{2.1}$$

If the system is not uniquely ergodic, there exists $\omega \in \mathcal{E}_T(X)$ such that $\omega \neq \nu$. Let p be a generic point for ω. Using (2.1), we have

$$\int_X f d\omega = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(p) = \int_X f d\nu < \int_X f d\omega.$$

It is a contradiction. So (X, T) is uniquely ergodic. □

Lemma 2.3. *Let $T : X \to X$ be a continuous transformation of a compact metric space. Given $f \in C(X)$. Then, $E(x,f) = 0$ for every $x \in X$ if and only if $\alpha(f) = \int_X f d\mu$, for all $\mu \in \mathcal{M}_T(X)$.*

Proof. By Proposition 2.1, we know that the set

$$H = \left\{ \nu \in \mathcal{M}_T(X) : \alpha(f) = \int_X f d\nu \right\},$$

is a non-empty closed face of $M_T(X)$. If $H \neq \mathcal{M}_T(X)$, then there is $\mu \in \mathcal{E}_T(X) \setminus H$. Let p be a generic point for μ, so

$$\int_X f d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(p) = \alpha(f), \tag{2.2}$$

the last equality in (2.2) is a consequence of the hypothesis $E(p,f) = 0$. Thus $\mu \in H$, which is absurd.

Conversely, given $x \in X$ we can find a sequence $\{N_k\}$ such that the inferior mean sojourn time is written as

$$\underline{\tau}(x,f) = \lim_{k\to\infty} \frac{1}{N_k} \sum_{i=0}^{N_k-1} f \circ T^i(x), \tag{2.3}$$

and also $\left\{ N_k^{-1} \sum_{i=0}^{N_k-1} \delta_{T^i(x)} \right\}$ is convergent to $\nu \in \mathcal{M}_T(X)$. Therefore $E(x,f) = 0$ since

$$\underline{\tau}(x,f) = \int_X f d\nu = \alpha(f). \qquad \square$$

A consequence of the above result is the following

Corollary 2.4. *The set $\{f \in C(X) : E(x,f) = 0$ for every $x \in X\}$ is a closed linear subspace of $C(X)$.*

Lemma 2.5. *Let $T : X \to X$ be a continuous transformation of a compact metric space. Given $x \in X$. Then, $E(x,f) = 0$ for each $f \in C(X)$ if and only if every ergodic measure is a limit point of the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \right\}$.*

Proof. Denote by Λ the set of the limit points of the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \right\}$. Suppose there is $\mu \in \mathcal{E}_T(X) \setminus \Lambda$. By Theorem 1 in [3], for every ergodic measure μ there exists an $f \in C(X)$ such that μ is the unique with the property

$$\int_X f d\mu = \alpha(f).$$

Since $E(x,f) = 0$, we have

$$\int_X f d\mu = \alpha(f) = \underline{\tau}(x,f).$$

Moreover, using (2.3), there is a sequence $\{m_k\}$ in $\mathcal{M}(X)$ such that

$$\underline{\tau}(x,f) = \lim_{k\to\infty} \int_X f dm_k.$$

We can assume that $\{m_k\}$ converges to $\nu \in \mathcal{M}_T(X)$. Then $\int_X f d\mu = \int_X f d\nu$ with $\nu \neq \mu$. It is a contradiction.

Conversely, given $f \in C(X)$ there exists an ergodic measure μ such that $\alpha(f) = \int_X f d\mu$. On the other hand, there is a sequence $\{N_k\}$ satisfying

$$\mu = \lim_{k\to\infty} N_k^{-1} \sum_{i=0}^{N_k-1} \delta_{T^i(x)}.$$

Therefore

$$\alpha(f) \leq \underline{\tau}(x,f) \leq \lim_{k\to\infty} \frac{1}{N_k} \sum_{i=0}^{N_k-1} f \circ T^i(x) = \int_X f d\mu = \alpha(f),$$

so $E(x,f) = 0$. $\square$

Now, we introduce the following auxiliary application

$$\begin{aligned} \varrho : \mathcal{T} \times \mathcal{C} \quad &\longrightarrow \quad [0,1] \\ (A, \mathcal{F}) \quad &\longmapsto \quad \varrho(A,\mathcal{F}) = \min_{\mu\in\mathcal{F}} \mu(A), \end{aligned}$$

where $\mathcal{T}$ denotes the topology associated with X and $\mathcal{C}$ denotes the space of all closed subsets of $\mathcal{M}_T(X)$. We write $\varrho(A) = \varrho(A, \mathcal{M}_T(X))$. Note that $\varrho(A)$ can be interpreted as the capacity of an open set (see Lemma 4.1 in [5]).

Lemma 2.6. *Let $T : X \to X$ be a continuous transformation of a compact metric space. It holds that $\varrho(A) > 0$ for every non-empty open set A if and only if $\alpha(f) > 0$ for each non-zero $f \in C(X)$ with $f \geq 0$.*

Proof. Given a non-zero $f \in C(X)$ with $f \geq 0$. We can find a non-empty open A and a constant $c > 0$ that verify $f(x) \geq c$ for all $x \in A$. It follows that

$$\alpha(f) \geq \min_{\mu\in\mathcal{M}_T(X)} \int_A f d\mu \geq c\varrho(A) > 0.$$

Hence $\alpha(f) > 0$.

Conversely, let A be a non-empty open set in X. By Urysohn's Lemma choose $f \in C(X)$ with $0 \leq f \leq 1$, $f(p) = 1$ and $f = 0$ on A^c for some $p \in A$. If $\varrho(A) = 0$, there exists some $\nu \in \mathcal{M}_T(X)$ such that $\nu(A) = 0$, therefore

$$0 < \alpha(f) \leq \int_A f d\nu = 0.$$

It is a contradiction. $\square$

Lemma 2.7. *Let $T : X \to X$ be a continuous transformation of a compact metric space. If (X,T) is a minimal system, then $\varrho(A) > 0$ for every non-empty open set A.*

Proof. Since (X,T) is minimal, for every non-empty open set A we have

$$X = \bigcup_{i=0}^{n} T^{-i}(A),$$

for some $n \in \mathbb{N}$, therefore $\varrho(A) \geq \frac{1}{n+1}$, that is, $\varrho(A) > 0$. $\square$

Lemma 2.8. *Let $T : X \to X$ be a continuous transformation of a non-discrete compact metric space. If (X,T) is minimal, then given $r \in [0,1]$, there is an $f \in C(X)$ with $\|f\|_\infty = 1$ such that $\alpha(f) = r$.*

Proof. Note that the set $B = \{f \in C(X) : \|f\|_\infty = 1\}$ is connected in $(C(X), \|.\|_\infty)$. Given $r \in (0,1)$, by Lemma 2.7 and since (X,T) is non-discrete, we obtain a non-empty open set A with the following property $0 < \varrho(A) < r/2$. By Urysohn's Lemma choose $g \in C(X)$ with $0 \leq g \leq 1$, $g(p) = 1$ and $g = 0$ on A^c for some $p \in A$. Therefore

$$\alpha(g) = \min_{\mu \in \mathcal{M}_T(X)} \int_A g d\mu \leq \varrho(A) < r/2.$$

By Proposition 2.1, α is continuous on B and $\alpha(1) = 1$. So, there exists $f \in C(X)$ with $\|f\|_\infty = 1$ such that $\alpha(f) = r$. Now for the remaining cases it is sufficient to consider the constant functions $f \equiv 1$ and $g \equiv -1$. Then $\alpha(f) = 1$ and $\alpha(g) = -1$, it follows that there is $h \in B$ such that $\alpha(h) = 0$. $\square$

If we denote

$$\widetilde{E}(x,A) = \underline{\tau}(x, \chi_A) - \varrho_A,$$

this value represents the non-minimal mean sojourn time on A. Also we can obtain that $\widetilde{E}(x,A) \in [0,1]$.

Recall that a point $x \in X$ is periodic for $T : X \to X$ if $T^n(x) = x$ for some $n \in \mathbb{N}$ and the minimal such n is called the period of T. A point $x \in X$ is called pre-periodic if some iterate of x is periodic. We denote by $\mathcal{O}(x)$ the orbit of x.

Lemma 2.9. *Let $T : X \to X$ be a continuous transformation of a compact metric space. It holds that $\widetilde{E}(x,A) \equiv 0$ for every $x \in X$ and $A \in \mathcal{T}$ if and only if every point in X is pre-periodic and there is only one periodic orbit.*

Proof. It is enough to prove the sufficiency. First, we claim that $\widetilde{E} \equiv 0$ implies that each measure in $M_T(X)$ is atomic. Suppose there is a non-atomic μ invariant measure. Given $z \in X$, we can find open sets $\{V_n^z\}_{n \in \mathbb{N}}$ such that $T^n(z) \in V_n^z$ and $\mu(V_n^z) < 1/2^{n+1}$. Therefore, the open set $A_z = \bigcup_n V_n^z$ contains the orbit of z, so $\underline{\tau}(z, A_z) = 1$. Thus $\widetilde{E}(z, A_z) > 1/2$ since $\varrho(A_z) \leq \mu(A_z) < 1/2$. This proves our claim. Let ν be an ergodic measure. There is $p \in X$ with $\nu(p) > 0$. By the Poincaré's Recurrence (Theorem 1.2.4 in [8]), the point p is periodic. Given $x \in X$, if $X = \mathcal{O}(p)$, then there is nothing to prove. Otherwise, the open set $B = X \setminus \mathcal{O}(p)$ satisfies $\varrho_B = 0$, so $\underline{\tau}(x, B) = 0$. This implies that $\mathcal{O}(x) \not\subset B$. Hence, there exists a periodic point p such that for each $x \in X$ there exists $k \in \mathbb{N}$ satisfying $T^k(x) \in \mathcal{O}(p)$. □

3 Proof of the theorems

Proof of Theorem 1.1. The proof of this result is actually contained in the Lemmas 2.2, 2.3 and 2.5. □

Proof of Theorem 1.2. To prove that Item (1) implies Item (2), we use Lemma 2.7. To prove that Item (2) implies Item (1), assume that (X, T) is not a minimal system. There exists some point $x \in X$ whose orbit is not dense in X. We consider the non-empty open set $A = X \setminus \overline{\mathcal{O}(x)}$, so $\varrho(A) > 0$. On the other hand, there are a sequence $\{N_k\}$ and a measure $\mu \in \mathcal{M}_T(X)$ satisfying

$$\mu = \lim_{k\to\infty} N_k^{-1} \sum_{i=0}^{N_k - 1} \delta_{T^i(x)},$$

therefore

$$\varrho(A) \leq \mu(A) \leq \liminf_{k\to\infty} \frac{1}{n_k} |\{0 \leq i \leq N_k - 1 : T^i(x) \in A\}| = 0.$$

It is a contradiction. Finally, the Lemma 2.6 proves the equivalence between Item (2) and Item (3). □

Proof of Theorem 1.3. Suppose that there exists $r \in (0, 1)$ such that $\varrho(A, \mathcal{F}) \neq r$ for every open set A . We consider the set

$$\mathcal{Z} = \{B : B \text{ is open in } X \text{ and } 0 < \varrho(B, \mathcal{F}) < r\}.$$

By Lemma 2.7, we obtain that $\mathcal{Z}$ is non-empty since (X, T) is non-discrete. We partially order $\mathcal{Z}$ by inclusion. Assume $\{B_i\}_{i\in I} \subset \mathcal{Z}$ is a totally ordered subset of $\mathcal{Z}$ where I is infinite. An upper bound for the B_i's in $\mathcal{Z}$ is the open set $\mathfrak{B} = \bigcup_{i\in I} B_i$. Since I is infinite, we can suppose that $\mathbb{N} \subset I$. We choose an increasing sequence $\{A_j\}$ such that $\mathfrak{B} = \bigcup_{j\in\mathbb{N}} A_j$. If $\mathcal{F} = \{\mu_\ell\}_{\ell=1}^N$, then there exists ℓ such that the set $K_\ell = \{j \in \mathbb{N} : \varrho(A_j, \mathcal{F}) = \mu_\ell(A_j)\}$ is infinite. Thus, given $\nu \in \mathcal{F}$ we have

$$\nu(\mathfrak{B}) = \lim_{j\in K_\ell} \nu(A_j) \geq \lim_{j\in K_\ell} \mu_\ell(A_j) = \mu_\ell(\mathfrak{B}),$$

then $\varrho(\mathfrak{B}, \mathcal{F}) = \mu_\ell(\mathfrak{B})$. On the other hand, since $\mathfrak{B} = \bigcup_{j\in\mathbb{N}} A_j$ using the regularity of the measure we have that there are $j \in \mathbb{N}$ and a compact K such that $K \subset A_j$ and $\mu_\ell(K) \leq \mu_\ell(A_j) < r$. So, $\mu_\ell(\mathcal{B}) \leq r$ but for the hypothesis $\varrho(A, \mathcal{F}) \neq r$. Hence $\varrho(\mathfrak{B}, \mathcal{F}) = \mu_\ell(\mathfrak{B}) < r$, therefore $\mathfrak{B} \in \mathcal{Z}$. Zorn's lemma now tells us that $\mathcal{Z}$ contains a maximal element $\mathfrak{A}$. Let $\mu \in \mathcal{F}$ such that $\mu(\mathfrak{A}) = \varrho(\mathfrak{A}, \mathcal{F}) < r$. Given $x \in X$, there is an open set U_x with $\mu(U_x) < r - \mu(\mathfrak{A})$. Hence

$$\varrho(\mathfrak{A} \cup U_x, \mathcal{F}) \leq \mu(\mathfrak{A} \cup U_x) \leq \mu(\mathfrak{A}) + \mu(U_x) < r.$$

Using the maximality of $\mathfrak{A}$ it is concluded that $U_x \subset \mathfrak{A}$ for every $x \in X$, so $\mathfrak{A} = X$. It implies $\varrho(\mathfrak{A}, \mathcal{F}) = 1 > r$, which is absurd. $\square$

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References

[1] S. Addas-Zanata and F. A. Tal, "Support of maximizing measures for typical C^0 dynamics on compact manifolds", Discrete Contin. Dyn. Syst., vol. 26, no. 3, pp. 795–804, 2010.

[2] W. Huang, Z. Lian, S. Shao and X. Ye, "Minimal systems with finitely many ergodic measures", J. Funct. Anal., vol. 280, no. 12, Paper No. 109000, 42 pages, 2021.

[3] O. Jenkinson, "Every ergodic measure is uniquely maximizing", Discrete Contin. Dyn. Syst., vol. 16, no. 2, pp. 383–392, 2006.

[4] O. Jenkinson, "Ergodic optimization in dynamical systems", Ergodic Theory Dynam. Systems, vol. 39, no. 10, pp. 2593–2618, 2019.

[5] K. Liu, L. Xu and R. Zhang, "Time-restricted sensitivity and entropy", J. Differential Equations, vol. 293, pp. 70–85, 2021.

[6] I. Morris, "Lyapunov-maximizing measures for pairs of weighted shift operators", Ergodic Theory Dynam. Systems, vol. 39, no. 1, pp. 225–247, 2019.

[7] I. Morris, "Prevalent uniqueness in ergodic optimisation", Proc. Amer. Math. Soc., vol. 149, no. 4, pp. 1631–1639, 2021.

[8] M. Viana and K. Oliveira, *Foundations of ergodic theory*, Cambridge studies in advanced mathematics 151, Cambridge: Cambridge University Press, 2016.

[9] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics 79, New York: Springer New York, 1982.

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Positive solutions of nabla fractional boundary value problem

N. S. Gopal[1]
J. M. Jonnalagadda[1, ✉]

[1] *Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad - 500078, Telangana, India.*
nsgopal94@gmail.com
j.jaganmohan@hotmail.com ✉

ABSTRACT

In this article, we consider the following two-point discrete fractional boundary value problem with constant coefficient associated with Dirichlet boundary conditions.

$$\begin{cases} -\big(\nabla^{\nu}_{\rho(a)}u\big)(t) + \lambda u(t) = f(t, u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases}$$

where $1 < \nu < 2$, $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_3$, $\mathbb{N}^{b}_{a+2} = \{a+2, a+3, \ldots, b\}$, $|\lambda| < 1$, $\nabla^{\nu}_{\rho(a)}u$ denotes the ν^{th}-order Riemann–Liouville nabla difference of u based at $\rho(a) = a - 1$, and $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}^{+}$.
We make use of Guo–Krasnosels'kiĭ and Leggett–Williams fixed-point theorems on suitable cones and under appropriate conditions on the non-linear part of the difference equation. We establish sufficient requirements for at least one, at least two, and at least three positive solutions of the considered boundary value problem. We also provide an example to demonstrate the applicability of established results.

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RESUMEN

En este artículo consideramos el siguiente problema de valor en la frontera de dos puntos discreto fraccional con coeficientes constantes asociado a condiciones de frontera de tipo Dirichlet

$$\begin{cases} -\big(\nabla^{\nu}_{\rho(a)}u\big)(t) + \lambda u(t) = f(t, u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases}$$

donde $1 < \nu < 2$, $a, b \in \mathbb{R}$ con $b - a \in \mathbb{N}_3$, $\mathbb{N}^{b}_{a+2} = \{a+2, a+3, \ldots, b\}$, $|\lambda| < 1$, $\nabla^{\nu}_{\rho(a)}u$ denota la nabla diferencia de Riemann–Liouville de u de orden ν basada en $\rho(a) = a - 1$, y $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}^{+}$.
Usamos los teoremas de punto fijo de Guo–Krasnosels'kiĭ y Leggett–Williams en conos adecuados y bajo condiciones apropiadas en la parte nolineal de la ecuación en diferencias. Establecemos requerimientos suficientes para al menos una, al menos dos, y al menos tres soluciones positivas del problema de valor en la frontera considerado. También entregamos un ejemplo para mostrar la aplicabilidad de los resultados.

Keywords and Phrases: Nabla fractional difference, boundary value problem, Dirichlet boundary conditions, positive solution, existence, fixed-point.

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1 Introduction

Nabla fractional calculus is a branch of mathematics that deals with arbitrary order differences and sums in the backward sense. The theory of nabla fractional calculus is still in its early stages, with the most important contributions appearing in the last two decades. Gray & Zhang [15] and Miller & Ross in [34] first introduced the concept of nabla fractional difference and sum. Atici & Eloe [2] developed the Riemann–Liouville type nabla fractional difference operator. They also studied the nabla fractional initial value problem, and established the exponential law, product rule, and nabla Laplace transform in this line. Several mathematicians [2, 3, 4, 5, 6, 7, 8, 16, 17, 21, 22] have contributed to the development of the theory of discrete fractional calculus in line with the theory of continuous fractional calculus. For historical references on continuous fractional calculus, see [28, 31, 32]. As a result of their works, today discrete fractional calculus has turned into a fruitful field of research in science and engineering. We refer here to recent monographs [9, 12, 29] and the references therein, which are important resources pertaining to this field of work.

The study of boundary value problems (BVPs) has a long past and can be followed back to the work of Euler and Taylor on vibrating strings. On the discrete fractional side, there is a sudden growth in interest for the development of nabla fractional BVPs. Many authors have studied nabla fractional BVPs recently. To name a few, Goar [11] and Ikram [18] worked with self-adjoint Caputo nabla BVPs. Gholami *et al.* [10] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla BVP with Dirichlet boundary conditions. Jonnalagadda [19, 20, 23] analysed some qualitative properties of two-point non-linear Riemann–Liouville nabla fractional BVPs associated with a variety of boundary conditions.

As pointed out earlier, many authors have studied the discrete fractional two-point boundary value problem like in [4, 19] and recently authors in [23] have worked with general nabla fractional difference equation with constant coefficients coupled with Dirichlet conditions, which resulted in for the first time Green's function in terms of discrete Mittag–Leffler function along with a few properties of the same. Compared to discrete Taylor monomial, discrete Mittag–Leffler function is an infinite series because of which it poses a challenge while proving positivity of Green's function. In the article, [23] the authors have overcome this challenge of proving positivity of Green's function. In the present article, we use the positivity of Green's function and prove an important lemma which helps us deal with conical mappings by proving that a ratio of infinite series is increasing or decreasing with respect to the ratio of its coefficient. To the best of our knowledge, no work has been done with Leggett–Williams fixed-point theorem in the nabla setting.

We consider the following boundary value problem

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)}u\right)(t) + \lambda u(t) = f(t, u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where $1 < \nu < 2$, $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_3$, $\mathbb{N}_{a+2}^b = \{a+2, a+3, \ldots, b\}$, $|\lambda| < 1$, $\nabla_{\rho(a)}^\nu u$ denotes the ν^{th}-order Riemann–Liouville nabla difference of u based at $\rho(a) = a - 1$, and $f : \mathbb{N}_{a+2}^b \times \mathbb{R} \to \mathbb{R}^+$.

The present paper is organized as follows: Section 2 contains preliminaries on nabla fractional calculus. In Section 3, we establish some properties of the Green's function associated with the nabla fractional boundary value problem (1.1) and construct the existence of at least one, at least two and at least three positive solutions with the help of Guo–Krasnosel'skiĭ and Leggett–Williams fixed-point theorems on suitable cones and under appropriate conditions on the non-linear part of the difference equation. Finally, we conclude this article with an example to demonstrate the applicability of our results.

2 Preliminaries

Denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{Z}^+$, respectively. We use the following notations, definitions and known results of nabla fractional calculus [12]. Assume empty sums and products are 0 and 1, respectively.

Definition 2.1. *For $a \in \mathbb{R}$, the sets $\mathbb{N}_a$ and $\mathbb{N}_a^b$, where $b - a \in \mathbb{Z}^+$, are defined by*

$$\mathbb{N}_a = \{a, a+1, a+2, \ldots\}, \quad \mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}.$$

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t)$, for $t \in \mathbb{N}_{a+N}$.

Definition 2.2 ([12])**.** *For $t \in \mathbb{R}\setminus\{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R}\setminus\{\ldots, -2, -1, 0\}$, the generalized rising function (many authors employ the Pochhammer symbol [33] to denote the same) is defined by*

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Also, if $t \in \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then we use the convention that $t^{\overline{r}} = 0$.

Definition 2.3 ([12])**.** *Let t, $a \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}$. The μ^{th}-order nabla fractional Taylor monomial is given by*

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)},$$

provided the right-hand side exists.

We observe the following properties of the nabla fractional Taylor monomials.

Lemma 2.4 ([18, 19])**.** *Let $\mu > -1$ and $s \in \mathbb{N}_a$. Then the following hold:*

(1) *If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geq 0$ and if $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$.*

(2) *If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is an increasing function of s.*

(3) *If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of t.*

(4) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of s.*

(5) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a non-decreasing function of t.*

(6) *If $t \in \mathbb{N}_s$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is an increasing function of t.*

(7) *If $0 < v \leq \mu$, then $H_v(t, a) \leq H_\mu(t, a)$, for each fixed $t \in \mathbb{N}_a$.*

Definition 2.5 ([12])**.** *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u is given by*

$$\left(\nabla_a^{-\nu} u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6 ([12])**.** *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by*

$$\left(\nabla_a^{\nu} u\right)(t) = \Big(\nabla^N \left(\nabla_a^{-(N-\nu)} u\right)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Lemma 2.7 ([13])**.** *Let a, b be two real numbers such that $0 < a \leq b$ and $1 < \alpha < 2$. Then $\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Lemma 2.8 ([12])**.** *Assume the successive fractional nabla Taylor monomials are well defined.*

(1) *Let $\nu > 0$ and $\alpha \in \mathbb{R}$. Then, $\nabla_a^{-\nu} H_\alpha(t, a) = H_{\alpha+\nu}(t, a)$, for $t \in \mathbb{N}_a$.*

(2) *Let ν, $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_1$ such that $n - 1 < \nu \leq n$. Then, $\nabla_a^{\nu} H_\alpha(t, a) = H_{\alpha-\nu}(t, a)$, for $t \in \mathbb{N}_{a+n}$.*

Finally, we present the definition of the nabla Mittag–Leffler function which is the nabla analogue of classical Mittag-Leffler function [14, 30].

Definition 2.9 ([12])**.** *Let α, β, $\lambda \in \mathbb{R}$ such that $\alpha > 0$ and $|\lambda| < 1$. The nabla Mittag–Leffler function is defined by*

$$E_{\lambda,\alpha,\beta}(t, a) = \sum_{n=0}^{\infty} \lambda^n H_{\alpha n+\beta}(t, a), \quad \text{for} \;\; t \in \mathbb{N}_a.$$

Theorem 2.10 ([23])**.** *Assume $1 < \nu < 2$, $-1 < \lambda < 1$ and $h : \mathbb{N}_{a+2} \to \mathbb{R}$. The unique solution of the nabla fractional boundary value problem*

$$\begin{cases} -\big(\nabla^{\nu}_{\rho(a)} u\big)(t) + \lambda u(t) = h(t), \quad t \in \mathbb{N}^b_{a+2}, \\ u(a) = u(b) = 0, \end{cases} \tag{2.1}$$

is given by

$$u(t) = \sum_{s=a+2}^{b} G(t,s)h(s), \quad t \in \mathbb{N}^b_a, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} G_1(t,s) = \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)), & s \in \mathbb{N}^b_{t+1}, \\ G_2(t,s) = \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - E_{\lambda,\nu,\nu-1}(t,\rho(s)), & s \in \mathbb{N}^t_{a+2}. \end{cases} \tag{2.3}$$

Now, we state some positive properties of the Green's function (2.3).

Lemma 2.11 ([23])**.** *Assume $1 < \nu < 2$ and $t \in \mathbb{N}_{a+2}$. For each $0 \le \lambda < 1$, denote by*

$$g(\lambda) = \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-3}(t,\rho(a)) \tag{2.4}$$

$$= \sum_{n=0}^{\infty} \lambda^n \frac{\Gamma(t-a+\nu n+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu n+\nu-2)}. \tag{2.5}$$

Then there exists a unique $\bar{\lambda} = \bar{\lambda}(t) \in (0,1)$ such that

$$g(\bar{\lambda}) = 0. \tag{2.6}$$

Take $\lambda^* = \min\limits_{t \in \mathbb{N}^b_{a+2}} \bar{\lambda}(t)$. Then, $0 < \lambda^* < 1$.

We observe the following properties of the nabla Mittag-Leffler function

Lemma 2.12 ([23])**.** *Assume $1 < \nu < 2$ and $0 \le \lambda < 1$. Then,*

(1) $0 < H_{\nu-1}(t,\rho(a)) \le E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *for* $t \in \mathbb{N}_a$;

(2) $E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *is an increasing function with respect to t for $t \in \mathbb{N}_a$;*

(3) $0 < H_{\nu-2}(t,\rho(a)) \le \nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *for* $t \in \mathbb{N}_{a+1}$;

(4) $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *is a decreasing function with respect to t for $t \in \mathbb{N}_{a+1}$ and $\lambda \in (0,\lambda^*]$;*

(5) $E_{\lambda,\nu,\nu-1}(t,\rho(s)) \le E_{\lambda,\nu,\nu-1}(t,a)$ *for $t \in \mathbb{N}_s$ and $s \in \mathbb{N}_{a+1}$;*

(6) $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) \ge \nabla E_{\lambda,\nu,\nu-1}(t,a)$ *for $t \in \mathbb{N}_s$, $s \in \mathbb{N}_{a+1}$ and $\lambda \in (0,\lambda^*]$.*

Lemma 2.13 ([27]). *Let (a_n) and (b_n) $(n = 0, 1, 2, \dots)$ be real numbers and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < r$. If $b_n > 0$, $n = 0, 1, 2, \dots$ and the sequence $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ is (strictly) increasing (decreasing), then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing (decreasing) on $[0, r)$.*

Theorem 2.14 ([23]). *Assume $1 < \nu < 2$ and $0 \leq \lambda < 1$ such that $\lambda \in (0, \lambda^*]$. The Green's function $G(t, s)$ defined in (2.3) satisfies $G(t, s) \geq 0$ for each $(t, s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. In particular, $G(a, s) = G(b, s) = 0$ and $G(t, s) > 0$ for each $(t, s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^b$.*

3 Multiple Positive Solutions

In this section, we establish sufficient conditions on existence of at least one, at least two and at least three positive solutions of (1.1) using Guo–Krasnosel'skiĭ and Leggett–Williams fixed-point theorems on conical shells.

Definition 3.1. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty convex set $K \subset \mathcal{B}$ is called a cone provided,*

(i) $\lambda_1 u \in K$, for all $u \in K$ and $\lambda_1 \geq 0$.

(ii) $u \in K$ and $-u \in K$ implies $u = 0$.

Definition 3.2. *A functional α_2 is said to be a non-negative continuous concave functional on a cone K of a real Banach space β, if $\alpha_2 : K \to [0, \infty)$ is continuous and*

$$\alpha_2(tx + (1-t)y) \geq t\alpha_2(x) + (1-t)\alpha_2(y),$$

for all $x, y \in K$ and $t \in [0, 1]$.

Definition 3.3. *An operator is called completely continuous, if it is continuous and maps bounded sets into precompact sets.*

Theorem 3.4 (Guo–Krasnosel'skiĭ fixed-point theorem, [24]). *Let $\mathcal{B}$ be a Banach space and $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume further that $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow \mathcal{K}$ is a completely continuous operator. If, either*

(1) *$\|Tu\| \leq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$; or*

(2) *$\|Tu\| \geq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$;*

holds, then T has at least one fixed-point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The following results are useful for the main results of this section.

Lemma 3.5. *Let a, b be two real numbers such that $0 < a \leq b$ and $1 < \nu < 2$. Then $\dfrac{E_{\lambda,\nu,\nu-1}(a,\rho(s))}{E_{\lambda,\nu,\nu-1}(b,\rho(s))}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Proof. For each $s \in \mathbb{N}_0^{a-1}$, denote by

$$a_n = H_{\nu n+\nu-1}(a,\rho(s)) \quad \text{and} \quad b_n = H_{\nu n+\nu-1}(b,\rho(s)), \quad n \in \mathbb{N}_0.$$

Clearly, a_n and b_n for $n \in \mathbb{N}_0$ are real numbers. Further, denote by

$$A(\lambda) = E_{\lambda,\nu,\nu-1}(a,\rho(s)) \quad \text{and} \quad B(\lambda) = E_{\lambda,\nu,\nu-1}(b,\rho(s)).$$

We know that the power series $A(\lambda)$ and $B(\lambda)$ are convergent for $|\lambda| < 1$. Also, $b_n > 0,\ n \in \mathbb{N}_0$ and the sequence

$$\left(\frac{a_n}{b_n}\right)_{n\geq 0} = \left(\frac{H_{\nu n+\nu-1}(a,\rho(s))}{H_{\nu n+\nu-1}(b,\rho(s))}\right)_{n\geq 0}$$

is strictly decreasing, by Lemma 2.7. Then, by Lemma 2.13, the function

$$\frac{A(\lambda)}{B(\lambda)} = \frac{E_{\lambda,\nu,\nu-1}(a,\rho(s))}{E_{\lambda,\nu,\nu-1}(b,\rho(s))}$$

is also strictly decreasing on $[0,1)$ for each $s \in \mathbb{N}_0^{a-1}$. The proof is complete. ☐

Theorem 3.6. *There exists a number $\gamma \in (0,1)$, such that*

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma \max_{t\in\mathbb{N}_a^b} G(t,s) = \gamma G(s-1,s), \tag{3.1}$$

for $\lambda \in (0,\lambda^*]$ and $c, d \in \mathbb{N}_{a+1}^{b-1}$ such that $c = a + \left\lceil \dfrac{b-a+1}{4} \right\rceil$ and $d = a + 3\left\lfloor \dfrac{b-a+1}{4} \right\rfloor$.

Proof. It follows from the proof of Theorem 2.14 in [23] that for each $\lambda \in (0,\lambda^*]$, $G(t,s)$ is an increasing function of t for $\in \mathbb{N}_a^{s-1}$ and is a decreasing function of t for $\in \mathbb{N}_s^b$. Thus, we have

$$\max_{t\in\mathbb{N}_a^b} G(t,s) = G(s-1,s) \text{ for } s \in \mathbb{N}_{a+2}^b.$$

Consider

$$\frac{G(t,s)}{G(s-1,s)} = \begin{cases} \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)}, & s \in \mathbb{N}_{t+1}^b, \\ \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \dfrac{E_{\lambda,\nu,\nu-1}(t,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}, & s \in \mathbb{N}_{a+2}^t. \end{cases}$$

Now, for $s > t$ and $c \leq t \leq d$, $G_1(t,s)$ is an increasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_1(t,s) = G_1(c,s) = \frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)), \quad s \in \mathbb{N}_{t+1}^b.$$

For $t > s$ and $c \leq t \leq d$, $G_2(t,s)$ is a decreasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_2(t,s) = G_2(d,s) = \frac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - E_{\lambda,\nu,\nu-1}(d,\rho(s)), \quad s \in \mathbb{N}_{a+2}^t.$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) = \begin{cases} G_1(c,s), & \text{for } s\in\mathbb{N}_d^b, \\ \min\{G_2(d,s), G_1(c,s)\}, & \text{for } s\in\mathbb{N}_{c+1}^{d-1}, \\ G_2(d,s), & \text{for } s\in\mathbb{N}_{a+2}^{c}, \end{cases}$$
$$= \begin{cases} G_2(d,s), & \text{for } s\in\mathbb{N}_{a+2}^{r}, \\ G_1(c,s), & \text{for } s\in\mathbb{N}_r^b, \end{cases}$$

where $c < r < d$. Consider

$$\frac{\min_{t\in\mathbb{N}_c^d} G(t,s)}{G(s-1,s)} = \begin{cases} \dfrac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)}, & s\in\mathbb{N}_r^b, \\ \dfrac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \dfrac{E_{\lambda,\nu,\nu-1}(d,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}, & s\in\mathbb{N}_{a+2}^r. \end{cases}$$

Hence,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma(s) \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.2}$$

where

$$\gamma(s) = \min\left[\frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)}, \frac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \frac{E_{\lambda,\nu,\nu-1}(d,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}\right].$$

For $s\in\mathbb{N}_r^b$, denote by

$$\gamma_1(s) = \frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} \geq \frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)}.$$

Similarly, for $s\in\mathbb{N}_{a+2}^r$, we take

$$\gamma_2(s) = \frac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \frac{E_{\lambda,\nu,\nu-1}(d,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}.$$

By Lemma 3.5, we see that $\dfrac{E_{\lambda,\nu,\nu-1}(d,\rho(s))}{E_{\lambda,\nu,\nu-1}(b,\rho(s))}$ is a decreasing function for $s\in\mathbb{N}_{a+2}^r$. Then,

$$\begin{aligned}\gamma_2(s) &\geq \frac{1}{E_{\lambda,\nu,\nu-1}(s-1,a)}\left[E_{\lambda,\nu,\nu-1}(d,a) - \frac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)}\right] \\ &> \frac{1}{E_{\lambda,\nu,\nu-1}(d,a)}\left[E_{\lambda,\nu,\nu-1}(d,a) - \frac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)}\right].\end{aligned}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.3}$$

where

$$\gamma = \min\left[\frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)}, 1 - \frac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)E_{\lambda,\nu,\nu-1}(d,a)}\right].$$

Since $G_1(c,s) > 0$ and $G_2(d,s) > 0$, we have $\gamma(s) > 0$ for all $s \in \mathbb{N}_{a+2}^b$, implying that $\gamma > 0$. It would be suffice to prove that one of the terms $\dfrac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)}, 1-\dfrac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)E_{\lambda,\nu,\nu-1}(d,a)}$ is less than 1. It follows from Lemma 2.12 that

$$\frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)} < 1.$$

Therefore, we conclude that $\gamma \in (0,1)$. The proof is complete. $\square$

By Theorem 2.10, we observe that u is a solution of (1.1) if and only if u is a solution of the summation equation

$$u(t) = \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.4}$$

Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (1.1) can be viewed as a real $(b-a+1)$-tuple vector. Consequently, $u \in \mathbb{R}^{b-a+1}$. Define the operator $T : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$ by

$$\big(Tu\big)(t) = \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.5}$$

Clearly, u is a fixed-point of T if and only if u is a solution of (1.1). We use the fact that $\mathbb{R}^{b-a+1}$ is a Banach space equipped with the maximum norm $\|u\| = \max_{t\in\mathbb{N}_a^b} |u(t)|$, for any $u \in \mathbb{R}^{b-a+1}$. Denote by

$$\mathcal{B} = \{u : \mathbb{N}_a^b \to \mathbb{R} \mid u(a) = u(b) = 0\} \subseteq \mathbb{R}^{b-a+1}. \tag{3.6}$$

Clearly $\mathcal{B}$ is a Banach space equipped with the maximum norm *i.e.*

$$\|u\| = \max_{t\in\mathbb{N}_a^b} |u(t)|.$$

Since T is defined on a discrete finite domain, it is trivially completely continuous. Define the cone

$$K = \{u \in \mathcal{B} : u(t) \geq 0 \text{ for } t \in \mathbb{N}_a^b, \text{ and } \min_{t\in\mathbb{N}_c^d} u(t) \geq \gamma\|u\|\}. \tag{3.7}$$

Lemma 3.7. *For $\lambda \in (0,\lambda^*]$ the operator T maps K into itself.*

Proof. Let $u \in K$. Clearly, $(Tu)(t) \geq 0$, whenever $u \in K$. Consider

$$\begin{aligned}
\min_{t\in\mathbb{N}_c^d} (Tu)(t) &= \min_{t\in\mathbb{N}_c^d} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d} [G(t,s)]\, f(s,u(s)) \\
&\geq \sum_{s=a+2}^{b} \gamma \max_{t\in\mathbb{N}_a^b} [G(t,s)]\, f(s,u(s)) \geq \gamma \max_{t\in\mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \\
&= \gamma \max_{t\in\mathbb{N}_a^b} \left| \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \right| \\
&= \gamma \|Tu\|.
\end{aligned}$$

Thus, we have $T : K \to K$ and it is completely continuous. The proof is complete. $\square$

Take

$$\eta = \frac{1}{\sum_{s=a+2}^{b} G(s-1,s)}.$$

Theorem 3.8. *Assume $f(t,u(t))$ satisfies the following conditions for $0 < r_1 < r_2$*

(i) There exists a number $r_1 > 0$ such that $f(t,u(t)) \leq \eta r_1$, whenever $0 \leq u \leq r_1$.

(ii) There exists a number $r_2 > 0$ such that $f(t,u(t)) \geq \frac{\eta r_2}{\gamma}$, whenever $\gamma r_2 \leq u \leq r_2$.

Then, for $\lambda \in (0,\lambda^*]$ the BVP (1.1) has at least one positive solution.

Proof. We know that $T : K \to K$ is completely continuous. Define the set

$$\Omega_1 = \{u \in K : \|u\| < r_1\}.$$

Clearly, $\Omega_1 \subseteq \beta$ is an open set with $0 \in \Omega_1$. Since $\|u\| = r_1$ for $u \in \partial\Omega_1$, condition (i) holds for all $u \in \partial\Omega_1$. So, it follows that

$$\begin{aligned}\|Tu\| &= \max_{t\in\mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) \leq \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b} G(t,s)f(s,u(s)) \leq \eta r_1 \sum_{s=a+2}^{b} G(s-1,s)\\ &= r_1 = \|u\|.\end{aligned}$$

implying that $\|Tu\| \leq \|u\|$ whenever $u \in K \cap \partial\Omega_1$. On the other hand, define the set

$$\Omega_2 = \{u \in K : \|u\| < r_2\}.$$

Clearly, $\Omega_2 \subseteq \beta$ is an open set and $\overline{\Omega}_1 \subseteq \Omega_2$. Since $\|u\| = r_2$ for $u \in \partial\Omega_2$, condition (ii) holds for all $u \in \partial\Omega_2$.

Thus, we have

$$\begin{aligned}\|Tu\| &\geq \min_{t\in\mathbb{N}_c^d} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) \geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d} G(t,s)f(s,u(s))\\ &\geq \gamma \sum_{s=a+2}^{b} G(s-1,s)f(s,u(s)) \geq \eta r_2 \sum_{s=a+2}^{b} G(s-1,s)\\ &= r_2 = \|u\|\end{aligned}$$

implying that $\|Tu\| \geq \|u\|$ whenever $u \in K \cap \partial\Omega_2$. Hence by part 1 of Theorem 3.4, T has at least one fixed-point in $K \cap (\overline{\Omega}_1 \backslash \Omega_1)$, say u_0 satisfying $r_1 < \|u_0\| < r_2$ $\square$

Theorem 3.9. *Assume $f(t,u(t))$ satisfies the following conditions*

(i) There exists a number $r_2 > 0$ such that $f(t,u(t)) \leq \eta r_2$, whenever $0 \leq u \leq r_2$.

(ii) $\displaystyle\lim_{u\to 0^+} \min_{t\in\mathbb{N}_a^b} \frac{f(t,u(t))}{u} = \infty, \quad \lim_{u\to\infty} \min_{t\in\mathbb{N}_a^b} \frac{f(t,u(t))}{u} = \infty.$

Then, for $\lambda \in (0,\lambda^*]$ the BVP (1.1) has at least two positive solution.

Proof. Let us choose a number $N > 0$ such that

$$\frac{N\gamma}{\eta} > 1,$$

by condition (ii) there exists a number $r^* > 0$ such that $r^* < r_1 < r_2$ and $f(t,u(t)) \geq Nu$ for $u \in [0,r^*]$ and $t \in \mathbb{N}_a^b$. Define the set $\Omega_{r^*} = \{u \in \mathcal{K} : \|u\| < r^*\}$. It can easily be shown that $\|Tu\| > \|u\|$, for $u \in \partial\Omega_{r^*} \cap \mathcal{K}$.

Next for the same N, we can find a number $R_1 > 0$ such that $f(t,u) \geq Nu$ for $u \geq R_1$ and $t \in \mathbb{N}_a^b$. Choose R such that $R = \max\left\{r_2, \frac{R_1}{\gamma}\right\}$. Define the set $\Omega_R = \{u \in \mathcal{K} : \|u\| < R\}$. We can show that $\|Tu\| > \|u\|$, for $u \in \partial\Omega_R \cap \mathcal{K}$.
Finally define the set

$$\Omega_2 = \{u \in \mathcal{K} : \|u\| < r_2\}.$$

Since $\|u\| = r_2$ condition (i) holds for all $u \in \partial\Omega_2$. Then, we have

$$\begin{aligned} \|Tu\| = \max_{t\in\mathbb{N}_b^a} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) &\leq \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a} [G(t,s)]\, f(s,u(s) \\ \leq r_2\eta \sum_{s=a+2}^{b} G(s-1,s) = r_2. \end{aligned}$$

Implying $\|Tu\| \leq \|u\|$, for $u \in \partial\Omega_{r_2} \cap \mathcal{K}$. Hence, we conclude that T has at least two fixed-points say $u_1 \in \Omega_2 \backslash \hat{\Omega}_{r^*}$ and $u_2 \in \Omega_R \backslash \hat{\Omega}_2$, where $\hat{\Omega}$ denoted the interior of the set Ω. In particular (1.1) has at least two positive solutions, say u_1 and u_2 satisfying $0 < \|u_1\| < r_2 < \|u_2\|$. The proof is complete. □

We state here the Leggett–Williams fixed-point theorem as follows. The proof can be found in [26] and also, we would like to refer here a paper by Kwong [25] on the same.

Denote

$$\begin{aligned} K_c =& \{u \in K : \|u\| < c\}, \\ K_{\alpha_2}(a,b) =& \{u \in K : a \leq \alpha_2(u), \|u\| \leq b\}, \end{aligned}$$

where α_2 is defined as in Definition 3.2.

Theorem 3.10 ([1])**.** *Let $T : \bar{K}_c \to \bar{K}_c$ be completely continuous and α_2 be a non-negative continuous concave functional on K, such that $\alpha_2(u) \leq \|u\|$, for all $u \in \bar{K}_c$. Suppose there exists $0 < d < a < b \leq c$, such that*

(1) $\{u \in K_{\alpha_2}(a,b) : \alpha_2(u) > a\} \neq \emptyset$ *and* $\alpha_2(Tu) > a$, *for* $u \in K_{\alpha_2}(a,b)$;

(2) $\|Tu\| < d$, *for* $\|u\| \leq d$;

(3) $\alpha_2(Tu) > a$, *for* $u \in K_{\alpha_2}(a,c)$ *with* $\|Tu\| > b$.

Then, T has at least three fixed-points u_1, u_2, u_3 satisfying

$$\|u_1\| < d, \quad a < \alpha_2(u_2),$$

and

$$\|u_3\| > d \text{ and } \alpha_2(u_3) < a.$$

We introduce here growth conditions on the non-linear function f in line with [1].

Theorem 3.11. *Suppose there exists numbers $a', b', d' \in \mathbb{R}^+$, where $0 < d' < a' < \gamma b' < b'$, such that f satisfies the following*

(1) $f(t,u(t)) > \dfrac{a'\eta}{\gamma}$, *if* $u \in [a', b']$;

(2) $f(t,u(t)) < d'\eta$, *if* $u \in [0, d']$;

(3) *There exists c' such that $c' > b'$ and if $u \in [0, c']$ then $f(t,u(t)) < c'\eta$;*

Then, the boundary value problem (1.1) for $\lambda \in (0, \lambda^*]$ has at least three positive solutions.

Proof. Define a non-negative continuous concave functional $\alpha_2 : K \to [0, \infty)$ with $\alpha_2(u) \leq \|u\|$, for all $u \in K$, by

$$\alpha_2(u) = \min_{t \in \mathbb{N}_c^d} u(t).$$

Claim 1: If there exists a positive number r such that $u \in [0, r]$ implies $f(u) < r\eta$, then $T : \bar{K}_r \to K_r$.

Suppose that $u \in \bar{K}_r$. Then,

$$\begin{aligned}
\|Tu\| = \max_{t \in \mathbb{N}_a^b} \left[\sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \right] &\leq \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} [G(t,s)]\, f(s,u(s)) \\
&= \sum_{s=a+2}^{b} G(s-1,s) f(s,u(s)) \\
&< r\eta \sum_{s=a+2}^{b} G(s-1,s) = r.
\end{aligned}$$

Thus, $T : \bar{K}_r \to K_r$. Hence, we have that if condition (3) holds, then there exists a number c' such that $c' > b'$ and $T : \bar{K}_{c'} \to K_{c'}$. Note that with $r = d'$ and using condition (2), we get that $T : \bar{K}_{d'} \to K_{d'}$.

Claim 2: $\{u \in K_{\alpha_2}(a', b') : \alpha_2(u) > a'\} \neq \emptyset$ and $\alpha_2(Tu) > a'$ for $u \in K_{\alpha_2}(a', b')$.

Since $u = \frac{a'+b'}{2} \in \{u \in K_{\alpha_2}(a', b') : \alpha_2(u) > a'\} \neq \emptyset$. Let $u \in K_{\alpha_2}(a', b')$. By using condition (1), we have

$$\begin{aligned}\alpha_2(Tu) &= \min_{t \in \mathbb{N}_c^d} \left[\sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \right] \geq \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} \left[G(t,s) \right] f(s,u(s)) \\ &\geq \gamma \sum_{s=a+2}^{b} G(s-1,s) f(s,u(s)) > a'\end{aligned}$$

Thus, if $u \in K_{\alpha_2}(a', b')$, then $\alpha_2(Tu) > a'$.

Claim 3: If $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$ then $\alpha_2(Tu) > a'$.

Suppose $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$. Then,

$$\begin{aligned}\alpha_2(Tu) &= \min_{t \in \mathbb{N}_c^d} \left[\sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \right] \geq \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} \left[G(t,s) \right] f(s,u(s)) \\ &\geq \gamma \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} \left[G(t,s) \right] f(s,u(s)) \geq \gamma \max_{t \in \mathbb{N}_c^d} \left[\sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \right] \\ &= \gamma \|Tu\| > \gamma b' > a'.\end{aligned}$$

Thus, $\alpha_2(Tx) > a'$.

Hence all the hypothesis of the Theorem 3.10 are satisfied. Therefore, the boundary value problem (1.1) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\|u_1\| < d', \quad a' < \alpha_2(u_2),$$

and

$$\|u_3\| > d' \text{ and } \alpha_2(u_3) < a'.$$

The proof is complete. $\square$

Example

In this section, we have constructed a suitable example to illustrate the applicability of the established results.

Example 3.12. *Take* $\nu = 1.5$, $a = 0, b = 5$, *and* $f(t, u(t)) = \frac{1}{20}\left(\sqrt{u} + u^2\right)$. *Then,* (1.1) *becomes*

$$\begin{cases} -\left(\nabla^{1.5}_{\rho(0)} u\right)(t) + \lambda u(t) = \dfrac{1}{20}\left(\sqrt{u} + u^2\right), \quad t \in \mathbb{N}_2^5, \\ u(0) = 0 = u(5). \end{cases} \tag{3.8}$$

Choose $\lambda^* = 0.007$. *Then, we get*

$$\eta = \frac{1}{\sum\limits_{s=2}^{5} G(s-1, s)} = \frac{E_{\lambda,1.5,0.5}(5,0)}{\sum\limits_{s=2}^{5} E_{\lambda,1.5,0.5}(s-1,0)E_{\lambda,1.5,0.5}(5,s-1)} = 0.2473.$$

By taking $r_2 = 2$, *we have*

$$f(t,u) = \frac{1}{20}\left(\sqrt{u} + u^2\right) \le \frac{1}{20}\left(\sqrt{r_2} + r_2^2\right) = 0.270 < \eta r_2 = 0.4946,$$

implying that $f(t, u)$ *satisfies conditions (i) and (ii) of Theorem 3.9. Thus, all conditions of Theorem 3.9 are satisfied. Hence,* (3.8) *has at least two positive solutions* u_1 *and* u_2 *such that* $0 < \|u_1\| < 2 < \|u_2\|$.

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References

[1] D. Anderson, R. Avery and A. Peterson, "Three positive solutions to a discrete focal boundary value problem", J. Comput. Appl. Math., vol. 88, no. 1, pp. 103–118, 1998.

[2] F. M. Atıcı and P. W. Eloe, "Discrete fractional calculus with the nabla operator", Electron. J. Qual. Theory Differ. Equ., Special Edition I, Paper No. 3, 12 pages, 2009.

[3] F. M. Atici and P. W. Eloe, "Linear systems of fractional nabla difference equations", Rocky Mountain J. Math., vol. 41, no. 2, pp. 353–370, 2011.

[4] F. M. Atıcı and P. W. Eloe, "Two-point boundary value problems for finite fractional difference equations", J. Difference Equ. Appl., vol. 17, no. 4, pp. 445–456, 2011.

[5] F. M. Atıcı and P. W. Eloe, "Gronwall's inequality on discrete fractional calculus", Comput. Math. Appl., vol. 64, no. 10, pp. 3193–3200, 2012.

[6] M. Bohner and A. Peterson, *Dynamic equations on time scales. An introduction with applications*, Boston: Birkhäuser Boston, 2001.

[7] P. Eloe and J. Jonnalagadda, "Mittag-Leffler stability of systems of fractional nabla difference equations", Bull. Korean Math. Soc. vol. 56, no. 4, pp. 977–992, 2019.

[8] P. Eloe and Z. Ouyang, "Multi-term linear fractional nabla difference equations with constant coefficients", Int. J. Difference Equ., vol. 10, no. 1, pp. 91–106, 2015.

[9] R. A. C. Ferreira, *Discrete fractional calculus and fractional difference equations*, Springer Briefs in Mathematics. Cham: Springer, 2022.

[10] Y. Gholami and K. Ghanbari, "Coupled systems of fractional ∇-difference boundary value problems", Differ. Equ. Appl., vol. 8, no. 4, pp. 459–470, 2016.

[11] J. St. Goar, "A Caputo boundary value problem in nabla fractional calculus", Ph. D. dissertation, Univ. Nebraska–Lincoln, Nebraska, 2016.

[12] C. Goodrich and A. C. Peterson, *Discrete fractional calculus*, Cham: Springer, 2015.

[13] N. S. Gopal and J. M. Jonnalagadda, "Existence and uniqueness of solutions to a nabla fractional difference equation with dual nonlocal boundary conditions", Foundations, vol. 2, pp. 151–166, 2022.

[14] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer Monographs in Mathematics, 2nd. ed., Berlin: Springer, 2020.

[15] H. L. Gray and N. F. Zhang, "On a new definition of the fractional difference", Math. Comp., vol. 50, no. 182, pp. 513–529, 1988.

[16] J. Henderson, "Existence of local solutions for fractional difference equations with Dirichlet boundary conditions", J. Difference Equ. Appl., vol. 25, no. 6, pp. 751–756, 2019.

[17] J. Henderson and J. T. Neugebauer, "Existence of local solutions for fractional difference equations with left focal boundary conditions", Fract. Calc. Appl. Anal., vol. 24, no. 1, pp. 324–331, 2021.

[18] A. Ikram, "Lyapunov inequalities for nabla Caputo boundary value problems", J. Difference Equ. Appl., vol. 25, no. 6, pp. 757–775, 2019.

[19] J. M. Jonnalagadda, "On two-point Riemann-Liouville type nabla fractional boundary value problems", Adv. Dyn. Syst. Appl., vol. 13, no. 2, pp. 141–166, 2018.

[20] J. M. Jonnalagadda, "Existence results for solutions of nabla fractional boundary value problems with general boundary conditions", Adv. Theory Non-linear Anal. Appl., vol. 4, no. 1, pp. 29–42, 2020.

[21] J. M. Jonnalagadda and N. S. Gopal, "On hilfer-type nabla fractional differences", Int. J. Differ. Equ., 2020, vol. 15, no. 1, pp. 91–107, 2020.

[22] J. M. Jonnalagadda and N. S. Gopal. "Linear Hilfer nabla fractional difference equations", Int. J. Dyn. Syst. Differ. Equ., vol. 11, no. 3–4, pp. 322–340, 2021.

[23] J. M. Jonnalagadda and N. S. Gopal, "Green's function for a discrete fractional boundary value problem", Differ. Equ. Appl., vol. 14, no. 2, pp. 163–178, 2022.

[24] M. A. Krasnosel'skiĭ, *Positive solutions of operator equations*, The Netherlands: P. Noordhoff Ltd., 1964.

[25] M. K. Kwong, "On Krasnoselskii's cone fixed point theorem", Fixed Point Theory Appl., Art. ID 164537, 18 pages, 2008.

[26] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces". Indiana Univ. Math. J., vol. 28, no. 4, pp. 673–688, 1979.

[27] K. Mehrez and S. M. Sitnik, "Functional inequalities for the Mittag-Leffler functions". Results Math., vol. 72, no. 1–2, pp. 703–714, 2017.

[28] K. S. Miller and B. Ross, "Fractional difference calculus" in *Univalent functions, fractional calculus, and their applications*, Ellis Horwood Series in Mathematics and its Applications, H. M. Srivastava and S. Owa, Chichester: Ellis Horwood Limited, 1989, pp. 139–152.

[29] P. Ostalczyk, *Discrete fractional calculus: Applications in control and image processing*, Singapore: World Scientific Publishing Co. Pte. Ltd, 2016.

[30] J. D. Paneva-Konovska, *From Bessel to multi-index Mittag-Leffler functions. Enumerable families, series in them and convergence*, London: World Scientific Publishing, 2017.

[31] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering 198, San Diego: Academic Press, Inc., 1999.

[32] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Switzerland: Gordon & Breach Science Publishers, 1993.

[33] J. Spanier and K. B. Oldham, "The Pochhammer Polynomials $(x)_n$", in *An Atlas of functions*, Washington, DC: Hemisphere Publishing Corporation, 1987, pp. 149–165.

[34] H. M. Srivastava and S. Owa (Eds.), *Univalent functions, fractional calculus, and their applications*, Ellis Horwood Series in Mathematics and its Applications, Chichester: Ellis Horwood Limited, 1989.

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Einstein warped product spaces on Lie groups

Buddhadev Pal[1]
Santosh Kumar[1,✉]
Pankaj Kumar[1]

[1] *Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India*
pal.buddha@gmail.com
thakursantoshbhu@gmail.com ✉
pankaj.kumar14@bhu.ac.in

ABSTRACT

We consider a compact Lie group with bi-invariant metric, coming from the Killing form. In this paper, we study Einstein warped product space, $M = M_1 \times_{f_1} M_2$ for the cases, (i) M_1 is a Lie group (ii) M_2 is a Lie group and (iii) both M_1 and M_2 are Lie groups. Moreover, we obtain the conditions for an Einstein warped product of Lie groups to become a simple product manifold. Then, we characterize the warping function for generalized Robertson-Walker spacetime, $(M = I \times_{f_1} G_2, -dt^2 + f_1^2 g_2)$ whose fiber G_2, being semi-simple compact Lie group of $\dim G_2 > 2$, having bi-invariant metric, coming from the Killing form.

RESUMEN

Consideramos un grupo de Lie compacto con métrica bi-invariante, que proviene de la forma de Killing. En este artículo estudiamos espacios productos alabeados de Einstein, $M = M_1 \times_{f_1} M_2$ para los casos (i) M_1 es un grupo de Lie (ii) M_2 es un grupo de Lie y (iii) ambos M_1 y M_2 son grupos de Lie. Más aún, obtenemos condiciones para que un producto alabeado de Einstein de grupos de Lie sea una variedad producto simple. Luego, caracterizamos la función de alabeo para el espacio-tiempo generalizado de Robertson-Walker, $(M = I \times_{f_1} G_2, -dt^2 + f_1^2 g_2)$ cuya fibra G_2 es un grupo de Lie compacto semi-simple de $\dim G_2 > 2$ con una métrica bi-invariante, que proviene de la forma de Killing.

Keywords and Phrases: Einstein space, warped product, Lie group, bi-invariant metric, Killing form.

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1 Introduction

R. L. Bishop and B. O'Neill [3], introduced the notion of warped product space to study the examples of complete Riemannian manifolds of negative sectional curvature. Authors proved that the completeness of warped space is followed by the completeness of base and fiber spaces. Further, the results for isometrically immersed warped product manifold into some Riemannian manifold were considered in [5, 6, 7]. In [9, 19], authors studied the conditions for the warping function to become a constant by using the relation between the scalar curvatures of a warped manifold with its base and fiber spaces.

The concept of the warped product has been generalized to the twisted warped product [11, 28], the doubly warped product and the multiply warped product [25, 32, 33]. A multiply warped product is a product manifold $M = B \times M_1 \times M_2 \times \cdots \times M_k$, equipped with the metric

$$g = \pi^*(g_B) + (f_1 \circ \pi_1)^2 \pi_2^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2) + \cdots + (f_1 \circ \pi_1)^2 \pi_2^*(g_k),$$

where (B, g_B) and (M_i, g_i), $i \in \{1, \ldots, k\}$, are pseudo-Riemannian manifolds, f_i are smooth functions on (M_i, g_i) and π_i are projections from M to M_i. In particular, if $B = (a, b)$, $k = 1$ and $g_B = -dt^2$, then M is known as a generalized Robertson-Walker spacetime [1, 10, 31]. A generalized Robertson-Walker spacetime with a fiber of constant scalar curvature is known as a Robertson-Walker spacetime. The simplest example for Robertson-Walker spacetime is an Einstein static universe. The product manifold $M = M_1 \times M_2$ with metric $g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2)$ is known as a doubly warped product space.

A pseudo-Riemannian manifold M with metric g is an Einstein manifold provided $Ric = cg$, where Ric is a Ricci curvature and c is some real constant. The Einstein metric g is of much interest, both in geometry and physics. A warped product with a constant warping function is considered as simply Riemannian product. In [2, p. 265], A. L. Besse proposed the question, "Does there exist a compact Einstein warped product with non-constant warping function?". Some answers to the question were given in [16, 30]. If M is an Einstein warped product space of nonpositive scalar curvature with a compact base manifold, then the warped product space is reduced to a simply Riemannian product [16]. In [24, 26], authors studied Einstein warped product space by using quarter and semi symmetric connections. The triviality results for Einstein warped product space with non-compact base manifold were studied in [30].

In 1976, Milnor investigated the curvature properties of left-invariant metrics in Lie groups [20]. Most of the Lie groups carry the more than one left-invariant metric, because in [18], authors showed that for a non-Abelian Lie group with a unique left-invariant metric up to homothety, the group is either the hyperbolic space H^n, or $R^{n-3} \times H_3$, where H_3 is a Heisenberg group. The Heisenberg group H_3 has a unique Riemannian metric up to homothety, whereas it has three metrics in the Lorentzian case [29]. Classifications for four-dimensional nilpotent Lie groups were considered in [4, 17]. The class of Lie groups obtaining a bi-invariant metric is smaller than that of Lie groups with a left-invariant metric. In [14, 15], authors study the warped product Einstein metrics on spaces of constant scalar curvature and homogeneous spaces. The classifications of warped product Einstein metric were studied in [13]. In [8, 22], the authors study the general helices and slant helices in three dimensional Lie group equipped with a bi-invariant metric.

In our paper, we discuss the few possible answers to the question “Does there exist a compact Einstein warped product with non-constant warping function?” for a compact Einstein warped product of Lie groups. We know that every compact Lie group has a bi-invariant metric and bi-invariant metric is much easier to handle than the left invariant metric. That is why, we use the bi-invariant metric in our paper. Now the results of the left invariant metric are still open to study. Section 2, of this paper includes some of the basic results. The central part of our paper is section 3, where we prove our main results for a warped product having either base manifold or fiber manifold is a compact Lie group with bi-invariant metric, coming from the Killing form. We show that an Einstein warped product space of nonnegative scalar curvature with a one-dimensional base manifold (Riemannian manifold) and fiber being a compact Lie group with bi-invariant metric, coming from the Killing form does not exist. Also, the characteristic of warping function in generalized Robertson-Walker spacetime is studied in Theorem 3.9. Finally, we give examples of warped products, obtained using a semi-simple compact Lie group taking bi-invariant metric from the Killing form.

2 Preliminaries

A Lie group G_1 is a smooth manifold with a group structure such that the multiplicative and inverse maps are smooth. To study the geometry of G_1, it becomes necessary to associate a left invariant metric with it. A metric in which left multiplication behaves as an isometry is known as a left invariant metric, and for a metric in which right multiplication behaves as an isometry is known as a right invariant metric. Left multiplication and right multiplication on G_1, are defined as $L_{a_1} : G_1 \mapsto G_1$, $L_{a_1}x_1 = a_1x_1$ and $R_{a_1} : G_1 \mapsto G_1$, $R_{a_1}x_1 = x_1a_1$, for all $a_1, x_1 \in G_1$. Let $\mathfrak{g}_1$ be the Lie algebra of G_1, then an adjoint representation, $Ad : G_1 \mapsto \mathfrak{g}_1$, of a Lie group G_1 is a map such that $Ad_{a_1} : \mathfrak{g}_1 \mapsto \mathfrak{g}_1$ is linear isomorphism given by $Ad_{a_1} = d(R_{a_1^{-1}} \circ L_{a_1})_{e_1}$ for all $a_1 \in G_1$. An inner product g_1 on $\mathfrak{g}_1$ is said to be Ad-invariant if

$$g_1(Ad_{a_1}X_1, Ad_{a_1}Y_1) = g_1(X_1, Y_1),$$

for all $a_1 \in G_1$ and $X_1, Y_1 \in \mathfrak{g}_1$.

A metric g_1, which is both left invariant and right invariant is said to be a bi-invariant metric.

The metric g_1 is bi-invariant if and only if

$$g_1([S_1, K_1], T_1) = g_1(K_1, [T_1, S_1]) = g_1(S_1, [K_1, T_1]),$$

for all $S_1, K_1, T_1 \in \mathfrak{g}_1$. Also, using the Koszul formula and above equation, we obtain

$$\nabla_{S_1} K_1 = \frac{1}{2}[S_1, K_1], \ \forall \ S_1, K_1 \in \mathfrak{g}_1.$$

Corresponding to bi-invariant metric g_1 on m_1- dimensional Lie group G_1, the Riemann curvature tensor R, and the Ricci tensor Ric, are given by

$$R(X_1, Y_1)Z_1 = \frac{1}{4}\big[[X_1, Y_1], Z_1\big],$$

$$Ric(X_1, Y_1) = \frac{1}{4} g_1\big([X_1, E_i], [Y_1, E_i]\big),$$

where $\{E_1, \ldots, E_{m_1}\}$, is an orthonormal frame for $\mathfrak{g}_1$. From [12, p. 622], we get the existence of bi-invariant metric on Lie group.

Proposition 2.1. *Let G_1 be a Lie group with Lie algebra $\mathfrak{g}_1$ and metric g_1, then g_1 induces a bi-invariant metric if and only if $\overline{Ad(G_1)}$ is compact. In other words, every compact Lie group has a bi-invariant metric.*

Also, for a connected Lie group G_1, the metric g_1 induce a bi-invariant metric if and only if $Ad_{a_1} : \mathfrak{g}_1 \mapsto \mathfrak{g}_1$, is skew adjoint for all $a_1 \in G_1$, which means

$$g_1(Ad_{a_1} X_1, Y_1) = -g_1(X_1, Ad_{a_1} Y_1), \qquad \forall \ X_1, Y_1 \in \mathfrak{g}_1.$$

Definition 2.2 ([2, 23]). *The Killing form $B : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{R}$ is a symmetric $B(X_1, Y_1) = B(Y_1, X_1)$, $Ad(G_1)$-invariant $B([X_1, Y_1], Z_1) = B(X_1, [Y_1, Z_1])$ and bilinear form, defined by*

$$B(X_1, Y_1) = tr(ad(X_1) \circ ad(Y_1)),$$

where $ad(X_1) : \mathfrak{g}_1 \mapsto \mathfrak{g}_1$ is a map, sending each Z_1 to $[X_1, Z_1]$, for all $X_1, Y_1, Z_1 \in \mathfrak{g}_1$.

A Killing form on a Lie group G_1 is nondegenerate if and only if G_1 is semisimple. In case of compact semisimple Lie group, the Killing form is always negative definite. From [23, p. 304–306], we have

Corollary 2.3. *Let G_1 be a semisimple compact Lie group with bi-invariant metric g_1, then*

$(a.1)$ *For nondegenerate plane spanned by S and K in $\mathfrak{g}_1$, the sectional curvature is given by*

$$\mathcal{K} = \frac{1}{4}\Big(\frac{g_1([S,K],[S,K])}{g_1(S,S)g_1(K,K) - g_1(S,K)g_1(S,K)}\Big).$$

$(a.2)$ *If the metric g_1 is induced from the Killing form, then G_1 is an Einstein $(Ric_1 = -\frac{1}{4}g_1)$ and the scalar curvature (τ), is given by*

$$\tau = \frac{1}{4}\dim(G).$$

It is clear from $(a.1)$, that if g_1 is a Riemannian metric then $\mathcal{K} \geq 0$ and $\mathcal{K} = 0$, if G_1 is an Abelian group.

Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds of dimensions m_1, m_2 and f_1 be a positive smooth function on M_1. Then for natural projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$, the warped product $(M = M_1 \times_{f_1} M_2, g)$ is a product manifold $M_1 \times M_2$ with the metric

$$g = \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2),$$

where * representing the pull-back operator and f_1 is a warping function on M. Whereas M_1 and M_2 are known as the base, and the fiber of (M, g), respectively. Let Ric, Ric_1 and Ric_2 are Ricci tensors on M, M_1 and M_2, respectively. Then from [23, p. 211], we have

Proposition 2.4. *Let $M = M_1 \times_{f_1} M_2$ be a warped product space, then Ricci tensors on M, M_1 and M_2, satisfies*

$$Ric = Ric_1 - \frac{m_2}{f_1}H^{f_1} + Ric_2 - f^\sharp g_2, \tag{2.1}$$

where $f^\sharp = -f_1\Delta f_1 + (m_2-1)g_1(grad\ f_1, grad\ f_1)$. Here $grad\ f_1$, H^{f_1} and Δf_1 denote the gradient of f_1, the Hessian of f_1 and the Laplacian of f_1, defined as $\Delta f_1 = -trH^{f_1}$.

Corollary 2.5. *The warped product $M = M_1 \times_{f_1} M_2$ is an Einstein with $Ric = \lambda g$ if and only if*

$(a.3)$ $Ric_1 = \lambda g_1 + \dfrac{m_2}{f_1}H^{f_1}$,

$(a.4)$ *(M_2, g_2) is an Einstein, such that $Ric_2 = \nu g_2$, where $\nu = f^\sharp + \lambda f_1^2$.*

3 Main results

Proposition 3.1. *Let (M_2, g_2) be a pseudo-Riemannian manifold and (G_1, g_1) be a semi-simple compact Lie group whose bi-invariant metric coming from the Killing form. Then warped product* *manifold* $(M = G_1 \times_{f_1} M_2, g)$*, is an Einstein manifold* $(Ric = \lambda g)$ *if and only if*

$(a.5)$ $H^{f_1} = -\frac{(1+4\lambda)f_1}{4m_2} g_1,$

$(a.6)$ (M_2, g_2) *is an Einstein with* $Ric_2 = \nu g_2$*, where*

$$\nu = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2.$$

Proof. Let $(M = G_1 \times_{f_1} M_2, g)$ be an Einstein manifold $(Ric = \lambda g)$, where (M_2, g_2) is a pseudo-Riemannian manifold and (G_1, g_1) is a semi-simple compact Lie group taking bi-invariant metric from the Killing form. Then from (2.1), we have

$$\lambda g_1 + f_1^2 \lambda g_2 = Ric_1 - \frac{m_2}{f_1} H^{f_1} + Ric_2 - f^\sharp g_2, \tag{3.1}$$

where λ is some constant and $f^\sharp = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1)$. Now, by restricting the argument (horizontal and vertical vectors) on G_1, M_2, and taking $Ric_1 = -\frac{1}{4} g_1$ in (3.1), we get

$$\begin{cases} \lambda g_1 & = -\frac{1}{4} g_1 - \frac{m_2}{f_1} H^{f_1}, \\ f_1^2 \lambda g_2 & = Ric_2 - f^\sharp g_2. \end{cases} \tag{3.2}$$

Conversely, assume that $(M = G_1 \times_{f_1} M_2, g)$ be a warped product with conditions $(a.5)$ and $(a.6)$. Then from (2.1), we get

$$Ric = \lambda g_1 + \frac{m_2}{f_1} H^{f_1} - \frac{m_2}{f_1} H^{f_1} + \nu g_2 - f^\sharp g_2. \tag{3.3}$$

Since $\nu = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2$, so from (3.3), we have

$$Ric = \lambda(g_1 + f_1^2 g_2) = \lambda g. \tag{3.4}$$

$\square$

Proposition 3.2. *Let* (M_1, g_1) *be a pseudo-Riemannian manifold and* (G_2, g_2) *be a semi-simple compact Lie group whose bi-invariant metric coming from the Killing form. Then warped product manifold* $(M = M_1 \times_{f_1} G_2, g)$*, is an Einstein manifold* $(Ric = \lambda g)$ *if and only if*

$(a.7)$ $Ric_1 = \lambda g_1 + \frac{m_2}{f_1} H^{f_1},$

$(a.8)$ (M_2, g_2) *is an Einstein with* $Ric_2 = \nu g_2$*, where*

$$\nu = -\frac{1}{4} = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2.$$

Proof. Since (G_2, g_2) is a semi-simple compact Lie group taking bi-invariant metric from the Killing form, so using $Ric_2 = -\frac{1}{4}g_2$ in $(a.6)$, we have

$$Ric_2 = -\frac{1}{4}g_2 = \nu g_2 = (f^\sharp + \lambda f_1^2)g_2. \quad \square$$

Lemma 3.3 ([16])**.** *Let f_1 be a smooth function on semi-Riemannian manifold M_1, then the divergence of Hessian tensor satisfies*

$$div(H^{f_1})(X_1) = Ric_1(grad\ f_1, X_1) - d(\Delta f_1)(X_1), \tag{3.5}$$

for all $X_1 \in \Gamma TM_1$.

Theorem 3.4. *Let (G_1, g_1) be a semi-simple compact Lie group of dimension $m_1 > 2$ and whose bi-invariant metric coming from the Killing form. If $4m_2H^{f_1} + (1+4\lambda)f_1g_1 = 0$, where $\lambda \in \mathbb{R}$, $m_2 \in \mathbb{N}$ and f_1 is a non constant smooth function on G_1, then f_1 satisfy the condition*

$$\nu = -f_1\Delta f_1 + (m_2 - 1)g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2,$$

where $\nu \in \mathbb{R}$.

Proof. The trace of $(a.5)$, provide us

$$\frac{m_2}{f_1}\Delta f_1 + \frac{(1+4\lambda)m_1}{4} = 0. \tag{3.6}$$

On differentiating (3.6), we get

$$\frac{m_2}{f_1^2}\left(\Delta f_1 df_1 - f_1 d(\Delta f_1)\right) = 0. \tag{3.7}$$

By the definition of divergence and Hessian for any vector field X_1 and g_1-orthonormal frame $\{E_1, \ldots, E_{m_1}\}$ on G_1, we have

$$\begin{aligned} div\left(\frac{1}{f_1}H^{f_1}\right)(X_1) &= \sum_i \epsilon_i \Big(D_{E_i}\big(\frac{1}{f_1}H^{f_1}\big)\Big)(E_i, X_1) \\ &= -\frac{1}{f_1^2}H^{f_1}(grad\ f_1, X_1) + \frac{1}{f_1}div(H^{f_1})(X_1), \end{aligned} \tag{3.8}$$

where $\epsilon_i = g_1(E_i, E_i)$. Using the fact that $Ric_1 = -\frac{1}{4}g_1$, in equation (3.5), the divergence of Hessian becomes

$$div(H^{f_1})(X_1) = -\frac{1}{4}g_1(grad\ f_1, X_1) - d(\Delta f_1)(X_1). \tag{3.9}$$

Also, from $(a.5)$ and $H^{f_1}(grad\ f_1, X_1) = (D_{X_1}df_1)(grad\ f_1) = \frac{1}{2}d(g_1(grad\ f_1, grad\ f_1))$, we have

$$-\frac{1}{4}g_1(grad\ f_1, X_1) = \frac{m_2}{2f_1}d(g_1(grad\ f_1, grad\ f_1))(X_1) + \lambda df_1(X_1). \tag{3.10}$$

In view of equations (3.9) and (3.10), the equation (3.8) becomes

$$div\left(\frac{1}{f_1}H^{f_1}\right) = \frac{1}{2f_1^2}\Big((m_2-1)d(g_1(grad\ f_1, grad\ f_1)) + 2\lambda_1 f_1 df_1(X_1) - 2f_1 d(\Delta f_1)\Big). \tag{3.11}$$

But the divergence of $(a.5)$, implies that $div\left(\frac{1}{f_1}H^{f_1}\right) = 0$. Hence from (3.11), we get

$$(m_2-1)d(g_1(grad\ f_1, grad\ f_1)) + 2\lambda_1 f_1 df_1 - 2f_1 d(\Delta f_1) = 0. \tag{3.12}$$

Therefore from equations (3.7) and (3.12), we obtain

$$d\Big((m_2-1)(g_1(grad\ f_1, grad\ f_1)) + \lambda_1 f_1^2 - f_1(\Delta f_1)\Big) = d(\nu) = 0. \tag{3.13}$$

Hence from equation (3.13), we can conclude that for a compact Einstein manifold (M_2, g_2) with dimension m_2 and $Ric_2 = \nu g_2$, the construction of an Einstein warped manifold $M = G_1 \times_{f_1} M_2$ is possible. □

Corollary 3.5. *Let $M = G_1 \times_{f_1} M_2$ be an Einstein warped product space with semi-simple compact Lie group G_1 of dimension $m_1 > 2$ and whose bi-invariant metric coming from the Killing form. Then M reduces to a simply Riemannian product.*

Proof. Rearranging the equation (3.6), we have

$$\Delta f_1 = \frac{(1+4\lambda)m_1}{4m_2}f_1. \tag{3.14}$$

As λ is a constant, so for $\lambda \leq -\frac{1}{4}$, equation (3.14) implies that $\Delta f_1 \leq 0$, hence f_1 is constant. Similarly if $\lambda \geq -\frac{1}{4}$, then $\Delta f_1 \geq 0$. Since according to the weak maximum principle, if f_1 is subharmonic or superharmonic *i.e.* ($\Delta f_1 \geq 0$ or $\Delta f_1 \leq 0$), then f_1 is constant [27, p. 75]. Hence M is a simply Riemannian product. □

In our next result, we prove that if fiber space of warped space is also a semi-simple compact Lie group of dimension $m_2 > 2$ and inherits the bi-invariant metric from the Killing form, then the only possible values for f_1 are ± 1.

Corollary 3.6. *Let G_1 and G_2 be semi-simple compact Lie groups of dimensions $m_1, m_2 > 2$ and bi-invariant metric tensors coming from their respective Killing forms. Then $M = G_1 \times_{f_1} G_2$ is an Einstein if and only if $f_1 = \pm 1$.*

Proof. Let $M = G_1 \times_{f_1} G_2$ be an Einstein, then from Proposition 3.1, Corollary 3.5 and using the fact that $Ric_2 = -\frac{1}{4}g_2$, we obtain, $\nu = \lambda = -\frac{1}{4}$. Therefore $f_1^2 = 1$.

Now conversely assume that $f_1 = \pm 1$, then $Ric = Ric_1 + Ric_2 = -\frac{1}{4}(g_1 + g_2) = -\frac{1}{4}g$, hence $M = G_1 \times G_2$ is an Einstein. □

Next, we consider those warped product spaces whose base is any pseudo-Riemannian manifold and fiber space is a semi-simple compact Lie group of dimension $m_2 > 2$, taking bi-invariant metric from the Killing form.

Theorem 3.7. *Let $M = M_1 \times_{f_1} G_2$ be an Einstein warped product space with fiber G_2 as a semi-simple compact Lie group of dimension $m_2 > 2$ and having bi-invariant metric coming from the Killing form. If M has negative scalar curvature, then the warped product becomes a simply Riemannian product.*

Proof. Let $M = M_1 \times_{f_1} G_2$ be an Einstein warped product space with fiber G_2 as a semi-simple compact Lie group of dimension $m_2 > 2$, having bi-invariant metric is coming from the Killing form. Then from $(a.4)$, we can say that

$$-f_1\Delta f_1 + (m_2 - 1)g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2 = -\frac{1}{4}. \tag{3.15}$$

Since M is an Einstein, therefore the trace of $Ric = \lambda g$, implies that

$$\tau = \lambda(m_1 + m_2), \tag{3.16}$$

where τ is a scalar curvature of M. Now assume that p_1 and p_2 are maximum and minimum points of f_1 on M_1. Therefore $grad\ f_1(p_1) = grad\ f_1(p_2) = 0$, $\Delta f_1(p_1) \geq 0$ and $\Delta f_1(p_2) \leq 0$. From (3.16) it is clear that $\tau \leq 0$, implies $\lambda \leq 0$, therefore

$$f_1(p_1)^2 \geq f_1(p_2)^2 \implies \lambda f_1(p_1)^2 \leq \lambda f_1(p_2)^2 \implies \lambda f_1(p_1)^2 + \frac{1}{4} \leq \lambda f_1(p_2)^2 + \frac{1}{4}, \tag{3.17}$$

where ν is some constant. Since $\Delta f_1(p_2)f(p_2) \leq 0$ and $\Delta f_1(p_1)f(p_1) \geq 0$, therefore from (3.15), $\lambda f_1(p_2)^2 + \frac{1}{4} \leq 0$ and $\lambda f_1(p_1)^2 + \frac{1}{4} \geq 0$, gives us

$$\lambda f_1(p_2)^2 + \frac{1}{4} \leq \lambda f_1(p_1)^2 + \frac{1}{4}. \tag{3.18}$$

Comparing equations (3.17) and (3.18), we have $f_1(p_1) = f_1(p_2)$ for $\lambda < 0$. □

Theorem 3.8. *Let $M = I \times_{f_1} G_2$ be an Einstein warped product space with the metric $g = dt^2 + f_1^2(t)g_2$, where I is an open interval in $\mathbb{R}$ and G_2 is a semi-simple compact Lie group of dimension $m_2 > 2$ and having bi-invariant metric coming from the Killing form. If M has non* *negative scalar curvature, then there does not exist any such f_1, so that $M = I \times_{f_1} G_2$ is an Einstein warped product space.*

Proof. Let $M = I \times_{f_1} G_2$ have positive scalar curvature ($\lambda > 0$). Then taking $f_1 = e^{\frac{u}{2}}$, the Hessian of f_1,

$$H^{f_1} = \frac{u''}{2}e^{\frac{u}{2}} + \frac{(u')^2}{4}e^{\frac{u}{2}}.$$

Using the above equation in (a.7), we have

$$\frac{u''}{2} + \frac{(u')^2}{4} = -\frac{\lambda}{m_2}. \tag{3.19}$$

Also, from (a.8), we get

$$\Big(\frac{u''}{2} + \frac{(u')^2}{4}\Big) + (m_2 - 1)\frac{(u')^2}{4} + \lambda = -\frac{1}{4}e^{-u}. \tag{3.20}$$

Thus from (3.19) and (3.20), we obtain

$$(u')^2 = -\Big(\frac{1}{m_2 - 1}e^{-u} + \frac{4}{m_2}\lambda\Big). \tag{3.21}$$

The possible solutions for (3.21) (with the help of Maple), are

$$\begin{cases} u = -\ln\Big(- \frac{4\lambda(m_2-1)}{m_2}\Big), \\ u = -\ln\Big(- \frac{4(m_2-1)}{m_2}\lambda\Big(1 + \tan^2\big(- \sqrt{\frac{\lambda}{m_2}}t + c\sqrt{\frac{\lambda}{m_2}}\big)\Big)\Big), \end{cases} \tag{3.22}$$

where c is some constant. It is clear from (3.22) that the function u is not well defined. Furthermore, as u is a real valued function, therefore $(u')^2 \geq 0$ and $-\Big(e^{-u}\frac{1}{m_2-1} + \frac{4}{m_2}\lambda\Big) < 0$, for any point on I. Therefore from equation (3.21), we can conclude that there does not exist any real solution for the equation.

For $\lambda = 0$, (a.7) and (a.8), imply that $f_1'' = 0$ and $f_1 f_1'' + (m_2 - 1)(f_1')^2 = -\frac{1}{4}$, respectively. Hence

$$f_1 = at + b \implies (m_2 - 1)(a)^2 = -\frac{1}{4}, \tag{3.23}$$

where a and b are some real constants. Thus from (3.22) and (3.23), we can say that there does not exist any such f_1 such that $M = I \times_{f_1} G_2$ be an Einstein warped product space of non negative scalar curvature. □

Next, we find the characteristic of warping function in generalized Robertson-Walker spacetime, whose fiber is semi-simple and compact Lie group of dimension $m_2 > 2$.

Theorem 3.9. *Let $M = I \times_{f_1} G_2$ be an Einstein warped product space with the metric $g = -dt^2 + f_1^2(t)g_2$, where I is an open interval in $\mathbb{R}$ and G_2 is a semi-simple compact Lie group of dimension $m_2 > 2$ and having bi-invariant metric coming from the Killing form. Then*

(*i*) *If M is Ricci flat, then there exists a non-constant function f_1 on I such that $f_1 = \frac{1}{2\sqrt{m_2-1}}t + b$, where b is some constant.*

(*ii*) *If M has positive scalar curvature ($\tau > 0$) or negative scalar curvature ($\tau < 0$), then there does not exist any such f_1, so that $M = I \times_{f_1} G_2$ be an Einstein warped product space.*

Proof. Let $M = I \times_{f_1} G_2$ be an Einstein warped product space with the metric $g = -dt^2 + f_1^2(t)g_2$, then from Proposition 3.2, we get

$$f_1'' = \frac{\lambda f_1}{m_2}, \quad \text{and} \quad f_1 f_1'' - (m_2 - 1)(f_1')^2 + \lambda f_1^2 = -\frac{1}{4}. \tag{3.24}$$

From these two differential equations, we obtain

$$(f_1')^2 - \frac{\lambda(1+m_2)}{m_2(m_2-1)} f_1^2 = \frac{1}{4(m_2-1)}. \tag{3.25}$$

As λ is constant, therefore to obtain the solutions for differential equation (3.25), we have to consider all possible values of λ.

(*i*) If $\lambda = 0$, then from (3.25), we obtain

$$f_1 = \frac{1}{2\sqrt{m_2-1}} t + b, \tag{3.26}$$

where b is some constant. Since f_1 is also satisfying (3.24), hence in the Ricci flat manifold case, it is possible to find a non-constant function on I.

(*ii*) (*a*) Let M be an Einstein manifold with positive scalar curvature $\lambda > 0$, then from (3.25), the possible solutions are

$$\begin{cases} f_1 = \pm\sqrt{\frac{-m_2}{4\lambda(1+m_2)}}, \\ f_1 = \frac{\sqrt{m_2(m_2-1)}}{2\sqrt{\lambda(1+m_2)}}\Big(-\frac{1}{4(m_2-1)}e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(c_1-t)} + e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(t-c_1)}\Big), \\ f_1 = \frac{\sqrt{m_2(m_2-1)}}{2\sqrt{\lambda(1+m_2)}}\Big(-\frac{1}{4(m_2-1)}e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(t-c_1)} + e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(c_1-t)}\Big), \end{cases} \tag{3.27}$$

where c_1 is some constant. As $m_2 > 2$, so $f_1 = \pm\sqrt{\frac{-m_2}{4\lambda(1+m_2)}} \notin \mathbb{R}$, hence constant solution of f_1 is not possible. From second and third part of (3.27), we have

$$f_1'' = \frac{\lambda(1+m_2)}{m_2(m_2-1)} f_1. \tag{3.28}$$

Equation (3.28), showing that second and third part of (3.27), is not satisfying (3.24). Hence there does not exist such type of f_1 which satisfies the equation (3.24) for $\lambda > 0$.

(b) Let M be an Einstein manifold with negative scalar curvature $\lambda < 0$, then (3.25), reduced to

$$(f_1')^2 + \frac{a(1+m_2)}{m_2(m_2-1)} f_1^2 = \frac{1}{4(m_2-1)}, \tag{3.29}$$

where $\lambda = -a$ and a is some positive real number. The solutions for differential equation (3.29), are

$$\begin{cases} f_1 = \pm\sqrt{\frac{m_2}{4a(1+m_2)}}, \\ f_1 = \pm\sqrt{\frac{m_2}{4a(1+m_2)}} \sin\left(\sqrt{\frac{a(1+m_2)}{m_2(m_2-1)}}(-t+c_1)\right). \end{cases} \tag{3.30}$$

Since solutions obtained in (3.30), are not satisfying the equation (3.24), hence there is no solution for (3.24). □

Examples for warped product of Lie groups

The Lie groups $\mathbf{SU}(n)$, $n \geq 2$, and $SO(n)$, $n \geq 3$ are examples of semi-simple compact Lie groups. The Lie algebra $\mathfrak{su}(n)$ of $\mathbf{SU}(n)$, set of $n \times n$ skew hermitian matrices with zero trace. For $n = 2$, the elements of $\mathfrak{su}(2)$,

$$X_1 = \begin{pmatrix} a_1\iota & a_2 + a_3\iota \\ -a_2 + a_3\iota & -a_1\iota \end{pmatrix}, \qquad a_1, a_2, a_3 \in \mathbb{R}.$$

Similarly, If $Y_1 \in \mathfrak{su}(2)$, then

$$Y_1 = \begin{pmatrix} b_1\iota & b_2 + b_3\iota \\ -b_2 + b_3\iota & -b_1\iota \end{pmatrix}, \qquad b_1, b_2, b_3 \in \mathbb{R}.$$

The basis E_1, E_2 and E_3, for $\mathfrak{su}(2)$, can be chosen as

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix}.$$

Hence Ad_{X_1} and Ad_{X_2}, are obtained as

$$Ad_{X_1} = \begin{pmatrix} 0 & -2a_1 & 2a_3 \\ 2a_1 & 0 & -2a_2 \\ -2a_3 & 2a_2 & 0 \end{pmatrix}, \qquad Ad_{X_2} = \begin{pmatrix} 0 & -2b_1 & 2b_3 \\ 2b_1 & 0 & -2b_2 \\ -2b_3 & 2b_2 & 0 \end{pmatrix}.$$

Thus, the Killing form $B(X_1, Y_1)$ on $\mathfrak{su}(2)$, will be

$$B(X_1, Y_1) = tr(Ad_{X_1} \circ Ad_{Y_1}) = -8a_1b_1 - 8a_2b_2 - 8a_3b_3 = 4tr(X_1Y_1).$$

So, we can made the following examples from all the above discussions.

1. The warped product manifold $M = \mathbf{SU}(2) \times_{f_1} M_2$, with metric $g = B + f_1^2 g_2$, where (M_2, g_2) is any pseudo-Riemannian manifold and non constant function f_1 on $\mathbf{SU}(2)$, is not an Einstein.
2. The product manifold $M = \mathbf{SU}(2) \times \mathbf{SO}(2)$, with metric $g = B_1 + B_2$, is an Einstein manifold, where B_1 and B_2 are Killing forms on $\mathfrak{su}(2)$ and $\mathfrak{so}(2)$, respectively.

Conclusion 3.10. *In [21], Mustafa proved that for every compact manifold G_1 there exist a metric on it such that non trivial Einstein warped products with base G_1 cannot be constructed. In our paper, from Corollary 3.5, we can say that bi-invariant metric generated by the Killing form on semi-simple compact Lie group G_1 is one in which we cannot construct non trivial Einstein warped product with base G_1.*

Data availability statement

No new data were created and analyzed in this study.

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References

[1] L. J. Alías, A. Romero and M. Sánchez, "Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes", Gen. Relativity Gravitation, vol. 27, no. 1, pp. 71–84, 1995.

[2] A. L. Besse, *Einstein manifolds*, Berlin: Springer-Verlag, 2007.

[3] R. Bishop and B. O'Neill, "Manifolds of negative curvature", Trans. Am. Math. Soc., vol. 145, pp. 1–49, 1969.

[4] N. Bokan, T. Šukilović and S. Vukmirović, "Lorentz geometry of 4-dimensional nilpotent Lie groups", Geom. Dedicata, vol. 177, pp. 83–102, 2015.

[5] B. Y. Chen, "Twisted product CR-submanifolds in Kaehler manifolds", Tamsui Oxf. J. Math. Sci., vol. 16, no. 2, pp. 105–121, 2000.

[6] B. Y. Chen, "Geometry of warped product CR-submanifolds in Kaehler manifolds", Monatsh. Math., vol. 133, no. 3, pp. 177–195, 2001.

[7] B. Y. Chen, "Geometry of warped products as Riemannian submanifolds and related problems", Soochow J. Math., vol. 28, no. 2, pp. 125–156, 2002.

[8] Ü. Ciftci, "A generalization of Lancret's theorem", J. Geom. Phys., vol. 59, no. 12, pp. 1597–1603, 2009.

[9] F. Dobarro and E. Lami Dozo, "Scalar curvature and warped products of Riemann manifolds", Trans. Amer. Math. Soc., vol. 303, no. 1, pp. 161–168, 1987.

[10] P. E. Ehrlich, Y. T. Jung and S. B. Kim, "Constant scalar curvatures on warped product manifolds", Tsukuba J. Math, vol. 20, no. 1, pp. 239–256, 1996.

[11] M. Fernández-López, E. García-Río, D. N. Kupeli and B. Ünal, "A curvature condition for a twisted product to be a warped product", Manuscripta Math., vol. 106, no. 2, pp. 213–217, 2001.

[12] J. Gallier and J. Quaintance, *Differential geometry and Lie groups: a computational perspective*, Geometry and Computing 12, Cham: Springer, 2020.

[13] C. He, P. Petersen and W. Wylie, "On the classification of warped product Einstein metrics", Comm. Anal. Geom., vol. 20, no. 2, pp. 271–311, 2012.

[14] C. He, P. Petersen and W. Wylie, "Warped product Einstein metrics over spaces with constant scalar curvature", Asian J. Math., vol. 18, no. 1, pp. 159–189, 2014.

[15] C. He, P. Petersen and W. Wylie, "Warped product Einstein metrics on homogeneous spaces and homogeneous Ricci solitons", J. Reine Angew. Math., vol. 707, pp. 217–245, 2015.

[16] D. Kim and Y. Kim, "Compact Einstein warped product spaces with nonpositive scalar curvature", Proc. Amer. Math. Soc., vol. 131, no. 8, pp. 2573–2576, 2003.

[17] J. Lauret, "Homogeneous nilmanifolds of dimensions 3 and 4", Geom. Dedicata, vol. 68, no. 2, pp. 145–155, 1997.

[18] J. Lauret, "Degenerations of Lie algebras and geometry of Lie groups", Differential Geom. Appl., vol. 18, no. 2, pp. 177–194, 2003.

[19] J. Meléndez and M. Hernández, "A note on warped products", J. Math. Anal. Appl., vol. 508, no. 2, pp. 161–168, 2022.

[20] J. Milnor, "Curvatures of left invariant metrics on Lie groups", Advances in Math., vol. 21, no. 3, pp. 293–329, 1976.

[21] M. T. Mustafa, "A non-existence result for compact Einstein warped products", J. Physics A., vol. 38, no. 47, pp. L791–L793, 2005.

[22] O. Z. Okuyucu, I. Gök, Y. Yayli and N. Ekmekci, "Slant helices in three dimensional Lie groups", Appl. Math. Comput., vol. 221, pp. 672–683, 2013.

[23] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, New York: Academic Press, 1983.

[24] S. Pahan, B. Pal and A. Bhattacharyya, "On Ricci flat warped products with a quarter-symmetric connection", J. Geom., vol. 107, no. 3, pp. 627–634, 2016.

[25] B. Pal and P. Kumar, "Compact Einstein multiply warped product space with nonpositive scalar curvature", Int. J. Geom. Methods Mod. Phys., vol. 16, no. 10, 14 pages, 2019.

[26] B. Pal and P. Kumar, "On Einstein warped product space with respect to semi symmetric metric connection", Hacet. J. Math. Stat., vol. 50, no. 50, pp. 1477–1490, 2021.

[27] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics 171, New York: Springer, 2006.

[28] R. Ponge, H. Reckziegel, "Twisted products in pseudo-Riemannian geometry", Geom. Dedicata, vol. 48, no. 1, pp. 15–25, 1993.

[29] J. Rahmani, "Métriques de Lorentz sur les groupes de Lie unimodulaires, de dimension trois", J. Geom. Phys., vol. 9, no. 3, pp. 295–302, 1992.

[30] M. Rimoldi, “A remark on Einstein warped products”, Pacific J. Math., vol. 252, no. 1, pp. 207–218, 2011.

[31] M. Sánchez, “On the geometry of generalized Robertson-Walker spacetimes: curvature and Killing fields”, J. Geom. Phys., vol. 31, no. 1, pp. 1–15, 1999.

[32] B. Ünal, “Multiply warped products”, J. Geom. Phys., vol. 34, no. 3–4, pp. 287–301, 2000.

[33] B. Ünal, “Doubly warped products”, Differential Geom. Appl., vol. 15, no. 3, pp. 253–263, 2001.

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Infinitely many solutions for a nonlinear Navier problem involving the p-biharmonic operator

Filippo Cammaroto[1, ✉]

[1] *Department of Mathematical and Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, Viale F. Stagno d'Alcontres, 31, 98166 Messina, Italy.*
`fdcammaroto@unime.it` ✉

ABSTRACT

In this paper we establish some results of existence of infinitely many solutions for an elliptic equation involving the p-biharmonic and the p-Laplacian operators coupled with Navier boundary conditions where the nonlinearities depend on two real parameters and do not satisfy any symmetric condition. The nature of the approach is variational and the main tool is an abstract result of Ricceri. The novelty in the application of this abstract tool is the use of a class of test functions which makes the assumptions on the data easier to verify.

RESUMEN

En este artículo establecemos algunos resultados sobre la existencia de infinitas soluciones para una ecuación elíptica que involucra los operadores p-biarmónico y p-Laplaciano acoplados con condiciones de borde de Navier, donde las nolinealidades dependen de dos parámetros reales y no satisfacen ninguna condición simétrica. La naturaleza del enfoque es variacional y la herramienta principal es un resultado abstracto de Ricceri. La novedad de la aplicación de esta herramienta abstracta es el uso de una clase de funciones test que hacen que las hipótesis sobre la data sean más fáciles de verificar.

Keywords and Phrases: p-biharmonic operator, p-Laplacian operator, Navier problem, multiplicity.

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1 Introduction

In this paper we investigate the existence of infinitely many solutions to the following p-biharmonic elliptic equation with Navier conditions,

$$\begin{cases} \Delta_p^2 u - \Delta_p u + V(x)|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \Omega \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $p > \max\left\{1, \frac{n}{2}\right\}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator, $\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $V \in C(\overline{\Omega})$ satisfying $\inf_{\overline{\Omega}} V > 0$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions with suitable behaviors, $\lambda \in \mathbb{R}$ and $\mu > 0$.

In the last years several authors have showed their interest in fourth-order differential problems involving biharmonic and p-biharmonic operators, motivated by the fact that this type of equations finds applications in fields such as the elasticity theory, or more in general, in continuous mechanics. In particular, the fourth-order elliptic equations can describe the static form change of beam or the motion of rigid body, so they are widely applied in physics and engineering. In 1990 Lazer and Mckenna, in a large paper in which they investigated the oscillatory phenomena that led to the collapse of the Tacoma Narrows bridge, considered fourth-order problems with the nonlinearity $(u+1)^+ - 1$; this nonlinearity is useful to study traveling waves in suspension bridges. Anyway the same authors observed that this kind of problems are interesting also when this particular nonlinearity is replaced by a somewhat more general function $F(\cdot, u)$ (see [24, 31, 32]).

As regards fourth-order differential problems involving biharmonic and p-biharmonic operators, a non-negligible part of the literature is devoted to the study of the existence of infinitely many solutions to problems involving only the biharmonic or p-biharmonic operator (see, for instance, [2, 4, 5, 6, 9, 10, 17, 18, 19, 29, 30, 40]) or considering also the presence of Laplacian or p-Laplacian operator ([22, 26, 38, 42, 43]) and/or a term with a potential function ([11, 12, 13, 25, 28]); some authors have also recently considered the case in which a nonlocal term is present ([16, 41]).

Unlike some papers concerning problems set in an unbounded domain (see [2, 4, 11, 12, 13, 18, 19, 30] and above all [25] which inspired us in the choice of this type of problem), most of the literature is devoted to the bounded case. In this case, different approaches have been adopted for obtaining infinitely many solutions. In a lot of papers symmetry conditions on the nonlinearities are assumed together with the use of the symmetric mountain pass theorem of Ambrosetti Rabinowitz (see [26, 40]) or with the use of the fountain theorem ([38, 42, 43]).

In our investigation the approach is variational. More precisely we will apply the following critical point theorem that Ricceri established in 2000 ([34, Theorem 2.5]), recalled below for the reader's convenience.

Theorem 1.1. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is (strongly) continuous and coercive. For each $r > \inf_X \Psi$, we put*

$$\varphi(r) := \inf_{x \in \Psi^{-1}(]-\infty, r[)} \frac{\Phi(x) - \inf_{\overline{\Psi^{-1}(]-\infty,r[)}_{\omega}} \Phi}{r - \Psi(x)}$$

where $\overline{\Psi^{-1}(]-\infty, r[)}_w$ is the closure of $\Psi^{-1}(]-\infty, r[)$ in the weak topology. Fixed $\lambda \in \mathbb{R}$, then

a) if $\{r_k\}$ is a real sequence such that $\lim\limits_{k\to\infty} r_k = +\infty$ and $\varphi(r_k) < \lambda$, for each $k \in \mathbb{N}$, the following alternative holds: either $\Phi + \lambda\Psi$ has a global minimum or there exists a sequence $\{x_k\}$ of critical points of $\Phi + \lambda\Psi$ such that $\lim\limits_{k\to\infty} \Psi(x_k) = +\infty$;

b) if $\{s_k\}$ is a real sequence such that $\lim\limits_{k\to\infty} s_k = (\inf\limits_x \Psi)^+$ and $\varphi(s_k) < \lambda$ for each $k \in \mathbb{N}$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi + \lambda\Psi$ or there exists a sequence $\{x_k\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$ with $\lim\limits_{k\to\infty} \Psi(x_k) = \inf\limits_X \Psi$, which weakly converges to a global minimum of Ψ.

Since its appearance in 2000 until our days, it has been a powerful tool to get multiplicity results for different kinds of problems. In particular, it has been widely applied to obtain theorems of existence of infinitely many solutions to problems associated with a vast range of differential equations. In each of these applications, in order to guarantee that $\varphi(r_k) < \lambda$ (or $\varphi(s_k) < \lambda$), for each $k \in \mathbb{N}$, and that the functional $\Phi + \lambda\Psi$ has no global minimum, it is necessary to use some sequences of functions defined *ad hoc*. Generally, in these functions the norm of the variable is raised to a suitable power which depends on the nature of the problem and that gives them the requested regularity properties: in some applications the norm is used without power (see, for instance, [3, 7, 14, 15, 23, 27, 39]), in some others it is raised to the second ([9, 10, 29, 33, 35, 36]) or to the third ([22, 28]) or to the fourth power ([1]); in [20, 21] the authors combined the norm with trigonometric functions.

The choice of a particular sequence of functions inside the proof reflects heavily on the assumptions and while there are some cases in which probably the choice is optimal, in some other cases it could happen that a different choice of the sequence would make the result applicable in a greater number of cases. This is the reason we have introduced an abstract class of test functions serving our purpose. We will clarify this fact in Section 3, showing some examples. A similar line of reasoning can be found in [8] and above all in [37] where the author does not choose the test functions arbitrarily during the proof but he uses two generic functions whose properties are described in the statement of his result.

2 Preliminaries

In this section we describe the variational framework in which we will work in our investigations.

To begin with, we denote by $\omega := \pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}+1\right)$ the measure of the unit ball in $\mathbb{R}^n$. If X is a Banach space, the symbol $B(x,r)$ stands for the open ball centered at $x \in X$ and of radius $r > 0$.

Let Ω be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 1$, $p > \max\left\{1, \frac{n}{2}\right\}$ and let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$. Put $E = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$; it is a reflexive Banach space when endowed with the standard norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}}.$$

Moreover, the assumptions on V assure that the position

$$\|u\|_V = \left(\int_\Omega \left(|\Delta u|^p + |\nabla u|^p + V(x)|u|^p\right) dx\right)^{\frac{1}{p}}$$

for any $u \in E$, defines a norm equivalent to the standard one. Being $p > \frac{n}{2}$, the Rellich-Kondrachov theorem assures that E is compactly embedded in $C^0(\overline{\Omega})$; in particular, there exists a constant $c_\infty > 0$ such that

$$\|u\|_\infty \leq c_\infty \|u\| \leq c_\infty \|u\|_V \tag{2.1}$$

for every $u \in E$. Now, motivated by the reasons that we have illustrated in the Introduction, let us introduce the following class of functions. If $\{a_k\}$, $\{b_k\}$, $\{\sigma_k\}$ are three real sequences with $0 < a_k < b_k$ and $\sigma_k > 0$, for each $k \in \mathbb{N}$, let us denote by $\mathcal{H}(\{a_k\},\{b_k\},\{\sigma_k\})$ the space of all sequences $\{\chi_k\} \subset W^{2,p}(]a_k, b_k[)$ satisfying

i) $0 \leq \chi_k(x) \leq \sigma_k$ for a.e. $x \in]a_k, b_k[$;

ii) $\lim\limits_{x\to a_k^+} \chi_k(x) = \sigma_k, \quad \lim\limits_{x\to b_k^-} \chi_k(x) = 0$;

iii) $\lim\limits_{x\to a_k^+} \chi_k'(x) = \lim\limits_{x\to b_k^-} \chi_k'(x) = 0$;

iv) for all $j \in \{1,2\}$ there exists $c_j > 0$, independent of k, such that

$$|\chi_k^{(j)}(x)| \leq c_j \frac{\sigma_k}{(b_k - a_k)^j} \tag{2.2}$$

for a.e. $x \in]a_k, b_k[$ and for all $k \in \mathbb{N}$.

Now, we show how the space $\mathcal{H}(\{a_k\},\{b_k\},\{\sigma_k\})$ help us to build some sequences in E that play a crucial role in the proof of the main result.

If $x_0 \in \Omega$, $\{b_k\} \subset]0, +\infty[$ such that $B(x_0, b_k) \subset \Omega$, for each $k \in \mathbb{N}$, and $\{\chi_k\} \in \mathcal{H}(\{a_k\},\{b_k\},\{\sigma_k\})$, consider the function $u_k : \Omega \to \mathbb{R}$ defined by setting

$$u_k(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(x_0, b_k), \\ \sigma_k & \text{in } B(x_0, a_k), \\ \chi_k(|x - x_0|) & \text{in } B(x_0, b_k) \setminus B(x_0, a_k) \end{cases}$$

for each $k \in \mathbb{N}$.

Simple computations show that, fixed $k \in \mathbb{N}$, for each $i \in \{1, \ldots, n\}$, we have

$$\frac{\partial u_k}{\partial x_i}(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(x_0, b_k), \\ 0 & \text{in } B(x_0, a_k), \\ \chi_k'(|x - x_0|) \dfrac{x_i - x_i^0}{|x - x_0|} & \text{in } B(x_0, b_k) \setminus B(x_o, a_k) \end{cases}$$

and

$$\frac{\partial^2 u_k}{\partial x_i^2}(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(x_0, b_k), \\ 0 & \text{in } B(x_0, a_k), \\ \chi_k''(|x - x_0|) \dfrac{(x_i - x_i^0)^2}{|x - x_0|^2} + \chi_k'(|x - x_0|) \dfrac{|x - x_0|^2 - (x_i - x_i^0)^2}{|x - x_0|^3} & \text{in } B(x_0, b_k) \setminus B(x_0, a_k) \end{cases}$$

Using these computations together with (2.2), we get the following inequalities

$$|\nabla u_k(x)| \leqslant |\chi_k'(|x - x_0|)| \leq c_1 \frac{\sigma_k}{(b_k - a_k)},$$

and

$$|\Delta u_k(x)| \leqslant |\chi_k''(|x - x_0|)| + |\chi_k'(|x - x_0|)| \frac{(n-1)}{|x - x_0|} \leq c_2 \frac{\sigma_k}{(b_k - a_k)^2} + c_1 \frac{\sigma_k}{(b_k - a_k)} \frac{(n-1)}{a_k}.$$

These inequalities allow us to estimate the norm of the functions u_k as follows

$$\begin{aligned} \|u_k\|_V^p &= \int_\Omega (|\Delta u_k|^p + |\nabla u_k|^p + V(x)|u_k(x)|^p)\, dx \\ &= \int_{B(x_0,b_k) \setminus B(x_0,a_k)} |\Delta u_k(x)|^p dx + \int_{B(x_0,b_k) \setminus B(x_0,a_k)} |\nabla u_k(x)|^p dx + \int_{B(x_0,b_k)} V(x)|u_k(x)|^p dx \\ &\leq \omega \sigma_k^p \left\{ \left[\frac{c_2}{(b_k - a_k)^2} + \frac{c_1(n-1)}{a_k(b_k - a_k)} \right]^p (b_k^n - a_k^n) + \left[\frac{c_1}{(b_k - a_k)} \right]^p (b_k^n - a_k^n) + b_k^n \max_{B(x_0,b_k)} V \right\}. \end{aligned}$$

Let us denote by $\mathcal{C}$ the class of all Carathéodory functions $\eta : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying $\sup_{|t| \leq \xi} |\eta(\cdot, t)| \in L^1(\Omega)$ for all $\xi > 0$ and let $f, g \in \mathcal{C}$.

We say that a function $u \in E$ is a weak solution to $(P_{\lambda,\mu})$ if

$$\int_\Omega \left(|\Delta u|^{p-2}\Delta u \Delta v + |\nabla u|^{p-2}\nabla u \nabla v + V(x)|u|^{p-2}uv\right) dx = \lambda \int_\Omega f(x,u(x))v(x)dx + \mu \int_\Omega g(x,u(x))v(x)dx$$

for each $v \in E$. Obviously the weak solutions to $(P_{\lambda,\mu})$ are exactly the critical points in E of the energy functional defined, for each $u \in E$, by

$$\mathcal{E}(u) := \frac{1}{p}\Psi(u) + \lambda \Phi_F(u) + \mu \Phi_G(u),$$

where

$$\Psi(u) := \|u\|_V^p, \quad \Phi_F(u) := -\int_\Omega F(x,u(x))dx, \quad \Phi_G(u) := -\int_\Omega G(x,u(x))dx,$$

where, for each $(x,t) \in \Omega \times \mathbb{R}$,

$$F(x,t) := \int_0^t f(x,s)ds, \quad G(x,t) := \int_0^t g(x,s)ds.$$

3 Results

The first multiplicity result deals with the case in which f has a global $(m-1)$-sublinear growth, with $m < p$, while different cases are considered for the behaviour of function g.

Theorem 3.1. *Let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$ and let $f, g \in \mathcal{C}$ such that:*

(i_1) there exist $1 < m < p$ and $h \in L^1(\Omega)$ such that $|f(x,t)| \leq h(x)\left(1 + |t|^{m-1}\right)$ for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$,

(i_2) $G(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$,

(i_3) there exists $x_0 \in \Omega$ and $\rho > 0$, $p_1, p_2 > 1$ such that $B(x_0,\rho) \subseteq \Omega$ and

$$\liminf_{t\to+\infty} \frac{\int_\Omega \max_{|\xi|\leq t} G(x,\xi)dx}{t^{p_1}} := a < +\infty, \quad \limsup_{t\to+\infty} \frac{\int_{B(x_0,\rho)} G(x,t)dx}{t^{p_2}} := b > 0.$$

Then the following facts hold:

(r_1) if $p_1 < p < p_2$, for all $\lambda \in \mathbb{R}$ and for all $\mu > 0$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;

(r_2) if $p_1 < p = p_2$, there exists $\mu_1 > 0$ such that for all $\lambda \in \mathbb{R}$ and for all $\mu > \mu_1$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;

(r_3) if $p_1 = p < p_2$, there exists $\mu_2 > 0$ such that for all $\lambda \in \mathbb{R}$ and for all $\mu \in]0, \mu_2[$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;

(r_4) if $p_1 = p_2 = p$, there exists $\gamma > 1$ and $C_{V,\gamma,\rho} > 0$ such that, if

$$C_{V,\gamma,\rho} < \frac{b}{\omega c_\infty^p a}, \tag{3.1}$$

(the previous inequality always being satisfied whether $a = 0$ or $b = +\infty$) then $\mu_1 < \mu_2$ and for all $\lambda \in \mathbb{R}$ and for all $\mu \in]\mu_1, \mu_2[$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

Proof. To prove (r_1), let us apply part $a)$ of Theorem 1.1 choosing $X = E$, Ψ defined as in the Preliminaries and $\Phi = \lambda\Phi_F + \mu\Phi_G$. As we have already observed the critical points of the functional $\Phi + \frac{1}{p}\Psi$ are precisely the weak solution of problem $(P_{\lambda,\mu})$. The functionals Φ and Ψ are sequentially weak lower semicontinuous and moreover Ψ is strongly continuous and coercive. In our case the function φ is defined by setting

$$\varphi(r) = \inf_{\|u\|_V^p < r} \frac{\Phi(u) + \sup_{\|w\|_V^p \le r}(-\Phi)}{r - \|u\|_V^p}$$

for each $r > 0$. Now, we wish to find a sequence $\{r_k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} r_k = +\infty$ and $\varphi(r_k) < \frac{1}{p}$ for each $k \in \mathbb{N}$. To this aim it suffices to prove that for each $k \in \mathbb{N}$ there exists a function $u_k \in X$, with $\|u_k\|_V^p < r_k$, such that

$$\sup_{\|w\|_V^p \le r_k} \left\{\lambda \int_\Omega F(x, w(x))dx + \mu \int_\Omega G(x, w(x))dx\right\} - \lambda \int_\Omega F(x, u_k(x))dx +$$

$$-\mu \int_\Omega G(x, u_k(x))dx < \frac{1}{p}\left(r_k - \|u_k\|_V^p\right).$$

Thanks to (i_3), fixed $\overline{a} > a$, for each $k \in \mathbb{N}$ there exists $\alpha_k \ge k$ such that

$$\int_\Omega \max_{|\xi|\le\alpha_k} G(x, \xi)dx \le \overline{a}\alpha_k^{p_1}.$$

Now we choose $u_k = \theta_E$ and

$$r_k = \frac{1}{c_\infty^p}\alpha_k^p.$$

Obviously we have $\lim\limits_{k\to\infty} r_k = +\infty$. Before proving (3), observe that, for each $w \in X$ with $\|w\|_V^p \le r_k$, one has

$$\|w\|_\infty \le c_\infty \|w\|_V \le c_\infty r_k^{\frac{1}{p}} = \alpha_k$$

for each $k \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}\lambda\int_\Omega F(x,w(x))dx+\mu\int_\Omega G(x,w(x))dx &\le |\lambda|\int_\Omega |h(x)|\left(|w(x)|+\frac{|w(x)|^m}{m}\right)dx\\ &\quad+\mu\int_\Omega \max_{|\xi|\le\alpha_k} G(x,\xi)dx \le |\lambda|\|h\|_1\left(\alpha_k+\frac{\alpha_k^m}{m}\right)+\mu\bar{a}\alpha_k^{p_1}\\ &\le |\lambda|\|h\|_1 c_\infty r_k^{\frac{1}{p}}+\frac{|\lambda|}{m}\|h\|_1 c_\infty^m r_k^{\frac{m}{p}}+\mu\bar{a}c_\infty^{p_1} r_k^{\frac{p_1}{p}}<\frac{1}{p}r_k\end{aligned}$$

for k large enough, being $1<m<p$ and $p_1<p$. So, thanks to part $a)$ of Theorem 1.1, the functional $\Phi+\frac{1}{p}\Psi$ has a global minimum, or there exists a sequence of weak solutions $\{u_k\}\subset E$ such that $\lim_{k\to\infty}\|u_k\|=+\infty$. This part of the proof will end if we show that the functional $\Phi+\frac{1}{p}\Psi$ has no global minimum. To this aim, using (i_3), fixed $0<\bar{b}<b$, we get $\beta_k\in]0,+\infty[$ with $\beta_k\ge k$, such that

$$\int_{B(x_0,\rho)} G(x,\beta_k)dx\ge \bar{b}\beta_k^{p_2}$$

for each $k\in\mathbb{N}$. After choosing $\gamma>1$ such that $B(x_0,\gamma\rho)\subseteq\Omega$ and a sequence $\{\chi_k\}\in\mathcal{H}(\rho,\gamma\rho,\{\alpha_k\})$, we consider

$$w_k(x)=\begin{cases}0, & \text{in } \Omega\setminus B(x_0,\gamma\rho),\\ \beta_k, & \text{in } B(x_0,\rho),\\ \chi_k(|x-x_0|) & \text{in } B(x_0,\gamma\rho)\setminus B(x_0,\rho).\end{cases}$$

Using the estimation of the norm made in the previous section, we get

$$\|w_k\|_V^p\le\omega\beta_k^p\left[\frac{2^{p-1}(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^{2p}}c_2^p+\frac{(2^{p-1}(n-1)^p+\rho^p)(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^p}c_1^p+\gamma^n\rho^n\max_{B(x_0,\gamma\rho)}V\right].$$

If we put

$$C_{V,\gamma,\rho}=\frac{2^{p-1}(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^{2p}}c_2^p+\frac{(2^{p-1}(n-1)^p+\rho^p)(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^p}c_1^p+\gamma^n\rho^n\max_{B(x_0,\gamma\rho)}V$$

we have

$$\begin{aligned}\Phi(w_k)+\frac{1}{p}\Psi(w_k)&=-\lambda\int_\Omega F(x,w_k(x))dx-\mu\int_\Omega G(x,w_k(x))dx+\frac{1}{p}\|w_k\|_V^p\\ &\le|\lambda|\int_\Omega|h(x)|\left(|w_k(x)|+\frac{|w_k(x)|^m}{m}\right)dx-\mu\int_{B(x_0,\rho)}G(x,\beta_k)dx+\frac{\omega C_{V,\gamma,\rho}}{p}\beta_k^p\\ &\le|\lambda|\|h\|_1\beta_k+|\lambda|\|h\|_1\frac{\beta_k^m}{m}-\mu\bar{b}\beta_k^{p_2}+\frac{\omega C_{V,\gamma,\rho}}{p}\beta_k^p\end{aligned}$$

and, since $1<m<p<d_2$ and $\lim_{k\to\infty}\beta_k=+\infty$, the functional $\Phi+\frac{1}{p}\Psi$ has no global minimum, being $\lim_{k\to\infty}\Phi(w_k)+\dfrac{1}{p}\Psi(w_k)=-\infty$. This concludes the proof of (r_1).

The proof of (r_2) is similar. If $p_1 < p$ and $p_2 = p$, we choose $\mu_1 = \frac{\omega C_{V,\gamma,\rho}}{pb}$ (obviously if $b = +\infty$ we choose $\mu_1 = 0$). Therefore, if $\lambda \in \mathbb{R}$ and $\mu > \mu_1$, choosing $\bar{b}$ such that $\frac{\omega C_{V,\gamma,\rho}}{p\mu} < \bar{b} < b$, in a similar way we have

$$\Phi(w_k) + \frac{1}{p}\Psi(w_k) \leq |\lambda| \|h\|_1 \beta_k + |\lambda| \|h\|_1 \frac{\beta_k^m}{m} - \left(\mu \bar{b} - \frac{\omega C_{V,\rho,\gamma}}{p}\right) \beta_k^p$$

and, thanks to the choice of $\bar{b}$, also in this case the functional $\Phi + \frac{1}{p}\Psi$ has no global minimum. This concludes (r_2).

As for the proof of (r_3), if $p_1 = p$ and $p_2 > p$, we choose $\mu_2 = \frac{1}{pc_\infty^p a}$ (obviously if $a = 0$ we choose $\mu_2 = +\infty$). Then, fixing $\lambda \in \mathbb{R}$ and $0 < \mu < \mu_2$, we can choose $\bar{a}$ such that $a < \bar{a} < \frac{1}{pc_\infty^p \mu}$. Similar computations give

$$\lambda \int_\Omega F(x, w(x))dx + \mu \int_\Omega G(x, w(x))dx \leq |\lambda| \|h\|_1 c_\infty r_k^{\frac{1}{p}} + \frac{|\lambda|}{m} \|h\|_1 c_\infty^m r_k^{\frac{m}{p}} + \mu \bar{a} c_\infty^p r_k < \frac{1}{p} r_k$$

for k large enough, being $1 < m < p$ and $\mu \bar{a} c_\infty^p < \frac{1}{p}$.

Finally, the proof of (r_4) relies on the considerations made in the previous two cases. We have only to prove that $\mu_1 < \mu_2$, but this is guaranteed by the assumption (3.1). $\square$

Now, we are interested in the existence of infinitely many weak solutions in the case that the nonlinearities f and g have a particular form.

Theorem 3.2. *Let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$, $m < p$, $h \in L^1(\Omega)$, and $r \in L^1(\Omega) \setminus \{0\}$ with $r \geq 0$ a.e. in Ω. Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\int_0^t s(\xi)d\xi \geq 0$, for all $t \geq 0$. Moreover assume that there exists $p_1, p_2 > 1$, $\alpha, \beta > 0$ and $\{\alpha_k\}$, $\{\beta_k\}$ satisfying $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = +\infty$, such that*

$$\max_{|\xi| \leq \alpha_k} \int_0^\xi s(t)dt \leq \alpha \alpha_k^{p_1}, \qquad \int_0^{\beta_k} s(t)dt \geq \beta \beta_k^{p_2}$$

for each $k \in \mathbb{N}$. Then, for the problem

$$\begin{cases} \Delta_p^2 u - \Delta_p u + V(x)|u|^{p-2}u = \lambda h(x)|u|^{m-2}u + \mu r(x)s(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \Omega \end{cases} \qquad (\overline{P}_{\lambda,\mu})$$

the following facts hold:

- $(\bar{r}_1)$ *if $p_1 < p < p_2$, for all $\lambda \in \mathbb{R}$ and for all $\mu > 0$, the problem $(\overline{P}_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

- $(\bar{r}_2)$ *if $p_1 < p = p_2$, there exists $\mu_1 > 0$ such that for all $\lambda \in \mathbb{R}$ and for all $\mu > \mu_1$, the problem $(\overline{P}_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

($\overline{r}_3$) *if* $p_1 = p < p_2$, *there exists* $\mu_2 > 0$ *such that for all* $\lambda \in \mathbb{R}$ *and for all* $\mu \in]0, \mu_2[$, *the problem* $(\overline{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions;*

($\overline{r}_4$) *if* $p_1 = p_2 = p$, *there exist* $x_0 \in \Omega$, $\rho > 0$, $\gamma > 1$ *and* $C_{V,\gamma,\rho} > 0$, *such that, if*

$$C_{V,\gamma,\rho} < \frac{\beta \|r\|_{L^1(B(x_0,\rho))}}{\alpha \omega c_\infty^p \|r\|_{L^1(\Omega)}}, \tag{3.2}$$

then $\mu_1 < \mu_2$ *and for all* $\lambda \in \mathbb{R}$ *and for all* $\mu \in]\mu_1, \mu_2[$, *the problem* $(\overline{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions.*

Proof. We want to apply Theorem 3.1 choosing $f(x,t) = h(x)|t|^{m-2}t$ and $g(x,t) = r(x)s(t)$ for all $(x,t) \in \Omega \times \mathbb{R}$. The hypotheses $(i_1), (i_2)$ are obviously verified. Since $r \not\equiv 0$ we can choose $x_0 \in \Omega$ and $\rho > 0$ such that $B(x_0,\rho) \subset \Omega$ and $r > 0$ in $B(x_0,\rho)$. Then we have:

$$\int_\Omega \max_{|\xi|\le\alpha_k} G(x,\xi)dx = \int_\Omega \max_{|\xi|\le\alpha_k} \left(\int_0^\xi r(x)s(t)dt\right) dx = \|r\|_{L^1(\Omega)} \max_{|\xi|\le\alpha_k} \int_0^\xi s(t)dt \le \|r\|_{L^1(\Omega)} \alpha \alpha_k^{p_1}$$

and

$$\int_{B(x_0,\rho)} G(x,\beta_k)dx = \int_{B(x_0,\rho)} \left(\int_0^{\beta_k} r(x)s(t)dt\right) dx = \|r\|_{L^1(B(x_0,\rho))} \int_0^{\beta_k} s(t)dt \ge \|r\|_{L^1(B(x_0,\rho))} \beta \beta_k^{p_2}.$$

Therefore

$$\liminf_{t\to+\infty} \frac{\int_\Omega \max_{|\xi|\le t} G(x,\xi)dx}{t^{p_1}} \le \|r\|_{L^1(\Omega)} \alpha < +\infty$$

and

$$\limsup_{t\to+\infty} \frac{\int_{B(x_0,\rho)} G(x,t)dx}{t^{p_2}} \ge \|r\|_{L^1(B(x_0,\rho))} \beta > 0.$$

So, (i_3) is also verified with $a = \alpha\|r\|_{L^1(\Omega)}$ and $b = \beta\|r\|_{L^1(B(x_0,\rho))}$. Therefore we can apply the Theorem 3.1 and obtain the conclusions $(\overline{r}_1)$–$(\overline{r}_4)$. □

Now, we want to exhibit two examples. In the first one we present a function s verifying the hypotheses of Theorem 3.2.

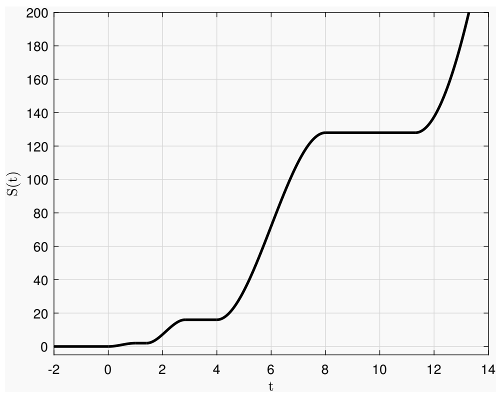
Example 3.3. *Let* $p > 1$, $\delta > 1$ *and let* $s : \mathbb{R} \to \mathbb{R}$ *be the function such that*

$$S(t) = \int_0^t s(\xi)d\xi = \begin{cases} 0, & \text{in }]-\infty, 0], \\ -2\delta t^3 + 3\delta t^2, & \text{in }]0,1], \\ 2^{p(k-1)}\delta^k & \text{in } \left]2^{k-1}\delta^{\frac{k-1}{p}}, 2^{k-1}\delta^{\frac{k}{p}}\right] \quad k \ge 1, \\ A_k t^3 + B_k t^2 + C_k t + D_k & \text{in } \left]2^{k-1}\delta^{\frac{k}{p}}, 2^k\delta^{\frac{k}{p}}\right] \quad k \ge 1 \end{cases}$$

where

$$A_k := -2^{(p-3)k+4}\delta^{\frac{(p-3)k}{p}}\left(\delta - 2^{-p}\right), \quad B_k := 9 \cdot 2^{(p-2)k+2}\delta^{\frac{(p-2)k}{p}}\left(\delta - 2^{-p}\right),$$
$$C_k := -3 \cdot 2^{(p-1)k+3}\delta^{\frac{(p-1)k}{p}}\left(\delta - 2^{-p}\right), \quad D_k := 2^{pk}\delta^k\left(5\delta - 2^{2-p}\right).$$

Using MATLAB by MathWorks, we have plotted the graph of the function S (for $\delta = 2$ and $p = 2$), showed in the following image.



The function s satisfies all the assumption of Theorem 3.2 with $\alpha = 1$, $\beta = \delta$, $\alpha_k = 2^{k-1}\delta^{\frac{k}{p}}$ and $\beta_k = 2^k\delta^{\frac{k}{p}}$, for each $k \in \mathbb{N}$. In particular

$$\max_{|\xi|\leq\alpha_k}\int_0^{\xi} s(t)dt = \int_0^{2^{k-1}\delta^{\frac{k}{p}}} s(t)dt = 2^{p(k-1)}\delta^k = \alpha_k^p$$

and

$$\int_0^{\beta_k} s(t)dt = 2^{pk}\delta^{k+1} = \delta\beta_k^p$$

for all $k \in \mathbb{N}$.

In Theorems 3.1 and 3.2, inequalities (3.1) and (3.2) serve to assure that $\mu_1 < \mu_2$; moreover the value of $C_{V,\gamma,\rho}$ depends heavily also on constants c_j and then on the choice of the sequence $\{\chi_k\}$. Obviously, fixed the nonlinearity, the smaller the constant $C_{V,\gamma,\rho}$ the easier the inequalities (3.1) and (3.2) will be verified. The next example is in this direction.

Example 3.4. *Let $p > 1$, $\Omega = B(0,1)$ in $\mathbb{R}^n$, $x_0 = 0$, $r \in L^1(\Omega) \setminus 0$, with $r \geq 0$, $V(x) = |x|^2_{\mathbb{R}^2} + 1$, for all $x \in B(0,1)$, $\rho = \frac{1}{2}$, $\gamma = 2$ and $\{\sigma_k\} \subset]0, +\infty[$ with $\lim_{k\to\infty}\sigma_k = +\infty$. Let $\left\{\chi_k^1\right\}, \left\{\chi_k^2\right\} \in \mathcal{H}(\frac{1}{2}, 1, \{\sigma_k\})$ the sequences defined by*

$$\chi_k^1(x) = 4\sigma_k(4x^3 - 9x^2 + 6x - 1)$$

and

$$\chi_k^2(x) = \frac{\sigma_k}{2}\cos(\pi(2x-1)+1)$$

for all $x \in]\frac{1}{2}, 1[$ *and for each* $k \in \mathbb{N}$. *We observe that, for each* $x \in]\frac{1}{2}, 1[$,

$$|\chi_k^{1\prime}(x)| \leq 3\sigma_k, \qquad |\chi_k^{1\prime\prime}(x)| \leq 24\sigma_k$$

and then the constants $c_j(\{\chi_k^1\})$, *defined in (2.2), are respectively* $c_1(\{\chi_k^1\}) = \frac{3}{2}$ *and* $c_2(\{\chi_k^1\}) = 6$. *In a similar way, for each* $x \in]\frac{1}{2}, 1[$, *we have*

$$|\chi_k^{2\prime}(x)| \leq \pi\sigma_k, \qquad |\chi_k^{2\prime\prime}(x)| \leq 2\pi^2\sigma_k$$

and, in this case, the constants $c_j(\{\chi_k^2\})$ *are respectively* $c_1(\{\chi_k^2\}) = \frac{\pi}{2}$ *and* $c_2(\{\chi_k^2\}) = \frac{\pi^2}{2}$. *Now let us consider a sequence of functions that, in combination with the norm, raises it to the second power; namely*

$$\chi_k^3(x) = \begin{cases} \sigma_k\left(-8x^2+8x-1\right) & \text{in }]\frac{1}{2}, \frac{3}{4}[\\ \alpha_k(8x^2-16x+8) & \text{in }]\frac{3}{4}, 1[\end{cases} \tag{3.3}$$

for each $k \in \mathbb{N}$. *In this case*

$$|\chi_k^{3\prime}(x)| \leq 4\sigma_k, \quad |\chi_k^{3\prime\prime}(x)| \leq 16\sigma_k$$

and then $c_1(\{\chi_k^3\}) = 2$ *and* $c_2(\{\chi_k^3\}) = 4$. *With respect to these three sequences of test functions the smallest* $C_{V,\gamma,\rho}$ *(among the three) depends on the values of* n *and* p. *For instance, for* $n = 3$ *and* $p = 2$ *the smallest* $C_{V,\gamma,\rho}$ *is the one in correspondence with the sequence* $\{\chi_k^3\}$; *in fact, using MATLAB again to compute these constants, one has*

$$C_{V,\gamma,\rho}(\{\chi_k^1\}) \approx 1270, \quad C_{V,\gamma,\rho}(\{\chi_k^2\}) \approx 969, \quad C_{V,\gamma,\rho}(\{\chi_k^3\}) = 912.$$

But, for instance, for $n = 4$ *and* $p = 3$, *the smallest* $C_{V,\gamma,\rho}$ *is the one in correspondence with the sequence* $\{\chi_k^2\}$ *being*

$$C_{V,\gamma,\rho}(\{\chi_k^1\}) \approx 73737, \quad C_{V,\gamma,\rho}(\{\chi_k^2\}) \approx 53988, \quad C_{V,\gamma,\rho}(\{\chi_k^3\}) = 67262.$$

Obviously if we consider the function s *of Example 3.3, taking a posteriori* $\delta > \frac{\omega c_\infty^p \|r\|_{L^1(\Omega)} C_{V,\gamma,\rho}}{\|r\|_{L^1(B(0,\frac{1}{2})}}$ *the corresponding problem admits a sequence of non-zero weak solutions; but if* δ *is fixed a priori, Theorems 3.1 and 3.2 could be always applied as long as one manages to find an appropriate sequence* $\{\chi_k\}$ *while it is not sure that a generic application of Theorem 1.1 can be applied because the assumptions depends heavily by the particular sequence of test functions fixed during the proof.*

The last theorem concerns the case in which the growth exponent of nonlinearity $f(x,t)$ is exactly $p-1$. In this situation the existence of infinite weak solutions will be obtained not for each $\lambda \in \mathbb{R}$ but in an appropriate interval.

Theorem 3.5. *Let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$ and let $f, g \in \mathcal{C}$ such that (i_2) and (i_3) are verified. Moreover, suppose that:*

$(\tilde{i}_1)$ *there exist $h \in L^1(\Omega)$ such that $|f(x,t)| = h(x)\left(1+|t|^{p-1}\right)$ for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$.*

Then the following facts hold:

$(\tilde{r}_1)$ *if $p_1 < p < p_2$, for all λ such that $|\lambda| < \frac{1}{\|h\|_1 c_\infty^p}$ (for all λ if $h = 0$) and for all $\mu > 0$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\tilde{r}_2)$ *if $p_1 < p = p_2$, there exists $\mu_1 > 0$ such that, for all $\mu > \mu_1$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\tilde{r}_3)$ *if $p_1 = p < p_2$, there exists $\mu_2 > 0$ such that, for all $\mu \in]0, \mu_2[$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\tilde{r}_4)$ *if $p_1 = p_2 = p$, there exists $\gamma > 1$ and $C_{V,\gamma,\rho} > 0$ such that, if*

$$C_{V,\gamma,\rho} < \frac{b}{\omega c_\infty^p a} \tag{3.4}$$

then $\mu_1 < \mu_2$ and for all $\mu \in]\mu_1, \mu_2[$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$ the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

Proof. The proof is similar to that of Theorem 3.1. In fact, computing the two main evaluations for $m = p$, we get:

$$\lambda \int_\Omega F(x,w(x))dx + \mu \int_\Omega G(x,w(x))dx \le |\lambda|\|h\|_1 c_\infty r_k^{\frac{1}{p}} + \frac{|\lambda|}{p}\|h\|_1 c_\infty^p r_k + \mu \bar{a} c_\infty^{p_1} r_k^{\frac{p_1}{p}} \tag{3.5}$$

and

$$\Phi(w_k) + \frac{1}{p}\Psi(w_k) \le |\lambda|\|h\|_1 \beta_k + |\lambda|\|h\|_1 \frac{\beta_k^p}{p} - \mu \bar{b} \beta_k^{p_2} + \frac{\omega C_{V,\gamma,\rho}}{p}\beta_k^p. \tag{3.6}$$

To prove $(\tilde{r}_1)$, fix λ such that $|\lambda| \le \frac{1}{\|h\|_1 c_\infty^p}$ and $\mu > 0$. Thanks to the choice of λ and to the fact that $p_1 < p$ then, from (3.5) we get

$$\lambda \int_\Omega F(x,w(x))dx + \mu \int_\Omega G(x,w(x))dx < \frac{1}{p} r_k \tag{3.7}$$

for k large enough (remember that $\lim\limits_{k\to\infty} r_k = +\infty$); moreover, from (3.6) we obtain

$$\lim_{k\to\infty} \Phi(w_k) + \frac{1}{p}\Psi(w_k) = -\infty \tag{3.8}$$

because $p < p_2$.

To prove $(\tilde{r}_2)$, it is sufficient to choose $\mu_1 = \frac{\omega C_{V,\gamma,\rho}}{pb}$. Fixed $\mu > \mu_1$ and $\bar{b}$ in a similar way as done in Theorem 3.1, we define $\lambda_\mu = \min\left\{\frac{1}{\|h\|_1 c_\infty^p}, \frac{\mu p\bar{b} - \omega C_{V,\gamma,\rho}}{\|h\|_1}\right\}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (3.5), we get (3.7) (for k large enough) because $p_1 < p$ and thanks to the choice of λ. Moreover, using (3.6), the choice of λ and μ guarantees that (3.8) holds.

To prove $(\tilde{r}_3)$, it is sufficient to choose $\mu_2 = \frac{1}{pc_\infty^p a}$. Fixed $\mu \in]0, \mu_2[$ and $\overline{a}$ in a similar way as done in Theorem 3.1, we choose $\lambda_\mu = \frac{1-\mu p c_\infty^p \overline{a}}{\|h\|_1 c_\infty^p}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (3.6), we get (3.8) because $p < p_2$. Moreover, using (3.5), the choice of λ and μ guarantees that (3.7) holds.

In the last case, to prove $(\tilde{r}_4)$, we observe that, thanks to (3.4), we have $\mu_1 < \mu_2$. So, fixed $\mu \in]\mu_1, \mu_2[$, and choosing $\overline{a}$ and $\bar{b}$ in a similar way as done in Theorem 3.1, we define $\lambda_\mu = \min\left\{\frac{1-\mu c_\infty^p \overline{a}}{\|h\|_1}, \frac{\mu p\bar{b} - \omega C_{V,\gamma,\rho}}{\|h\|_1}\right\}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (3.5), we get (3.7) (for k large enough) because of the choice of λ and μ. Moreover, using (3.6), the choice of λ and μ guarantees that (3.8) holds. □

We conclude with an example related to case $(\tilde{r}_4)$ of Theorem 3.5. In this case we consider the one-dimensional setting, providing an explicit estimate of the constant c_∞ in (3.4).

Example 3.6. *Let $n = 1$, $\Omega =]-1, 1[$, $p_1 = p_2 = p = 2$, $V(x) = x^2 + 1$ for all $x \in]-1, 1[$, $h \in L^1(]-1,1[)$, $r \in L^1(]-1,1[) \setminus \{0\}$ with $r \geq 0$ in $]-1,1[$ and $\int_{-1/2}^{1/2} r(x)dx > 0$. It is well-known that, for all $u \in W^{2,2}(]-1,1[) \cap W_0^{1,2}(]-1,1[)$, one has*

$$\max_{x\in]-1,1[} |u(x)| \leq \frac{\sqrt{2}}{2} \|u'\|_{L^2(]-1,1[)}$$

and

$$\|u'\|_{L^2(]-1,1[)} \leq \frac{2}{\pi} \|u''\|_{L^2(]-1,1[)},$$

so

$$\max_{x\in]-1,1[} |u(x)| \leq \frac{\sqrt{2}}{\pi} \|u''\|_{L^2(]-1,1[)} \leq \frac{\sqrt{2}}{\pi} \|u\|_V$$

and then $c_\infty = \frac{\sqrt{2}}{\pi}$. Now choosing $x_0 = 0$, $\rho = \frac{1}{2}$, $\gamma = 2$, $\delta > \frac{1064\|r\|_{L^1(]-1,1[)}}{\pi^2 \|r\|_{L^1(]-\frac{1}{2},\frac{1}{2}[)}}$, and $g(t,x) = r(x)s(t)$ (where the function s is that of Example 3.3), assumptions (i_2) and (i_3) are satisfied with $a = \|r\|_{L^1(]-1,1[)}$ and $b = \delta\|r\|_{L^1(]-\frac{1}{2},\frac{1}{2}[)}$. Using the sequence $\{\chi_k^3\}$ of Example 3.4 as test function, we compute $C_{V,\gamma,\rho} = 266$ (lower than those associated with the other two sequences). It is easy to see that

$$\frac{b}{\omega c_\infty^p a} = \frac{\delta\pi^2 \|r\|_{L^1(]-\frac{1}{2},\frac{1}{2}[)}}{8\|r\|_{L^1(]-1,1[)}} > 266$$

then (3.4) is satisfied and then the fact $(\tilde{r}_4)$ holds. In particular, for all $\mu \in \left]\frac{266}{\delta}, \frac{\pi^2}{4\|r\|_{L^1(]-1,1[)}}\right[$, there exists $\lambda_\mu > 0$ (defined inside the proof of Theorem 3.5) such that, for all $|\lambda| < \lambda_\mu$ the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

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References

[1] G. A. Afrouzi and S. Shokooh, "Existence of infinitely many solutions for quasilinear problems with a $p(x)$-biharmonic operator", Electron. J. Differ. Equations, Paper No. 317, 14 pages, 2015.

[2] G. Arioli, F. Gazzola and H. C. Grunau, "Entire solutions for a semilinear fourth order elliptic problem with exponential nonlinearity", J. Differential Equations, vol. 230, no. 2, pp. 743–770, 2006.

[3] C. Bai. "Infinitely many solutions for a perturbed nonlinear fractional boundary-value problem", Electron. J. Differ. Equations, Paper No. 136, 12 pages, 2013.

[4] M. Ben Ayed and A. Selmi, "Concentration phenomena for a fourth-order equation on $\mathbb{R}^n$", Pacific J. Math., vol. 242, no. 1, pp. 1–32, 2009.

[5] H. Bueno, L. Paes-Leme and H. Rodrigues, "Multiplicity of solutions for p-biharmonic problems with critical growth", Rocky Mountain J. Math., vol. 48, no. 2, pp. 425–442, 2018.

[6] F. Cammaroto, "Sequences of weak solutions to a fourth-order elliptic problem", Atti Accad. Peloritana Pericolanti Cl. di Sci. Fis. Mat. Natur., vol. 98, suppl. 2, pp. A3-1–A3-19, 2020.

[7] F. Cammaroto, A. Chinnì and B. Di Bella, "Infinitely many solutions for the Dirichlet problem involving the p-Laplacian", Nonlinear Anal., vol. 61, no. 1–2, pp. 41–49, 2005.

[8] F. Cammaroto and L. Vilasi, "Sequences of weak solutions for a Navier problem driven by the $p(x)$-biharmonic operator", Minimax Theory Appl., vol. 4, no. 1, pp. 71–85, 2019.

[9] P. Candito, L. Li and R. Livrea, "Infinitely many solutions for a perturbed nonlinear Navier boundary value problem involving the p-biharmonic", Nonlinear Anal., vol. 75, no. 17, pp. 6360–6369, 2012.

[10] P. Candito and R. Livrea, "Infinitely many solution for a nonlinear Navier boundary value problem involving the p-biharmonic", Stud. Univ. Babeş-Bolyai Math., vol. 55, no. 4, pp. 41–51, 2010.

[11] P. C. Carrião, R. Demarque and O. H. Miyakagi. "Nonlinear biharmonic problems with singular potentials", Commun. Pure Appl. Anal., vol. 13, no. 6, pp. 2141–2154, 2014.

[12] Q. Chen and C. Chen, "Infinitely many solutions for a class of p-biharmonic equation in $\mathbb{R}^n$". Bull. Iranian Math. Soc., vol. 43, no. 1, pp. 205–215, 2017.

[13] B. Cheng and X. Tang, "High energy solutions of modified quasilinear fourth-order elliptic equations with sign-changing potential", Comput. Math. Appl., vol. 73, no. 1, pp. 27–36, 2017.

[14] G. Dai, “Infinitely many non-negative solutions for a Dirichlet problem involving $p(x)$-Laplacian”, Nonlinear Anal., vol. 71, no. 11, pp. 5840–5849, 2009.

[15] G. Dai and J. Wei. “Infinitely many non-negative solutions for a $p(x)$-Kirchhoff-type problem with Dirichlet boundary condition”, Nonlinear Anal., vol. 73, no. 10, pp. 3420–3430, 2010.

[16] G. M. Figuereido and R. G. Nascimento, “Multiplicity of solutions for equations involving a nonlocal term and the biharmonic operator”, Electron. J. Differential Equations, Paper No. 217, 15 pages, 2016.

[17] Y. Furusho and K. Takaŝi. “Positive entire solutions to nonlinear biharmonic equations in the plane”, J. Comput. Appl. Math., vol. 88, no. 1, pp. 161–173, 1998.

[18] Z. Guo and J. Wei, “Qualitative properties of entire radial solutions for a biharmonic equation with supercritical nonlinearity”, Proc. Amer. Math. Soc., vol. 138, no. 11, pp. 3957–3964, 2010.

[19] Z. Guo, J. Wei and W. Yang, “On nonradial singular solutions of supercritical biharmonic equations”, Pacific J. Math., vol. 284, no. 2, pp. 395–430, 2016.

[20] A. Hadjian and M. Ramezani, “Existence of infinitely many solutions for fourth-order equations depending on two parameters”, Electron. J. Differential Equations, Paper No. 117, 10 pages, 2017.

[21] M. R. Heidari Tavani and A. Nazari, “Infinitely many weak solutions for a fourth-order equation with nonlinear boundary conditions”, Miskolc Math. Notes, vol. 20, no. 1 pp. 525–538, 2019.

[22] S. Heidarkhani, “Infinitely many solutions for systems of n fourth order partial differential equations coupled with Navier boundary conditions”, Arab J. Math. Sci., vol. 20, no. 1, pp. 77–100, 2014.

[23] A. Kristály, “Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$”, J. Differential Equations, vol. 220, no. 2, pp. 511–530, 2006.

[24] A. C. Lazer and P. J. McKenna, “Large amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis”, SIAM Rev., vol. 32, no. 4, pp. 537–578, 1990.

[25] L. Liu and C. Chen, “Infinitely many solutions for p-biharmonic equation with general potential and concave-convex nonlinearity in $\mathbb{R}^N$”, Bound. Value Probl., Paper No. 6, 9 pages, 2016.

[26] C. Liu and J. Wang, “Existence of multiple solutions for a class of biharmonic equations”, Discrete Dyn. Nat. Soc., Art. ID 809262, 5 pages, 2013.

[27] M. Makvand Chaharlang and A. Razani, “Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition”, Commun. Korean Math. Soc., vol. 34, no. 1, pp. 155–167, 2019.

[28] M. Makvand Chaharlang and A. Razani, “Infinitely many solutions for a fourth order singular elliptic problem”, Filomat, vol. 32, no. 14, pp. 5003–5010, 2018.

[29] M. Massar, E. M. Hssini and N. Tsouli, “Infinitely many solutions for class of Navier boundary (p, q)-biharmonic systems”, Electron. J. Differential Equations, Paper No. 163, 9 pages, 2012.

[30] P. J. McKenna and W. Reichel, “Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry”, Electron. J. Differential Equations, Paper No. 37, 13 pages, 2003.

[31] P. J. McKenna and W. Walter, “Nonlinear oscillations in a suspension bridge”, Arch. Rational Mech. Anal., vol. 98, no. 2, pp. 167–177, 1987.

[32] P. J. McKenna and W. Walter. “Travelling waves in a suspension bridge”, SIAM J. Appl. Math., vol. 50, no. 3, pp. 703–715, 1990.

[33] Q. Miao, “Multiple solutions for nonlocal elliptic systems involving $p(x)$-biharmonic operator”, Mathematics, vol. 7, no. 8, Paper No. 756, 10 pages, 2019.

[34] B. Ricceri, “A general variational principle and some of its applications”, J. Comput. Appl. Math., vol. 113, no. 1–2, pp. 401–410, 2000.

[35] S. Shokooh and G. A. Afrouzi, “Infinitely many solutions for a class of fourth-order impulsive differential equations”, Adv. Pure Appl. Math., vol. 10, no. 1, pp. 7–16, 2019.

[36] S. Shokooh, G. A. Afrouzi and H. Zahmatkesh, “Infinitely many weak solutions for fourth-order equations depending on two parameters”, Bol. Soc. Parana Mat., vol. 36, no. 4, pp. 131–147, 2018.

[37] Y. Song, “A nonlinear boundary value problem for fourth-order elastic beam equations”, Bound. Value Probl., Paper No. 191, 11 pages, 2014.

[38] F. Sun, L. Liu and Y. Wu, Appl. Math. Lett., vol. 73, pp. 128–135, 2017.

[39] K. Teng, “Infinitely many solutions for a class of fractional boundary value problems with nonsmooth potential”, Abst. Appl. Anal., Art. ID 181052, 6 pages, 2013.

[40] Y. Wang and Y. Shen, “Infinitely many sign-changing solutions for a class of biharmonic equation without symmetry”, Nonlinear Anal., vol. 71, no. 3–4, pp. 967–977, 2009.

[41] M. Xu and C. Bai, “Existence of infinitely many solutions for pertubed Kirchhoff type elliptic problems with Hardy potential”, Electron. J. Differential Equations, Paper No. 268, 9 pages, 2015.

[42] J. Zhang, “Infinitely many nontrivial solutions for a class of biharmonic equations via variant fountain theorems”, Electron. J. Qual. Theory Diff. Equ., Paper No. 9, 14 pages, 2011.

[43] J. Zhang and Z. Wei, “Infinitely many nontrivial solutions for a class of biharmonic equations via variant fountain theorems”, Nonlinear Anal., vol. 74, no. 18, pp. 7474–7485, 2011.

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A derivative-type operator and its application to the solvability of a nonlinear three point boundary value problem

René Erlin Castillo[1,] ✉
Babar Sultan[2]

[1] *Universidad Nacional de Colombia, Departamento de Matemáticas, Bogotá, Colombia*
`recastillo@unal.edu.co` ✉

[2] *Department of Mathematics, Quaid-I-Azam University, Islamabad 45320, Pakistan.*
`babarsultan40@yahoo.com`

ABSTRACT

In this paper we introduce an operator that can be thought as a derivative of variable order, *i.e.* the order of the derivative is a function. We prove several properties of this operator, for instance, we obtain a generalized Leibniz's formula, Rolle and Cauchy's mean theorems and a Taylor type polynomial. Moreover, we obtain its inverse operator. Also, with this derivative we analyze the existence of solutions of a nonlinear three-point boundary value problem of "variable order".

RESUMEN

En este artículo introducimos un operador que puede ser pensado como una derivada de orden variable, *i.e.* el orden de la derivada es una función. Demostramos varias propiedades de este operador, por ejemplo, obtenemos una fórmula generalizada de Leibniz, teoremas de valor medio de Rolle y Cauchy y un polinomio de tipo Taylor. Más aún, obtenemos su operador inverso. También con esta derivada analizamos la existencia de soluciones de un problema no lineal de valor en la frontera de tres puntos de "orden variable".

Keywords and Phrases: Fractional Derivative, boundary value problem, Hammerstein-Volterra integral equation.

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1 Motivation

Derivatives of non-integer order have been studied since the celebrated question of L'Hospital to Leibniz about the meaning of $\frac{d^n f}{dx^n}$ when $n = 1/2$. There are several definitions of derivatives of fractional order, *e.g.*, derivative of *Riemann-Louville, Caputo, Hadamard, Erdélyi-Kober, Grünwald-Letnikov* and *Riesz*, among others. Typically, these derivatives are defined using an integral form of the classical derivative, as a consequence of it, some basic properties of the usual derivative, as the product rule and chain rule are lost. For a more comprehensible information about these notions we recommend [17, 20, 30].

Despite of the lack of some properties, derivatives of fractional order appear in many real world applications as, for instance, in *memory effects and future dependence, control theory of dynamical systems, nanotechnology, viscoelasticity* and *financial modeling* see, *e.g.*, [8, 12, 18, 19, 21, 24, 25, 31, 32]. Thus, due to this development, in the last decades a lot of research has been devoted to the study of the existence of solutions for several kinds of boundary value problems of fractional type, see, for instance, [2, 3, 4, 5, 9, 11, 26, 28, 29] and references therein.

In order to overcome the limitations of the classical derivative, in [16] it is introduced a new limit-based definition of derivative, the so-called *conformable fractional derivative*, which can be seen as a natural extension of the fractional derivative, although as it is stated in [7], *it is best to consider the conformable derivative in its own right, independent of fractional derivative theory.* Some of the basic properties, physical interpretation and some boundary value problems for conformable differential equations can be found in [1, 6, 10, 14, 15, 33, 34] and its references.

In this article, based on the idea of conformable fractional derivative and in ideas from [13], we consider an extension of the conformable fractional derivative of order α and develop some of its properties. Additionally, we study the existence and uniqueness of solutions for a nonlinear three-point boundary value problem in this new setting.

2 Derivative of variable order

We now introduce the notion of (φ, ω)-derivative.

Definition 2.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$. The (φ, ω)-derivative at the point $x \in (a, b)$ $(\varphi(x) \neq 0)$ is defined as*

$$D^{\varphi}_{\omega} f(x) = D^{\varphi}_{\omega}(f)(x) = D^{(\varphi,1)}_{\omega} f(x) = \lim_{h\to 0} \frac{f(x + h\varphi(x)) - f(x)}{\omega(x + h) - \omega(x)}. \tag{2.1}$$

Where ω is a strictly increasing function and φ is a function. At the point $x \in (a, b)$ such that

$\varphi(x) = 0$ *we define the* (φ, ω)*-derivative as*

$$D_\omega^\varphi f(x) = \lim_{\xi \to x} D_\omega^\varphi f(\xi),$$

when the limit exists.

Taking $\varphi(x) = x^{1-\alpha}$ and $\omega(x) = x$ we obtain the conformable fractional derivative of order α, *cf.* [16].

Theorem 2.2. *Let* f, g *be* (φ, ω)*-differentiable. Then:*

(a) The function f *is continuous.*

(b) $D_\omega^\varphi(a) = 0$*,* a *is a constant.*

(c) $D_\omega^\varphi(af + g) = aD_\omega^\varphi(f) + D_\omega^\varphi(g)$*.*

(d) $D_\omega^\varphi(fg) = fD_\omega^\varphi(g) + fD_\omega^\varphi(g)$*.*

(e) $D_\omega^\varphi\left(\dfrac{f}{g}\right) = \dfrac{fD_\omega^\varphi(g) - fD_\omega^\varphi(g)}{g^2}$*.*

(f) If f *and* ω *are differentiable, we have*

$$D_\omega^\varphi(f)(t) = \varphi(t)\frac{f'(t)}{\omega'(t)}. \tag{2.2}$$

(g) If f, g *and* ω *are differentiable, we have*

$$D_\omega^\varphi(f \circ g)(t) = f'(g(t)) \cdot D_\omega^\varphi(g)(t).$$

Proof. It is a matter of direct calculations. □

Formula (2.2) enables us to calculate in a straightforward way some (φ,ω)-derivatives. For example, letting $\varphi(x) = \sin(x)$, $f(x) = \cos(x)$, and $\omega(x) = x$, we have

$$D_\omega^\varphi \varphi(x) = \sin(x)\cos(x), \quad D_\omega^\varphi f(x) = -\sin^2(x),$$

whereas taking φ and f as above with $\omega(x) = e^x - 1$ we get

$$D_\omega^\varphi \varphi(x) = \frac{\sin(x)\cos(x)}{e^x}, \quad D_\omega^\varphi f(x) = \frac{-\sin^2(x)}{e^x}.$$

We now introduce the n-iterated (φ, ω)-derivative.

Definition 2.3. *By $D_\omega^{(\varphi,n)}f(x)$ we define the n-iterated (φ,ω)-derivative of the function f, i.e.*

$$D_\omega^{(\varphi,n)}f(x) = D_\omega^\varphi\left(D_\omega^{(\varphi,n-1)}f\right)(x),$$

with the convention $D_\omega^{(\varphi,0)}f(x) := f(x)$.

Theorem 2.4 (Generalized Leibniz's formula)**.** *We have*

$$D_\omega^{(\varphi,n)}(f_1 f_2\cdots f_m) = \sum_{\substack{i_1+i_2+\cdots+i_m=n\\ i_j=\overline{0,n}}} n!\frac{D_\omega^{(\varphi,i_1)}(f_1)D_\omega^{(\varphi,i_2)}(f_2)\cdots D_\omega^{(\varphi,i_m)}(f_m)}{i_1!i_2!\cdots i_m!}, \tag{2.3}$$

where we suppose that all is well-defined.

Proof. For $m = 2$, equation (2.3) is obtained by induction on n and using the formula for the (φ,ω)-derivative of the product, in this case we obtain

$$D_\omega^{(\varphi,n)}(f_1 f_2) = \sum_{j=0}^{n}\binom{n}{j}D_\omega^{(\varphi,n-j)}(f_1)D_\omega^{(\varphi,j)}(f_2). \tag{2.4}$$

By the well-known method to prove the multinomial theorem from the binomial theorem we can, in the same way, obtain (2.3) from (2.4). □

Theorem 2.5 (Fermat's Theorem)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ have a local maximum or minimum at $x = c \in (a,b)$ and $D_\omega^\varphi(f)(c)$ exists. Then $D_\omega^\varphi(f)(c) = 0$.*

Proof. Let us suppose, without loss of generality, that $x = c$ is a minimum of f. We have, for sufficiently small $h \neq 0$, that

$$\operatorname{sgn}(h\varphi(c))\frac{f(c+h\varphi(c)) - f(c)}{\omega(c+h)-\omega(c)} \geqslant 0. \tag{2.5}$$

From (2.5) and the hypothesis of the existence of $D_\omega^\varphi(f)(c)$ the result follows. □

Theorem 2.6 (Rolle's theorem)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function in $[a,b]$ and (φ,ω)-differentiable in (a,b) such that $f(a) = f(b) = 0$. Then there exists $c \in (a,b)$ such that $D_\omega^\varphi(f)(c) = 0$.*

Proof. Supposing, without loss of generality, that there exists $\xi \in (a,b)$ such that $f(\xi) \geq 0$. Then by Weierstraß theorem, there exists $c \in (a,b)$ which is a maximum. Invoking Fermat's theorem 2.5 we end the proof. □

Theorem 2.7 (Cauchy mean-value theorem)**.** *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be both continuous on the closed interval $[a, b]$ and (φ, ω)-differentiable in the open interval (a, b). Then there exists a number $\xi \in (a, b)$ such that*

$$[f(b) - f(a)]D^{\varphi}_{\omega}(g)(\xi) = [g(b) - g(a)]D^{\varphi}_{\omega}(f)(\xi). \tag{2.6}$$

Proof. The proof follows, as in the classical case, from Rolle's theorem 2.6 applied to the function

$$F(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]. \qquad \square$$

3 Integration of variable order

Definition 3.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$. We define the (φ, ω)-integral of the function f as*

$$I^{\varphi}_{\omega}(f)(t) = \int_a^t \frac{f(\xi)}{\varphi(\xi)}\, \mathrm{d}\omega(\xi), \tag{3.1}$$

where the integral is understood in the Lebesgue-Stieltjes sense.

Notice that for $f \in L^{\infty}([a, b])$ and $\frac{1}{\varphi} \in L^1([a, b], \mathrm{d}w)$ the integral (3.1) is finite.

When f, φ and ω' are continuous functions, it is straightforward the relation $D^{\varphi}_{\omega}(I^{\varphi}_{\omega} f)(t) = f(t)$, since

$$D^{\varphi}_{\omega}(I^{\varphi}_{\omega} f)(t) = \frac{\varphi(t)}{\omega'(t)} D\left(\int_a^t \frac{f(\xi)}{\varphi(\xi)}\, \mathrm{d}\omega(\xi)\right)(t) = f(t),$$

using (2.2).

In the case φ and ω' are continuous functions, the following *Lagrange mean-value theorem*

$$D^{\varphi}_{\omega}(f)(\xi) = \frac{f(b) - f(a)}{I^{\varphi}_{\omega}(1)(b) - I^{\varphi}_{\omega}(1)(a)}, \quad \xi \in (a, b) \tag{3.2}$$

is valid, when $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on the closed interval $[a, b]$ and (φ, ω)-differentiable in the open interval (a, b). The equation (3.2) follows from (2.6) taking $g(x) = I^{\varphi}_{\omega}(1)(x)$ (note that $I^{\varphi}_{\omega}(1)(a) = 0$, but we leave it in (3.2) just for keeping with the parallel in the classical case).

By $I^{(\varphi,n)}_{\omega}\varphi(x)$ we define the n-iterated (φ, ω)-integral of the function f, *i.e.*

$$I^{(\varphi,n)}_{\omega} f(x) = I^{\varphi}_{\omega}(I^{(\varphi,n-1)}_{\omega} f)(x),$$

with the convention $I^{(\varphi,0)}_{\omega} f(x) := f(x)$.

4 Taylor formula

In this section we will obtain a Taylor type formula using the (φ,ω)-derivative with a remainder which generalizes well-know remainders, *i.e.* Cauchy, Lagrange, Peano, Schlömilch, among others, *cf.* [22, 23, 27] for similar remainders for the classical derivative.

Theorem 4.1. *Let $f : I \longrightarrow \mathbb{R}$ be a continuous function in the open interval I and n-times (φ,ω)-differentiable function in I. We also require that φ, ω' and $I_\omega^{(\varphi,j)}(1)(x)$ are continuous functions, for $j = \overline{1,n}$. Moreover, let $g : I \longrightarrow \mathbb{R}$ be a n-times (φ,ω)-differentiable function such that $D_\omega^{(\varphi,j)} g(a) = 0$ for $j = \overline{1, n-1}$ and $D_\omega^{(\varphi,k)} g(y) \neq 0$ for all y different from a and x and $j = \overline{1, n-1}$. Then, for all $x \in I$ we have*

$$f(x) = \sum_{j=0}^{n} D_\omega^{(\varphi,j)}(f)(a)\; I_\omega^{(\varphi,j)}(1)(x) + R_n(x), \tag{4.1}$$

with

$$R_n(x) = \frac{g(x) - g(a)}{D_\omega^{(\varphi,n)} g(\xi)} \left(D_\omega^{(\varphi,n)}(f)(\xi) - D_\omega^{(\varphi,n)}(f)(a) \right), \tag{4.2}$$

where $x \neq a$ and ξ is between a and x.

Proof. We first note that, since

$$D_\omega^{(\varphi,n)}(I_\omega^{(\varphi,j)} 1)(x) = \begin{cases} I_\omega^{(\varphi,j-n)}(1)(x), & j > n, \\ 1, & n = j, \\ 0, & n > j. \end{cases}$$

we have

$$R_n(a) = D_\omega^{(\varphi,1)}(R_n)(a) = \cdots = D_\omega^{(\varphi,n-1)}(R_n)(a) = 0. \tag{4.3}$$

By the Cauchy type finite increment formula (2.6), relations (4.3) and the hypothesis on g we have

$$\begin{aligned} \frac{R_n(x) - R_n(a)}{g(x) - g(a)} &= \frac{D_\omega^{(\varphi,1)}(R_n)(\theta_1) - D_\omega^{(\varphi,1)}(R_n)(a)}{D_\omega^{(\varphi,1)}(g)(\theta_1) - D_\omega^{(\varphi,1)}(g)(a)} = \ldots = \frac{D_\omega^{(\varphi,n-1)}(R_n)(\theta_{n-1}) - D_\omega^{(\varphi,n-1)}(R_n)(a)}{D_\omega^{(\varphi,n-1)}(g)(\theta_{n-1}) - D_\omega^{(\varphi,n-1)}(g)(a)} \\ &= \frac{D_\omega^{(\varphi,n)}(R_n)(\xi)}{D_\omega^{(\varphi,n)}(g)(\xi)}, \end{aligned} \tag{4.4}$$

where $\xi := \theta_n$. On the other hand, (φ,ω)-differentiating the equality (4.1) n-times we obtain $D_\omega^{(\varphi,n)}(f)(x) - D_\omega^{(\varphi,n)}(f)(a) = D_\omega^{(n)}(R_n)(x)$ which, together with (4.4), entails (4.2). □

5 Three-point boundary value problems of variable order

Inspired in [10], we are interested in the use of the (φ, ω)-derivative to study the solutions of the following nonlinear boundary value problem

$$D_{\omega}^{\varphi}(D+\lambda)x(t) = f(t, x(t)), \qquad t \in [0,1] \tag{5.1}$$

$$x(0) = 0, \quad x'(0) = \alpha, \quad x(1) = \beta x(\eta), \tag{5.2}$$

where D_{ω}^{φ} is the derivative of variable order, D is the ordinary derivative, $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a known function, β, λ and α are real numbers, $\lambda \neq 0$ and $\eta \in (0,1)$. Notice that in virtue of Theorem 2.2, a sufficient condition for the well posedness of equation (5.1) is, by considering $\omega \in C^1[0,1]$, and $x \in C^2[0,1]$. Thus, in the sequel we consider these conditions on the functions ω and x. In addition, in order to use the (φ, ω)-integral, we are going to assume that φ is continuous and bounded away from zero.

From the conditions on the functions ω and φ we conclude that the following non negative numbers are finite

$$\Omega := \sup_{t \in [0,1]} \omega'(t) < \infty, \quad M := \sup_{t \in [0,1]} \left| \frac{1}{\varphi(t)} \right| < \infty.$$

We will use these numbers in the sequel to establish the existence results.

First, as usual, we will consider the linear boundary value problem:

$$D_{\omega}^{\varphi}(D+\lambda)x(t) = g(t), \qquad t \in [0,1], \quad g \in C[0,1] \tag{5.3}$$

$$x(0) = 0, \quad x'(0) = \alpha, \quad x(1) = \beta x(\eta), \quad \alpha, \beta, \lambda \in \mathbb{R}, \quad \lambda \neq 0, \quad \eta \in (0,1). \tag{5.4}$$

To obtain a solution for the boundary value problem, we apply the (φ, ω)-integral to equation (5.3):

$$(D+\lambda)x(t) + (D+\lambda)x(0) = I_{\omega}^{\varphi}(g)(t), \tag{5.5}$$

where, using the boundary condition (5.4), $(D+\lambda)x(0) = \alpha$. Then, equation (5.5) simplifies as

$$(D+\lambda)x(t) + \alpha = I_{\omega}^{\varphi}(g)(t). \tag{5.6}$$

Let $y(t) = e^{\lambda t}x(t)$, Then we rewrite (5.6) as

$$Dy(t) = e^{\lambda t} I_{\omega}^{\varphi}(g)(t) - \alpha e^{\lambda t}.$$

Integrating from 0 to t we obtain

$$y(t) - y(0) = \int_0^t e^{\lambda s} I_{\omega}^{\varphi}(g)(s)\, \mathrm{d}s - \frac{\alpha}{\lambda}(e^{\lambda t} - 1)$$

$$y(t) = \int_0^t e^{\lambda s} \int_0^s \frac{g(r)}{\varphi(r)} \omega'(r)\,\mathrm{d}r\,\mathrm{d}s - \frac{\alpha}{\lambda}(e^{\lambda t} - 1), \qquad (y(0) = x(0) = 0).$$

Now, notice that

$$\int_0^t e^{\lambda s} \int_0^s \frac{g(r)}{\varphi(r)} \omega'(r)\,\mathrm{d}r\,\mathrm{d}s = \frac{e^{\lambda t}}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{1}{\lambda} \int_0^t \frac{e^{\lambda s} g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s, \quad 0 \le s \le t \le 1.$$

From here we have that

$$y(t) = \frac{e^{\lambda t}}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{1}{\lambda} \int_0^t \frac{e^{\lambda s} g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{\alpha}{\lambda}(e^{\lambda t} - 1), \quad 0 \le s \le t \le 1.$$

Thus,

$$x(t) = \frac{1}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{e^{-\lambda t}}{\lambda} \int_0^t \frac{e^{\lambda s} g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda}(e^{-\lambda t} - 1).$$

Finally, from the condition $\beta x(\eta) = x(1)$ we get

$$\frac{\beta}{\lambda} \int_0^\eta \frac{g(s)}{\varphi(s)}(1 - e^{\lambda(s-\eta)}) \omega'(s)\,\mathrm{d}s + \frac{\alpha\beta}{\lambda}(e^{-\lambda\eta} - 1) - \frac{1}{\lambda} \int_0^1 \frac{g(s)}{\varphi(s)}(1 - e^{\lambda(s-1)})\,\mathrm{d}s - \frac{\alpha}{\lambda}(e^{-\lambda} - 1) = 0.$$

Therefore, introducing this equality into the formula of function x above, we obtain the following expression for x satisfying boundary value problem (5.3)–(5.4)

$$\begin{aligned} x(t) = &\frac{1}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)}(1 - e^{\lambda(s-t)}) \omega'(s)\,\mathrm{d}s + \frac{\beta}{\lambda} \int_0^\eta \frac{g(s)}{\varphi(s)} \left(1 - e^{\lambda(s-\eta)}\right) \omega'(s)\,\mathrm{d}s \\ &- \frac{1}{\lambda} \int_0^1 \frac{g(s)}{\varphi(s)} \left(1 - e^{\lambda(s-1)}\right) \omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda}\left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right). \end{aligned}$$

Notice that actually we just proved the following result.

Theorem 5.1. *The linear boundary value problem* (5.3)–(5.4) *has a unique solution given by*

$$\begin{aligned} x(t) = &\frac{1}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s) k(s,t)\,\mathrm{d}s + \frac{\beta}{\lambda} \int_0^\eta \frac{g(s)}{\varphi(s)} \omega'(s) k(s,\eta)\,\mathrm{d}s - \frac{1}{\lambda} \int_0^1 \frac{g(s)}{\varphi(s)} k(s,1) \omega'(s)\,\mathrm{d}s \\ &+ \frac{\alpha}{\lambda}\left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right). \end{aligned}$$

where, $k(s,t) = 1 - e^{\lambda(s-t)}$.

Now, we are going to analyze the existence of solutions for the nonlinear boundary value problem:

$$D_\omega^\varphi(D + \lambda)x(t) = f(t, x(t)), \quad t \in [0,1], \quad \lambda \in (-1,\infty) \setminus \{0\} \tag{5.7}$$

$$x(0) = 0, \quad x'(0) = \alpha, \quad x(1) = \beta x(\eta). \tag{5.8}$$

As in Theorem 5.1, we can transform boundary value problem (5.7)–(5.8) into the nonlinear

Hammerstein-Volterra integral equation

$$
\begin{aligned}
x(t) = {} & \frac{1}{\lambda}\int_0^t \frac{f(s,x(s))}{\varphi(s)}k(s,t)\omega'(s)\,\mathrm{d}s + \frac{\beta}{\lambda}\int_0^\eta \frac{f(s,x(s))}{\varphi(s)}k(s,\eta)\omega'(s)\,\mathrm{d}s \\
& - \frac{1}{\lambda}\int_0^1 \frac{f(s,x(s))}{\varphi(s)}k(s,1)\omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda}\left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right).
\end{aligned}
$$

where, $k(s,t) = 1 - e^{\lambda(s-t)}$.

In order to investigate the existence of a solution for this integral equation, we analyze it as a fixed point problem; that is, letting

$$
\begin{aligned}
T : (C^2[0,1], \|\cdot\|_\infty) & \longrightarrow (C^2[0,1], \|\cdot\|_\infty) \\
x(t) & \longmapsto Tx(t)
\end{aligned}
$$

$$
\begin{aligned}
Tx(t) := {} & \frac{1}{\lambda}\int_0^t \frac{f(s,x(s))}{\varphi(s)}k(s,t)\omega'(s)\,\mathrm{d}s + \frac{1}{\lambda}\int_0^1 \frac{f(s,x(s))}{\varphi(s)}\left(\chi_{(0,\eta)}(s)\beta k(s,\eta) - k(s,1)\right)\omega'(s)\,\mathrm{d}s \\
& + \frac{\alpha}{\lambda}\left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right), \qquad (5.9)
\end{aligned}
$$

(with $\chi_{(0,\eta)}(s)$ the characteristic function of the interval $(0,\eta)$), we have that the existence of the solution of the integral equation is equivalent to the existence of a fixed point of the operator T. To assure that the operator T applies $C^2[0,1]$ into itself, we assume that $f(t,x(t))$ is continuous and differentiable in the first variable.

We are going to use metric fixed point theory (Banach's contraction principle) to provide conditions to guarantee that the boundary value problem (5.7)–(5.8) has a unique solution.

Theorem 5.2. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function satisfying that*

$$
|f(t,x) - f(t,y)| \le K|x-y|, \quad K > 0, \quad \text{for all } t \in [0,1], \quad x,y \in \mathbb{R}.
$$

Then, the nonlinear boundary value problem (5.7)–(5.8) has a unique solution provide that

$$
\frac{(|\beta|+1)MK\Omega}{|\lambda|} < \frac{1}{4},
$$

where $M := \sup_{t\in[0,1]} \dfrac{1}{|\varphi(t)|}$ and $\Omega := \sup_{t\in[0,1]} w'(t)$.

Proof. As we saw, it is sufficient to show that the operator T defined by the formula (5.9) has a unique fixed point. Let x and y be two functions in $C^2[0,1]$. Then,

$$
\begin{aligned}
|Tx(t)-Ty(t)| &= \bigg| \frac{1}{\lambda}\int_0^t \frac{(f(s,x(s))-f(s,y(s))}{\varphi(s)} k(s,t)\omega'(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\lambda}\int_0^1 \frac{(f(s,x(s))-f(s,y(s))}{\varphi(s)} \Big(\chi_{(0,\eta)}(s)\beta k(s,\eta)-k(s,1)\Big)\,\omega'(s)\,\mathrm{d}s \bigg| \\
&\le \frac{1}{|\lambda|}\int_0^t \frac{|k(s,t)|}{|\varphi(s)|}|f(s,x(s))-f(s,y(s))|\omega'(s)\,\mathrm{d}s \\
&\quad + \frac{1}{|\lambda|}\int_0^1 \frac{\Big|\chi_{(0,\eta)}(s)\beta k(s,\eta)-k(s,1)\Big|}{|\varphi(s)|}|f(s,x(s))-f(s,y(s))|\omega'(s)\,\mathrm{d}s.
\end{aligned}
$$

On the other hand,

$$|k(s,t)| = |1-e^{\lambda(s-t)}| \le 1+e^{\lambda(s-t)}.$$

Notice that for $-1<\lambda<0$, the inequality $|e^x-1|<7/4|x|$, for $0<|x|<1$, gives us the estimate

$$|1-e^{\lambda(s-t)}| < \frac{7}{4}\lambda(s-t) < \frac{7}{4} < 2.$$

Then,

$$\sup_{s\in[0,t]}(1+e^{\lambda(s-t)}) \le 2, \quad -1<\lambda<0.$$

Now, for $\lambda>0$,

$$\sup_{s\in[0,t]}(1+e^{\lambda(s-t)}) = 1+e^{-\lambda t} \le 2, \qquad \text{for any } t\in[0,1].$$

Therefore, we obtain the following bound

$$|k(s,t)| \le 2. \tag{5.10}$$

Moreover,

$$
\begin{aligned}
\Big|\chi_{(0,\eta)}(s)\beta k(s,\eta)-k(s,1)\Big| &= \Big|-\chi_{(0,\eta)}(s)\beta e^{\lambda(s-\eta)}+e^{\lambda(s-1)}+\chi_{(0,\eta)}(s)\beta-1\Big| \\
&\le |-\chi_{(0,\eta)}(s)\beta e^{\lambda(s-\eta)}| + |e^{\lambda(s-1)}| + |\beta| + 1,
\end{aligned}
$$

where, for $-1<\lambda<0$, we have that

$$\sup_{s\in[0,\eta]} e^{\lambda(s-\eta)} = 1, \quad \sup_{s\in[0,1]} e^{\lambda(s-1)} = 1.$$

In the case $\lambda>0$, we get

$$\sup_{s\in[0,\eta]} e^{\lambda(s-\eta)} = e^{-\lambda\eta} \le 1, \quad \sup_{s\in[0,1]} e^{\lambda(s-1)} = e^{-\lambda} \le 1.$$

With these bounds we obtain the following estimation

$$\left|\chi_{(0,\eta)}(s)\beta k(s,\eta) - k(s,1)\right| \leq 2(|\beta|+1). \tag{5.11}$$

We introduce the bounds (5.10) and (5.11) into the difference $|Tx(t) - Ty(t)|$:

$$\begin{aligned} |Tx(t) - Ty(t)| \leq & \frac{2}{|\lambda|}\int_0^t \frac{1}{|\varphi(s)|}|f(s,x(s)) - f(s,y(s))|\omega'(s)\,\mathrm{d}s \\ & + \frac{2(|\beta|+1)}{|\lambda|}\int_0^1 \frac{1}{|\varphi(s)|}|f(s,x(s)) - f(s,y(s))|\omega'(s)\,\mathrm{d}s. \end{aligned} \tag{5.12}$$

Since $f(s,x(s))$ is Lipschitz in the second variable, then

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \frac{2}{|\lambda|}\int_0^t \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s + \frac{2(|\beta|+1)}{|\lambda|}\int_0^1 \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s \\ &\leq \frac{2(|\beta|+1)}{|\lambda|}\int_0^t \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s \\ &+ \frac{2(|\beta|+1)}{|\lambda|}\int_0^1 \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s. \end{aligned}$$

Taking the maximum over $t \in [0,1]$ we obtain

$$\|Tx - Ty\|_\infty \leq 2\frac{2(|\beta|+1)}{|\lambda|}KM\Omega\|x - y\|_\infty.$$

Therefore, T is a contraction operator, since $\mu = 2\frac{2(|\beta|+1)}{|\lambda|}KM\Omega < 1$. Thus from the Banach contraction principle, T has a unique fixed point as desired. □

Now, we are going to use topological fixed point theory, more precisely Schaefer's fixed point theorem, to establish the existence of at least one solution of boundary value problem (5.7)–(5.8), dropping the Lipschitzianity of the function f.

First, we prove that the operator T is compact.

Theorem 5.3. *The operator* $T : (C^2[0,1], \|\cdot\|_\infty) \longrightarrow (C^2[0,1], \|\cdot\|_\infty)$ *is compact.*

Proof. We start by proving the continuity of T. Let $(x_n) \subset C^2[0,1]$, $x \in C^2[0,1]$ be such that $\|x_n - x\|_\infty \to 0$. We have to show that $\|Tx_n - Tx\|_\infty \to 0$. Fixed $\varepsilon > 0$, there exists $K \geq 0$ such that

$$\begin{aligned} \|x_n\|_\infty \leq K, &\qquad \forall n \in \mathbb{N} \\ \|x\|_\infty \leq K. & \end{aligned}$$

Since $f : [0,1] \times [-K,K] \longrightarrow \mathbb{R}$ is continuous, then it is uniformly continuous on $[0,1] \times [-K,K]$.

Thus there exists $\delta(\varepsilon) > 0$ such that

$$|f(s_1, x(s_1)) - f(s_2, y(s_2))| \leq \varepsilon,$$

for every $(s_1, x(s_1)), (s_2, y(s_2)) \in [0,1] \times [-K, K]$ such that $\|(s_1 - s_2, x(s_1) - y(s_2))\|_2 < \delta(\varepsilon)$.

From the fact that $\|x_n - x\|_\infty \to 0$, it follow that there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{t\in[0,1]} |x_n(t) - x(t)| < \delta,$$

for every $n \geq N(\varepsilon)$. Consequently, from (5.12),

$$\begin{aligned}
\|Tx_n - Tx\|_\infty &= \sup_{t\in[0,1]} |Tx_n(t) - Tx(t)| \\
&\leq \sup_{t\in[0,1]} \left\{ \frac{2}{|\lambda|} \int_0^t \frac{|f(s, x_n(s)) - f(s, x(s))|}{|\varphi(s)|} \omega'(s)\, \mathrm{d}s \right. \\
&\quad \left. + \frac{2(|\beta| + 1)}{|\lambda|} \int_0^1 \frac{|f(s, x_n(s)) - f(s, x(s))|}{|\varphi(s)|} \omega'(s)\, \mathrm{d}s \right\} \\
&< \frac{2|\beta| + 4}{|\lambda|} M\Omega\varepsilon, \qquad M := \sup_{t\in[0,1]} \frac{1}{|\varphi(t)|}, \quad \Omega := \sup_{t\in[0,1]} \omega'(t).
\end{aligned}$$

Therefore, the operator T is continuous. To prove the compactness we consider a bounded set $X \subset C^2[0,1]$ and we will show that $T(X)$ is relatively compact in $(C^2[0,1], \|\cdot\|_\infty)$ by using the Arzela-Ascoli theorem. Let $K \geq 0$ be such that

$$\|x\|_\infty \leq K,$$

for every $x \in X$.

From the bounds (5.10) and (5.11) we have

$$\begin{aligned}
|Tx(t)| \leq& \frac{1}{|\lambda|} \int_0^t \frac{|f(s, x(s))|}{|\varphi(s)|} |k(s,t)| \omega'(s)\, \mathrm{d}s + \frac{1}{|\lambda|} \int_0^1 \frac{|f(s, x(s))|}{|\varphi(s)|} |\chi_{(0,\eta)} \beta k(s, \eta) - k(s,1)| \omega'(s)\, \mathrm{d}s \\
&+ \left|\frac{\alpha}{\lambda}\right| |e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta| \\
\leq& \frac{2M\Omega}{|\lambda|} \int_0^t |f(s, x(s))|\, \mathrm{d}s + \frac{2(|\beta| + 1)}{|\lambda|} M\Omega \int_0^1 |f(s, x(s))|\, \mathrm{d}s \\
&+ \left|\frac{\alpha}{\lambda}\right| |e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta| \\
\leq& \frac{2|\beta| + 4}{|\lambda|} M\Omega \int_0^1 |f(s, x(s))|\, \mathrm{d}s + \left|\frac{\alpha}{\lambda}\right| |e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta|.
\end{aligned}$$

We obtain a upper bound for $|e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta|$, namely

$$|e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta| \leq \Delta, \quad t \in [0,1],$$

where

$$\Delta := \begin{cases} (|\beta|+1)e^{-\lambda}, \ -1 < \lambda < 0 \\ 2(|\beta|+1), \ \lambda > 0. \end{cases}$$

On the other hand, since the function f is uniformly continuous on the compact set $[0,1]\times[-K,K]$, then there exists, and it is finite, the positive number

$$R_K = \|f\|_\infty = \sup_{x\in X}\sup_{s\in[0,1]} |f(s,x(s))| < \infty, \quad (s,x(s)) \in [0,1]\times[-K,K].$$

Thus, we have

$$\|Tx\|_\infty \leq \frac{2|\beta|+4}{|\lambda|} M\Omega R_K + \left|\frac{\alpha}{\lambda}\right|\Delta, \tag{5.13}$$

for every $x \in X$. That means, the set $T(X)$ is bounded in $C^2[0,1]$. Now, if $t_1, t_2 \in [0,1]$, are such that $t_1 \leq t_2$ and satisfy $|t_1 - t_2| < \delta$, then

$$\begin{aligned} |Tx(t_1) - Tx(t_2)| &= \left|\frac{1}{\lambda}\int_0^{t_1} \frac{f(s,x(s))}{\varphi(s)}\omega'(s)\,\mathrm{d}s - \frac{1}{\lambda}\int_0^{t_2} \frac{f(s,x(s))}{\varphi(s)}\omega'(s)\,\mathrm{d}s - \frac{\alpha}{\lambda}e^{-\lambda t_1} + \frac{\alpha}{\lambda}e^{-\lambda t_2}\right| \\ &= \left|\frac{1}{\lambda}\int_{t_1}^{t_2} \frac{f(s,x(s))}{\varphi(s)}\omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda}(e^{-\lambda t_2} - e^{-\lambda t_1})\right| \\ &\to 0, \quad \text{as } |t_1 - t_2| \to 0 \end{aligned}$$

for every $x \in X$, so the set $T(X) \subset C^2[0,1]$ satisfies the hypotheses of Arzela-Ascoli's theorem, so $T(X)$ is relatively compact in $C^2[0,1]$. Therefore, the operator T is compact. □

Now, we establish the following existence result.

Theorem 5.4. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function, and let us assume that there exist $C, D \geq 0$ and $q \in (0,1)$ such that*

$$|f(s,r)| \leq C|r|^q + D.$$

For every $(s,r) \in [0,1]\times\mathbb{R}$. Then, the nonlinear boundary value problem (5.7)–(5.8) *has at least one solution.*

Proof. The theorem is proved once we assure the existence of at least a fixed point of the operator T. Let

$$\mathcal{S} = \{x \in C^2[0,1] \ : \ \exists\sigma \in [0,1] \text{ such that } x = \sigma Tx\}.$$

To apply Schaefer's fixed point theorem we should show that $\mathcal{S}$ is bounded. Let $x \in \mathcal{S}$,

$$\|x\|_\infty = \sigma\|Tx\|_\infty.$$

Now, from (5.13) we have

$$|Tx(t)| \leq \frac{2|\beta|+4}{|\gamma|}\int_0^1 \frac{|f(s,x(s))|}{|\varphi(s)|}\omega'(s)\,\mathrm{d}s + \left|\frac{\alpha}{\beta}\right|\Delta \leq \frac{2|\beta|+4}{|\gamma|}M\Omega(C\|x\|_\infty^q + D) + \left|\frac{\alpha}{\beta}\right|\Delta < \infty.$$

Then

$$\|x\|_\infty = \sigma\|Tx\|_\infty \leq \sigma\frac{2|\beta|+4}{|\gamma|}M\Omega(C\|x\|_\infty^q + D) + \left|\frac{\alpha}{\beta}\right|\Delta\sigma < \infty.$$

This inequality and the fact $q \in (0,1)$ shows that $\mathcal{S}$ is bounded. Thus, from Schaefer's fixed point theorem, the operator T has a fixed point, which implies that boundary value problem (5.7)–(5.8) has a solution. □

Notice from the proof of the theorem above, that we can use the functions φ and ω given in the definition of the (φ,ω)-derivative to rewrite Theorem 5.4 as:

Theorem 5.5. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function, and let us assume that there exist $C, D \geq 0$ and $q \in (0,1)$ such that*

$$\frac{|f(s,r)|}{|\varphi(s)|}\omega'(s) \leq C|r|^q + D.$$

For every $(s,r) \in [0,1]\times\mathbb{R}$. Then, the nonlinear boundary value problem (5.7)–(5.8) has at least one solution.

Schaefer's theorem is a consequence of the Schauder fixed point theorem, which is a localization fixed point result. We will use Schauder's theorem to give a localization result for the solutions of boundary value problem (5.7)–(5.8).

Theorem 5.6. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function and, in addition, let us assume that $f \in L^1([0,1]\times\mathbb{R})$. Let $\overline{B}(r)$ be the closed ball with radius r. Then, the nonlinear boundary value problem (5.7)–(5.8) has at least one solution for every closed ball $\overline{B}(r)$ such that*

$$r \geq \frac{2|\beta|+4}{|\lambda|}M\Omega\|f\|_1 + \left|\frac{\alpha}{\lambda}\right|\Delta, \tag{5.14}$$

with,

$$\Delta := \begin{cases}(|\beta|+2)e^{-\lambda}, \ -1<\lambda<0 \\ 2(|\beta|+1), \ \lambda>0\end{cases}, \quad M := \sup_{t\in[0,1]}\left|\frac{1}{\varphi(t)}\right|, \quad \Omega = \sup_{t\in[0,1]}\omega'(t).$$

Proof. Since the operator T is continuous and compact, we can apply Schauder's fixed point theorem, once we prove that $T(\overline{B}(r)) \subset \overline{B}(r)$.

From (5.13) and the hypotheses, we have

$$\begin{aligned} |Tx(t)| &\leq \frac{2|\beta|+4}{|\gamma|}\int_0^1 \frac{|f(s,x(s))|}{|\varphi(s)|}\omega'(s)\,\mathrm{d}s + \left|\frac{\alpha}{\beta}\right|\Delta \\ &\leq \frac{2|\beta|+4}{|\gamma|}M\Omega\|f\|_1 + \left|\frac{\alpha}{\beta}\right|\Delta \\ &\leq r. \end{aligned} \tag{5.15}$$

Thus, $\|Tx\|_\infty \leq r$. Consequently $T(\overline{B}(r)) \subset \overline{B}(r)$. Finally, from Schauder's theorem, T has a fixed point, as so boundary value problem (5.7)–(5.8) has at least one solution, for every closed ball $\overline{B}(r)$ with radius r as in (5.14). ☐

We can control the growth behavior of the nonlinear function f and still guarantee the existence of solutions for BVP (5.7)–(5.8). Some of these behaviors, as we will see, can be controlled in terms of the functions φ and ω given in the definition of the (φ,ω)-derivative, which can be interpreted as behaviors scaled for the (φ,ω)-derivative.

The main idea is to replace the integral term (5.15) with some condition which allows found a bound for it. For instance if we assume that f is uniformly bounded by $A>0$ on $[0,1]\times\mathbb{R}$, then use the estimate $|f(s,x(s))| \leq A$ in (5.15) and obtain the radius $r \geq \frac{2|\beta|+4}{|\lambda|}M\Omega AK + \left|\frac{\alpha}{\lambda}\right|\Delta$.

If we assume that $|f(s,y)| \leq A\frac{|\varphi(s)|}{w'(s)}$, for some $A>0$, for all $(s,x)\in[0,1]\times\mathbb{R}$, the integral term is less or equal to A and the radius is $r \geq \frac{2|\beta|+4}{|\lambda|}A + \left|\frac{\alpha}{\lambda}\right|\Delta$.

Finally, if $|f(s,x(s))| \leq \dfrac{|\gamma|}{2(|\beta|+2)M\Omega}s|x(s)|$, and we assume that

$$\left|\frac{\alpha}{\lambda}\right|\Delta \leq \frac{r}{2}, \quad \text{for each } r>0 \text{ given.}$$

Then, estimate (5.15) is rewrite as

$$|Tx(t)| \leq \frac{\|x\|_\infty}{2} + \left|\frac{\alpha}{\beta}\right|\Delta \leq \frac{r}{2}+\frac{r}{2} = r.$$

This proves that T applies any ball of radius r into itself. Therefore, we conclude that BVP (5.7)–(5.8) as at leat one solution on each ball of radius r.

In similar fashion it can be proved that for

$$|f(s,y)| \leq \frac{|\gamma|}{4(|\beta|+2)M\Omega(\frac{|a_n|}{n+1}+\cdots+|a_0|)}|a_ns^n+\cdots+a_0|)|x(s)|,$$

and

$$\left|\frac{\alpha}{\lambda}\right|\Delta \leq \frac{r}{2}, \quad \text{for each } r > 0,$$

the same conclusion holds.

References

[1] T. Abdeljawad, "On conformable fractional calculus", J. Comput. Appl. Math, vol. 279, pp. 57–66, 2015.

[2] H. Afshari, H. Marasi and H. Aydi, "Existence and uniqueness of positive solutions for boundary value problems of fractional differential equations", Filomat, vol. 31, no. 9, pp. 2675–2682, 2017.

[3] H. Afshari, H. Shojaat and M. S. Siahkali, "Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions", Results in Nonlinear Analysis, vol. 4, no. 3, pp. 186–199, 2021.

[4] B. Ahmad, A. Alsaedi, S. K. Ntouyas and J. Tariboon, *Hadamard-type fractional differential equations, inclusions and inequalities*, Cham: Springer, 2017.

[5] B. Ahmad and S. Ntouyas, "A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations", Fract. Calc. Appl. Anal., vol. 17, no. 2, pp. 348–360, 2014.

[6] D. R. Anderson, "Taylor's formula and integral inequalities for conformable fractional derivatives" in *Contributions in Mathematics and Engineering*, Cham: Springer, 2016, pp. 25–43.

[7] D. R. Anderson and D. J. Ulness, "Newly defined conformable derivatives", Adv. Dyn. Syst. Appl., vol. 10, no. 2, pp. 109–137, 2015.

[8] D. Baleanu, Z. B. Güvenç and J. A. Tenreiro Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, New York: Springer, 2010.

[9] T. Bashiri, S. M. Vaezpour and J. J. Nieto, "Approximating solution of Fabrizio-Caputo Volterra's model for population growth in a closed system by homotopy analysis method", J. Funct. Spaces, Art. ID 3152502, 10 pages, 2018.

[10] H. Batarfi, J. Losada, J. J. Nieto and W. Shammakh, "Three-point boundary value problems for conformable fractional differential equations", J. Funct. Spaces, Art. ID 706383, 6 pages, 2015.

[11] M. Benchohra, J. R. Graef and S. Hamani, "Existence results for boundary value problems with non-linear fractional differential equations", Appl. Anal., vol. 87, no. 7, pp. 851–863, 2008.

[12] R. Caponetto, G. Dongola, L. Fortuna and I. Petras, *Fractional order systems: modeling and control applications*, World Scientific Series on Nonlinear Science Series A, vol. 72, Singapore: World Scientific Publishing Co, Pte. Ltd., 2010.

[13] R. E. Castillo and S. A. Chapinz, "The fundamental theorem of calculus for the Riemann-Stieljes integral", Lect. Mat., vol. 29, no. 2, pp. 115–122, 2008.

[14] X. Dong, Z. Bai and S. Zhang, "Positive solutions to boundary value problems of p-Laplacian with fractional derivative", Bound. Value Probl., Paper No. 5, 15 pages, 2017.

[15] O. S. Iyiola and E. R. Nwaeze, "Some new results on the new conformable fractional calculus with applications using D'Alambert approach", Prog. Fract. Differ. Appl., vol. 2, no. 2, pp. 115–122, 2016.

[16] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, "A new definition of fractional derivative", J. Comput. Appl. Math, vol. 264, pp. 65–70, 2014.

[17] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Amsterdam: Elsevier B. V., 2006.

[18] F. Mainardi, *Fractional calculus and waves in linear viscoelesticity*, London: Imperial College Press, 2010.

[19] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue and V. Feliu, *Fractional-order systems and controls*, Advances in Industrial Control, London: Springer, 2010.

[20] K. B. Oldham and J. Spanier, *The fractional calculus*, Mathematics in Science and Engineering 111, New York-London: Academic Press, 1974.

[21] M. D. Ortigueira, *Fractional calculus for scientists and engineers*, Lecture Notes in Electrical Engineering 84, Dordrecht: Springer, 2011.

[22] L.-E. Persson and H. Rafeiro, "On a Taylor remainder", Acta Math. Acad. Paedagog. Nyházy, vol. 33, no. 2, pp. 195–198, 2017.

[23] L.-E. Persson, H. Rafeiro and P. Wall, "Historical synopsis of the Taylor remainder", Note Mat., vol. 37, no. 1, pp. 1–21, 2017.

[24] C. Pinto and A. R. M. Carvalho, "New findings on the dynamics of HIV and TB coinfection models", Appl. Math. and Comput., vol. 242, pp. 36–46, 2014.

[25] C. Pinto and A. R. M. Carvalho, "Fractional modeling of typical stages in HIV epidemics with drug-resistance", Progr. Fract. Differ. Appl., vol. 1, no. 2, pp. 111–122, 2015.

[26] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering 198, San Diego: Academic Press, Inc., 1999.

[27] H. Rafeiro and S. Kim, "Revisiting the first mean value theorem for integrals", Teach. Math., vol. 25, no. 1, pp. 30–35, 2022.

[28] M. N. Sahlan and H. Afshari, "Three new approaches for solving a class of strongly nonlinear two-point boundary value problems", Bound. Value Probl., Paper No. 60, 21 pages, 2021.

[29] M. N. Sahlan and H. Afshari, "Lucas polynomials based spectral methods for solving the fractional order electrohydrodynamics flow model", Commun. Nonlinear Sci. Numer. Simul., vol. 107, Paper No. 106108, 21 pages, 2022.

[30] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Switzerland: Gordon & Breach Science Publishers, 1993.

[31] E. Scalas, R. Gorenflo, F. Mainardi and M. Meerschaert, "Speculative option valuation and the fractional diffusion equation", in *Proceedings of the IFAC Workshop on Fractional Differentiation and its Applications*, J. Sabatier and J. Tenreiro Machado, Bordeaux, 2004.

[32] J. A. Tenreiro Machado, "And I say to myself: "What a fractional world!" ", Fract. Calc. Appl. Anal., vol. 14, no. 4, Paper No. 635, 2011.

[33] D. Zhao and M. Luo, "General conformable fractional derivative and its physical interpretation", Calcolo, vol. 54, no. 3, pp. 903–917, 2017.

[34] W. Zhong and L. Wang, "Positive solutions of conformable fractional differential equations with integral boundary conditions", Bound. Value Probl., Paper No. 136, 12 pages, 2018.

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Estimates for the polar derivative of a constrained polynomial on a disk

Gradimir V. Milovanović[1, ✉]
Abdullah Mir[2]
Adil Hussain[2]

[1] *Serbian Academy of Sciences and Arts, 11000 Belgrade, Serbia University of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia.*
gvm@mi.sanu.ac.rs ✉

[2] *Department of Mathematics, University of Kashmir, Srinagar, 190006, India.*
mabdullah_mir@uok.edu.in
malikadil6909@gmail.com

ABSTRACT

This work is a part of a recent wave of studies on inequalities which relate the uniform-norm of a univariate complex coefficient polynomial to its derivative on the unit disk in the plane. When there is a limit on the zeros of a polynomial, we develop some additional inequalities that relate the uniform-norm of the polynomial to its polar derivative. The obtained results support some recently established Erdős-Lax and Turán-type inequalities for constrained polynomials, as well as produce a number of inequalities that are sharper than those previously known in a very large literature on this subject.

RESUMEN

Este trabajo es parte de una reciente ola de estudios sobre desigualdades que relacionan la norma uniforme de un polinomio univariado con coeficientes complejos con su derivada en el disco unitario en el plano. Cuando existe un límite sobre los ceros de un polinomio, desarrollamos algunas desigualdades adicionales que relacionan la norma uniforme del polinomio con su derivada polar. Los resultados obtenidos satisfacen desigualdades de tipo Erdős-Lax y Turán para polinomios restringidos recientemente establecidas, y también producen desigualdades que son más estrictas que aquellas conocidas previamente en la larga literatura dedicada a este tema.

Keywords and Phrases: Complex domain, Constrained polynomial, Rouché's theorem, Zeros.

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1 Introduction

Experimental data is converted into mathematical notation and mathematical models in scientific inquiries. In order to solve these, it may be necessary to know how large or small the maximum modulus of the derivative of an algebraic equation can be in terms of maximum modulus of the polynomial. In practise, setting boundaries for these circumstances is crucial. The only information available in the literature is in the form of approximations, and there are no closed formulae for calculating these limitations precisely. These approximate boundaries are quite accurate when computed effectively adequate for the demands of investigators and scientists. As a result, there is a constant desire to find boundaries that are superior to those described in the literature. We were inspired to write this note because there is a need for updated and more precise bounds. The inequalities for polynomials and their derivatives, which generalise the classical inequalities for different norms and with different constraints on utilising various methods of geometric function theory, are a fertile topic in analysis. In the literature, for proving the inverse theorems in approximation theory, many inequalities in both directions relating the norm of the derivative and the polynomial itself play a significant role and, of course, have their own intrinsic appeal. As shown by various recent studies, numerous research papers have been published on these inequalities for constrained polynomials (for example, see [11, 13, 17, 19, 20, 21]). We begin with the well-known Bernstein inequality [4] for the uniform norm on the unit disk in the plane: namely, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

If we only consider polynomials without zeros in $|z| < 1$, the above inequality (1.1) can then be emphasised. In fact, Erdős conjectured and later Lax [14] proved that, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The inequality (1.2) is sharp and equality holds if $P(z)$ has all of its zeros on $|z| = 1$.

When there is a restriction on the polynomial's zeros, Turán's classical inequality [25] offers a lower bound estimate for the size of the derivative of the polynomial on the unit circle in relation to the size of the polynomial. It states that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Aziz and Dawood [2] improved inequality (1.3) to take the form

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\right\}. \tag{1.4}$$

Any polynomial that has all of its zeros on $|z| = 1$ holds true for (1.3) and (1.4).

The inequalities (1.3) and (1.4) have been generalised and expanded in a number of ways over time. For a polynomial $P(z)$ of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, Govil [8], proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.5}$$

As is easy to see that (1.5) becomes an equality if $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomials having all zeros on $|z| = k$, then it may be possible to improve the bound in (1.5). In this direction, it was shown by Govil [10] that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n}\left\{\max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)|\right\}. \tag{1.6}$$

As an extension of (1.2), Malik [15] proved that, if $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.7}$$

The result is sharp and equality in (1.7) holds for $P(z) = (z+k)^n$.

On the other hand, if $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, the precise estimate of maximum of $|P'(z)|$ on $|z| = 1$ does not seem to be known in general, and this problem is still open. However, some special cases in this direction have been considered by many people where some partial extensions of (1.2) are established. In 1980, it was shown by Govil [9] that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{1.8}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$. Under the same hypothesis as in (1.8), Aziz and Ahmad [1] established an improved inequality in the form

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n}\left\{\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)|\right\}. \tag{1.9}$$

In the literature, more generalised variations of Bernstein and Turán inequalities have emerged, in which the underlying polynomial is replaced with more general classes of functions. One such generalisation is moving from the domain of ordinary derivatives of polynomials to the domain of their polar derivatives. Before drawing any more conclusions, let us first discuss the idea of the polar derivative. For a polynomial $P(z)$ of degree n, we define

$$D_\beta P(z) := nP(z) + (\beta - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point β. The polynomial $D_\beta P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\beta\to\infty}\left\{\frac{D_\beta P(z)}{\beta}\right\} = P'(z),$$

uniformly with respect to z for $|z| \le R,\ R > 0$.

The comprehensive books by Marden [16], Milovanović *et al.* [18], Rahman and Schmeisser [23], and the most recent one by Gardner *et al.* [7] all provide access to the extensive literature on the polar derivative of polynomials.

In 1998, Aziz and Rather [3] established the polar derivative analogue of (1.5) by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \ge k$,

$$\max_{|z|=1}|D_\beta P(z)| \ge n\left(\frac{|\beta|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|. \tag{1.10}$$

In the same publication, Aziz and Rather extended the inequality (1.4) to the polar derivative of a polynomial. In fact, they proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le 1$, then for any complex number β with $|\beta| \ge 1$,

$$\max_{|z|=1}|D_\beta P(z)| \ge \frac{n}{2}\left\{(|\beta|-1)\max_{|z|=1}|P(z)| + (|\beta|+1)\min_{|z|=1}|P(z)|\right\}. \tag{1.11}$$

The corresponding polar derivative analogue of (1.6) and a refinement of (1.10) was given by Dewan *et al.* [5]. They proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for any complex number β with $|\beta| \ge k$,

$$\max_{|z|=1}|D_\beta P(z)| \ge \frac{n}{1+k^n}\left\{(|\beta|-k)\max_{|z|=1}|P(z)| + \left(|\beta| + \frac{1}{k^{n-1}}\right)\min_{|z|=k}|P(z)|\right\}. \tag{1.12}$$

Singh and Chanam [24] most recently developed the following generalisation and strengthening of (1.10).

Theorem A. *Let* $P(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \le s \le n$, *be a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \ge 1$, *then for every complex number* β *with* $|\beta| \ge k$,

$$\max_{|z|=1} |D_\beta P(z)| \ge (|\beta| - k)\left\{\frac{n+s}{1+k^n} + \frac{k^{(n-s)/2}\sqrt{|a_{n-s}|} - \sqrt{|a_0|}}{(1+k^n)k^{(n-s)/2}\sqrt{|a_{n-s}|}}\right\} \max_{|z|=1} |P(z)|. \tag{1.13}$$

The improvement of inequality (1.8) as a result of Govil [9] was demonstrated by Singh and Chanam in the same paper in the form of the subsequent outcome.

Theorem B. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *having no zeros in* $|z| < k$, $k \le 1$, *and let* $Q(z) = z^n \overline{P(1/\overline{z})}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *attain their maximum at the same point on* $|z| = 1$,

$$\max_{|z|=1} |P'(z)| \le \left\{\frac{n}{1+k^n} - \frac{\left(\sqrt{|a_0|} - k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|}}\right\} \max_{|z|=1} |P(z)|. \tag{1.14}$$

The result is sharp and equality holds in (1.14) for $P(z) = z^n + k^n$.

The study of these inequalities for a certain class of polynomials having a zero of order $s \ge 0$ at the origin is continued in this paper, and we set some new upper and lower bounds for the derivative and polar derivative of a polynomial on the unit disk while taking into account the location of the zeros and extremal coefficients of the underlying polynomial.

2 Main results

We begin this section by proving the following Turán-type inequality giving generalisations and refinements of (1.10)–(1.13) and related inequalities.

Theorem 2.1. *Let* $P(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \le s \le n$, *be a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \ge 1$, *then for every complex number* β *with* $|\beta| \ge k$,

$$\begin{aligned}\max_{|z|=1} |D_\beta P(z)| \ge \frac{n}{1+k^n}&\left\{(|\beta| - k)\max_{|z|=1}|P(z)| + \left(|\beta| + \frac{1}{k^{n-1}}\right)m_k\right\}\\ &+ \left(\frac{|\beta|-k}{1+k^n}\right)\left\{s + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - m_k}}\right\}\left(\max_{|z|=1}|P(z)| - \frac{m_k}{k^n}\right),\end{aligned} \tag{2.1}$$

where $m_k = \min_{|z|=k} |P(z)|$.

Setting $s = 0$ in (2.1), we get the following refinement of (1.12) and hence of (1.10) and (1.11) as well.

Corollary 2.2. *Let $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z|\leq k$, $k\geq 1$, then for every complex number β with $|\beta|\geq k$,*

$$\begin{aligned}\max_{|z|=1}|D_\beta P(z)|\geq &\frac{n}{1+k^n}\left\{(|\beta|-k)\max_{|z|=1}|P(z)|+\left(|\beta|+\frac{1}{k^{n-1}}\right)m_k\right\}\\ &+\left(\frac{|\beta|-k}{1+k^n}\right)\left\{\frac{\sqrt{k^n|a_n|-m_k}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-m_k}}\right\}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),\end{aligned} \tag{2.2}$$

where m_k is as defined in Theorem 2.1.

By taking $k=1$ in (2.2), we easily get a refinement of (1.11). If we divide both sides of (2.1) and (2.2) by $|\beta|$ and let $|\beta|\to\infty$, we get the following results:

Corollary 2.3. *Let $P(z)=z^s\sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0\leq s\leq n$, be a polynomial of degree n having all its zeros in $|z|\leq k,\ k\geq 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)|\geq &\frac{n}{1+k^n}\left(\max_{|z|=1}|P(z)|+m_k\right)\\ &+\left\{\frac{s}{1+k^n}+\frac{\sqrt{k^{n-s}|a_{n-s}|-m_k}-\sqrt{|a_0|}}{(1+k^n)\sqrt{k^{n-s}|a_{n-s}|-m_k}}\right\}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),\end{aligned} \tag{2.3}$$

where m_k is as defined in Theorem 2.1. Equality in (2.3) holds for $P(z)=z^n+k^n$.

Corollary 2.4. *Let $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z|\leq k$, $k\geq 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)|\geq &\frac{n}{1+k^n}\left(\max_{|z|=1}|P(z)|+m_k\right)\\ &+\frac{\sqrt{k^n|a_n|-m_k}-\sqrt{|a_0|}}{(1+k^n)\sqrt{k^n|a_n|-m_k}}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),\end{aligned} \tag{2.4}$$

where m_k is as defined in Theorem 2.1. Equality in (2.4) holds for $P(z)=z^n+k^n$.

Remark 2.5. *It is clear that, in general for any polynomial of degree n of the form $P(z)=z^s(a_0+a_1z+\cdots+a_{n-s}z^{n-s}),\ 0\leq s\leq n$, having all its zeros in $|z|\leq k,\ k\geq 1$, the inequality (2.1) improves the inequality (1.13) considerably, excepting the case when $P(z)$ has a zero on $|z|=k$. For the class of polynomials having a zero on $|z|=k$, the inequality (2.2) will give bounds that are sharper than the bound obtained from the inequality (1.12). One can also observe that the inequality (2.4) improves inequality (1.6) considerably when $\sqrt{k^n|a_n|-m_k}-\sqrt{|a_0|}\neq 0$.*

As an application of Corollary 2.4, we prove the following result for the class of polynomials having no zeros in $|z|<k,\ k\leq 1$, which in turn provides a generalization and refinement to Theorem B.

Theorem 2.6. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n having no zeros in $|z|<k,\ k\le 1$, and let $Q(z)=z^n\overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z|=1$, then for every complex number β with $|\beta|\ge 1$,*

$$\begin{aligned}\max_{|z|=1}|D_\beta P(z)|\le&\frac{n(|\beta|+k^n)}{1+k^n}\max_{|z|=1}|P(z)|-\frac{nm_k(|\beta|-1)}{1+k^n}\\&-\frac{(|\beta|-1)\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\},\end{aligned}\tag{2.5}$$

where m_k is as defined in Theorem 2.1. Equality in (2.5) holds for $P(z)=z^n+k^n$, with real $\beta\ge 1$.

If we divide both sides of inequality (2.5) by $|\beta|$ and let $|\beta|\to\infty$, we get the following result.

Corollary 2.7. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n having no zeros in $|z|<k,\ k\le 1$, and let $Q(z)=z^n\overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z|=1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)|\le&\frac{n}{1+k^n}\max_{|z|=1}|P(z)|-\frac{nm_k}{1+k^n}\\&-\frac{\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\},\end{aligned}\tag{2.6}$$

where m_k is as defined in Theorem 2.1. Equality in (2.6) holds for $P(z)=z^n+k^n$.

Remark 2.8. It may be remarked here that, in general for any polynomial of degree n of the form $P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$, having no zeros in $|z|<k$, $k\le 1$, the inequality (2.6) improves the inequality (1.14), excepting the case when $P(z)$ has a zero on $|z|=k$. For the class of polynomials having a zero on $|z|=k$, the inequality (2.5) sharpens a result of Mir and Breaz [20, Corollary 2] considerably.

3 Lemmas

In order to prove our results, we need the following lemmas. The first lemma is a simple deduction from the Maximum Modulus Principle (see [22]).

Lemma 3.1. *If $P(z)$ is a polynomial of degree at most n, then for $R\ge 1$,*

$$\max_{|z|=R}|P(z)|\le R^n\max_{|z|=1}|P(z)|.$$

The following lemma is due to Dewan and Upadhye [6].

Lemma 3.2. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k,\ k \geq 1$, then*

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)| + \frac{k^n-1}{k^n+1} \min_{|z|=k} |P(z)|.$$

Lemma 3.3. *If $P(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu,\ 0 \leq s \leq n$, is a polynomial of degree n having all zeros in $|z| \leq 1$, then for any complex number β with $|\beta| \geq 1$ and $|z| = 1$,*

$$|D_\beta P(z)| \geq (|\beta|-1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s}|} - \sqrt{|a_0|}}{2\sqrt{|a_{n-s}|}}\right\}|P(z)|.$$

The above lemma is due to Singh and Chanam [24].

Lemma 3.4. *If $P(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \leq s \leq n$, is a polynomial of degree n having all zeros in $|z| \leq 1$, then for any complex number β with $|\beta| \geq 1$ and $|z| = 1$,*

$$\begin{aligned}|D_\beta P(z)| \geq &\frac{n}{2}\Big((|\beta|-1)|P(z)| + (|\beta|+1)m_1\Big)\\ &+ \left(\frac{|\beta|-1}{2}\right)\left\{s + \frac{\sqrt{|a_{n-s}|-m_1} - \sqrt{|a_0|}}{\sqrt{|a_{n-s}|-m_1}}\right\}(|P(z)| - m_1),\end{aligned}$$

where $m_1 = \min_{|z|=1} |P(z)|$.

Proof. By hypothesis $P(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \leq s \leq n$, has all its zeros in $|z| \leq 1$. If the polynomial $h(z) = \sum_{\nu=0}^{n-s} a_\nu z^\nu$ has a zero on $|z| = 1$, then $m_1 = \min_{|z|=1} |P(z)| = 0$ and the result follows by Lemma 3.3 in this case. Henceforth, we assume that all the zeros of $P(z) = z^s h(z)$ lie in $|z| < 1$, so that $m_1 > 0$. Therefore, we have $m_1 \leq |P(z)|$ for $|z| = 1$. This implies for any complex number μ with $|\mu| < 1$, that

$$m_1|\mu z^n| < |P(z)| \quad \text{for} \quad |z| = 1.$$

Since all the zeros of $P(z)$ lie in $|z| < 1$, it follows by Rouché's theorem that all the zeros of

$$P(z) - \mu m_1 z^n = z^s\Big(a_0 + a_1 z + \cdots + (a_{n-s} - \mu m_1) z^{n-s}\Big)$$

also lie in $|z| < 1$. Hence, by Lemma 3.3, we get for $|\beta| \geq 1$ and $|z| = 1$,

$$|D_\beta(P(z) - \mu m_1 z^n)| \geq (|\beta|-1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s} - \mu m_1|} - \sqrt{|a_0|}}{2\sqrt{|a_{n-s} - \mu m_1|}}\right\}|P(z) - \mu m_1 z^n|. \tag{3.1}$$

For every $\mu \in \mathbb{C}$, we have

$$|a_{n-s} - \mu m_1| \geq |a_{n-s}| - |\mu| m_1,$$

and since the function $\psi(x) = \frac{\left(\sqrt{x} - \sqrt{|a_0|}\right)}{\sqrt{x}}$, $x > 0$, is a non-decreasing function of x, it follows from (3.1) that for every μ with $|\mu| < 1$ and $|z| = 1$,

$$|D_\beta(P(z) - \mu m_1 z^n)| \geq (|\beta| - 1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s}| - |\mu| m_1} - \sqrt{|a_0|}}{2\sqrt{|a_{n-s}| - |\mu| m_1}}\right\}|P(z) - \mu m_1 z^n|. \tag{3.2}$$

It is a simple deduction of Laguerre theorem (see [16, p. 52]) on the polar derivative of a polynomial that for any β with $|\beta| \geq 1$, the polynomial

$$D_\beta(P(z) - \mu m_1 z^n) = D_\beta P(z) - \mu \beta n m_1 z^{n-1}$$

has all its zeros in $|z| < 1$. This implies that

$$|D_\beta P(z)| \geq m_1 n |\beta| |z|^{n-1} \quad \text{for} \quad |z| \geq 1. \tag{3.3}$$

Now choosing the argument of μ suitably on the left hand side of (3.2) such that

$$\left|D_\beta P(z) - \mu \beta n m_1 z^{n-1}\right| = |D_\beta P(z)| - |\mu||\beta| n m_1 \quad \text{for} \quad |z| = 1,$$

which is possible by (3.3), we get for $|z| = 1$

$$|D_\beta P(z)| - m_1 n |\mu||\beta| \geq (|\beta| - 1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s}| - |\mu| m_1} - \sqrt{|a_0|}}{2\sqrt{|a_{n-s}| - |\mu| m_1}}\right\}\big(|P(z)| - |\mu| m_1\big). \tag{3.4}$$

If in (3.4), we make $|\mu| \to 1$, we easily get for $|z| = 1$,

$$\begin{aligned}|D_\beta P(z)| \geq\ & \frac{n}{2}\Big((|\beta| - 1)|P(z)| + (|\beta| + 1) m_1\Big) \\ & + \left(\frac{|\beta| - 1}{2}\right)\left\{s + \frac{\sqrt{|a_{n-s}| - m_1} - \sqrt{|a_0|}}{\sqrt{|a_{n-s}| - m_1}}\right\}(|P(z)| - m_1).\end{aligned}$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. *If $P(z)$ is a polynomial of degree n and, $Q(z) = z^n \overline{P(1/\overline{z})}$, then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|.$$

The above lemma is due to Govil and Rahman [12].

4 Proofs of the main results

Proof of Theorem 2.1. Recall that $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore, all the zeros of the polynomial $E(z) = P(kz)$ lie in $|z| \leq 1$. Applying Lemma 3.4 to the polynomial $E(z)$ and noting that $|\beta|/k \geq 1$, we get

$$\begin{aligned}\max_{|z|=1} \left|D_{\beta/k} E(z)\right| \geq \frac{n}{2}\Bigg\{ \left(\frac{|\beta|}{k} - 1\right) \max_{|z|=1} |E(z)| + \left(\frac{|\beta|}{k} + 1\right) m^* \Bigg\} \\ + \left(\frac{|\beta|}{k} - 1\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m^*} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m^*}} \right\} \left(\max_{|z|=1} |E(z)| - m^*\right), \qquad (4.1)\end{aligned}$$

where $m^* = \min_{|z|=1} |E(z)| = \min_{|z|=1} |P(kz)| = \min_{|z|=k} |P(z)| = m_k$.

The above inequality (4.1) is equivalent to

$$\begin{aligned}\max_{|z|=1} \left| nP(kz) + \left(\frac{\beta}{k} - z\right) kP'(kz) \right| \geq \frac{n}{2}\Bigg\{ \left(\frac{|\beta| - k}{k}\right) \max_{|z|=1} |P(kz)| + \left(\frac{|\beta|}{k} + 1\right) m_k \Bigg\} \\ + \left(\frac{|\beta| - k}{k}\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m_k}} \right\} \\ \times \left(\max_{|z|=1} |P(kz)| - m_k\right).\end{aligned}$$

The last inequality yields

$$\begin{aligned}\max_{|z|=k} |D_\beta P(z)| \geq \frac{n}{2}\Bigg\{ \left(\frac{|\beta| - k}{k}\right) \max_{|z|=k} |P(z)| + \left(\frac{|\beta|}{k} + 1\right) m_k \Bigg\} \\ + \left(\frac{|\beta| - k}{k}\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m_k}} \right\} \left(\max_{|z|=k} |P(z)| - m_k\right). \qquad (4.2)\end{aligned}$$

Since $D_\beta P(z)$ is a polynomial of degree at most $n-1$, we have by Lemma 3.1 for $R = k \geq 1$,

$$\max_{|z|=k} |D_\beta P(z)| \leq k^{n-1} \max_{|z|=1} |D_\beta P(z)|.$$

On using this and Lemma 3.2, the above inequality (4.2) clearly gives

$$\begin{aligned}k^{n-1} \max_{|z|=1} |D_\beta P(z)| \geq \frac{n}{2}\Bigg\{ \left(\frac{|\beta| - k}{k}\right) \left(\frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)| + \left(\frac{k^n - 1}{k^n + 1}\right) m_k\right) + \left(\frac{|\beta|}{k} + 1\right) m_k \Bigg\} \\ + \left(\frac{|\beta| - k}{k}\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m_k}} \right\} \\ \times \left\{ \frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)| + \left(\frac{k^n - 1}{k^n + 1}\right) m_k - m_k \right\}.\end{aligned}$$

After rearranging the terms, we get

$$\max_{|z|=1}|D_\beta P(z)| \geq \frac{n}{1+k^n}\left\{(|\beta|-k)\max_{|z|=1}|P(z)| + \left(|\beta|+\frac{1}{k^{n-1}}\right)m_k\right\} + \left(\frac{|\beta|-k}{1+k^n}\right)\left\{s+\frac{\sqrt{k^{n-s}|a_{n-s}|-m_k}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-m_k}}\right\}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),$$

which is exactly (2.1). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.6. Let $Q(z) = z^n\overline{P(1/\overline{z})}$. Since $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ in $|z| < k$, $k \leq 1$, the polynomial $Q(z)$ of degree n has all its zeros in $|z| \leqslant 1/k$, $1/k \geq 1$. On applying inequality (2.4) of Corollary 2.4 to $Q(z)$, we get

$$\max_{|z|=1}|Q'(z)| \geq \frac{n}{1+\frac{1}{k^n}}\left(\max_{|z|=1}|Q(z)|+m_k'\right) + \frac{\sqrt{\frac{1}{k^n}|a_0|-m_k'}-\sqrt{|a_n|}}{(1+\frac{1}{k^n})\sqrt{\frac{1}{k^n}|a_0|-m_k'}}\left\{\max_{|z|=1}|Q(z)|-k^n m_k'\right\}. \tag{4.3}$$

Now,

$$m_k' = \min_{|z|=1/k}|Q(z)| = \min_{|z|=1/k}\left|z^n\overline{P\left(\frac{1}{\overline{z}}\right)}\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m_k}{k^n}$$

and

$$\max_{|z|=1}|Q(z)| = \max_{|z|=1}|P(z)|.$$

Using these observations in (4.3), we get

$$\max_{|z|=1}|Q'(z)| \geq \frac{nk^n}{1+k^n}\left(\max_{|z|=1}|P(z)|+\frac{m_k}{k^n}\right) + \frac{\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\}. \tag{4.4}$$

Since $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we have

$$\max_{|z|=1}(|P'(z)|+|Q'(z)|) = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{4.5}$$

On combining (4.4), (4.5) and Lemma 3.5, we get

$$n\max_{|z|=1}|P(z)| \geq \max_{|z|=1}|P'(z)| + \frac{nk^n}{1+k^n}\left(\max_{|z|=1}|P(z)|+\frac{m_k}{k^n}\right) + \frac{\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\},$$

which gives

$$\max_{|z|=1}|P'(z)| \le n\max_{|z|=1}|P(z)| - \frac{nk^n}{1+k^n}\left(\max_{|z|=1}|P(z)| + \frac{m_k}{k^n}\right) - \frac{\left(\sqrt{|a_0|-m_k} - k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)| - m_k\right\}. \tag{4.6}$$

Also, it is easy to verify that for $|z|=1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{4.7}$$

Note that for any complex number β, and $|z|=1$, we have

$$|D_\beta P(z)| = |nP(z) + (\beta - z)P'(z)| \le |nP(z) - zP'(z)| + |\beta||P'(z)|,$$

which gives by (4.7) and $|\beta| \ge 1$, that

$$\begin{aligned} |D_\beta P(z)| &\le |Q'(z)| + |\beta||P'(z)| = |Q'(z)| + |P'(z)| - |P'(z)| + |\beta||P'(z)| \\ &\le n\max_{|z|=1}|P(z)| + (|\beta|-1)|P'(z)| \qquad \text{(by Lemma 3.5)} \\ &\le n\max_{|z|=1}|P(z)| + (|\beta|-1)\max_{|z|=1}|P'(z)|. \end{aligned} \tag{4.8}$$

Inequality (4.8), in conjunction with (4.6), gives for $|z|=1$,

$$|D_\beta P(z)| \le n|\beta|\max_{|z|=1}|P(z)| - \frac{nk^n(|\beta|-1)}{1+k^n}\left(\max_{|z|=1}|P(z)| + \frac{m_k}{k^n}\right) - \frac{(|\beta|-1)\left(\sqrt{|a_0|-m_k} - k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)| - m_k\right\},$$

which is equivalent to (2.5). This completes the proof of Theorem 2.6. □

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References

[1] A. Aziz and N. Ahmad, “Inequalities for the derivative of a polynomial”, Proc. Indian Acad. Sci. Math. Sci., vol. 107, no. 2, pp. 189–196, 1997.

[2] A. Aziz and Q. M. Dawood, “Inequalities for a polynomial and its derivative”, J. Approx. Theory, vol. 54, no. 3, pp. 306–313, 1988.

[3] A. Aziz and N. A. Rather, “A refinement of a theorem of Paul Turán concerning polynomials”, Math. Inequal. Appl., vol. 1, no. 2, pp. 231–238, 1998.

[4] S. Bernstein, *Sur l’ordre de la meilleure approximation des functions continues par des polynômes de degré donné*, Mémoires de l’Académie Royale de Belgique 4, Brussels: Hayez, imprimeur des académies royales, 1912.

[5] K. K. Dewan, N. Singh, A. Mir and A. Bhat, “Some inequalities for the polar derivative of a polynomial”, Southeast Asian Bull. Math., vol. 34, no. 1, pp. 69–77, 2010.

[6] K. K. Dewan and C. M. Upadhye, “Inequalities for the polar derivative of a polynomial”, JIPAM. J. Inequal. Pure Appl. Math., vol. 9, no. 4, Article 119, 9 pages, 2008.

[7] R. B. Gardner, N. K. Govil and G. V. Milovanović, *Extremal problems and inequalities of Markov-Bernstein type for algebraic polynomials*, Mathematical Analysis and Its Applications, London: Elsevier/Academic Press, 2022.

[8] N. K. Govil, “On the derivative of a polynomial”; Proc. Amer. Math. Soc., vol. 41, no. 2, pp. 543–546, 1973.

[9] N. K. Govil, “On a theorem of S. Bernstein”, Proc. Nat. Acad. Sci. India Sec. A., vol. 50, no. 1, pp. 50–52, 1980.

[10] N. K. Govil, “Some inequalities for derivative of polynomials”, J. Approx. Theory, vol. 66, no. 1, pp. 29–35, 1991.

[11] N. K. Govil and P. Kumar, “On sharpening of an inequality of Turán”, Appl. Anal. Discrete Math., vol. 13, no. 3, pp. 711–720, 2019.

[12] N. K. Govil and Q. I. Rahman, “Functions of exponential type not vanishing in a half plane and related polynomials”, Trans. Amer. Math. Soc., vol. 137, pp. 501–517, 1969.

[13] P. Kumar and R. Dhankhar, “Some refinements of inequalities for polynomials”, Bull. Math. Soc. Sci. Math. Roumanie (N.S), vol. 63(111), no. 4, pp. 359–367, 2020.

[14] P. D. Lax, “Proof of a conjecture of P. Erdős on the derivative of a polynomial”, Bull. Amer. Math. Soc., vol. 50, no. 8, pp. 509–513, 1944.

[15] M. A. Malik, "On the derivative of a polynomial", J. London Math. Soc.(2), vol. 1, no. 1, pp. 57–60, 1969.

[16] M. Marden, *Geometry of Polynomials*, 2nd edition, Mathematical Surveys 3, Providence, R. I.: American Mathematical Society, 1966.

[17] G. V. Milovanović, A. Mir and A. Hussain, "Extremal problems of Bernstein-type and an operator preserving inequalities between polynomials", Sib. Math. J., vol. 63, no. 1, pp. 138–148, 2022.

[18] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in polynomials, extremal problems, inequalities, zeros*, River Edge, NJ: World Scientific Publishing, 1994.

[19] A. Mir, "On an operator preserving inequalities between polynomials", Ukrainian Math. J., vol. 69, no. 8, pp. 1234–1247, 2018.

[20] A. Mir and D. Breaz, "Bernstein and Turán-type inequalities for a polynomial with constraints on its zeros", Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, vol. 115, no. 3, Paper No. 124, 12 pages, 2021.

[21] A. Mir and I. Hussain, "On the Erdős-Lax inequality concerning polynomials", C. R. Math. Acad. Sci. Paris, vol. 355, no. 10, pp. 1055–1062, 2017.

[22] G. Pólya and G. Szegő, "Aufgaben und Lehrsätze aus der Analysis", Grundlehren der mathematischen Wissenschaften 19–20, Berlin: Springer, 1925.

[23] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs New Series 26, New York: Oxford University Press, Inc., 2002.

[24] T. B. Singh and B. Chanam, "Generalizations and sharpenings of certain Bernstein and Turán types of inequalities for the polar derivative of a polynomial", J. Math. Inequal., vol. 15, no. 4, pp. 1663–1675, 2021.

[25] P. Turán, "Über die Ableitung von Polynomen", Compositio Math., vol. 7, pp. 89–95, 1940.

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