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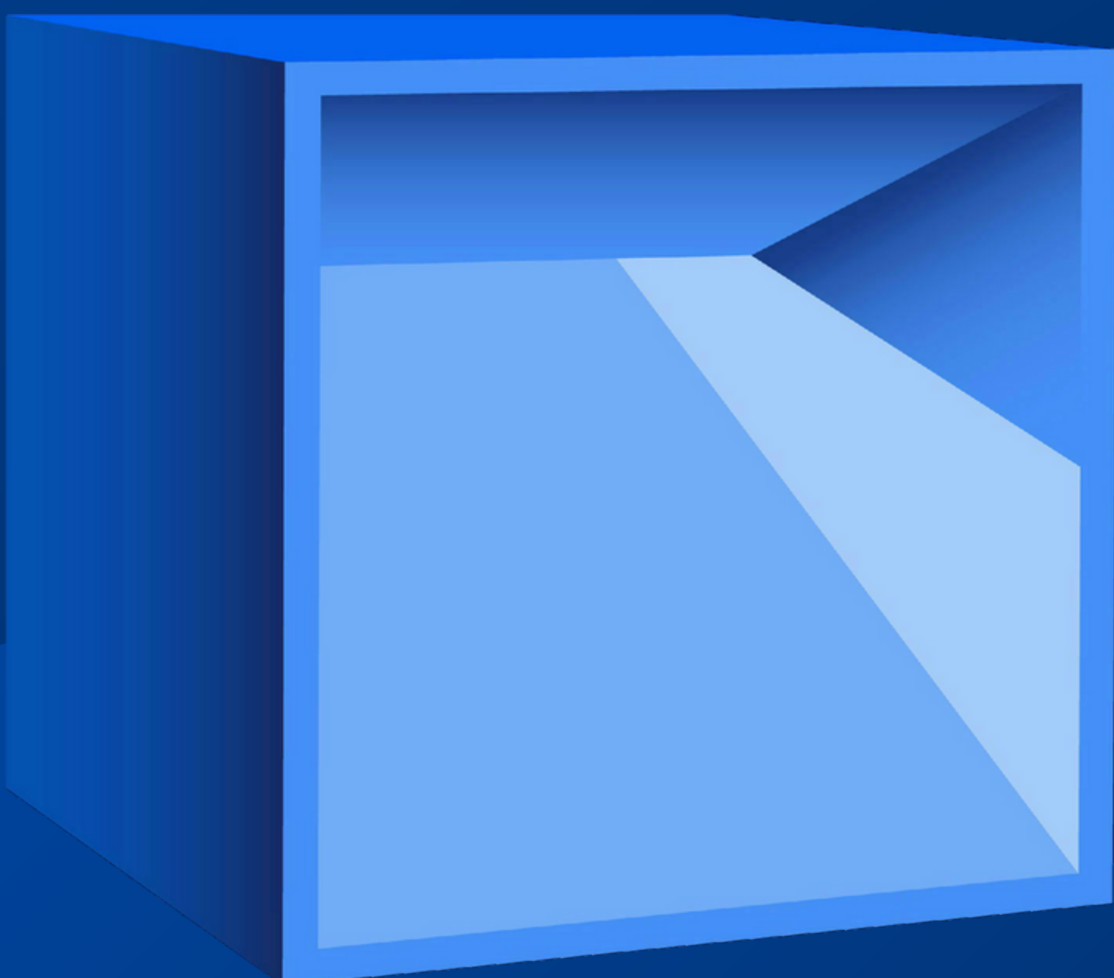

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SUMMARY

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Existence results for a class of local and nonlocal nonlinear elliptic problems

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ABSTRACT

In this paper, we study the p-Laplacian problems in the case where p depends on the solution itself. We consider two situations, when p is a local and nonlocal quantity. By using a singular perturbation technique, we prove the existence of weak solutions for the problem associated to the following equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) + |u|^{p(u)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and also for its nonlocal version. The main goal of this paper is to extend the results established by M. Chipot and H. B. de Oliveira (Math. Ann., 2019, 375, 283-306).

RESUMEN

En este artículo, estudiamos los problemas p-Laplacianos en el caso donde p depende de la solución misma. Consideramos dos situaciones, cuando p es una cantidad local y no-local. Usando una técnica de perturbación singular, demostramos la existencia de soluciones débiles para el problema asociado a la siguiente ecuación

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) + |u|^{p(u)-2}u = f & \text{en } \Omega \\ u = 0 & \text{sobre } \partial\Omega, \end{cases}$$

y también para su versión no-local. El principal objetivo de este artículo es extender los resultados establecidos por M. Chipot y H. B. de Oliveira (Math. Ann., 2019, 375, 283-306).

Keywords and Phrases: $p(u)$-Laplacian; elliptic problems; variable nonlinearity; generalised Sobolev spaces.

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1 Introduction

The study of partial differential equations involving the p-Laplacian generalised several types of problems not only in physics, but also in biophysics, plasma physics, and in the study of chemical reactions. These problems appear, for example, in a general reaction-diffusion system:

$$u_t = -\operatorname{div}\left(a|\nabla u|^{p(\cdot)-2}\nabla u\right) + |u|^{p(\cdot)-2}u,$$

where $a \in \mathbb{R}^+$ is a positive constant, the function u generally describes the concentration, the term $\operatorname{div}\left(a|\nabla u|^{p(\cdot)-2}\nabla u\right)$ corresponds to the diffusion with coefficient $D(u) = a|\nabla u|^{p(\cdot)-2}$, and $|u|^{p(\cdot)-2}u$ is the reaction term related to source and loss processes. In general, the reaction term has a polynomial form with respect to the concentration u.

Because of the importance of this kind of problems, many authors have investigated the existence and uniqueness of their different types of solutions [1, 4, 10].

Our main interest in this work is to extend these results to the case when p may depend both on the space variable x and on the unknown solution u. We first consider the case where the dependency of p on u is a local quantity. Namely, we study the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) + |u|^{p(u)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, f is a given data and p is the nonlinear exponent function $p : \mathbb{R} \to [1, +\infty)$ such that

$$p \text{ is continuous and } \quad 1 < r \leq p \leq s < \infty \quad \text{for some constants } r, s. \tag{1.2}$$

In the second part of this work, we consider also the following nonlocal problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(b(u))-2}\nabla u) + |u|^{p(b(u))-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $p : \mathbb{R} \longrightarrow [1, +\infty)$ and $b : W_0^{1,r}(\Omega) \longrightarrow \mathbb{R}$ are the functions involved in the exponent of nonlinearity, for some constant exponent r such that $1 < r < \infty$.

The fact that in reality physical measurements of certain quantities are not made in a punctual way but through local averages is always the motivation to study non-local problems. This kind of problems appear in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision [5, 6, 17]. J. Türola in [17] presented several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u. A numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$ is presented in [5]. To our best knowledge, there are only a few important contributions concerning the well-posedness of the solutions of this $p(u)$-Laplacian problems. The study of these problems was recently developed by Andreianov *et al.* [3]. They established the partial existence and uniqueness result to the weak solution in the cases of homogeneous Dirichlet boundary condition for the following problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2}\nabla u) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

S. Ouaro and N. Sawadogo in [14] and [15] considered the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \mathrm{div}\, a(x, u, \nabla u) = f & \text{in } \Omega \\ a(x, u, \nabla u) \cdot \eta + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness results of entropy and weak solutions are established by an approximation method and convergent sequences in terms of Young measure.

We were inspired by the work of M. Chipot and H. B. de Oliveira in [7], where the authors have proved the existence of the $p(u)$-problem (1.1) without the second term in the left-hand side, the existence proofs of [2] and [7] are based on the Schauder fixed-point theorem. They considered for the first time in the literature the nonlocal exponent of nonlinearity p as we consider here.

This paper is organized as follows. In Section 2 we introduce the basic assumptions and we recall some definitions and basic properties of generalised Sobolev spaces. Section 3 is devoted to show the existence of a solution to the local problem (1.1) using a singular perturbation technique. In Section 4, we prove the existence of weak solutions to the nonlocal problem (1.3) by using the Minty trick together with the technique of Zhikov (see [18]) for passing to the limit in our sequence of $p(u_n)$-Laplacian problems .

2 Preliminaries

The fact that the function p depends on the solution u and therefore it depends on the space variable x, allows us to look for the weak solutions in a Sobolev space with variable exponents.

Let Ω be a bounded domain of $\mathbb{R}^N$ with $\partial\Omega$ Lipschitz-continuous, we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous in Ω (for more details, see [9]) if

$$\exists C > 0 : \ |p(x) - p(y)| \leq \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \quad \forall x, y \in \Omega, \quad |x - y| < \frac{1}{2}. \tag{2.1}$$

For any Lebesgue-measurable function $p : \Omega \to [1, \infty)$, we define

$$p_- := \operatorname{ess}\inf_{x \in \Omega} p(x) \ , \ p_+ := \operatorname{ess}\sup_{x \in \Omega} p(x), \tag{2.2}$$

and we introduce the variable exponent Lebesgue space by:

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ / \ \rho_{p(\cdot)}(u) \ := \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}. \tag{2.3}$$

Equipped with the Luxembourg norm

$$\|u\|_{p(\cdot)} \ := \inf \left\{ \lambda > 0 \ : \ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}, \tag{2.4}$$

$L^{p(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < p_- \leq p_+ < \infty, \tag{2.5}$$

$L^{p(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where $p'(x)$ is the generalised Hölder conjugate of $p(x)$,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

The next proposition shows that there is a gap between the modular and the norm in $L^{p(\cdot)}(\Omega)$.

Proposition 2.1 (See [11])**.** *If* (2.5) *holds, for $u \in L^{p(x)}(\Omega)$, then the following assertions hold*

$$\min\left\{ \|u\|_{p(\cdot)}^{p_-}, \ \|u\|_{p(\cdot)}^{p_+} \right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{ \|u\|_{p(\cdot)}^{p_-}, \ \|u\|_{p(\cdot)}^{p_+} \right\},$$

$$\min\left\{ \rho_{p(\cdot)}(u)^{\frac{1}{p_-}} \ , \ \rho_{p(\cdot)}(u)^{\frac{1}{p_+}} \right\} \leq \|u\|_{p(\cdot)} \leq \max\left\{ \rho_{p(\cdot)}(u)^{\frac{1}{p_-}} \ , \ \rho_{p(\cdot)}(u)^{\frac{1}{p_+}} \right\}, \tag{2.6}$$

$$\|u\|_{p(\cdot)}^{p_-} - 1 \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p_+} + 1. \tag{2.7}$$

Proposition 2.2 (Generalised Hölder's inequality. See [13]).

- *For any functions* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have:*

$$\int_{\Omega} uv\,dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

- *For all* p *satisfying* (2.5), *we have the following continuous embedding,*

$$L^{p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ whenever } p(x) \geq r(x) \text{ for a.e. } x \in \Omega. \tag{2.8}$$

In generalised Lebesgue spaces, there holds a version of Young's inequality,

$$|uv| \leq \delta \frac{|u|^{p(x)}}{p(x)} + C(\delta)\frac{|v|^{p'(x)}}{p(x)},$$

for some positive constant $C(\delta)$ and any $\delta > 0$.

We define also the generalised Sobolev space by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}. \tag{2.9}$$

The space $W^{1,p(\cdot)}(\Omega)$ is separable and is reflexive when (2.5) is satisfied. We also have

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega) \text{ whenever } p(x) \geq r(x) \text{ for a.e. } x \in \Omega. \tag{2.10}$$

Now, we introduce the following function space

$$W_0^{1,p(\cdot)}(\Omega) := \{u \in \mathrm{W}_0^{1,1}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega)\},$$

endowed with the following norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} := \|u\|_1 + \|\nabla u\|_{p(\cdot)}. \tag{2.11}$$

If $p \in C(\overline{\Omega})$, then the norm in $W_0^{1,p(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{p(\cdot)}$. When p is log-Hölder continuous, then $C_0^{\infty}(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.

If p is a measurable function in Ω satisfying $1 \leq p_- \leq p_+ < N$ and the log-Hölder continuity property (2.1), then

$$\|u\|_{p^*(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega),$$

for some positive constant C, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

On the other hand, if p satisfies (2.1) and $p_- > N$, then

$$\|u\|_\infty \leq C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega) \ ,$$

where C is another positive constant.

Lemma 2.3 ([7]). *Assume that*

$$\begin{gathered} 1 < r \leq p_n(x) \leq s < \infty \quad \forall n \in \mathbb{N}, \\ \textit{for a.e. } x \in \Omega, \textit{ for some constants } r \textit{ and } s, \end{gathered} \tag{2.12}$$

$$p_n \to p \ \ \textit{a.e. in } \Omega, \ \textit{as } n \to \infty, \tag{2.13}$$

$$\nabla u_n \to \nabla u \ \textit{in } L^1(\Omega)^N, \ \textit{as } n \to \infty, \tag{2.14}$$

$$\||\nabla u_n|^{p_n(x)}\|_1 \leq C, \ \textit{for some positive constant } C \textit{ not depending on } n. \tag{2.15}$$

Then $\nabla u \in L^{p(\cdot)}(\Omega)^N$ *and*

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n(x)} dx \geq \int_\Omega |\nabla u|^{p(x)} dx. \tag{2.16}$$

Lemma 2.4 ([8, 12]). *For all* ξ, $\eta \in \mathbb{R}^N$, *the following assertions hold true:*

$$2 \leq p < \infty \Rightarrow \frac{1}{2^{p-1}}|\xi - \eta|^p \leq \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta), \tag{2.17}$$

$$1 < p < 2 \Rightarrow (p-1)|\xi - \eta|^2 \leq \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \left(|\xi|^p + |\eta|^p\right)^{\frac{2-p}{p}} . \tag{2.18}$$

3 Existence for the local problem

In this section, we prove the existence of weak solutions for the local problem (1.1). Firstly, we define the following space:

$$W_0^{1,p(u)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(u)} dx < \infty \right\} \text{ such that } 1 < p(u) < \infty \text{ for all } u \in \mathbb{R}.$$

It is a Banach space for the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ defined at (2.11) which is equivalent to $\|\nabla u\|_{p(u)}$ when $p(u) \in C(\overline{\Omega})$. Since p is continuous then from the fact that $1 < r \leq p$, $W_0^{1,p(u)}(\Omega)$ is separable and reflexive.

Definition 3.1. *Assume that p verifies* (1.2) *and*

$$f \in W^{-1,r'}(\Omega). \tag{3.1}$$

A function $u \in W_0^{1,p(u)}(\Omega)$ is said to be a weak solution to the problem (1.1)*, if*

$$\int_\Omega |\nabla u|^{p(u)-2} \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p(u)-2} uv \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,p(u)}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(W_0^{1,p(u)}(\Omega))'$ and $W_0^{1,p(u)}(\Omega)$.

Theorem 3.2. *Assume that* (1.2) *and* (3.1) *hold together with*

$$N < r \leq p(u) \leq s < +\infty \tag{3.2}$$

and

$$p : \mathbb{R} \longrightarrow [1, +\infty) \text{ is a Lipschitz-continuous function.} \tag{3.3}$$

Then there exists at least one weak solution to problem (1.1) *in the sense of Definition 3.1.*

The proof of Theorem 3.2 is divided into several steps.

Step 1: Approximate problems

For each $\varepsilon > 0$, we consider the following auxiliary problem (namely, the regularized problem)

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2} \nabla u) + |u|^{p(u)-2} u + \varepsilon \left(|u|^{s-2} u - \operatorname{div}(|\nabla u|^{s-2} \nabla u) \right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where

$$N < r \le p(u) \le s < \infty \quad \forall u \in \mathbb{R}.$$

Proposition 3.3. *For each $\varepsilon > 0$, the problem* (3.4) *admits a weak solution u_ε.*

Proof. Let $w \in L^2(\Omega)$, then

$$N < r \le p(w) \le s < \infty \quad \text{for a.e.} \quad x \in \Omega. \tag{3.5}$$

Recalling that $f \in W^{-1,r'}(\Omega) \subset W^{-1,s'}(\Omega)$. Now, we focus on the operator $T_\varepsilon : W_0^{1,s}(\Omega) \to W^{-1,s'}(\Omega)$ defined by

$$\langle T_\varepsilon(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(w)-2}\nabla u \cdot \nabla v dx + |u|^{p(w)-2}uv\right) dx + \varepsilon \left[\int_\Omega \left(|\nabla u|^{s-2}\nabla u \cdot \nabla v dx + |u|^{s-2}uv\right) dx\right],$$

for all $u, v \in W_0^{1,s}(\Omega)$. We can establish that:

(i) T_ε is continuous, bounded;

(ii) T_ε is strictly monotone (the strict monotonicity follows by Lemma 2.4);

(iii) T_ε is coercive.

According to (i), (ii) and (iii), the operator T_ε is continuous, strictly monotone (hence, maximal monotone too), and coercive. It follows that T_ε is a strictly monotone surjective operator (see Corollary 2.8.7, p. 135, [16]). Therefore, there exists a unique solution $u_w \in W_0^{1,s}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega |\nabla u_w|^{p(w)-2}\nabla u_w \cdot \nabla v dx + \int_\Omega |u_w|^{p(w)-2}u_w v dx + \\ &\varepsilon \left(\int_\Omega |\nabla u_w|^{s-2}\nabla u_w \cdot \nabla v dx + \int_\Omega |u_w|^{s-2}u_w v dx\right) = \langle f, v\rangle \quad \forall v \in W_0^{1,s}(\Omega). \end{aligned} \tag{3.6}$$

We take $v = u_w$ in (3.6) to derive that

$$\int_\Omega |\nabla u_w|^{p(w)}dx + \int_\Omega |u_w|^{p(w)}dx + \varepsilon\left(\int_\Omega |u_w|^s dx + \int_\Omega |\nabla u_w|^s dx\right) \le \|f\|_{-1,r'}\|\nabla u_w\|_r \le C\|\nabla u_w\|_s,$$

where $C = C(r, s, \Omega, f)$, and $\|\cdot\|_{-1,r'}$ is the operator norm associated to the norm $\|\nabla \cdot\|_r$. Therefore

$$\varepsilon\|u_w\|_{1,s}^s \le C\|\nabla u_w\|_s \le C\|u_w\|_{1,s}.$$

Hence

$$\|u_w\|_{1,s} \le C, \tag{3.7}$$

where $C = C(r, s, \Omega, \varepsilon, f)$ is a positive constant without w-dependence. From the fact that $s > N \geq 2$, we can deduce that

$$\|u_w\|_{L^2(\Omega)} \leq C. \tag{3.8}$$

Next, we introduce the self-map $T : B \to B$ defined by $T(w) = u_w$, over the set

$$B := \left\{ v \in L^2(\Omega) : \|v\|_{L^2(\Omega)} \leq C \right\}.$$

The compact embedding $W_0^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$ implies that $T(B)$ is relatively compact in B. Appealing to the Schauder fixed-point theorem, we know that the continuity of T is required in obtaining a fixed point of T.

With the assumption that we work on a sequence $\{w_n\}$ in $L^2(\Omega)$ satisfying

$$w_n \to w \text{ in } L^2(\Omega) \quad \text{as} \quad n \to \infty, \tag{3.9}$$

we denote by u_n, for all $n \in \mathbb{N}$, the solution of (3.6) related to $w := w_n$. Therefore, the inequality in (3.7) leads to

$$\|u_n\|_{1,s} \leq C, \quad \text{for some positive constant (without } n\text{-dependence)}.$$

Passing to a subsequence if necessary (namely again $\{u_n\}$), for a certain $u \in W_0^{1,s}(\Omega)$ we get

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,s}(\Omega), \quad \text{as} \quad n \to \infty, \tag{3.10}$$

$$u_n \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty. \tag{3.11}$$

We return to (3.6), so that considering (u_n, w_n) instead of (u, w), we get

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(w_n)-2} \nabla u_n + \varepsilon |\nabla u_n|^{s-2} \nabla u_n \right) \cdot \nabla v \, dx + \\ &\int_\Omega \left(|u_n|^{p(w_n)-2} u_n + \varepsilon |u_n|^{s-2} u_n \right) v \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,s}(\Omega). \end{aligned} \tag{3.12}$$

Since the operator on the left-hand side of (3.12) is monotone, then

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(w_n)-2} \nabla u_n + \varepsilon |\nabla u_n|^{s-2} \nabla u_n \right) \cdot \nabla (u_n - v) dx + \\ &\int_\Omega \left(|u_n|^{p(w_n)-2} u_n + \varepsilon |u_n|^{s-2} u_n \right) (u_n - v) dx - \end{aligned} \tag{3.13}$$

$$\int_\Omega \left(|\nabla v|^{p(w_n)-2}\nabla v + \varepsilon|\nabla v|^{s-2}\nabla v\right)\cdot\nabla(u_n - v)dx -$$
$$\int_\Omega \left(|v|^{p(w_n)-2}v + \varepsilon|v|^{s-2}v\right)(u_n - v)dx \geq 0 \quad \forall v \in W_0^{1,s}(\Omega).$$

Considering (3.12) with $v = u_n - v$ as a test function, we use (3.13) to get

$$\langle f, u_n - v\rangle - \int_\Omega \left(|\nabla v|^{p(w_n)-2}\nabla v + \varepsilon|\nabla v|^{s-2}\nabla v\right)\cdot\nabla(u_n - v)dx - \tag{3.14}$$
$$\int_\Omega \left(|v|^{p(w_n)-2}v + \varepsilon|v|^{s-2}v\right)(u_n - v)dx \geq 0 \quad \forall v \in W_0^{1,s}(\Omega).$$

The convergence in (3.9) implies

$$w_n \to w \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty.$$

Since p is a continuous function, we can apply Lebesgue's theorem (in $L^{s'}(\Omega)^N$), therefore

$$|\nabla v|^{p(w_n)-2}\nabla v \to |\nabla v|^{p(w)-2}\nabla v \quad \text{strongly in} \quad L^{s'}(\Omega)^d, \quad \text{as} \quad n \to \infty \tag{3.15}$$

and

$$|v|^{p(w_n)-2}v \to |v|^{p(w)-2}v \quad \text{strongly in} \quad L^s(\Omega), \quad \text{as} \quad n \to \infty, \tag{3.16}$$

for all $v \in W_0^{1,s}(\Omega)$. Finally, by the weak convergence in (3.10) and using (3.15) and (3.16) we can pass to the limit in (3.14) to obtain

$$\langle f, u - v\rangle - \int_\Omega \left(|\nabla v|^{p(w)-2}\nabla v + \varepsilon|\nabla v|^{s-2}\nabla v\right)\cdot\nabla(u - v)dx -$$
$$\int_\Omega \left(|v|^{p(w)-2}v + \varepsilon|v|^{s-2}v\right)(u - v)dx \geq 0 \quad \forall v \in W_0^{1,s}(\Omega). \tag{3.17}$$

Next, choosing $v = u \pm \delta\varphi$, where $\varphi \in W_0^{1,s}(\Omega)$ and $\delta > 0$, we get

$$\pm\Big[\langle f, \varphi\rangle - \int_\Omega \big(|\nabla(u \pm \delta\varphi)|^{p(w)-2}\nabla(u \pm \delta\varphi) + \varepsilon|\nabla(u \pm \delta\varphi)|^{s-2}\nabla(u \pm \delta\varphi)\big)\cdot\nabla\varphi\, dx -$$
$$\int_\Omega \left(|u \pm \delta\varphi|^{p(w)-2}(u \pm \delta\varphi) + \varepsilon|v|^{s-2}(u \pm \delta\varphi)\right)\varphi\, dx\Big] \geq 0. \tag{3.18}$$

We pass to the limit as δ goes to zero in (3.18), and deduce that

$$\int_\Omega \left(|\nabla(u)|^{p(w)-2}\nabla u + \varepsilon|\nabla u|^{s-2}\nabla u\right)\cdot\nabla\varphi dx + \int_\Omega \left(|u|^{p(w)-2}u + \varepsilon|v|^{s-2}u\right)\varphi dx = \langle f, \varphi\rangle \quad \forall \varphi \in W_0^{1,s}(\Omega).$$

Consequently $u = u_w$. In view of (3.11) and by the strong convergence in (3.11), we conclude that

$$u_{w_n} \to u_w \quad \text{strongly in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty,$$

It follows that T is continuous, and this establishes the existence of the fixed point which is the exact weak solution to (3.4). $\square$

Step 2: Passage to the limit as $\varepsilon \to 0$

From Proposition 3.3, for each $\varepsilon > 0$ there exists $u_\varepsilon \in W_0^{1,s}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega |\nabla(u_\varepsilon)|^{p(u_\varepsilon)-2}\nabla u_\varepsilon \nabla v dx + \int_\Omega |u_\varepsilon|^{p(u_\varepsilon)-2}u_\varepsilon v dx + \\ &\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^{s-2}\nabla u_\varepsilon \cdot \nabla v dx + \int_\Omega |u_\varepsilon|^{s-2}u_\varepsilon v dx\right) = \langle f, v\rangle \quad \forall v \in W_0^{1,s}(\Omega)\end{aligned} \tag{3.19}$$

and

$$N < r \le p(u_\varepsilon(x)) \le s < \infty \quad \forall \varepsilon > 0, \quad \text{for a.e.} \quad x \in \Omega.$$

Next, we choose $v = u_\varepsilon$ as a test function in (3.19) to obtain

$$\int_\Omega \left(|\nabla u_\varepsilon|^{p(u_\varepsilon)} + |u_\varepsilon|^{p(u_\varepsilon)}\right) dx + \varepsilon\left(\|\nabla u_\varepsilon\|_s^s + \|u_\varepsilon\|_s^s\right) = \langle f, u_\varepsilon\rangle. \tag{3.20}$$

From (2.7), we deduce that

$$\|u\|_{q(\cdot)} \le (\rho_{q(\cdot)}(u)+1)^{\frac{1}{q_-}} = \left(\int_\Omega |\nabla u|^{q(x)}dx + 1\right)^{\frac{1}{q_-}}.$$

By using the Hölder inequality, we get

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^r dx &\le C\||\nabla u_\varepsilon|^r\|_{\frac{p(u_\varepsilon)}{r}} \le C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right)^{\frac{1}{\left(\frac{p(u_\varepsilon)}{r}\right)_-}} \\ &\le C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right),\end{aligned} \tag{3.21}$$

where $C = C(r, s, \Omega)$. Therefore

$$\langle f, u_\varepsilon\rangle \le \|f\|_{-1,r'}||\nabla u_\varepsilon||_r \le C\|f\|_{-1,r'}\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right)^{\frac{1}{r}}. \tag{3.22}$$

From (3.20), (3.22) and by using Young's inequality, we obtain

$$\int_\Omega \left(|\nabla u_\varepsilon|^{p(u_\varepsilon)} + |u_\varepsilon|^{p(u_\varepsilon)}\right) dx + \varepsilon\left(\|\nabla u_\varepsilon\|_s^s + \|u_\varepsilon\|_s^s\right) \le C. \tag{3.23}$$

Using (3.21) and (3.22), we can deduce the estimate

$$||u_\varepsilon||_{1,r} \le C, \tag{3.24}$$

where C is a positive constant without ε-dependence.

Now we consider a sequence $\{\varepsilon_n\}$ of positive real numbers. For every $n \in \mathbb{N}$, let u_{ε_n} be the solution to the problem (3.4) associated to ε_n. Since $W_0^{1,r}(\Omega) \hookrightarrow L^2(\Omega)$ compactly, then after passing to a subsequence if needed, for some $u \in W_0^{1,r}(\Omega)$ we have

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{in} \quad W_0^{1,r}(\Omega), \quad \text{as} \quad n \to \infty \tag{3.25}$$

$$\nabla u_{\varepsilon_n} \rightharpoonup \nabla u \quad \text{in} \quad L^r(\Omega)^N, \quad \text{as} \quad n \to \infty \tag{3.26}$$

$$u_{\varepsilon_n} \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty$$

$$u_{\varepsilon_n} \to \quad u \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty. \tag{3.27}$$

The constraints on the exponent range in (3.2) imply that u is Hölder-continuous, then from the condition (3.3), the same conclusion holds for $p(u)$. From (3.27), we deduce that

$$p(u_{\varepsilon_n}) \to p(u) \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty. \tag{3.28}$$

Clearly, the following chain of inequalities is satisfied

$$N < r \leq p(u_{\varepsilon_n}) \leq s < \infty \quad \forall n \in \mathbb{N}, \quad \text{for a.e.} \quad x \in \Omega. \tag{3.29}$$

Using (3.23) written for u_{ε_n}, together with (3.26), (3.28) and (3.29), we conclude that (by Lemma 2.3)

$$u \in W_0^{1,p(u)}(\Omega). \tag{3.30}$$

From the theory of monotone operators, we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2}\nabla u_{\varepsilon_n} + \varepsilon_n|\nabla u_{\varepsilon_n}|^{s-2}\nabla u_{\varepsilon_n}\right) \cdot \nabla(u_{\varepsilon_n} - v)dx + \\
&\int_\Omega \left(|u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2}u_{\varepsilon_n} + \varepsilon_n|u_{\varepsilon_n}|^{s-2}u\right)(u_{\varepsilon_n} - v)dx - \\
&\Big(\int_\Omega \left(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + \varepsilon_n|\nabla v|^{s-2}\nabla v\right) \cdot \nabla(u_{\varepsilon_n} - v)dx + \\
&\int_\Omega \left(|v|^{p(u_{\varepsilon_n})-2}v + \varepsilon_n|v|^{s-2}v\right)(u_{\varepsilon_n} - v)dx\Big) \geq 0 \quad \forall v \in W_0^{1,s}(\Omega).
\end{aligned} \tag{3.31}$$

By replacing u_ε with u_{ε_n} and choosing $u_{\varepsilon_n} - v$ as a test function in (3.19), we can reduce (3.31) to the form

$$\langle f, u_{\varepsilon_n} - v\rangle - \Big(\int_\Omega \Big(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + \varepsilon |\nabla v|^{s-2}\nabla v \Big) \cdot \nabla (u_{\varepsilon_n} - v) dx + \int_\Omega \Big(|v|^{p(u_{\varepsilon_n})-2} v + \varepsilon |v|^{s-2} v \Big) (u_{\varepsilon_n} - v) dx \Big) \geq 0, \tag{3.32}$$

for all $v \in C_0^\infty(\Omega)$. By using (3.28) and the Lebesgue theorem, we have

$$|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v \to |\nabla v|^{p(u)-2}\nabla v \quad \text{in} \quad L^{r'}(\Omega)^d, \quad \text{as} \quad n \to \infty \tag{3.33}$$

and

$$|v|^{p(u_{\varepsilon_n})-2} v \to |v|^{p(u)-2} v \quad \text{in} \quad L^{r}(\Omega), \quad \text{as} \quad n \to \infty. \tag{3.34}$$

We take the limit as n goes to infinity in (3.32), and use (3.24), (3.25), (3.33) and (3.34), therefore

$$\langle f, u - v\rangle - \Big(\int_\Omega |\nabla v|^{p(u)-2}\nabla v \cdot \nabla (u - v) dx + \int_\Omega |v|^{p(u)-2} v (u - v) dx \Big) \geq 0 \quad \forall v \in C_0^\infty(\Omega). \tag{3.35}$$

From the assumptions (3.2) and (3.3), the functions $p(u)$ is Hölder-continuous which implies that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(u)}(\Omega)$. Thus, (3.34) holds true also for all $v \in W_0^{1,p(u)}(\Omega)$.

So we can take $v = u \pm \delta\varphi$, where $\varphi \in W_0^{1,p(u)}(\Omega)$ and $\delta > 0$, as a test function in (3.34) we get

$$\pm \Big(\langle f, \varphi\rangle - \Big(\int_\Omega |\nabla u|^{p(u)-2}\nabla u \cdot \nabla\varphi \, dx + \int_\Omega |u|^{p(u)-2} u\varphi \, dx \Big) \Big) \geq 0. \tag{3.36}$$

This implies that,

$$\int_\Omega |\nabla u|^{p(u)-2}\nabla u \cdot \nabla\varphi \, dx + \int_\Omega |u|^{p(u)-2} u\varphi \, dx = \langle f, \varphi\rangle \quad \forall \varphi \in W_0^{1,p(u)}(\Omega). \tag{3.37}$$

Finally, we arrived to a solution for our local problem (1.1) (See Definition 3.1).

4 Nonlocal problems

Along with problem (1.1), we consider in this section its nonlocal version. Firstly, we assume that the function p satisfies the conditions in (1.2). We denote by b a mapping from $W_0^{1,r}(\Omega)$ into $\mathbb{R}$ such that

$$b \text{ is continuous}, \ \ b \text{ is bounded}. \tag{4.1}$$

The next theorem needs the following revised definition of a weak solution.

Definition 4.1. *A function u is said to be a weak solution to the problem* (1.3) *if*

$$\begin{cases} u \in W_0^{1,p(b(u))}(\Omega) \ , \\ \displaystyle\int_\Omega |\nabla u|^{p(b(u))-2} \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p(b(u))-2} uv \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,p(b(u))}(\Omega), \end{cases} \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\left(W_0^{1,p(b(u))}(\Omega)\right)'$ and $W_0^{1,p(b(u))}(\Omega)$.

Since $p(b(u))$ is here a real number and not a function, thus the Sobolev spaces involved are the classical ones.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a bounded domain and assume that* (1.2) *and* (4.1) *hold together with*

$$f \in W^{-1,q'}(\Omega) \quad \text{for} \quad q < r.$$

Then there exists at least one weak solution to the problem (1.3) *in the sense of Definition 4.1.*

To prove Theorem 4.2, we need the following Lemma.

Lemma 4.3. *For $n \in \mathbb{N}$, let u_n be the solution to the problem*

$$\begin{cases} u_n \in W_0^{1,p_n}(\Omega), \\ \displaystyle\int_\Omega |\nabla u_n|^{p_n-2} \nabla u_n \cdot \nabla v \, dx + \int_\Omega |u_n|^{p_n-2} u_n v \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,p_n}(\Omega), \end{cases} \tag{4.3}$$

where $\langle \cdot, \cdot \rangle$ denotes here the duality pairing between $\left(W_0^{1,p_n}(\Omega)\right)'$ and $W_0^{1,p_n}(\Omega)$.

Assume that

$$p_n \to p, \quad \text{as} \quad n \to \infty, \quad \text{where} \quad p \in (1, \ \infty), \tag{4.4}$$

$$f \in W^{-1,q'}(\Omega) \quad \text{for some} \quad q < p. \tag{4.5}$$

Then

$$u_n \to u \quad \text{in} \quad W_0^{1,q}(\Omega), \quad \text{as} \quad n \to \infty, \tag{4.6}$$

where u is the solution to the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} uv \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{4.7}$$

Proof of Lemma 4.3. The proof of Lemma 4.3 is divided into two steps.

Step 1: Weak convergence

In view of $p_n \to p$, as $n \to \infty$, and $q < p$, we may suppose that

$$p + 1 > p_n > q \quad \forall n \in \mathbb{N}. \tag{4.8}$$

We choose $v = u_n$ as a test function in (4.3) to obtain

$$\int_\Omega |\nabla u_n|^{p_n} dx + \int_\Omega |u_n|^{p_n} dx \leq \|f\|_{-1,q'} \|\nabla u_n\|_q. \tag{4.9}$$

From (4.8) and Hölder's inequality, we deduce that

$$\|\nabla u_n\|_q \leq C \|\nabla u_n\|_{p_n} \leq C \|u_n\|_{1,p_n}, \tag{4.10}$$

where $C = C(p, q, \Omega)$ is a positive constant. Therefore

$$\|u_n\|_{1,p_n} \leq C, \tag{4.11}$$

where $C = C(p, q, \Omega, f)$ is a positive constant. Combining (4.10) with (4.11), we get

$$\|\nabla u_n\|_q \leq C, \tag{4.12}$$

where C is a positive constant without n-dependence. Passing to a subsequence if necessary still denoted by u_n, for a certain $u \in W_0^{1,q}(\Omega)$ we get

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in} \quad L^q(\Omega), \quad \text{as} \quad n \to \infty. \tag{4.13}$$

On this basis, the convergences in (4.4), (4.8), (4.11) and (4.13) lead to the conclusion that (Lemma 2.3)

$$\liminf_{n \to \infty} \int_\Omega |\nabla u_n|^{p_n} dx \geq \int_\Omega |\nabla u|^p dx,$$

and hence

$$u \in W_0^{1,p}(\Omega). \tag{4.14}$$

We observe that, the second line in (4.3) is equivalent to

$$\int_\Omega |\nabla u_n|^{p_n - 2} \nabla u_n \cdot \nabla(v - u_n) dx + \int_\Omega |u_n|^{p_n - 2} u_n (v - u_n) dx \geq \langle f, v - u_n \rangle \quad \forall v \in W_0^{1,p_n}(\Omega),$$

using the Minty lemma, we have

$$\int_\Omega |\nabla v|^{p_n-2}\nabla v\cdot\nabla(v-u_n)dx+\int_\Omega |v|^{p_n-2}v(v-u_n)dx \geq \langle f, v-u_n\rangle \quad \forall v\in W_0^{1,p_n}(\Omega). \quad (4.15)$$

We choose $v\in C_0^\infty(\Omega)$, then we can take the limit as n goes to infinity in (4.15), and use (4.4) and (4.13), hence we obtain

$$\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla(v-u)dx+\int_\Omega |v|^{p-2}v(v-u)dx \geq \langle f, v-u\rangle \quad \forall v\in C_0^\infty(\Omega). \quad (4.16)$$

Since $C_0^\infty(\Omega)$ dense in $W_0^{1,p}(\Omega)$, we have (4.16) also holds for all $v\in W_0^{1,p}(\Omega)$. Now, choosing $v=u\pm\delta\varphi$, where $\varphi\in W_0^{1,p}(\Omega)$ and $\delta>0$, by passing to the limit as δ goes to zero, we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx+\int_\Omega |u|^{p-2}u\varphi dx = \langle f,\varphi\rangle \quad \forall\varphi\in W_0^{1,p}(\Omega),$$

Finally, it is sufficient to recall that $u\in W_0^{1,p}(\Omega)$ to conclude that we arrived to a solution for the problem (4.7).

Step 2: Strong convergence

In this step we will show that the convergence (4.13) is strong. Firstly, we take $v=u_n$ in (4.3) and using (4.13) to pass to the limit, we get

$$\int_\Omega |\nabla u_n|^{p_n}dx+\int_\Omega |u_n|^{p_n}dx=\langle f,v\rangle \to \int_\Omega |\nabla u|^{p}dx+\int_\Omega |u|^{p}dx=\langle f,v\rangle \quad \text{as} \quad n\to\infty. \quad (4.17)$$

Firstly, we consider the case when

$$p_n\geq p \quad \forall n\in\mathbb{N}.$$

By using Hölder's inequality, we have

$$\int_\Omega |\nabla u_n|^p dx \leq \left(\int_\Omega |\nabla u_n|^{p_n}dx\right)^{\frac{p}{p_n}} |\Omega|^{1-\frac{p}{p_n}}.$$

Thus by (4.17), we deduce that

$$\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^p dx \leq \int_\Omega |\nabla u|^p dx \leq \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^p dx,$$

which implies (from the fact that $\|\nabla u_n\|_p\to\|\nabla u\|_p$, as $n\to\infty$)

$$u_n\to u \quad \text{strongly} \quad \text{in} \quad W_0^{1,p}(\Omega), \quad \text{as} \quad n\to\infty. \quad (4.18)$$

From the fact that $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$, we conclude that

$$u_n \to u \quad \text{in} \quad W_0^{1,q}(\Omega), \qquad \text{as} \quad n \to \infty.$$

Now, we consider the case when

$$q < p_n < p \quad \forall n \in \mathbb{N}, \tag{4.19}$$

we set

$$A_n := \int_\Omega \left(|\nabla u_n|^{p_n-2}\nabla u_n - |\nabla u|^{p_n-2}\nabla u\right) \cdot (\nabla u_n - \nabla u) dx + \int_\Omega \left(|u_n|^{p_n-2}u_n - |u|^{p_n-2}u\right) \cdot (u_n - u) dx. \tag{4.20}$$

By the theory of monotone operators, we have $A_n \geq 0$, (4.3) imply that (4.20) reduces to the form

$$A_n = \langle f,\ u_n - u\rangle - \int_\Omega |\nabla u|^{p_n-2}\nabla u \cdot \nabla(u_n - u) dx - \int_\Omega |u|^{p_n-2}u(u_n - u) dx.$$

Due to (4.5) and the convergence in (4.13), we have

$$\langle f, u_n - u\rangle \to 0, \quad \text{as} \quad n \to \infty. \tag{4.21}$$

From the fact that $u \in W_0^{1,p}(\Omega)$ we get

$$||\nabla u|^{p_n-2}\nabla u| \leq \max\{1,\ |\nabla u|^{p-1}\} \in L^{p'}(\Omega), \tag{4.22}$$

$$||u|^{p_n-2}u| \leq \max\{1,\ |u|^{p-1}\} \in L^{p}(\Omega). \tag{4.23}$$

On this basis, we can conclude that

$$A_n \to 0, \quad \text{as} \quad n \to \infty. \tag{4.24}$$

We first consider the case when $p_n \geq 2$. By applying the Lemma 2.4 in (4.20), we get

$$A_n \geq \frac{1}{2^{p_n-1}} \left(\int_\Omega |\nabla(u_n - u)|^{p_n} dx + \int_\Omega |u_n - u|^{p_n} dx \right). \tag{4.25}$$

Since $p_n > q$, we can apply Hölder's inequality to obtain

$$\int_\Omega |\nabla(u_n - u)|^q dx + \int_\Omega |u_n - u|^q dx \leq \left[\left(\int_\Omega |\nabla(u_n - u)|^{p_n} dx \right)^{\frac{q}{p_n}} + \left(\int_\Omega |u_n - u|^{p_n} dx \right)^{\frac{q}{p_n}} \right] |\Omega|^{1-\frac{q}{p_n}}.$$

Hence, from (4.24) and (4.25) we get

$$\int_\Omega |\nabla(u_n-u)|^q dx + \int_\Omega |u_n-u|^q dx \longrightarrow 0, \quad \text{as} \quad n\to\infty.$$

Therefore,

$$u_n \to u \quad \text{in} \quad W_0^{1,q}(\Omega), \quad \text{as} \quad n\to\infty.$$

Now, we assume that $p_n < 2$:

By using the Hölder's inequality we obtain

$$\begin{aligned}
&\int_\Omega |\nabla(u_n-u)|^{p_n} dx + \int_\Omega |u_n-u|^{p_n} dx \\
&= \int_\Omega |\nabla(u_n-u)|^{p_n} (|\nabla u_n|+|\nabla u|)^{\frac{(p_n-2)p_n}{2}} (|\nabla u_n|+|\nabla u|)^{\frac{(2-p_n)p_n}{2}} dx \\
&+ \int_\Omega |u_n-u|^{p_n} (|u_n|+|u|)^{\frac{(p_n-2)p_n}{2}} (|u_n|+|u|)^{\frac{(2-p_n)p_n}{2}} dx \\
&\le \left[\int_\Omega |\nabla(u_n-u)|^2 (|\nabla u_n|+|\nabla u|)^{p_n-2} dx\right]^{\frac{p_n}{2}} \left[\int_\Omega (|\nabla u_n|+|\nabla u|)^{p_n} dx\right]^{1-\frac{p_n}{2}} \\
&+ \left[\int_\Omega |u_n-u|^2 (|u_n|+|u|)^{p_n-2} dx\right]^{\frac{p_n}{2}} \left[\int_\Omega (|u_n|+|u|)^{p_n} dx\right]^{1-\frac{p_n}{2}}.
\end{aligned} \tag{4.26}$$

From Lemma 2.4, one could deduce that

$$A_n \ge C(p_n)\left(\int_\Omega |\nabla(u_n-u)|^2 (|\nabla u_n|+|\nabla u|)^{p_n-2} dx + \int_\Omega |u_n-u|^2 (|u_n|+|u|)^{p_n-2} dx\right). \tag{4.27}$$

Since $\|u_n\|_{1,p_n} \le C$, then from (4.24), (4.26) and (4.27) we get

$$\int_\Omega |\nabla(u_n-u)|^{p_n} dx + \int_\Omega |u_n-u|^{p_n} dx \to 0, \quad \text{as} \quad n\to\infty.$$

Therefore,

$$u_n \to u \quad \text{in} \quad W_0^{1,q}(\Omega), \quad \text{as} \quad n\to\infty. \qquad \square$$

Proof of Theorem 4.2. For any $s > q$ we have, $f \in (W_0^{1,s}(\Omega))' \subset (W_0^{1,q}(\Omega))'$. Therefore, for each $\lambda \in \mathbb{R}$, the following $p(\lambda)$-Laplacian problem admits a unique solution u_λ,

$$\begin{cases} u \in W_0^{1,p(\lambda)}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(\lambda)-2}\nabla u\cdot\nabla v\, dx + \int_\Omega |u|^{p(\lambda)-2} uv\, dx = \langle f, v\rangle \quad \forall v \in W_0^{1,p(\lambda)}(\Omega). \end{cases} \tag{4.28}$$

The choice of test function u_λ in (4.28) implies that

$$\int_\Omega |\nabla u_\lambda|^{p(\lambda)} dx + \int_\Omega |u_\lambda|^{p(\lambda)} dx \leq \|f\|_{-1,r'} \|\nabla u_\lambda\|_r. \tag{4.29}$$

Now using the Hölder's inequality, one obtains

$$\|u_\lambda\|_{1,r} \leq \|u_\lambda\|_{1,p(\lambda)} |\Omega|^{\frac{1}{r} - \frac{1}{p(\lambda)}}. \tag{4.30}$$

From (4.29), it follows that

$$\|u_\lambda\|_{1,p(\lambda)}^{p(\lambda)-1} \leq \|f\|_{-1,r'} |\Omega|^{\frac{1}{r} - \frac{1}{p(\lambda)}}. \tag{4.31}$$

Combining (4.30) and (4.31), and using (1.2) to get

$$\|u_\lambda\|_{1,r} \leq \|f\|_{-1,r'}^{\frac{1}{p(\lambda)-1}} |z|^{\left(\frac{1}{r} - \frac{1}{p(\lambda)}\right)\frac{p(\lambda)}{p(\lambda)-1}} \leq \max_{p \in [r,s]} \|f\|_{1,r'}^{\frac{1}{p-1}} |\Omega|^{\left(\frac{1}{r} - \frac{1}{p}\right)\frac{p}{p-1}}. \tag{4.32}$$

Therefore

$$\|u_\lambda\|_{1,r} \leq C. \tag{4.33}$$

The inequality (4.33) and the fact that b is a bounded mapping, imply that there exists $K \in \mathbb{R}$ such that

$$b(u_\lambda) \in [-K,\ K] \quad \forall \lambda \in \mathbb{R}.$$

Next, we introduce the self-map $H : [-K,\ K] \to [-K,\ K]$ defined by $H(\lambda) = b(u_\lambda)$. We know that the continuity of H is required in obtaining a fixed point of H.

Assume that $\lambda_n \to \lambda$ as $n \to \infty$, because p is continuous, $p(\lambda_n) \to p(\lambda)$. Next, we apply Lemma 4.3, so that considering $p(\lambda_n)$ instead of p_n, we deduce that

$$u_{\lambda_n} \longrightarrow u_\lambda \quad \text{in} \quad W_0^{1,r}(\Omega), \quad \text{as} \quad n \to \infty.$$

We use the fact that b is continuous to deduce that $b(u_\lambda) \to b(u_\lambda)$, as n goes to infinity, which implies that the map H is continuous. This establishes the existence of the fixed point λ_0 and a weak solution u_{λ_0} for the problem (4.2). $\square$

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Existence, stability and global attractivity results for nonlinear Riemann-Liouville fractional differential equations

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ABSTRACT

Existence, attractivity, and stability of solutions of a nonlinear fractional differential equation of Riemann-Liouville type are proved using the classical Schauder fixed point theorem and a fixed point result due to Dhage. The results are illustrated with examples.

RESUMEN

Demostramos la existencia, atractividad y estabilidad de soluciones de la ecuación diferencial fraccional no-lineal de tipo Riemann-Liouville usando el clásico teorema de punto fijo de Schauder y un resultado de punto fijo de Dhage. Los resultados se ilustran con ejemplos.

Keywords and Phrases: Fractional differential equation; Asymptotic characterization of solution; Fixed point principle; Existence and stability theorem.

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1 The Problem

Dhage [5, 6, 7] and Dhage *et al.* [10] introduced the class of what they called pulling functions as follows. For $J_\infty = [t_0, \infty)$ with $t_0 \in \mathbb{R}_+ = [0, \infty)$ fixed, a continuous function $g : J_\infty \to (0, \infty)$ is a **pulling function** if $\lim\limits_{t\to\infty} g(t) = \infty$. We will denote the class of all pulling functions on J_∞ by $\mathcal{CRB}(J_\infty)$. We wish to point out that if g is a pulling function, then its reciprocal $\overline{g} = \overline{g}(t) = \frac{1}{g(t)}$ is continuous, bounded, and satisfies $\lim\limits_{t\to\infty} \overline{g}(t) = 0$. Using pulling functions, Dhage [6, 7, 8] proved some attractivity and stability results for nonlinear Caputo fractional differential equations. Instead, in this paper we consider fractional differential equations with a Riemann-Louville fractional derivative and use fixed point techniques, rather than the measure theoretic approach used in Dhage *et al.* [9].

Here we will study the nonlinear fractional differential equation

$$ {}^{RL}D^q_{t_0}\big[a(t)x(t)\big] = f(t, x(t)) \quad \text{a.e.} \quad t \in J_\infty, \tag{1.1}$$

together with the fractional integral initial condition (IC)

$$ \lim_{t\to t_0^+} I^{1-q}_{t_0^+}\big[a(t)x(t)\big] = b_0, \tag{1.2}$$

where $a \in \mathcal{CRB}(J_\infty) \cap L^1(J_\infty, \mathbb{R})$ is a pulling function, ${}^{RL}D^q$ is a Riemann-Liouville fractional derivative of order q with $0 < q < 1$, and $f : J_\infty \times \mathbb{R} \to \mathbb{R}$ is a Carathеódory function. Our goal is to characterize the attractivity and stability properties of the solutions of (1.1)–(1.2).

We begin with the following notions from the fractional calculus that are needed in our discussion; these can be found, for example, in Agarwal *et al.* [1], Podlubny [13] or Kilbas *et al.* [12]. Define the function space

$$ C(J_\infty, \mathbb{R}) = \big\{x : J_\infty \to \mathbb{R} \mid x \text{ is continuous}\big\}, $$

and let $L^1(J_\infty, \mathbb{R})$ denote the class of Lebesgue integrable functions. In what follows, Γ is the usual Euler's gamma function,

$$ \Gamma(q) = \int_0^\infty e^{-t} t^{q-1}\, dt, $$

and $[q]$ is the greatest integer less than or equal to q.

Definition 1.1. *Let $J_\infty = [t_0, \infty)$ for some $t_0 \geq 0$ in $\mathbb{R}$. For any $x \in L^1(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional integral of order $q > 0$ is defined as*

$$ I^q_{t_0} x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t \frac{x(s)}{(t-s)^{1-q}}\, ds, \ t \in J_\infty, $$

provided the right hand side is pointwise defined on (t_0, ∞).

Definition 1.2. *If $x \in L^1(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional derivative ${}^{RL}D^q_{t_0}x$ of x of order q is defined as*

$$ {}^{RL}D^q_{t_0}x(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_{t_0}^t (t-s)^{n-q-1}x(s)\,ds, \ n-1<q<n, \ n=[q]+1, $$

provided the right hand side exists.

Note that if a, $x \in L^1(J_\infty, \mathbb{R})$, then ${}^{RL}D^q_{t_0}[a(t)x(t)]$ exists on J_∞.

Definition 1.3. *A function x is called a classical solution of IVP* (1.1)–(1.2) *if*

(i) x is continuous on J_∞, and

(ii) x satisfies (1.1) *and* (1.2).

The fractional differential equation (1.1) is a scalar multiplicative perturbation of the second type obtained by multiplying the unknown function under the Riemann-Liouville derivative by a scalar function. This and other types of perturbations of a differential equation are described in Dhage [3].

2 Properties of solutions

We set our problem (1.1) in the Banach space $BC(J_\infty, \mathbb{R})$ of bounded continuous real-valued functions defined on J_∞ with the usual supremum norm

$$ \|x\| = \sup_{t \in J_\infty} |x(t)|. $$

We take $\mathcal{T} : BC(J_\infty, \mathbb{R}) \to BC(J_\infty, \mathbb{R})$ to be a continuous operator and we study the operator equation

$$ \mathcal{T}x(t) = x(t), \ t \in J_\infty. \tag{2.1} $$

Next, we describe various properties of solutions of the operator equation (2.1) in the space $BC(J_\infty, \mathbb{R})$.

First, we define the concepts of global attractivity and stability of the solutions as given in Banas and Dhage [2].

Definition 2.1. *A solution $x = x(t)$ of (2.1) is called globally attractive if*

$$ \lim_{t \to \infty} (x(t) - y(t)) = 0 \tag{2.2} $$

for each solution $y = y(t)$ of (2.1) in $BC(J_\infty, \mathbb{R})$.

That is, solutions of (2.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (2.1) in $BC(J_\infty, \mathbb{R})$, we have that condition (2.2) is satisfied. If (2.2) is satisfied uniformly in $BC(J_\infty, \mathbb{R})$ in the sense that for every $\epsilon > 0$ there exists $T > 0$ such that, for $t \geq T$,

$$|x(t) - y(t)| \leq \epsilon \tag{2.3}$$

for all solutions x, $y \in BC(J_\infty, \mathbb{R})$ of (2.1), then solutions of (2.1) are said to be *uniformly globally attractive* on J_∞.

Definition 2.2 (Banas and Dhage [2]). *A solution $x \in BC(J_\infty, \mathbb{R})$ of equation (2.1) is called asymptotic if $\lim_{t\to\infty} x(t) = 0$. If the limit is uniform with respect to the solution set of the operator equation (2.1) in $BC(J_\infty, \mathbb{R})$ (i.e., for each $\varepsilon > 0$ there exists $T > t_0 \geq 0$ such that $|x(t)| < \varepsilon$ for all solutions x of (2.1) in $BC(J_\infty, \mathbb{R})$ and for all $t \geq T$), we say that solutions of equation (2.1) are uniformly asymptotic on J_∞.*

Definition 2.3. *If all the solutions of the operator equation (2.1) are asymptotic and uniformly globally attractive, we will say that they are uniformly asymptotically attractive or stable on J_∞.*

In order to state the required fixed point techniques to be used in our proofs, we introduce the following concepts.

Definition 2.4 (Dhage [4]). *A nondecreasing upper semi-continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.*

Definition 2.5 (Dhage [4]). *Let X be a Banach space with norm $\|\cdot\|$. An operator $\mathcal{T} : X \to X$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}} \in \mathfrak{D}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi_{\mathcal{T}}\big(\|x - y\|\big) \tag{2.4}$$

for all x, $y \in X$.

If $\psi_{\mathcal{T}}(r) = kr$, $k > 0$, $\mathcal{T}$ is called a *Lipschitz operator* with Lipschitz constant k. Also, if $0 \leq k < 1$, then $\mathcal{T}$ is called a *contraction* on X and k is referred to as the contraction constant. In addition, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a *nonlinear $\mathcal{D}$-contraction* on X, and the set of all nonlinear $\mathcal{D}$-contractions will be denoted by $\mathcal{DN}$.

We say that an operator $\mathcal{T} : X \to X$ is *compact* if $\overline{\mathcal{T}(X)}$ is a compact subset of X. The operator $\mathcal{T}$ is called *totally bounded* if for any bounded subset S of X, $\mathcal{T}(S)$ is a totally bounded subset of X. Moreover, $\mathcal{T}$ is called *completely continuous* if $\mathcal{T}$ is continuous and totally bounded on X. We note that every compact operator is totally bounded, but the converse may not be true; the two notions are equivalent on bounded subsets of X. Additional details on different types of nonlinear contractions and compact and completely continuous operators can be found, for example, in Granas and Dugundji [11].

In an effort to prove our main existence results, we need the following fixed point theorems.

Theorem 2.6 (Schauder [11]). *Let S be a closed, convex, and bounded subset of a Banach space X, and let $\mathcal{T} : S \to S$ be a completely continuous operator. Then the operator equation $\mathcal{T}x = x$ has a solution.*

Theorem 2.7 (Dhage [3]). *Let X be a Banach space and let $\mathcal{T} : X \to X$ be a nonlinear $\mathcal{D}$-contraction. Then the operator equation $\mathcal{T}x = x$ has a unique solution.*

3 Existence, attractivity, and stability of solutions

Definition 3.1. *A function $\beta : J_\infty \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) the map $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto \beta(t,x)$ is continuous for each $t \in J_\infty$.

The following lemma is often used in the study of nonlinear differential equations.

Lemma 3.2 (Carathéodory). *Let $\beta : J_\infty \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the function $t \to \beta(t, x(t))$ is measurable for each $x \in C(J_\infty, \mathbb{R})$.*

We will make use of the following conditions in the remainder of our paper.

(H_1) The function f is bounded on $J_\infty \times \mathbb{R}$ with bound M_f.

(H_2) The function f is Carathédory on $J_\infty \times \mathbb{R}$.

(H_3) There exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$|f(t,x) - f(t,y)| \leq \psi_f(|x-y|)$$

for all x, $y \in \mathbb{R}$ and $t \in J_\infty$.

The following lemma will play an important role in obtaining our existence results.

Lemma 3.3. *For any function $h \in L^1(J_\infty, \mathbb{R})$, the function $x \in BC(J_\infty, \mathbb{R})$ is a solution of the fractional differential equation*

$$^{RL}D^q_{t_0}\big[a(t)x(t)\big] = h(t) \quad a.e. \quad t \in J_\infty, \tag{3.1}$$

satisfying the initial condition

$$\lim_{t \to t_0^+} I^{1-q}_{t_0^+}\big[a(t)x(t)\big] = b_0, \tag{3.2}$$

if and only if x satisfies the nonlinear fractional integral equation

$$x(t)=\frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)}+\frac{1}{a(t)\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds \tag{3.3}$$

for all $t\in J_\infty$.

Proof. Applying the Riemann-Liouville fractional integral operator $I_{t_0}^q$ to (3.1), we obtain

$$a(t)x(t)-\left.\frac{I_{t_0}^{1-q}[a(t)x(t)]}{\Gamma(q)}\right|_{t=t_0}(t-t_0)^{q-1}=I_{t_0}^q h(t)=\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds$$

for all $t\in J_\infty$, or

$$x(t)=\frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)}+\frac{1}{a(t)\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds.$$

That is, if $x(t)$ is a solution of (3.1)–(3.2), then $x(t)$ is a solution of (3.3).

Now let $x(t)$ be a solution of (3.3). Then,

$$a(t)x(t)=\frac{b_0}{\Gamma(q)}(t-t_0)^{q-1}+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds. \tag{3.4}$$

Applying the Riemann-Liouville fractional derivative operator to this expression gives

$${}^{RL}D_{t_0}^q\big[a(t)x(t)\big]={}^{RL}D_{t_0}^q\left[\frac{b_0}{\Gamma(q)}(t-t_0)^{q-1}\right]+h(t)$$

since ${}^{RL}D_{t_0}^q I_{t_0}^q h(t)=h(t)$. Also, since ${}^{RL}D_{t_0}^q(t-t_0)^{q-1}=0$, $x(t)$ satisfies equation (3.1).

From (3.4),

$$I_{t_0}^{1-q}\big[a(t)x(t)\big]=I_{t_0}^{1-q}\left[\frac{b_0}{\Gamma(q)}(t-t_0)^{q-1}\right]+I_{t_0}^{1-q}(I_{t_0}^q h(t)).$$

Now $I_{t_0}^{1-q}(I_{t_0}^q h)=I_{t_0}h=\int_{t_0}^{t}h(s)ds$ and $\lim\limits_{t\to t_0^+}\int_{t_0}^{t}h(s)ds=0$. Also, by [1, Proposition 1],

$$\begin{aligned}I_{t_0}^{1-q}\left[\frac{b_0}{\Gamma(q)}(t-t_0)^{q-1}\right]&=\frac{b_0}{\Gamma(q)}I_{t_0}^{1-q}(t-t_0)^{q-1}\\&=\frac{b_0}{\Gamma(q)}\frac{\Gamma(q)}{\Gamma(q+1-q)}(t-t_0)^{q+1-q-1}=\frac{b_0}{\Gamma(q)}\frac{\Gamma(q)}{\Gamma(1)}=b_0.\end{aligned}$$

Hence,

$$\lim_{t\to t_0^+}I_{t_0}^{1-q}\big[a(t)x(t)\big]=b_0,$$

and so (3.2) is satisfied. This proves the lemma. □

We need to introduce the following class of functions. Let

$$\mathcal{A} = \left\{ f \in C([t_0,\infty),(0,\infty)) : \lim_{t\to t_0} \frac{(t-t_0)^{q-1}}{f(t)} < \infty \text{ and } \lim_{t\to\infty} \frac{t^q}{f(t)} = 0 \right\},$$

and we assume in what follows that the function a in equation (1.1) belongs to the class $\mathcal{A} \cap \mathcal{CRB}(J_\infty)$.

Remark 3.4. *If $a \in \mathcal{CRB}(J_\infty)$, then $\overline{a} \in BC(J_\infty,\mathbb{R}_+)$ and so the number $\|\overline{a}\| = \sup_{t\in J_\infty} \overline{a}(t)$ exists. Also, the function $w : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $w(t) = \overline{a}(t)\, t^q$ is continuous on J_∞ and satisfies the relation $\lim_{t\to\infty} w(t) = 0$, so the number*

$$W = \sup_{t\geq t_0} w(t) \tag{3.5}$$

exists.

Our main existence and global attractivity result is contained in the following theorem.

Theorem 3.5. *Assume that conditions (H$_1$)–(H$_2$) hold. Then (1.1) has a solution defined on J_∞ and the solutions of (1.1) are uniformly globally asymptotically attractive.*

Proof. Since $a(t) \in \mathcal{A} \cap \mathcal{CRB}(J_\infty)$, there exists $d_0 > 0$ such that $\left| \frac{(t-t_0)^{q-1}}{a(t)} \right| \leq d_0$ on J_∞. Set $X = BC(J_\infty,\mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in X centered at the origin 0 with radius r given by

$$r = \frac{|b_0| d_0}{\Gamma(q)} + \frac{M_f\, W}{\Gamma(q+1)},$$

where M_f is from (H$_1$) and W is given in (3.5). By an application of Lemma 3.3, (1.1) is equivalent to the hybrid fractional integral equation

$$x(t) = \frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{1}{a(t)\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,x(s))\, ds \tag{3.6}$$

for all $t \in J_\infty$. Define the operator $\mathcal{T}$ on $\overline{B}_r(0)$ by

$$\mathcal{T}x(t) = \frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{1}{a(t)\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,x(s))\, ds, \quad t \in J_\infty. \tag{3.7}$$

Then (3.6) is transformed into the operator equation

$$\mathcal{T}x(t) = x(t),\ t \in J_\infty. \tag{3.8}$$

We will show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.6 with $S = \overline{B}_r(0) \subset BC(J_\infty,\mathbb{R})$. Now from the continuity of the integral, it follows that the function $t \to \mathcal{T}x(t)$ is continuous on J_∞ for each $x \in \overline{B}_r(0)$. Furthermore, by condition (H$_1$),

$$\begin{aligned}|\mathcal{T}x(t)| &\leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{a(t)\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\, ds \leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{\Gamma(q)} |\overline{a}(t)| \int_{t_0}^{t} (t-s)^{q-1} \\ &\leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{\Gamma(q+1)} |\overline{a}(t)| t^q \leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{\Gamma(q+1)} W\end{aligned}$$

for all $t \in J_\infty$ and all $x \in \overline{B}_r(0)$. Taking the supremum over t,

$$\|\mathcal{T}x\| \leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f W}{\Gamma(q+1)} = r$$

for all $x \in \overline{B}_r(0)$. As a result, $\mathcal{T}$ maps $\overline{B}_r(0)$ into itself.

To show that $\mathcal{T}$ is a completely continuous operator on $\overline{B}_r(0)$, we first show that it is continuous there. To do this, fix $\epsilon > 0$ and let $\{x_n\}$ be a sequence in $\overline{B}_r(0)$ converging to $x \in \overline{B}_r(0)$. Then,

$$\begin{aligned}|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)| &\leq \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \big| f(s, x_n(s)) - f(s, x(s)) \big|\, ds \\ &\leq \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \Big[|f(s, x_n(s))| + |f(s, x(s))| \Big]\, ds \\ &\leq \frac{2\, M_f \overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\, ds \leq \frac{2\, M_f}{\Gamma(q+1)}\, w(t). \end{aligned} \tag{3.9}$$

Since $a \in \mathcal{A}$, there exists $T > 0$ such that $w(t) \leq \dfrac{\epsilon \Gamma(q+1)}{2M_f}$ for $t \geq T$. Thus, for $t \geq T$, from (3.9), we see that

$$|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)| \leq \epsilon \quad \text{as} \quad n \to \infty.$$

Let $t \in [t_0, T]$. Then, by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[\frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x_n(s))\, ds \right] \\ &= \frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \Big[\lim_{n\to\infty} f(s, x_n(s)) \Big]\, ds \\ &= \mathcal{T}x(t) \end{aligned} \tag{3.10}$$

for all $t \in [t_0, T]$. Moreover, it can be shown as below that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in X. Now, using arguments similar to those given in Granas *et al.* [11], it follows that $\mathcal{T}$ is a continuous operator on $\overline{B}_r(0)$ into itself.

Next, we show that $\mathcal{T}$ is a compact operator on $\overline{B}_r(0)$. To accomplish this, it suffices to show that every sequence $\{\mathcal{T}x_n\}$ in $\mathcal{T}(\overline{B}_r(0))$ has a convergent subsequence. Similar to what we did above, we can show that $\|\mathcal{T}x_n\| \leq r$ for all $n \in \mathbb{N}$. This shows that $\{\mathcal{T}x_n\}$ is a uniformly bounded sequence in $\mathcal{T}(\overline{B}_r(0))$.

To show that $\{\mathcal{T}x_n\}$ is also an equicontinuous sequence in $\mathcal{T}(\overline{B}_r(0))$, let $\epsilon > 0$ be given. Since $\lim\limits_{t\to\infty} w(t) = 0$, there exists $T_1 > t_0 \geq 0$ such that

$$w(t) < \frac{\epsilon\Gamma(q+1)}{9M_f} \tag{3.11}$$

for all $t \geq T_1$.

Let t, $\tau \in J_\infty$ be arbitrary. If t, $\tau \in [t_0, T_1]$, then we have

$$\begin{aligned}
|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| &\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \left|\frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \left|\frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{t}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\quad + \left|\frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{t}(\tau-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \frac{M_f}{\Gamma(q)}\int_{t_0}^{t}\left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right|\,ds + \frac{M_f}{\Gamma(q)}\left|\int_{\tau}^{t}\left|\overline{a}(\tau)(\tau-s)^{q-1}\right|ds\right| \\
&\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \frac{M_f}{\Gamma(q)}\int_{t_0}^{T}\left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right|\,ds + \frac{M_f\,\|\overline{a}\|}{\Gamma(q+1)}\left|(\tau-t)^q\right|.
\end{aligned} \tag{3.12}$$

Since the function $t \mapsto \overline{a}(t)(t-s)^{q-1}$ is continuous on the compact interval $[t_0, T_1]$, it is uniformly continuous there. Therefore, for the above ϵ there exist $\delta_1 > 0$ and $\delta_2 > 0$, depending only on ϵ, such that

$$|t-\tau| < \delta_1 \quad \text{implies} \quad \left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right| < \min\left\{\frac{\epsilon\Gamma(q)}{9b_0}, \frac{\epsilon\Gamma(q)}{9M_fT_1}\right\},$$

and

$$|t-\tau| < \delta_2 \quad \text{implies} \quad |(t-\tau)^q| < \frac{\epsilon\Gamma(q+1)}{9M_f\|\overline{a}\|}.$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Then, if t, $\tau \in [t_0, T_1]$ with $|t-\tau| < \delta_3$, from (3.12) we have

$$|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| < \frac{\epsilon}{3}$$

for all $n \in \mathbb{N}$.

Now, if t, $\tau > T_1$, then there is a $0 < \delta_4 < \delta_3$ such that if $|t - \tau| < \delta_4$,

$$\begin{aligned}|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| &\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(t)}\right| \\ &\quad + \frac{\overline{a}(t)}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}f(s,x_n(s))\,ds\right| + \frac{\overline{a}(\tau)}{\Gamma(q)}\left|\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x_n(s))\,ds\right| \\ &\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(t)}\right| + \frac{M_f}{\Gamma(q+1)}\big[w(t) + w(\tau)\big] \\ &< \frac{\epsilon}{9} + \frac{2\epsilon}{9} = \frac{\epsilon}{3}\end{aligned}$$

for all $n \in \mathbb{N}$. Similarly, if t, $\tau \in \mathbb{R}_+$ with $t < T_1 < \tau$ and $|t - \tau| < \delta < \delta_4$, then

$$|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| \leq |\mathcal{T}x_n(t) - \mathcal{T}x_n(T_1)| + |\mathcal{T}x_n(T_1) - \mathcal{T}x_n(\tau)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}$$

for all $n \in \mathbb{N}$. As a result, $|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| < \epsilon$ for all t, $\tau \in J_\infty$ with $|t - \tau| < \delta$ and for all $n \in \mathbb{N}$. This shows that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence in $\overline{B}_r(0)$. An application of the Arzelà-Ascoli theorem implies that $\{\mathcal{T}x_n\}$ has a uniformly convergent subsequence on the compact set $\overline{B}_r(0)$.

Since $\mathcal{T}(\overline{B}_r(0))$ is closed, $\{\mathcal{T}x_n\}$ converges to a point in $\mathcal{T}(\overline{B}_r(0))$, so $\mathcal{T}(\overline{B}_r(0))$ is relatively compact. Therefore, $\mathcal{T}$ is a continuous and compact operator on $\overline{B}_r(0)$. An application of Theorem 2.6 shows that the operator equation $\mathcal{T}x = x$, and hence (1.1), has a solution on J_∞ belonging to $\overline{B}_r(0)$.

To prove the attractivity of solutions, let x, $y \in \overline{B}_r(0)$ be any two solutions of (1.1) on J_∞. Then,

$$\begin{aligned}|x(t) - y(t)| &\leq \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x(s)) - f(s,y(s))\big|\,ds \\ &\leq \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\big[|f(s,x(s))| + |f(s,y(s))|\big]\,ds \leq \frac{2\,M_f}{\Gamma(q+1)}w(t)\end{aligned}$$

for all $t \in J_\infty$. As in (3.11), for any $\epsilon > 0$ there exists $T_1 > t_0$ such that

$$w(t) < \frac{\epsilon\Gamma(q+1)}{2M_f}$$

for $t \geq T_1$. Thus,

$$|x(t) - y(t)| < \epsilon$$

for all $t \geq T$. Hence, the solutions of (1.1) are uniformly globally attractive on J_∞.

Finally, since a belongs to $\mathcal{A}$, for any $\epsilon > 0$, there exists $T_2 > T_1$ such that

$$\left|\frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)}\right| < \frac{\epsilon}{2}$$

for $t \geq T_2$. Then for any solution x of (1.1) defined on J_∞,

$$|x(t)| \leq \left|\frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)}\right| + \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|f(s,x(s))|\,ds \leq \frac{\epsilon}{2} + \frac{M_f}{\Gamma(q+1)}w(t) < \epsilon$$

for all $t \geq T_2$, that is, solutions are uniformly globally asymptotically attractive and stable on J_∞. This completes the proof of the theorem. □

In our next theorem, we wish to show that the uniformly globally asymptotically attractive solution of (1.1) obtained from Theorem 3.5 is unique.

Theorem 3.6. *Assume that conditions (H$_1$)–(H$_3$) hold with*

$$\frac{\sup_{t_0 \leq t}\overline{a}(t)\,t^q}{\Gamma(q)}\psi_f(r) < r, \quad r > 0. \tag{3.13}$$

Then (1.1) has a unique uniformly stable solution defined on J_∞.

Proof. Set $X = BC(J_\infty, \mathbb{R})$ and define the operator $\mathcal{T} : X \to X$ by (3.7). We want to show that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on X. Let x, $y \in X$; then by (H$_3$), we obtain

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}x(t)| &\leq \frac{\overline{a}(t)}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x(s)) - f(s,y(s))\big|\,ds\right| \\
&\leq \frac{\overline{a}(t)}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}\psi_f\big(|x(s)-y(s)|\big)\,ds\right| \\
&\leq \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\psi_f\big(|x-y|\big)\,ds \\
&\leq \frac{w(t)}{\Gamma(q+1)}\psi_f\big(|x-y|\big) \leq \frac{W}{\Gamma(q+1)}\psi_f\big(|x-y|\big)
\end{aligned}$$

for all $t \in J_\infty$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \frac{W}{\Gamma(q+1)}\psi_f\big(|x-y|\big)$$

for all $x, y \in X$, where $\dfrac{W}{\Gamma(q)}\psi_f(r) < r$ for $r > 0$ in view of condition (3.13). This shows that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on X. By Theorem 2.7, we obtain that the solution of (1.1) obtained in Theorem 3.5 is unique. □

Example 1. *Consider the initial value problem of fractional Riemann-Liouville type*

$$\begin{cases} {}^{RL}D_1^q\big[(t^q+1)e^t x(t)\big] = \dfrac{\ln(|x(t)|+1)}{x^2(t)+2}, \quad t \in J_\infty = [0,\infty), \\ \lim\limits_{t\to 0^+} I_{0^+}^{1-q}\big[(t^q+1)e^t x(t)\big] = 1. \end{cases} \tag{3.14}$$

Here we have $t_0 = 0$, $a(t) = (t^q+1)e^t$, *and* $f(t,x) = \dfrac{\ln(|x|+1)}{x^2+2}$ *for* $(t,x) \in [0,\infty) \times \mathbb{R}$. *Clearly,* $a(t) \to \infty$ *as* $t \to \infty$ *and* (H_1) *holds with* $M_f = 1$. *It is easy to see that* $\lim_{t\to 0} \dfrac{t^{q-1}}{(t^q+1)e^t} = 0$ *and* $\lim_{t\to\infty} \dfrac{t^q}{(t^q+1)e^t} = 0$, *so* $a \in \mathcal{A}$. *Hence, by Theorem 3.5, (3.15) has a solution and the solutions are uniformly globally asymptotically attractive and stable on* $[0,\infty)$.

Example 2. *Consider the problem*

$$\begin{cases} {}^{RL}D_1^q\big[(t^q+1)e^t x(t)\big] = f(t,x(t)), \quad t \in J_\infty = [0,\infty), \\ \lim\limits_{t\to 0^+} I_{0^+}^{1-q}\big[(t^q+1)e^t x(t)\big] = 1, \end{cases} \tag{3.15}$$

where

$$f(t,x) = \begin{cases} \ln(|x|+1), & \text{if } -5 \le x \le 5, \\ \ln 6, & \text{otherwise.} \end{cases}$$

Now, for $-5 \le x \le 5$,

$$\begin{aligned} |f(t,x) - f(t,y)| &= |\ln(|x|+1) - \ln(|y|+1)| = \ln\frac{|x|+1}{|y|+1} = \ln\frac{1+|y|+|x|-|y|}{|y|+1} \\ &= \ln\left(1 + \frac{|x|-|y|}{|y|+1}\right) \le \ln\left(1+\frac{|x-y|}{|y|+1}\right) \le \Psi_f(|x-y|). \end{aligned}$$

We can then take our $\mathcal{D}$*-function to be* $\psi_f(r) = \ln(1+r)$ *and* $M_f = \ln 6$. *Since*

$$\frac{\sup_{t\ge t_0} \overline{a}(t)\, t^q}{\Gamma(q)} \psi_f(r) \le \psi_f(r) = \ln(1+r) < r, \quad r > 0, \tag{3.16}$$

condition (3.13) *is satisfied. Therefore by Theorems 3.5 and 3.6, solutions of* (3.15) *exist, are unique, and are uniformly globally asymptotically attractive on* $\mathbb{R}_+$.

Data Availability

Data sharing is not applicable to this article since no datasets were generated or analysed during the current study.

Conflict of Interest

The authors declare that there are no conflicts of interest.

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Boundedness and stability in nonlinear systems of differential equations using a modified variation of parameters formula

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ABSTRACT

In this research we introduce a new variation of parameters for systems of linear and nonlinear ordinary differential equations. We use known mathematical methods and techniques including Gronwall's inequality and fixed point theory to obtain boundedness on all solutions and stability results on the zero solution.

RESUMEN

En esta investigación, introducimos un nuevo método de variación de parámetros para sistemas de ecuaciones diferenciales ordinarias lineales y no lineales. Usamos métodos y técnicas matemáticas conocidas incluyendo la desigualdad de Gronwall y teoría de punto fijo para obtener el acotamiento de todas las soluciones y resultados de estabilidad de la solución cero.

Keywords and Phrases: System, Ordinary differential equations, Linear, Nonlinear, Fundamental matrix, Boundedness, Stability, New variation of parameters.

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1 Introduction

A general approach to solving inhomogeneous linear ordinary differential equations in mathematics is variation of parameters, often known as variation of constants. In this paper we introduce new variation of parameters formula for systems of linear and nonlinear ordinary differential equations. Once the inversion is done, we apply known results such as Gronwall's inequality and the contraction mapping principle to obtain boundedness on all solutions and stability results on the zero solution. It is common practice to linearize around the equilibrium solution for nonlinear systems before drawing conclusions about the stability of the equilibrium solution for the original system using the signs of the linear system's eigenvalues. We demonstrate in our cases how this approach does not work. Utilizing Liapunov functions and functionals to analyze solutions is an additional well-liked technique. However, the method by which such Liapunov functions/functionals are created remains a mystery, and the type of Liapunov functions/functionals determines whether or not the conclusions reached are valid. In general, obtaining a variation of parameters formula relies on heuristics that require guessing and are not applicable to all inhomogeneous linear differential equations, it is typically possible to find solutions to first-order inhomogeneous linear differential equations using integrating factors or undetermined coefficients with a great deal less effort.

Hence, in this research our main intention is to be able to write totally nonlinear systems of the form

$$x' = f(t, x(t)),$$

into an integral system of equations, from which we obtain results concerning the behavior of solutions using fixed point theory. The absence of a linear term in $x' = f(t, x(t))$ is the sole cause for not being able to invert the system and obtain a variation of parameters formula for the solutions. For such systems, usually researchers borrow a linear term for the sake of inversion, and as a result, the resulting integral equation may not satisfy a contraction property.

In [18] the first author used Lyapunov functionals and studied the exponential stability of the zero solution of finite delay Volterra Integro-differential equation

$$x'(t) = Px(t) + \int_{t-\tau}^{t} C(t,s)g(x(s))\,ds. \tag{1.1}$$

Recently, in [5, 6], Burton used the notion of fixed point theory to alleviate some of the difficulties that arise from the use of Liapunov functionals and obtained results concerning the stability and asymptotic stability of the zero solution of (1.1) when it is scalar. We remark that the results of [5, 6, 18] were made possible due to the existence of the linear term Px.

To ease the reader into the main parts of this research, we begin with by considering the scalar differential equation

$$x'(t) = ax(t), \quad x(0) = x_0 \tag{1.2}$$

with the known solution

$$x(t) = x_0 e^{at}.$$

We notice that if

$$a < 0, \quad \text{then} \quad x(t) = x_0 e^{at} \to 0, \quad \text{as} \quad t \to \infty.$$

To further introduce our topic, we assume $a : \mathbb{R} \to \mathbb{R}$ is continuous and consider

$$x'(t) = a(t)x(t), \quad x(0) = x_0, \tag{1.3}$$

which has the solution

$$x(t) = x_0 e^{\int_0^t a(s)ds} \to 0, \quad \text{as} \quad t \to \infty$$

provided that

$$\int_0^t a(s)\,ds \to -\infty. \tag{1.4}$$

Condition (1.4) implies that the function $a(t)$ can be positive or oscillates for short time. Now assume the existence of a continuous function

$$v : [0, \infty) \to \mathbb{R}.$$

Multiply both sides of (1.2) by

$$e^{\int_0^t v(s)\,ds},$$

and then integrate from 0 to any $t \in [0, T)$. That is

$$\int_0^t e^{\int_0^u v(s)\,ds}\, x'(u)\,du = \int_0^t ax(u)e^{\int_0^u v(s)\,ds}\,du.$$

Perform an integration by parts on the left side and simplify to get

$$x(t) = x_0 e^{-\int_0^t v(s)ds} + \int_0^t x(u)(v(u) + a)e^{-\int_u^t v(s)\,ds}\,du. \tag{1.5}$$

Expression (1.5) is a *new variation of parameters formula for* (1.2) and of Volterra type integral equation. Note that if

$$v(t) = -a,$$

then (1.5) become the regular solution $x(t) = x_0 e^{at}$ of (1.2). In a similar fashion

$$x'(t) = a(t)x(t), \quad x(0) = x_0,$$

has the solution

$$x(t) = x_0 e^{-\int_0^t v(s)\, ds} + \int_0^t x(u)(v(u) + a(u)) e^{-\int_u^t v(s)\, ds}\, du. \tag{1.6}$$

Again, if we let

$$v(t) = -a(t),$$

then we get the regular known solution $x(t) = x_0 e^{\int_0^t a(s)\, ds}$, and $x(t) \to 0,$ as $t \to \infty$ provided that

$$\int_0^t a(s)\, ds \to -\infty.$$

Again (1.6) is a new variation of parameters formula that can be used to deduce qualitative properties about the solutions. In the mean time for (1.6), by setting up the proper spaces and using the contraction mapping principle, we can show that $x(t) \to 0$ as $t \to \infty$, provided that

$$\int_0^t |v(u) + a(u)| e^{-\int_u^t v(s)\, ds}\, du \le \alpha, \quad 0 < \alpha < 1,$$

and

$$\int_0^t v(s)\, ds \to \infty.$$

Suppose $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and consider the nonlinear differential equation

$$x'(t) = f(t, x(t)),\ x(0) = x_0 \quad \text{for a given constant} \quad x_0. \tag{1.7}$$

Then, multiplying by a function $\int_0^t v(s)ds$ then the solution of (1.7) is given by

$$x(t) = x_0 e^{-\int_0^t v(s)\, ds} + \int_0^t \big(x(u)v(u) + f(u, x(u))\big) e^{-\int_u^t v(s)\, ds}\, du. \tag{1.8}$$

Similarly, by setting up the proper space and assuming the right conditions on the function f one can obtain results regarding boundedness of solutions and the stability of the zero solution in the case $f(t, 0) = 0$. In [13] the authors studied obtained a new variation of parameters for the finite delay nonlinear differential equation

$$x'(t) = f(t, x(t - \tau))$$

and arrived at stability and periodicity results. For more on the use of the regular variation of parameters we refer to [5, 6, 7].

2 Homogeneous linear systems

Consider the time-varying homogeneous system

$$x'(t) = A(t)x(t), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{2.1}$$

where $A(t)$ is an $n \times n$ matrix of coefficients $a_{ij}(t)$ that are assumed to be continuous on an interval I. Recall that a solution $x(t)$ of (2.1) is an n-tuple of C^1 functions $x_i : I \to \mathbb{R}$. We adopt the notation that

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

The solution x maybe considered as a C^1 vector-valued functions $x : I \to \mathbb{R}^n$. Such space of functions is denoted by $C^1(I, \mathbb{R}^n)$. If $\mathcal{S}$ is the *solution space* of (2.1), then $\mathcal{S} \subset C^1(I, \mathbb{R}^n)$. We state the following definition regarding the fundamental matrix of (2.1).

Definition 2.1. *A set of n solutions of the linear differential system* (2.1) *all defined on the same open interval I, is called a fundamental set of solutions on I if the solutions are linearly independent functions on I.*

Now we state the following familiar theorem. For its proof we may refer to [9, 10, 11, 12].

Theorem 2.2. *If $\Phi(t)$ is a fundamental matrix of* (2.1) *on an interval I, then $\Phi(t)c$, with $c = \Phi^{-1}(t_0)x_0$ is a solution of* (2.1) *with $x(t_0) = x_0$. That is, the unique solution of* (2.1) *is given by*

$$x(t) = \Phi^{-1}(t)\Phi(t_0)x_0. \tag{2.2}$$

The literature is vast concerning the study of systems of differential equations using variation of parameters or Liapunov functionals. For emphasis, using the regular variation of parameters requires the presence of linear term in the form of $A(t)x$. For comprehensive work on such studies we refer to [1, 2, 3, 4, 5]. For results on comprehensive treatment of Liapunov functions/functionals, we refer to [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. It is worth noting that, in [16], the author constructed what we call today, the Resolvent matrix and used it in the form of variation of parameters to analyze solutions of linear Volterra integro-differential equations. Later on, Burton, in [5, 6, 7] generalized the notion of resolvent to nonlinear systems by borrowing linear terms and obtained results concerning boundedness, stability and periodicity. For various results concerning systems of differential equations, we refer to [13, 14, 15, 16]. As we have previously stated, there is a substantial body of scholarship on parameter variation in books, but not in refereed publications.

In [17], the authors considered considered the nonlinear matrix Lyapunov system

$$T^{(n)} = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} A^{n-r} T(t) B^{r},$$

where A and B are constant $n \times n$ matrices. They assumed the existence of the fundamental matrix of $T' = AT$ in order to obtain a variation of parameters formula for all solutions. Our work here does not require the existence of a linear term for the inversion. In addition, the authors in [8] consider different kinds of scalar linear and nonlinear first order differential equations and use Liapunov functions and fixed point theory to get results about the boundedness of solutions, the existence of periodic solutions, and the stability of the zero solution. Although they borrow a linear component in order to be able to invert nonlinear equations, this complicates the formula for the resulting variation of parameters and causes it to immediately encounter problems. Our purpose is to obtain a different variation of parameters that solves (2.1) and hopefully its characteristics are different from those of (2.2). We begin with the following lemma.

Lemma 2.3. *Let $\varphi(t)$ be an $n \times n$ differentiable matrix with continuous entries on the interval I. Assume $\varphi^{-1}(t)$ exists for all $t \in I$. Then $x(t)$ is a solution of* (2.1) *if and only if*

$$x(t) = \varphi^{-1}(t)\varphi(t_0)x_0 + \int_{t_0}^{t} \varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]x(s)\, ds, \quad t \geq t_0. \tag{2.3}$$

Proof. Multiply both sides of (2.1) from the left with the matrix $\varphi(t)$ and then integrate the resulting equation from t to t_0 and obtain

$$\int_{t_0}^{t} \varphi(s)x'(s)\, ds = \int_{t_0}^{t} \varphi(s)A(s)x(s)\, ds.$$

Integrating the left side by parts by letting

$$u = \varphi(s), dv = x'(s)ds,$$

we arrive at

$$\varphi(t)x(t) - \varphi(t_0)x_0 = \int_{t_0}^{t} \big[\varphi'(s) + \varphi(s)A(s)\big]x(s)\, ds.$$

Multiply from the left by $\varphi^{-1}(t)$ gives the desired result. Since every step is reversible, we have completed the proof. □

Remark 2.4. *We note that if*

$$\varphi'(t) = -\varphi(t)A(t), \quad \textit{for all} \quad t \in I,$$

then equation (2.3) *implies that*

$$x(t) = \varphi^{-1}(t)\varphi(t_0)x_0 \tag{2.4}$$

is a solution of (2.1). *To see this, set* $\varphi'(t) = -\varphi(t)A(t)$, *in* (2.3). *Then* (2.3) *reduces to* $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$ *with* $x(t_0) = x_0$. *Differentiating with respect to* t *we arrive at*

$$\begin{aligned} x'(t) &= \big(\varphi^{-1}(t)\big)'\varphi(t_0)x_0 = -\varphi^{-1}(t)\varphi'(t)\varphi^{-1}(t)\varphi(t_0)x_0 = -\varphi^{-1}(t)\big(-\varphi(t)A(t)\big)\varphi^{-1}(t)\varphi(t_0)x_0 \\ &= A(t)\varphi^{-1}(t)\varphi(t_0)x_0 = A(t)x(t). \end{aligned}$$

Note that a quick comparison of (2.2) *with* (2.4) *we see that*

$$\varphi(t) = \Phi^{-1}(t),$$

a result that is parallel to the scalar equations.

Thus one of the main advantages of using (2.3) with $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$, is that it enables us to find the desired matrix and hence a solution for a time-varying system. Usually finding the fundamental matrix solution of time-varying system (2.1) requires additional conditions that are hard to meet. For the rest of this work we consider system (2.1) over the interval $I = [0, \infty)$. We also assume $\|\cdot\|$ is a suitable matrix norm. Next we consider (2.1) such that $f(t, 0) = 0$.

Definition 2.5. *The zero solution* $(x = 0)$ *of* (2.1)*;*

- *(a) is stable (S) if for each* $\epsilon > 0$ *and* $t_0 \geq 0$, *there is a* $\delta = \delta(t_0, \epsilon) > 0$ *such that* $|x(t_0)| < \delta$ *implies* $|x(t, t_0, x_0)| < \varepsilon$,
- *(b) is uniformly stable (US) if* δ *independent of* t_0,
- *(c) is unstable if it is not stable,*
- *(d) is asymptotically stable (AS) if it is stable and* $\lim_{t\to\infty} |x(t, t_0, x_0)| = 0$.

We have the following theorem regarding boundedness of solutions and stability of the zero solution.

Theorem 2.6. *Assume the existence of a positive constant* K *such that*

$$\|\varphi^{-1}(t)\varphi(s)\| \leq K. \tag{2.5}$$

In addition, if there is a positive constant E *such that*

$$\int_0^\infty \|\varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]\|\, ds \leq E \tag{2.6}$$

then all solutions of (2.1) *are bounded and its zero solution is uniformly stable.*

Proof. Let $x(t)$ be given by (2.3) for all $t \geq t_0 \geq 0$. Since the constant K is independent of the initial time $t_0 \geq 0$ we have from (2.3) that

$$\begin{aligned} |x(t)| &= \|\varphi^{-1}(t)\varphi(t_0)\||x_0| + \int_{t_0}^{t} \|\varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]\| \, |x(s)| \, ds \\ &\leq K|x_0| + \int_{t_0}^{t} \|\varphi^{-1}(t)\big[\varphi'(s) + \varphi(s)A(s)\big]\| \, |x(s)| \, ds \\ &\leq K|x_0| e^{\int_{t_0}^{t} \|\varphi^{-1}(t)\left[\varphi'(s)+\varphi(s)A(s)\right]\| \, ds} \quad \text{(by Gronwall's inequality)} \\ &\leq K|x_0|e^{E} \end{aligned} \tag{2.7}$$

Hence inequality (2.7) implies all solutions are bounded. For the uniform stability of the zero solution, we let $\delta = \dfrac{\varepsilon}{Ke^{E}}$ so that for any $\varepsilon > 0$ we have from (2.7) for $|x_0| < \delta$, that $|x(t)| < \varepsilon$. This completes the proof. □

For the next theorems we assume that set $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$, so that the solution of (2.1) is given by $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$ as was indicated by Remark 2.4.

Theorem 2.7. *Let $\varphi(t)$ be as defined in Lemma 2.3 such that $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$. Then the zero solution of* (2.1) *is*

(a) stable if and only if there exists a positive constant M such that

$$\|\varphi^{-1}(t)\| \leq M, \quad t \geq 0,$$

(b) asymptotically stable if and only if

$$\|\varphi^{-1}(t)\| \to 0, \quad \textit{as} \quad t \to \infty.$$

Proof. (a) ($\Leftarrow$) Let $\varphi'(t) = -\varphi(t)A(t)$, for all $t \in I$. Then by the Remark 2.4, we have that $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$ is a solution of (2.1). Let $\epsilon > 0$ and set $\delta = \dfrac{\epsilon}{\|\varphi(t_0)\|M}$ such that for $|x_0| < \delta$ we have that

$$|x(t)| = |\varphi^{-1}(t)\varphi(t_0)x_0| \leq \|\varphi^{-1}(t)\|\|\varphi(t_0)\||x_0| \leq M\|\varphi(t_0)\|\delta = \epsilon.$$

($\Rightarrow$) Set $\epsilon = 1$ from the stability proof. Then

$$|x(t)| = |\varphi^{-1}(t)\varphi(t_0)x_0| < 1, \quad \text{for} \quad t \geq t_0 \quad \text{if} \quad |x_0| < \delta(1, t_0),$$

which implies that

$$\|\varphi^{-1}(t)\varphi(t_0)\| < \frac{1}{\delta(1, t_0)}.$$

Therefore,

$$\|\varphi(t)\| = \|\varphi^{-1}(t)\varphi(t_0)\varphi^{-1}(t_0)\| \le \|\varphi^{-1}(t)\varphi^{-1}(t_0)\|\|\varphi(t_0)\| \le \frac{1}{\delta(1,t_0)}|\varphi(t_0)| := M.$$

This completes the proof of (a).

Next we prove (b). We already know the zero solution is stable. Now,

$$|x(t)| = \|\varphi^{-1}(t)\varphi(t_0)x_0\| \to 0, \quad \text{as} \quad t \to \infty$$

if and only if

$$\|\varphi^{-1}(t)\| \to 0, \quad \text{as} \quad t \to \infty.$$

This completes the proof. □

Before we provide an example, we will have the following discussion. We integrate (2.1) from t_0 to t and get $x(t) = e^{\int_{t_0}^t A(s)\,ds}$. Now we let

$$\Phi(t) = e^{\int_{t_0}^t A(s)\,ds}. \tag{2.8}$$

For $\Phi(t)$ to be fundamental matrix solution, we must have

$$A(t)\Big(\int_{t_0}^t A(s)\,ds\Big) = \Big(\int_{t_0}^t A(s)\,ds\Big)A(t). \tag{2.9}$$

Let us see why. Let $J = \int_{t_0}^t A(s)\,ds$. Then

$$e^J = I + J + \frac{1}{2!}J^2 + \frac{1}{3!}J^3 + \cdots + \frac{1}{k!}J^k + \cdots .$$

and

$$\begin{aligned}
\frac{d}{dt}e^{\int_{t_0}^t A(s)\,ds} &= \frac{d}{dt}\Big(I + \int_{t_0}^t A(s)\,ds + \frac{1}{2!}\Big(\int_{t_0}^t A(s)\,ds\Big)^2 \\
&+ \cdots + \frac{1}{(k-1)!}\Big(\int_{t_0}^t A(s)\,ds\Big)^{k-1}A(t) + \frac{1}{k!}\Big(\int_{t_0}^t A(s)\,ds\Big)^k + \cdots\Big) \\
&= A(t) + \int_{t_0}^t A(s)\,ds\; A(t) + \frac{1}{2!}\Big(\int_{t_0}^t A(s)\,ds\Big)^2 A(t) \\
&+ \cdots + \frac{1}{(k-1)!}\Big(\int_{t_0}^t A(s)\,ds\Big)^{k-1}A(t) + \frac{1}{k!}\Big(\int_{t_0}^t A(s)\,ds\Big)^k A(t) + \cdots \\
&= \big[I + \int_{t_0}^t A(s)\,ds + \frac{1}{2!}\Big(\int_{t_0}^t A(s)\,ds\Big)^2 + \cdots \\
&+ \frac{1}{(k-1)!}\Big(\int_{t_0}^t A(s)\,ds\Big)^{k-1}A(t) + \frac{1}{k!}\Big(\int_{t_0}^t A(s)\,ds\Big)^k + \cdots\big]A(t)
\end{aligned}$$

$$\begin{aligned}
&\neq A(t)\big[I + \int_{t_0}^{t} A(s)\,ds + \frac{1}{2!}\Big(\int_{t_0}^{t} A(s)\,ds\Big)^2 \\
&+ \cdots + \frac{1}{(k-1)!}\Big(\int_{t_0}^{t} A(s)\,ds\Big)^{k-1} A(t) + \frac{1}{k!}\Big(\int_{t_0}^{t} A(s)\,ds\Big)^{k} + \cdots\big] \\
&= A(t)\Phi(t).
\end{aligned}$$

Thus, if (2.9) holds, then $\Phi'(t) = A(t)\Phi(t)$.

We have the following example.

Example 1. *For $t \geq 0$ we consider the linear system*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x(0) = x_0. \tag{2.10}$$

If we let

$$A(t) = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}$$

then it is clear that (2.9) *does not hold. Let*

$$\varphi(t) = \begin{pmatrix} 1+t^2 & -t \\ -t & 1 \end{pmatrix}.$$

Then one may easily verify that that the matrix φ satisfies $\varphi'(t) = -\varphi(t)A(t)$, for all $t \geq 0$, and hence every solution of (2.10) *satisfies $x(t) = \varphi^{-1}(t)\varphi(t_0)x_0$. In addition,*

$$\varphi^{-1}(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}.$$

Applying Theorem 2.7 we conclude solutions of (2.10) *are unbounded and its zero solution is unstable.*

In Example 1, it would have been difficult to find the fundamental matrix using the argument of eigenvalues and corresponding eigenfunctions since (2.9) does not hold. Moreover, the method of regular linearization does not work for time-varying systems. We are left with the notion of finding a suitable Liapunov function to prove the unboundedness of solutions and consequently the instability of the zero solution. This author could not find one that would do the job. In conclusion, the above discussion cements the usefulness of our method. A final note: the system may be solved using the Laplace transform. This can be done by writing the system as

$$x' = -tx_1 + x_2, \quad x_2' = (1-t^2)x_1 + tx_2,$$

subject to the initial conditions $x_1(0) = x_{01}, x_2(0) = x_{02}$. Looking forward, Laplace transforms can not be used in our next examples.

3 Nonlinear systems

We consider the general nonlinear system of ordinary differential equations

$$\begin{aligned}
x_1' &= f_1(t, x_1, \dots, x_n) \\
x_2' &= f_2(t, x_1, \dots, x_n) \\
&\vdots \\
x_n' &= f_n(t, x_1, \dots, x_n).
\end{aligned}$$

Using the vector notations

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$f(t,x) = \begin{pmatrix} f_1(t,x) \\ f_2(t,x) \\ \vdots \\ f_n(t,x) \end{pmatrix}$$

the above system can be written in the vector form

$$x' = f(t,x) \tag{3.1}$$

and assume $f \in C^1([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, is continuous in t and x. Let $\varphi(t)$ be an $n \times n$ matrix with continuous entries on $[0,\infty)$. Assume $\varphi^{-1}(t)$ exists for all $t \geq 0$. We multiply both sides of (3.1) with $\varphi(t)$. By similar work as before, we have

$$x(t) = \varphi^{-1}(t)\varphi(t_0)x_0 + \varphi^{-1}(t)\int_{t_0}^{t} \left[\varphi'(s)x(s) + \varphi(s)f(s,x(s))\right] ds, \quad t \geq t_0. \tag{3.2}$$

Now the advantage of our method is that (3.2) can be used on proper spaces to analyze the solutions of (3.1). In this work it is more convenient to use the following norms for a matrix and a vector.

For $x = (x_1, x_2, \ldots, x_n)$ we consider the norm

$$|x| = \sum_{i=1}^{n} |x_i|.$$

Similarly, we define the norm of a matrix B by

$$|B| = \sum_{i,j=1}^{n} |b_{ij}|,$$

for an $n \times n$ matrix $B = [b_{ij}]$. Under these two norms we have

$$|B(t)x| \leq |B(t)||x|,$$

and for any two $n \times n$ matrices B and K we have that

$$|B(t)K(t)| \leq |B(t)||K(t)|.$$

We have the following theorem regarding boundedness of solutions and stability of the zero solution of system (3.1).

Theorem 3.1. *Suppose there is a positive constant K and a continuous function $\lambda : [0, \infty) \to [0, \infty)$ such that*

$$|\varphi^{-1}(t)\varphi(t_0)| \leq K, \tag{3.3}$$

and

$$|f(t, x)| \leq \lambda(t)|x|. \tag{3.4}$$

In addition, if there is a positive constant E such that

$$\int_0^\infty \big[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]\, ds \leq E \tag{3.5}$$

then all solutions of (3.1) *are bounded and its zero solution is uniformly stable.*

Proof. Let $x(t)$ be given by (3.2) for all $t \geq t_0 \geq 0$. Since the constant K is independent of the initial time $t_0 \geq 0$ we have from (2.3) that

$$\begin{aligned}
|x(t)| &\leq |\varphi^{-1}(t)\varphi(t_0)||x_0| + \int_{t_0}^{t} |\varphi^{-1}(t)\big[\varphi'(s)x(s) + \varphi(s)f(s, x(s))\big]|\, ds \\
&\leq K|x_0| + \int_0^\infty \big[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]|x(s)|\, ds \\
&\leq K|x_0| e^{\int_{t_0}^{t} \left[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\right] ds} \quad \text{(by Gronwall's inequality)} \\
&\leq K|x_0| e^{E}.
\end{aligned} \tag{3.6}$$

Hence inequality (3.6) implies all solutions are bounded. For the uniform stability of the zero solution, we let $\delta = \dfrac{\varepsilon}{Ke^E}$ so that for any $\varepsilon > 0$ we have from (3.6) for $|x_0| < \delta$, that $|x(t)| < \varepsilon$. This completes the proof. $\square$

We provide the following example.

Example 2. *For $t \geq 0$ we consider the nonlinear system*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} \dfrac{x_1 \cos(x_2)\sin(t)}{(1+t)(x_1^2+1)} \\ \\ \dfrac{x_2 \sin(x_1)\cos(t)}{(1+t)(x_2^2+1)} \end{pmatrix}, \quad x(0) = x_0. \tag{3.7}$$

Note that

$$|f(t,x)| = \sum_{i=1}^{2} |f_i(t,x)| = \left| \frac{x_1 \sin(t)}{(1+t)(x_1^2+1)} \right| + \left| \frac{x_2 \cos(t)}{(1+t)(x_2^2+1)} \right| \leq \frac{1}{1+t} \left[|x_1| + |x_2| \right] = \frac{1}{1+t}|x|.$$

Hence

$$\lambda(t) = \frac{1}{1+t}.$$

To verify the rest of the conditions of Theorem 3.3, we let

$$\varphi(t) = \begin{pmatrix} \sqrt{1+t} & 0 \\ 0 & \sqrt{1+t} \end{pmatrix}.$$

Then

$$\varphi^{-1}(t) = \begin{pmatrix} \dfrac{1}{\sqrt{1+t}} & 0 \\ 0 & \dfrac{1}{\sqrt{1+t}} \end{pmatrix}.$$

One can easily compute that

$$\varphi^{-1}(t)\varphi(s) = \begin{pmatrix} \dfrac{\sqrt{1+s}}{\sqrt{1+t}} & 0 \\ 0 & \dfrac{\sqrt{1+s}}{\sqrt{1+t}} \end{pmatrix},$$

and

$$\varphi^{-1}(t)\varphi'(s) = \begin{pmatrix} \dfrac{1}{2\sqrt{1+s}\sqrt{1+t}} & 0 \\ 0 & \dfrac{1}{2\sqrt{1+s}\sqrt{1+t}} \end{pmatrix}.$$

Thus,

$$|\varphi^{-1}(t)\varphi(0)| \leq \frac{2}{\sqrt{1+t}} \leq 2 =: K, \quad \textit{for all} \quad t \geq 0.$$

Moreover,

$$\int_0^t |\varphi^{-1}(t)\varphi'(s)|\, ds \leq \frac{1}{\sqrt{1+t}} \int_0^t \frac{1}{\sqrt{1+s}}\, ds = 2 - \frac{2}{\sqrt{1+t}} \leq 2, \quad \textit{for all} \quad t \geq 0.$$

Similarly,

$$\int_0^t |\varphi^{-1}(t)\varphi(s)|\lambda(s)\, ds \leq \frac{2}{\sqrt{1+t}} \int_0^t \frac{1}{\sqrt{1+s}}\, ds = 4 - \frac{4}{\sqrt{1+t}} \leq 4, \quad \textit{for all} \quad t \geq 0.$$

Finally,

$$\int_0^\infty \left[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\right]\, ds \leq 6 =: E.$$

Thus, all conditions of Theorem 3.3 are satisfied which implies that all solutions of (3.7) *are bounded and its zero solutions is uniformly stable.*

Next, we use the contraction principle to show the solution is unique. Let $\mathcal{C}$ be the set of all real-valued continuous functions. Define the space

$$\mathcal{S} = \{\Phi : [0,\infty) \to \mathbb{R}^n \mid \Phi \in \mathcal{C}, |\Phi(t)| \leq M\},$$

for positive constant M. Then

$$(\mathcal{S}, |\cdot|)$$

is complete.

Theorem 3.2. *We assume the function f is locally Lipschitz on the set $\mathcal{S}$. That is, for any Φ_1 and $\Phi_2 \in \mathcal{S}$, we have*

$$|f(t,\Phi_1) - f(t,\Phi_2)| \leq \Lambda(t)|\Phi_1 - \Phi_2|, \tag{3.8}$$

for continuous $\Lambda : [0,\infty) \to (0,\infty)$. Suppose there is a positive constant $\alpha \in (0,1)$ such that

$$\int_0^\infty \left[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\Lambda(s)\right]\, ds \leq \alpha, \tag{3.9}$$

then (3.1) *has a unique solution. In addition if* (3.3) *holds then the unique solution is bounded and the zero solution of* (3.1) *is uniformly stable.*

Proof. For $\Phi \in \mathcal{S}$, define the mapping $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$, by

$$(\mathfrak{P}\Phi)(t) = \varphi^{-1}(t)\varphi(t_0)x_0 + \varphi^{-1}(t)\int_{t_0}^t [\varphi'(s)\Phi(s) + \varphi(s)f(s,\Phi(s))]\, ds, \quad t \geq t_0. \tag{3.10}$$

It is clear that $(\mathfrak{P}\Phi)(0) = x_0$ and $\mathfrak{P}$ is continuous in Φ. Let Φ_1 and $\Phi_2 \in \mathcal{S}$. Then

$$|(\mathfrak{P}\Phi_1)(t) - (\mathfrak{P}\Phi_2)(t)| \leq \int_0^\infty [|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\Lambda(s)]ds\, |\Phi_1 - \Phi_2| \leq \alpha|\Phi_1 - \Phi_2|.$$

This shows that $\mathfrak{P}$ is a contraction. By Banach's contraction mapping principle, $\mathfrak{P}$ has a unique fixed point $x \in \mathcal{S}$ which is a continuous function. The boundedness of the solution and the uniform stability of the zero solution follow from Theorem 3.2. This completes the proof. $\square$

We will need the following clarifications for the next example. Let $f : D \to \mathbb{R}^n$ where D is a subset of $[0,\infty) \times \mathbb{R}^n$. To check if a function $f : D \to \mathbb{R}^n$ is Lipschitz continuous on some subset D of $[0,\infty) \times \mathbb{R}^n$, it suffices to check that the component functions $f_i : D \to \mathbb{R}^n$ are Lipschitz continuous. This is due to the fact that

$$|f_i(t,z) - f_i(t,w)| \leq L_i |z-w| \;\text{ for } i = 1, \ldots, n$$

implies under our norm that

$$|f(t,z) - f(t,w)| = \sum_{i=1}^{n} |f_i(t,z) - f_i(t,w)| \leq \sum_{i=1}^{n} L_i |z-w|,$$

which shows that

$$|f(t,z) - f(t,w)| \leq L|z-w| \;\text{ with }\; L = \sum_{i=1}^{n} L_i.$$

Example 3. *For $t \geq 0$ we consider the nonlinear system*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} \dfrac{\cos(t)}{20(1+t)} \left[x_2 + \dfrac{x_1}{x_1^2+1} \right] \\ \\ \dfrac{\sin(t)}{20(1+t)} \left[x_1 + \dfrac{x_2}{x_2^2+1} \right] \end{pmatrix}, \quad x(0) = x_0. \tag{3.11}$$

Note that by a similar argument as in Example 2 we arrive at

$$|f(t,x)| \leq \frac{2}{20(1+t)} \Big[|x_1| + |x_2| \Big] = \frac{1}{10(1+t)} |x|.$$

Hence

$$\lambda(t) = \frac{1}{10(1+t)}.$$

Next we show f is Lipschitz continuous. Let $z = (z_1, z_2), w = (w_1, w_2) \in \mathcal{S}$ with $n = 2$. Then

$$|f_1(t,z) - f_1(t,w)| \leq \frac{1}{20(1+t)} \left[|z_2 - w_2| + \left| \frac{z_1}{z_1^2+1} - \frac{w_1}{w_1^2+1} \right| \right].$$

$$\left| \frac{z_1}{z_1^2+1} - \frac{w_1}{w_1^2+1} \right| = \frac{z_1 - w_1 + z_1 w_1 (w_1 - z_1)}{1 + z_1^2 w_1^2 + z_1^2 + w_1^2} \leq \frac{1 + |z_1 w_1|}{1 + z_1^2 w_1^2 + z_1^2 + w_1^2} |z_1 - w_1|.$$

We note that

$$1+|z_1w_1| \le (1+|z_1w_1|)^2 = 1+|z_1w_1|^2+2|z_1w_1| \le 1+|z_1w_1|^2+z_1^2+w_1^2.$$

Hence

$$\frac{1+|z_1w_1|}{1+z_1^2w_1^2+z_1^2+w_1^2} \le 1$$

and it follows that

$$\left|\frac{z_1}{z_1^2+1}-\frac{w_1}{w_1^2+1}\right| \le |z_1-w_1|.$$

This implies that

$$|f_1(t,z)-f_1(t,w)| \le \frac{1}{20(1+t)}[|z_2-w_2|+|z_1-w_1|].$$

In a symmetrical argument one can easily shows that

$$|f_2(t,z)-f_2(t,w)| \le \frac{1}{20(1+t)}\big[|z_1-w_1|+|z_2-w_2|\big].$$

Thus from the above discussion we arrive at

$$|f(t,z)-f(t,w)| = \sum_{i=1}^{2}|f_i(t,z)-f_i(t,w)| \le \sum_{i=1}^{2}L_i|z-w| = \frac{2}{20(1+t)}|z-w|.$$

Thus, $\Lambda(t) = \dfrac{1}{10(1+t)}$. *To verify the rest of the conditions of Theorem 3.2 we let*

$$\varphi(t) = \begin{pmatrix} e^{\frac{1}{10(1+t)}} & 0 \\ 0 & e^{\frac{1}{10(1+t)}} \end{pmatrix}.$$

Then

$$\varphi^{-1}(t) = e^{-\frac{1}{5(1+t)}}\begin{pmatrix} e^{\frac{1}{10(1+t)}} & 0 \\ 0 & e^{\frac{1}{10(1+t)}} \end{pmatrix}$$

and

$$\varphi'(t) = -\frac{1}{10(1+t)^2}\begin{pmatrix} e^{\frac{1}{10(1+t)}} & 0 \\ 0 & e^{\frac{1}{10(1+t)}} \end{pmatrix}.$$

One can easily compute that

$$\varphi^{-1}(t)\varphi(s) = e^{-\frac{1}{5(1+t)}}\begin{pmatrix} e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} & 0 \\ 0 & e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} \end{pmatrix},$$

and

$$\varphi^{-1}(t)\varphi'(s) = -\frac{e^{-\frac{1}{5(1+t)}}}{10(1+s)^2}\begin{pmatrix} e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} & 0 \\ 0 & e^{\frac{1}{10}(\frac{1}{1+t}+\frac{1}{1+s})} \end{pmatrix}.$$

Thus,

$$|\varphi^{-1}(t)\varphi(0)| \leq 2e^{\frac{t}{10(1+t)}} \leq 2e^{\frac{1}{10}} =: K, \quad \text{for all} \quad t \geq 0.$$

Moreover,

$$\int_0^t |\varphi^{-1}(t)\varphi'(s)|\, ds \leq \frac{1}{5}e^{-\frac{1}{10}\frac{1}{1+t}}\int_0^t \frac{1}{(1+s)^2}e^{\frac{1}{10}\frac{1}{1+s}}\, ds = 2e^{\frac{1}{10}-\frac{1}{10}\frac{1}{1+t}} - 2 \leq 2e^{\frac{1}{10}} - 2,$$

for all $t \geq 0$. *Similarly,*

$$\begin{aligned}\int_0^t |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]\, ds &\leq \frac{2}{5}e^{-\frac{1}{10(1+t)}}\int_0^t \frac{1}{(1+s)^2}e^{\frac{1}{10}\frac{1}{1+s}}\, ds \\ &= 4e^{\frac{1}{10}-\frac{1}{10}\frac{1}{1+t}} - 4 \leq 4e^{\frac{1}{10}} - 4, \quad \text{for all} \quad t \geq 0.\end{aligned}$$

Finally,

$$\int_0^\infty \big[|\varphi^{-1}(t)\varphi'(s)| + |\varphi^{-1}(t)\varphi(s)|\lambda(s)\big]\, ds \leq 6e^{\frac{1}{10}} - 6 \leq 0.31 =: \alpha.$$

Moreover, one can easily check that

$$|\varphi^{-1}(t)\varphi(0)| \leq e^{-\frac{1}{10}}.$$

Thus, all conditions of Theorems 3.2 and 3.1 are satisfied which implies that the unique solution of (3.7) *is bounded and its zero solution is uniformly stable.*

In Examples 2 and 3, the considered equations were totally nonlinear and therefore, the arguments of fundamental matrix solution, linearization or the use of Laplace transform would be impossible. Then, we are left with the construction of Liapunov function which is almost impossible. This shows the significance of our novel method.

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Some critical remarks on recent results concerning $\digamma-$contractions in b-metric spaces

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ABSTRACT

This paper aims to correct recent results on a generalized class of $\digamma-$contractions in the context of $b-$metric spaces. The significant work consists of repairing some novel results involving $\digamma-$contraction within the structure of b-metric spaces. Our objective is to take advantage of the property $(F1)$ instead of the four properties viz. $(F1)$, $(F2)$, $(F3)$ and $(F4)$ applied in the results of Nazam *et al.* ["Coincidence and common fixed point theorems for four mappings satisfying $(\alpha_s, F)-$contraction", Nonlinear Anal: Model. Control., vol. 23, no. 5, pp. 664–690, 2018]. Our approach of proving the results utilizing only the condition $(F1)$ enriches, improves, and condenses the proofs of a multitude of results in the existing state-of-art.

RESUMEN

Este artículo tiene por objetivo corregir resultados recientes sobre una clase generalizada de $\digamma-$contracciones en el contexto de $b-$espacios métricos. El trabajo significativo consiste en reparar algunos resultados nuevos que involucran $\digamma-$contracciones en la estructura de b-espacios métricos. Nuestro objetivo es aprovechar la propiedad $(F1)$ en vez de las cuatro propiedades viz. $(F1)$, $(F2)$, $(F3)$ y $(F4)$ aplicadas en los resultados de Nazam *et al.* ["Coincidence and common fixed point theorems for four mappings satisfying $(\alpha_s, F)-$contraction", Nonlinear Anal: Model. Control, vol. 23, no. 5, pp. 664–690, 2018]. Nuestro enfoque para probar los resultados usando solo la condición $(F1)$ enriquece, mejora y condensa las demostraciones de una multitud de resultados en el estado del arte existente.

Keywords and Phrases: $\digamma-$contraction, $b-$common fixed point; $b-$metric space; υ_s-complete.

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1 Introduction and preliminaries

Let $\aleph$ be a nonempty set and $\daleth$ be a mapping from $\aleph$ to itself, then the point x from $\aleph$, for which $\daleth x = x$ is called a fixed point of $\daleth$. Note that the fixed point of the map $\daleth$ is also the fixed point of each iteration $\daleth^n$ of the mapping $\daleth$ where n is any natural number. There are examples where the opposite is not true. The existence of a fixed point of a mapping $\daleth : \aleph \to \aleph$ is especially important to examine if the set $\aleph$ is supplied with some kind of distance $\bigwedge : \aleph \times \aleph \to [0, +\infty)$ or by some topology τ. Then, depending on that distance ([28], metrics for example [7]) or topology τ [6, 12, 13], the underlying mapping has one or more fixed points, or does not exist at all. The field that studies fixed points in metric spaces is called metric fixed point theory. If the study of fixed points is performed in topological spaces, then that area is called topological fixed point theory.

If $\daleth$ is a mapping of the metric space $(\aleph, \bigwedge)$ into itself then it is called a contraction if there is a $\lambda \in [0, 1)$ such that for every $x, y \in \aleph$ it holds $\bigwedge(\daleth(x), \daleth(y)) \leq \lambda \cdot \bigwedge(x, y)$. Almost a hundred year ago, S. Banach proved the following significant theorem:

Theorem 1.1 ([5])**.** *Each contraction $\daleth$ on $(\aleph, \bigwedge)$, a complete metric space, has exactly one fixed point. In addition, for each point $x \in \aleph$, the Picard sequence $\daleth^n x$ converges to that fixed point.*

Numerous mathematicians have attempted to propose the generalizations of Banach's theorem since then. These inferences were in two most important ways: either by changing the axioms of the metric space or by taking another condition instead of the right side in the definition of contraction.

In the first mentioned direction, there were generalized metric spaces, for example, b-metric space, dislocated metric space, rectangular metric space, partial metric space, dislocated b-metric space, and in the second direction, new contractions such as Kanann, Chatterjea, Reich, Hardy-Rogers, Ćirić, Boyd, Wong, etc.

In this paper, we will talk about $\digamma-$contractions in b-metric spaces, combined with various types of admissible mappings. We will first note that all types of admissibility in this paper are introduced in the same way as the corresponding ones introduced in [27]. So, putting in the condition,

$$\upsilon_s(r_1, r_2) \geq s^2 \quad \text{implies} \quad \upsilon_s(\Im(r_1), \Im(r_2)) \geq s^2 \quad \text{for all} \quad r_1, r_2 \in \aleph,$$

of [19, Definition 2] $\frac{1}{s^2} \cdot \upsilon_s = \upsilon$, we get that $\Im : \aleph \times \aleph \to [0, +\infty)$ is υ-admissible introduced in the sense of [27]. In the same way we get that the conditions given in [19, Definitions 2, 3, 4, 5, 6, 7, 8, 9, 10 and 12] can be reduced to the corresponding $\upsilon-$conditions considered in the setting of metric spaces. For further details see [21, 27].

It should be noted that A. I. Bakhtin [4] introduced the idea of b-metric spaces and later considered by S. Czerwik [9]. In fact, the axiom of a triangle in metric spaces is generalized by adding a coefficient $s \geq 1$ on the right-hand side, *i.e.*, $\bigwedge(x,z) \leq s\left[\bigwedge(x,y)+\bigwedge(y,z)\right]$ for all $x,y,z \in \aleph$, where $\bigwedge : \aleph \times \aleph \to [0,+\infty)$. Otherwise, there are significant differences between b-metric and ordinary metric. First, it does not have to be a continuous function with two variables such as metric, an open sphere does not have to be an open set. Note that convergence, Cauchyness and continuity of the mapping are defined in the same way as for metric spaces. Also, a convergent sequence can has only one limit value.

Generalizing Banach's principle of contraction [5], D. Wardowski [33] presented the notion of $\digamma-$contraction and manifested a new generalized result as a substitute of Banach's theorem.

Definition 1.2 ([33]). *Let $\Gamma : (0,+\infty) \to (-\infty,+\infty)$ be a mapping persuading the assertions described below:*

$(F1)$ *For all $\Upsilon_1, \Upsilon_2 \in (0,+\infty)$ if $\Upsilon_2 > \Upsilon_1$ implies $\Gamma(\Upsilon_2) > \Gamma(\Upsilon_1)$, that is, Γ is strictly increasing function in $(0,+\infty)$;*

$(F2)$ *If $\{\Upsilon_n\}_{n\in\mathbb{N}}$ is a positive sequence of real numbers, then the following is contented:*

$$\lim_{n\to+\infty} \Upsilon_n = 0 \quad \textit{if and only if} \quad \lim_{n\to+\infty} \Gamma(\Upsilon_n) = -\infty;$$

$(F3)$ $\lim_{t\to 0^+} t^{\lambda}\Gamma(t) = 0$, *where $\lambda \in (0,1)$.*

$\mathfrak{F}_\Gamma$ is the set of all functions that satisfy $(F1)-(F3)$.

The following functions $\Gamma_i : (0,+\infty) \to (-\infty,+\infty)$ are in $\mathfrak{F}_\Gamma$: $\Gamma_1(t) = \ln t$; $\Gamma_2(t) = t + \ln t$; $\Gamma_3(t) = -t^{-\frac{1}{2}}$; $\Gamma_4(t) = \ln\left(t+t^2\right)$. For further details on $\mathfrak{F}_\Gamma$ the reader can see [35, 36].

Definition 1.3 ([33]). *A mapping $\daleth : \aleph \to \aleph$ is termed as $\digamma-$contraction in the context of metric space $(\aleph, \bigwedge)$ if there exist $\Gamma \in \mathfrak{F}_\Gamma$ and $\tau > 0$ such that for all $\curlywedge, \Upsilon \in \aleph$,*

$$\bigwedge(\daleth(\curlywedge), \daleth(\Upsilon)) > 0 \quad \textit{implies} \quad \tau + \Gamma\left(\bigwedge(\daleth(\curlywedge), \daleth(\Upsilon))\right) \leq \Gamma\left(\bigwedge(\curlywedge, \Upsilon)\right). \tag{1.1}$$

Theorem 1.4 ([33]). *If $(\aleph, \bigwedge)$ is a complete metric space and let $\daleth : \aleph \to \aleph$ be an $\digamma-$contraction in the sense of Wardowsski. Then $\daleth$ possesses one and only one fixed point $\curlywedge^* \in X$. On the other hand, the sequence $\{\daleth^n \curlywedge\}_{n\in\mathbb{N}}$ converges to $\curlywedge^*$ for every $\curlywedge \in \aleph$.*

In [8], the authors introduce the following condition,

$(F4)$ If $(\curlywedge_n) \subset (0,+\infty)$ is a sequence such that $\tau + \Gamma(s \cdot \curlywedge_n) \leq \Gamma(\curlywedge_{n-1})$ for every $n \in \mathbb{N}$ and for some $\tau > 0$, then $\tau + \Gamma(s^n \cdot \curlywedge_n) \leq \Gamma\left(s^{n-1} \cdot \curlywedge_{n-1}\right)$, for all $n \in \mathbb{N}$.

$\mathfrak{F}_{\Gamma_s}$ stands for the family of all functions $\Gamma : (0,+\infty) \to (-\infty,+\infty)$ that satisfy $(F1)$, $(F2)$, $(F3)$ and $(F4)$.

Remark 1.5. *It is easy to verify that the condition (F4) implies $b-$Cauchyness of the sequence $\{r_n\}_{n\in\mathbb{N}}$. In other words, this condition is quite strong, but fortunately it can be avoided. We will not use it in our approach. It is therefore superfluous in the whole paper [19].*

The authors in [19] introduce and prove the following:

Let a b-metric space $(\aleph, \bigwedge, s \geq 1)$ be equipped with self-mappings $\hbar, \beth, \Im, \daleth : \aleph \to \aleph$, and υ_s be defined as in [19, Definition 2]. Then they define the next two sets of real numbers:

$$\gamma_{\hbar,\beth,\upsilon_s} = \Big\{(\upsilon, \varrho) \in \aleph \times \aleph : \upsilon_s\left(\Im(\upsilon), \daleth(\varrho)\right) \geq s^2 \text{ and } \bigwedge\left(\hbar(\upsilon), \beth(\varrho)\right) > 0\Big\} \tag{1.2}$$

and

$$\begin{aligned}\mathcal{M}_1(\upsilon, \varrho) = \max\Big\{&\bigwedge\left(\Im(\upsilon), \daleth(\varrho)\right), \bigwedge\left(\hbar(\upsilon), \Im(\upsilon)\right), \bigwedge\left(\beth(\varrho), \daleth(\varrho)\right),\\ &\frac{\bigwedge\left(\Im(\upsilon), \beth(\varrho)\right) + \bigwedge\left(\hbar(\upsilon), \daleth(\varrho)\right)}{2s}\Big\}.\end{aligned} \tag{1.3}$$

For more synthesis on the results based on $\digamma-$contractions, we refer the reader to the informative and notable articles [10, 11, 16, 17, 18, 19, 20, 21, 22, 24, 26, 29, 30, 31, 32, 33, 34].

Theorem 1.6. *Let $\aleph$ be a non-void set and υ_s as described in (1.2). Let the self-maps $\hbar, \beth, \Im, \daleth$ be υ_s-b-continuous on υ_s-complete b–metric space $(\aleph, \bigwedge, s \geq 1)$ such that $\hbar(\aleph) \subseteq \daleth(\aleph), \beth(\aleph) \subseteq \Im(\aleph)$. Assume that for every pair $(r_1, r_2) \in \gamma_{\hbar,\beth,\upsilon_s}$, there exist $\digamma \in \mathfrak{F}_{\digamma_s}$ and $\tau > 0$ with*

$$\tau + \digamma\left(s \cdot \bigwedge\left(\hbar(r_1), \beth(r_2)\right)\right) \leq \digamma\left(\mathcal{M}_1(r_1, r_2)\right). \tag{1.4}$$

Assume that the pairs $(\hbar, \Im), (\beth, \daleth)$ are υ_s-compatible and the pairs $(\hbar, \beth)$ and $(\beth, \hbar)$ are rectangular partially weakly υ_s-admissible with respect to $\daleth$ and $\Im$ respectively. Then the pairs $(\hbar, \Im), (g, \daleth)$ have the coincidence point (say) υ in $\aleph$. Moreover, if $\upsilon_s\left(\Im(\upsilon), \daleth(\upsilon)\right) \geq s^2$, then υ is a common fixed point of $\hbar, \beth, \Im, \daleth$.

To begin, we will utilize the following two findings to show that certain Picard sequences in b-metric spaces $(\aleph, \bigwedge, s \geq 1)$ are b-Cauchy. The proof is an exact replica of the equivalent result in [14] (see also [1]).

Lemma 1.7. *Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in b-metric space $(\aleph, \bigwedge, s \geq 1)$ such that*

$$\bigwedge(r_n, r_{n+1}) \leq \lambda \cdot \bigwedge(r_{n-1}, r_n) \tag{1.5}$$

for some $\lambda \in [0, \frac{1}{s})$ and for each $n \in \mathbb{N}$. Then $\{r_n\}_{n\in\mathbb{N}}$ is a b-Cauchy sequence.

Remark 1.8. *It is worth noting that the preceding Lemma holds for each $\lambda \in [0, 1)$ in the context of b-metric spaces. See [15] for additional information.*

Lemma 1.9. *Let $\{r_n\}_{n\in\mathbb{N}}$ be a Picard sequence in b-metric space $(\aleph, \bigwedge, s \geq 1)$ induced by a mapping $\urcorner : \aleph \to \aleph$ and let $r_0 \in \aleph$ be an initial point. If $\bigwedge(r_n, r_{n+1}) < \bigwedge(r_{n-1}, r_n)$ for all $n \in \mathbb{N}$ then $r_n \neq r_m$ whenever $n \neq m$.*

In the succeeding analysis, we make use of the following known lemma [3, 20, 23].

Lemma 1.10. *Suppose that $\{r_n\}_{n\in\mathbb{N}}$ belongs to a metric space $(\aleph, \bigwedge)$ and satisfies $\lim\limits_{n\to+\infty} \bigwedge(r_n, r_{n+1}) = 0$ is not a Cauchy sequence. Then, there exists $\varepsilon_1 > 0$ and sequences of positive integers $\{n_q\}$, $\{m_q\}$, $n_q > m_q > q$ such that each of the sequences,*

$$\bigwedge\left(r_{n_q}, r_{m_q}\right), \bigwedge\left(r_{n_q+1}, r_{m_q}\right), \bigwedge\left(r_{n_q}, r_{m_q-1}\right), \bigwedge\left(r_{n_q+1}, r_{m_q-1}\right), \bigwedge\left(r_{n_q+1}, r_{m_q+1}\right),$$

tends to ε_1^+ when $q \to +\infty$.

Remark 1.11. *Based on $\Gamma(a-) \leq \Gamma(a) \leq \Gamma(a+), a \in (0, +\infty),$ we conclude that $\lim\limits_{a\to b^-} \Gamma(a) = \Gamma(b-)$ and $\lim\limits_{a\to b^+} \Gamma(a) = \Gamma(b+)$. For particular details see [2] and [25].*

Likewise, if $\Gamma : (0, +\infty) \to (-\infty, +\infty)$ is a strictly increasing function, then either $\Gamma(0+) = \lim\limits_{a\to 0^+} \Gamma(a) = m$, $m \in \mathbb{R}$ or $\Gamma(0+) = \lim\limits_{a\to 0^+} \Gamma(a) = -\infty$.

Remark 1.12. *Before giving the proof of Theorem 1.6, we note that some parts of the formulations of all theorems and their consequences are incorrect. For example, "for each $(r_1, r_2) \in \gamma_{\hbar,\daleth,\upsilon_s}$ there exist $\Gamma \in \mathfrak{F}_{\Gamma_s}$ and $\tau > 0$ such that ...". It is evident that it should be "there is $\Gamma \in \mathfrak{F}_{\Gamma_s}$ and $\tau > 0$ such that for all $(r_1, r_2) \in \gamma_{\hbar,\daleth,\upsilon_s} \ldots$".*

2 Some improved results

To prove Theorem 1.6, the authors in [19] used all the four properties *viz.* $(F1)$, $(F2)$, $(F3)$ and $(F4)$ of the mapping Γ. In sharp contrast to this practice, in present article we prove the Theorem 1.6 by omitting properties $(F2)$, $(F3)$, $(F4)$ and we make use of $(F1)$ only, *i.e.*, we only require the strict growth of the mapping $\Gamma : (0, +\infty) \to (-\infty. + \infty)$. Additionally, we will distinguish two cases: $s > 1$ and $s = 1$.

Proof. First let $s > 1$.

Since $\Gamma : (0, +\infty) \to (-\infty, +\infty)$ is strictly increasing (satisfies $(F1)$) then inequality (1.4) implies

$$\bigwedge(\hbar(r_1), \daleth(r_2)) < \frac{1}{s} \cdot \mathcal{M}_1(r_1, r_2), \tag{2.1}$$

where $\mathcal{M}_1(r_1, r_2)$ is as in 1.3 with $\upsilon = r_1, \varrho = r_2$.

Otherwise the contractive condition (2.1) is well known in the setting of b-metric spaces. The sequence $\{j_n\}$ defined in [19] on page 671 is obviously b-Cauchy according to Lemma 1.7 and Remark 1.5.

Now, let $s=1$ where $\mathit{\Gamma}$ satisfies only $(F1)$ seems more difficult to prove in Theorem 1.6 than the case with $s>1$. This is because for $s=1$ we do not have the condition $\bigwedge(j_n,j_{n+1})\le\lambda\cdot\bigwedge(j_{n-1},j_n)$ for the Picard sequence $\{j_n\}_{n\in\mathbb{N}}$ defined by

$$j_{2n+1}=\hbar(r_{2n})=\daleth(r_{2n+1})\ \text{ and }\ j_{2n+2}=\beth(r_{2n+1})=\Im(r_{2n+2}), \tag{2.2}$$

for $\lambda\in[0,1)$, where $(\aleph,\bigwedge)$ is given metric space. However, if $s=1$ (in this case $\bigwedge=d$ is a metric) we get that 1.4 implies

$$\tau+\mathit{\Gamma}\left(\bigwedge(j_{2n},j_{2n+1})\right)\le\mathit{\Gamma}\left(\bigwedge(j_{2n-1},j_{2n})\right)$$

$$\text{and }\tau+\mathit{\Gamma}\left(\bigwedge(j_{2n-1},j_{2n})\right)\le\mathit{\Gamma}\left(\bigwedge(j_{2n-2},j_{2n-1})\right), \tag{2.3}$$

for all $n\in\mathbb{N}$, hence follows $\bigwedge(j_n,j_{n+1})<\bigwedge(j_{n-1},j_n)$ for each $n\in\mathbb{N}$. So there exist a limit $\delta\ge0$ of the sequence $\{\bigwedge(j_n,j_{n+1})\}_{n\in\mathbb{N}}$.

If we suppose that this limit $\delta>0$, then according to the property of the strictly increasing function $\mathit{\Gamma}$ (see Remark 1.8), we get $\tau+\mathit{\Gamma}(\delta+)\le\mathit{\Gamma}(\delta+)$, which is a contradiction since $\delta>0$.

Assume to the contrary that $\{j_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence, according to the Lemma 1.10 and inequality (1.4) with $s=1,\bigwedge=d,r_1=r_{2n_q},r_2=r_{2m_q-1}$, we get

$$\tau+\mathit{\Gamma}\left(\bigwedge\left(j_{2n_q+1},j_{2m_q}\right)\right)\le\mathit{\Gamma}\left(\mathcal{M}_1\left(r_{2n_q},r_{2m_q}\right)\right), \tag{2.4}$$

where

$$\mathcal{M}_1\left(r_{2n_q},r_{2m_q}\right)=\max_{q\to+\infty}\left\{\bigwedge\left(j_{2n_q},j_{2m_q-1}\right),\bigwedge\left(j_{2n_q+1},j_{2n_q}\right),\bigwedge\left(j_{2m_q},j_{2m_q-1}\right),\right.$$

$$\left.\frac{\bigwedge\left(j_{2n_q},j_{2m_q}\right)+\bigwedge\left(j_{2n_q+1},j_{2m_q-1}\right)}{2}\right\}\to\max\left\{\varepsilon_1^+,0,0,\frac{\varepsilon_1^++\varepsilon_1^+}{2}\right\}=\varepsilon_1^+. \tag{2.5}$$

By taking the limit in (2.4) with $q\to+\infty$, we acquire

$$\tau+\mathit{\Gamma}\left(\varepsilon_1^++\right)\le\mathit{\Gamma}\left(\varepsilon_1^++\right), \tag{2.6}$$

which is a contradiction with $\tau>0$. Hence, the sequence $\{j_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Until the end of the proof of Theorem 1.6 the function $\mathit{\Gamma}$ will no longer be used.

The continuation of the proof for both cases $(s=1,s>1)$ is exactly the same as in [19]. Of course, the application of the function Γ on page 674 in [19] as well as the use of its continuity on the same page is superfluous. Moreover, its continuity is not assumed in the formulation of Theorem 1.6 Also, the uniqueness of the common fixed point for the mappings $\hbar, \beth, \daleth$, and $\Im$ follows directly from (2.1) in both cases ($s > 1, s = 1$) without any use of the function Γ. □

Remark 2.1. *For both cases $s > 1$ and $s = 1$ there are different proofs, that the defined sequence $\{j_n\}_{n\in\mathbb{N}}$ is Cauchy. In the second case, the property about the left and right limit of the strictly increasing function Γ is used. Further, one known lemma is used if the sequence in the metric space is not a Cauchy but $\bigwedge(j_n, j_{n+1})$ tends to zero as $n \to +\infty$. The authors in [19] gave one proof for both cases, but applied all four properties of the function Γ. Our approach has improved their method and has shown, as in some already published papers, that $(F1)$ is sufficient to prove a fixed point under many contractive conditions. For the case of two mappings in metric spaces, but with all three properties of Γ, the reader can see [34].*

Remark 2.2. *Theorems 2, 3, 4, 5, 6, 7 and 8 from [19] can be corrected in the same way as Theorem 1 from [19], that is, as Theorem 1.6 in this paper. Of course, only property $(F1)$ can be used in their proofs instead of all four properties in [19]. In their proofs, two cases $s > 1$ and $s = 1$ can be also distinguished.*

We now state a simple example that supports our main result.

Example 1. *Let $X = [0, +\infty)$, $d(x, y) = (x - y)^2$, $Tx = kx$, $k \in [0, \frac{1}{2})$, because obviously $s = 2$. Taking further that $\Gamma(r) = \ln r$, $\tau = 1$, we get that the contractive condition $\tau + \Gamma(s \cdot d(Tx, Ty)) \leq \Gamma(d(x, y))$ is fulfilled whenever $d(Tx, Ty) > 0$.*

3 Conclusions

In this article we have showed, in sharp contrast to published articles, that a reduced set of requirements suffices for the proof of fixed point results regarding generalized class of $\digamma$−contractions in b-metric spaces. By using only the property $(F1)$, instead of the four properties $(F1)$, $(F2)$, $(F3)$ and $(F4)$ used in [19], we were able to produce improved and condesed version of the proofs.

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Inertial algorithm for solving split inclusion problem in Banach spaces

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ABSTRACT

The purpose of this paper is to propose an algorithm for finding a common element of the set of fixed points of relatively nonexpansive mapping and the set of solutions of split inclusion problem with a way of selecting the stepsize without prior knowledge of the operator norm in the framework of Banach spaces. Then, the main result is used to the common fixed point problems of a family of relatively nonexpansive mappings and split equilibrium problem. Finally, a numerical example is provided to illustrate the main result.

RESUMEN

El propósito de este artículo es proponer un algoritmo para encontrar un elemento común del conjunto de puntos fijos de aplicaciones relativamente no-expansivas y el conjunto de soluciones de problemas de inclusión escindidos con una manera de seleccionar el tamaño del paso sin conocimiento previo de la norma del operador en el contexto de espacios de Banach. Luego, el resultado principal se usa para los problemas de punto fijo común de una familia de aplicaciones relativamente no expansivas y el problemas del equilibrio escindido. Finalmente, se entrega un ejemplo numérico para ilustrar el resultado principal.

Keywords and Phrases: Strong convergence, split feasibility problem, uniformly convex, uniformly smooth, fixed point problem.

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1 Introduction

Let H_1 and H_2 be two Hilbert spaces. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. Consider the following split inclusion problem (SIP) introduced by Moudafi [25] in Hilbert space:

$$\text{To find} \quad x^* \in H_1 \quad \text{such that} \quad 0 \in B_1(x^*) \quad \text{and} \quad 0 \in B_2\,(Ax^*). \tag{1.1}$$

Let the solution set of (1.1) be denoted by Ω. In fact, we know that the SIP is a generalization of the inclusion problem and the split feasibility problem (SFP). Next, we have some special cases of SIP (1.1). Let $f : H_1 \to \mathbb{R} \cup \{\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{\infty\}$ be proper, lower semicontinuous and convex functions. If we take $B_1 = \partial f$ and $B_2 = \partial g$, where ∂f and ∂g are the sub-differential of f and g, then the SIP (1.1) becomes the following proximal split feasibility problem:

$$\text{To find} \quad x^* \in \arg\min f \quad \text{such that} \quad Ax^* \in \arg\min g, \tag{1.2}$$

where $\arg\min f = \{x \in H1 : f(x) \leq f(y), \forall y \in H_1\}$ and $\arg\min g = \{x \in H_2 : g(x) \leq g(y), \forall y \in H_2\}$. In particular, if we take $f(x) = \frac{1}{2}\|M(x) - b\|^2$ and $g(x) = \frac{1}{2}\|N(x) - c\|^2$, where M and N are matrices, and $b, c \in H_1$, then the (1.2) becomes the least square problem. This problem has been intensively studied, especially, in Hilbert spaces; see for instance [26].

Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. If $B_1 = N_C, B_2 = N_Q$, where N_C and N_Q are the normal cones of C and Q, respectively, then we have the SFP:

$$\text{To find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{1.3}$$

This problem was first introduced, in a finite dimensional Hilbert space, by Censor and Elfving [13] for modeling inverse problems in radiation therapy treatment, which arise from phase retrieval and in medical image reconstruction, especially intensity modulated therapy [12]. To solve the SIP (1.1) Byrne *et al.* [11] proved some weak convergence results in infinite dimensional Hilbert spaces and proposed the following algorithm for given $x_1 \in H_1$:

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad \forall n \geq 1, \tag{1.4}$$

where $\lambda > 0$, $\gamma \in \left(0, \dfrac{2}{\|A\|^2}\right)$ and $J_\lambda^{B_1}$, $J_\lambda^{B_2}$ are metric and resolvent operators of B_1 and B_2, respectively. In order to obtain strong convergence, Kazmi and Rizvi [19] proposed the following algorithm to solve SIP (1.1):

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n) \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, \forall\, n \geq 1, \end{cases}$$

where $\gamma \in \left(0, \dfrac{2}{\|A\|^2}\right)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. However, in order to achieve the solution, one has to obtain the operator norm $\|A\|$, which is not easy to calculate in general. To avoid this computation, López *et al.* [23] find a new way to select the stepsize as follows:

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1,$$

where P_Q is the metric projection of H_2 onto Q, $\rho_n \in (0,4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$. This method is a modification of the CQ method and is often called the self-adaptive method, which permits step-size being selected self adaptively, for more details see [30, 37].

To solve SIP (1.1) in p-uniformly convex and smooth Banach space, Bello Cruz *et al.* [9] proposed the following algorithm, for given $x_1 \in E_1$ and $\{\alpha_n\} \in (0,1)$:

$$\begin{cases} u_n = J_{E_1^*}^q \left[J_{E_1}^p(x_n) - t_n A^* J_{E_2}(I - J_\lambda^{B_2})Ax_n\right] \\ x_{n+1} = J_{E_1^*}^q \left[\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(J_\lambda^{B_1}(u_n))\right]. \end{cases} \tag{1.5}$$

Very recently, Cholamjiak *et al.* [14] proposed algorithm for finding common solution of fixed point problem of relatively nonexpansive mapping to solve SIP (1.1) in p-uniformly convex and smooth Banach space. An initial guess $u_1 \in E_1$, let $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by:

$$\begin{cases} x_n = J_{\lambda_1}^{B_1}\left(J_{E_1^*}^q(J_{E_1}^p(u_n) - \lambda_n A^* J_{E_2}^p(I - J_{\lambda_2}^{B_2})Au_n)\right) \\ u_{n+1} = J_{E_1^*}^q(\alpha_n J_{E_1}^p(\epsilon_n) + \beta_n J_{E_1}^p(x_n) + \gamma_n J_{E_1}^p(Tx_n)), \qquad n \geq 1, \end{cases} \tag{1.6}$$

where $J_{\lambda_1}^{B_1}$, $J_{\lambda_2}^{B_2}$ are metric and resolvent operators. The sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For more SIP related articles (see, [3, 6, 16, 28, 34, 36, 38, 42]).

In nonlinear analysis, to work with an algorithm that has a high rate of convergence is more useful, through adding inertial term in the algorithm. First it was proposed by Polyak [31]. The main purpose of this method is to make use of previous iterates to update the next iterate. Recently, many authors have shown interest to study inertial type algorithms, see [2, 4, 17, 37, 40, 41].

Intention of this paper is to propose an algorithm to solve SIP (1.1) and fixed point of relatively nonexpansive mapping in p-uniformly convex and uniformly smooth real Banach spaces, without prior knowledge of operator norm, so that it can be more efficiently implemented. As an application, we apply our result to the common fixed point problems of a family of relatively nonexpansive mappings and split equilibrium problem. A numerical example is given to illustrate the efficiency of our algorithm, also our results complement and extend many recent and important results in this direction.

2 Preliminaries

Let E be a real Banach space with dual E^* and let $A : E_1 \to E_2$ be a bounded linear operator and A^* is adjoint of A. The modulus of convexity $\delta_E : [0,2] \to [0,1]$ is defined as

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|, \|x-y\| \geq \varepsilon\right\}.$$

E is called uniformly convex if $\delta_E(\varepsilon) > 0$, for $\varepsilon \in (0,2]$ and p-uniformly convex if there exist a $C_p > 0$ such that $\delta_E(\varepsilon) \geq C_P \varepsilon^p$, for any $\varepsilon \in (0,2]$. The modulus of smoothness $\rho_E : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \sup\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

E is called uniformly smooth if $\lim\limits_{\tau\to 0} \dfrac{\rho_E(\tau)}{\tau} = 0$, $q-$uniformly smooth if there exist $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$, for any $\tau > 0$. The duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\| = \|x\|^{p-1}\}.$$

The duality mapping J_E^p is one-to-one and single-valued (see [5, 15]).

The metric projection for a nonempty, closed and convex subset C of Banach space E is given by

$$P_C x = \arg\min_{y\in C} \|x-y\|, \quad x \in E.$$

For a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the Bregman distance with respect to f is defined as

$$\Delta_{f(x,y)} = f(y) - f(x) - \langle f'(x), y-x\rangle, \quad x, y \in E.$$

Since the duality mapping J_E^p is the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$. Then the Bregman distance with respect to f_p is,

$$\Delta_p(x,y) = \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p = \frac{1}{q}(\|y\|^p - \|x\|^p) - \langle J_E^p x - J_E^p y, x\rangle. \tag{2.1}$$

We define the Bregman projection as the unique minimizer of the Bregman distance,

$$\Pi_C x = \arg\min_{y\in C} \Delta_p(x,y), \quad x \in E.$$

It can also be characterized by a variational inequality,

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0, \quad \forall z \in C, \tag{2.2}$$

also,

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x,z) - \Delta_p(x, \Pi_C x), \quad \forall z \in C. \tag{2.3}$$

In real Hilbert space $\Pi_C = P_C$, for more detail, see [1, 18]. The function $V_p : E^* \times E \to [0, +\infty)$ with f_p is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p, \quad \forall x \in E,\, \bar{x} \in E^*.$$

Then $V_p \geq 0$ and also satisfy following property:

$$V_p(\bar{x}, x) = \Delta_p(J_E^q(\bar{x}), x), \quad \forall x \in E,\, \bar{x} \in E^*. \tag{2.4}$$

Moreover,

$$V_p(\bar{x}, x) + \langle \bar{y}, J_E^q(\bar{x}) - x\rangle \leq V_p(\bar{x} + \bar{y}, x), \tag{2.5}$$

$\forall x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see [29]). Also, V_p is convex in the first variable. Thus, for all $z \in E$,

$$\Delta_p\left(J_E^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right), z\right) \leq \sum_{i=1}^N t_i \Delta_p(x_i, z), \tag{2.6}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\displaystyle\sum_{i=1}^N t_i = 1$, see [33].

Lemma 2.1 ([27])**.** *Let E be a p-uniformly convex and uniformly smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be bounded sequences in E, then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.2 ([43])**.** *Let $x, y \in E$. If E is q-uniformly smooth, then there is a $C_q > 0$ so that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

A point $x^* \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to x^* and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $\hat{F}(T)$ is the set of asymptotic fixed points. Similarly a point $x^* \in C$ is a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to x^* and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Set of strong asymptotic fixed points of

T is denoted by $\tilde{F}(T)$.

Definition 2.3 ([24]). *A mapping T from C to C is said to be,*

1. *Bregman relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and*

$$\Delta_p(x^*, Ty) \leq \Delta_p(x^*, y), \quad \forall y \in C, \, x^* \in F(T).$$

2. *Bregman weakly relatively nonexpansive if $\tilde{F}(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$ and*

$$\Delta_p(x^*, Ty) \leq \Delta_p(x^*, y), \quad \forall y \in C, \, x^* \in F(T).$$

For more details, see [32].

Definition 2.4 ([8]). *Let E be a p-uniformly convex and uniformly smooth Banach space and C a nonempty subset of E. A mapping $S : C \to E$ is said to be firmly nonexpansive-like if*

$$\langle J_E^p(x - Sx) - J_E^p(y - Sy), Sx - Sy \rangle \geq 0, \quad \forall x, y \in C. \tag{2.7}$$

If E is a Hilbert space, then S is firmly nonexpansive-like mapping if and only if it is firmly nonexpansive, *i.e.* $\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y \rangle$, $\forall x, y \in C$. We recall the following results:

Remark 2.5. *Let E be a p-uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Then the metric projection P_C is a firmly nonexpansive-like mapping.*

Lemma 2.6 ([8]). *Let E be a smooth Banach space, C be a closed and convex nonempty subset of E and $S : C \to E$ a firmly nonexpansive-like mapping then $F(S)$ is closed and convex and $\hat{F}(S) = F(S)$.*

Let $B : E \to 2^{E^*}$ be a mapping, the effective domain of B is denoted by $D(B)$, such that $D(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall x, y \in D(B), \quad u \in Bx \quad \text{and} \quad v \in By.$$

A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E.

For $\lambda_2 > 0$ and $x \in E_2$, consider the metric resolvent $M_{\lambda_2}^{B_2} : E_2 \to D(B_2)$ of B_2 defined by

$$M_{\lambda_2}^{B_2}(x) = (I + \lambda_2 (J_{E_2}^p)^{-1} B_2)^{-1}(x), \quad \forall x \in E_2.$$

Set of null points of B_2 is defined by ${B_2}^{-1}(0) = \{z \in E_2 : 0 \in Bz\}$. Since ${B_2}^{-1}(0)$ is closed and convex, then we have

$$0 \in J_{E_2}^{P}(M_{\lambda_2}^{B_2}(x) - x) + \lambda_2 B_2 M_{\lambda_2}^{B_2}(x).$$

Next, $F(M_{\lambda_2}^{B_2}) = B_2^{-1}(0)$, for $\lambda_2 > 0$, from [22] we also have,

$$\langle M_{\lambda_2}^{B_2}(x) - M_{\lambda_2}^{B_2}(y), J_{E_2}^{p}(x - M_{\lambda_2}^{B_2}(x)) - J_{E_2}^{p}(y - M_{\lambda_2}^{B_2}(y))\rangle \geq 0,$$

for all $x, y \in E_2$ and if $B_2^{-1}(0) \neq 0$, then

$$\langle J_{E_2}^{p}(x - M_{\lambda_2}^{B_2}(x)) - (M_{\lambda_2}^{B_2}(x) - z)\rangle \geq 0,$$

for all $x \in E_2$ and $z \in {B_2}^{-1}(0)$.

The monotonicity of B_2 implies that $M_{\lambda_2}^{B_2}$ is a firmly nonexpansive-like mapping.

Now, we can define a mapping $N_{\lambda_1}^{B_1} : E_1 \to D(B_1)$ called the relative resolvent of B_1 [20], for $\lambda_1 > 0$ as

$$N_{\lambda_1}^{B_1} = (J_{E_1}^{p} + \lambda_1 B_1)^{-1} J_{E_1}^{p}(x), \quad \forall x \in E_1.$$

Since $N_{\lambda_1}^{B_1}$ is relatively nonexpansive mapping and $F(N_{\lambda_1}^{B_1}) = {B_1}^{-1}(0)$ for $\lambda_1 > 0$.

Lemma 2.7 ([20])**.** *Let $B : E \to 2^{E^*}$ be a maximal monotone operator with $B^{-1} \neq \emptyset$ and let N_{λ}^{B} be a resolvent operator of B for $\lambda > 0$. Then*

$$\Delta_p(N_{\lambda}^{B}(x), z) + \Delta_p(N_{\lambda}^{B}(x), x) \leq \Delta_p(x, z) \quad \textit{for all} \quad x \in E \quad \textit{and} \quad z \in B^{-1}(0).$$

Lemma 2.8 ([35])**.** *Let E_1, E_2 be two p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. Let $N_{\lambda_1}^{B_1}$ be the resolvent operator associated with maximal monotone operator B_1 for $\lambda_1 > 0$ and $M_{\lambda_2}^{B_2}$ be a metric resolvent operator of maximal monotone operator B_2 for $\lambda_2 > 0$. Assume $\Omega \neq \emptyset$, $\lambda > 0$ and $x^* \in E_1$. Then x^* is a solution of problem* (1.1) *if and only if*

$$x^* = N_{\lambda_1}^{B_1}(J_{E_1^*}^{q}(J_{E_1}^{p}(x^*) - \lambda A^* J_{E_2}^{p}(I - M_{\lambda_2}^{B_2})Ax^*)).$$

3 Main results

We assume the following assumptions for the rest of the paper, let E_1, E_2 be two p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. Let $C = C_1$ be nonempty closed and convex subset of E_1. Let $B_1 : E_1 \to 2^{{E_1}^*}$ and $B_2 : E_2 \to 2^{{E_2}^*}$ be maximal monotone operators such that $B_1^{-1}(0) \neq 0$, $B_2^{-1}(0) \neq 0$. Let $N_{\lambda_1}^{B_1}$ be the resolvent operator of B_1 for $\lambda_1 > 0$ and $M_{\lambda_2}^{B_2}$ is the metric resolvent operator of B_2 for $\lambda_2 > 0$. Let $T : E_1 \to E_1$ be a Bregman relatively nonexpansive mapping. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$ and $\{\alpha_n\} \in (0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, $\theta_n \in (-\infty, +\infty)$ and assuming $\Omega \cap F(T) \ne \emptyset$.

Algorithm 3.1. *Select $x_0, x_1 \in E_1$ and assuming that the sequence x_n is generated via the formula*

$$\begin{cases} v_n = J^q_{E_1^*}[J^p_{E_1} x_n + \theta_n (J^p_{E_1} x_n - J^p_{E_1} x_{n-1})] \\ z_n = N^{B_1}_{\lambda_1}[J^q_{E_1^*}(J^p_{E_1}(v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n))] \\ y_n = J^q_{E_1^*}[\alpha_n J^p_{E_1}(z_n) + (1-\alpha_n) J^p_{E_1} T(z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \le \Delta_p(v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \ge 1, \end{cases} \tag{3.1}$$

where $f(v_n) = \frac{1}{p}\|(I - M^{B_2}_{\lambda_2})Av_n\|^p$, $g(v_n) = A^* J^p_{E_2}(I - M^{B_2}_{\lambda_2})Av_n$ and $\{\rho_n\} \in (0,\infty)$ satisfy $\liminf_{n\to\infty} \rho_n(pq - C_q \rho_n^{q-1}) > 0$. Suppose that the set $\Psi = \{n \in \mathbb{N} : (I - M^{B_2}_{\lambda_2})Av_n \ne 0\}$, otherwise $z_n = v_n$.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* = \Pi_{\Omega\cap F(T)} x_0$.*

Proof. We divide the proof into four steps:

Step 1: To show $\Omega \cap F(T) \subseteq C_n$, for all $n \ge 1$ and Algorithm 3.1 is well defined. Let C_k is closed and convex for $k \ge 1$. Then

$$\begin{aligned} C_{k+1} &= \{u \in C_k : \Delta_p(y_n, u) \le \Delta_p v_n, u)\} \\ &= \left\{u \in C_k : \frac{\|u\|^p}{p} + \frac{\|y_k\|}{q} - \langle J^p_{E_1} y_k, u\rangle \le \frac{\|u\|^p}{p} + \frac{\|v_k\|}{q} - \langle J^p_{E_1} v_k, u\rangle \right\} \\ &= \left\{u \in C_k : \|y_k\|^p - \|v_k\|^p \le q\langle J^p_{E_1} y_k - J^p_{E_1} v_k, u\rangle \right\}, \end{aligned}$$

which implies C_{k+1} is closed. Let $u_1, u_2 \in C_{k+1}$ and $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 + \lambda_2 = 1$. Then

$$\|y_k\|^p - \|v_k\|^p \le q\langle J^p_{E_1} y_k - J^p_{E_1} v_k, u_1\rangle \quad \text{and} \quad \|y_k\|^p - \|v_k\|^p \le q\langle J^p_{E_1} y_k - J^p_{E_1} v_k, u_2\rangle.$$

Combining these two, we get

$$\|y_k\|^p - \|v_k\|^p \le \langle J^p_{E_1} y_k - J^p_{E_1} v_k, \lambda_1 u_1 + \lambda_2 u_2\rangle.$$

By convexity $\lambda_1 u_1 + \lambda_2 u_2 \in C_k$. Therefore, $\lambda_1 u_1 + \lambda_2 u_2 \in C_{k+1}$ and C_{k+1} is convex. Thus C_n is convex, $\forall n \ge 1$. Let $x^* \in \Omega \cap F(T)$, then

$$\begin{aligned}\Delta_p(y_n, x^*) &= \Delta_p((1-\alpha_n)J_{E_1}^p z_n + \alpha_n J_{E_1}^p T(z_n), x^*)\\ &\leq (1-\alpha_n)\Delta_p(z_n, x^*) + \alpha_n \Delta_p(T(z_n), x^*) \leq \Delta_p(z_n, x^*).\end{aligned} \tag{3.2}$$

Set $w_n := J_{E_1^*}^q(J_{E_1}^p(v_n) - \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n))$, for all $n \geq 1$. From Lemma 2.2 and (2.1), we have

$$\begin{aligned}\Delta_p(z_n, x^*) &\leq \Delta_p(w_n, x^*)\\ &= \Delta_p\left(J_{E_1^*}^q\left[J_{E_1}^p(v_n) - \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n)\right], x^*\right)\\ &= \frac{1}{p}\|x^*\|^p + \frac{1}{q}\left\|J_{E_1}^p(v_n) - \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n)\right\|^q - \langle J_{E_1}^p(v_n), x^*\rangle\\ &+ \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^*, g(v_n)\rangle\\ &\leq \frac{1}{p}\|x^*\|^p + \frac{1}{q}\|v_n\|^p - \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle v_n, g(v_n)\rangle + \frac{C_q}{q}\rho_n^q \frac{f^p(v_n)}{\|g(v_n)\|^p}\\ &- \langle x^*, J_{E_1}^p v_n\rangle + \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^*, g(v_n)\rangle\\ &= \frac{1}{p}\|x^*\|^p + \frac{1}{q}\|v_n\|^p - \langle x^*, J_{E_1}^p v_n\rangle + \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^* - v_n, g(v_n)\rangle\\ &+ \frac{C_q}{q}\rho_n^q \frac{f^p(v_n)}{\|g(v_n)\|^p}\\ &= \Delta_p(v_n, x^*) + \rho_n \frac{f^{p-1}(v_n)}{\|g(v_n)\|^p}\langle x^* - v_n, g(v_n)\rangle + \frac{C_q}{q}\rho_n^q \frac{f^p(v_n)}{\|g(v_n)\|^p}.\end{aligned} \tag{3.3}$$

Since $g(v_n) = A^* J_{E_2}^p(I - M_{\lambda_2}^{B_2})Av_n$ and $\langle J_{E_2}^p(I - M_{\lambda_2}^{B_2})Av_n, M_{\lambda_2}^{B_2}Av_n - Ax^*\rangle \geq 0$, then

$$\begin{aligned}\langle g(v_n), x^* - v_n\rangle &= \langle A^* J_{E_2}^p(I - M_{\lambda_2}^{B_2})Av_n, x^* - v_n\rangle = \langle J_{E_2}^p(I - M_{\lambda_2}^{B_2})Av_n, Ax^* - Av_n\rangle\\ &= \langle J_{E_2}^p(I - M_{\lambda_2}^{B_2})Av_n, M_{\lambda_2}^{B_2}Av_n - Av_n\rangle\\ &+ \langle J_{E_2}^p(I - M_{\lambda_2}^{B_2})Av_n, Ax^* - M_{\lambda_2}^{B_2}Av_n\rangle\\ &\leq -\|Av_n - M_{\lambda_2}^{B_2}Av_n\|^p = -pf(v_n).\end{aligned} \tag{3.4}$$

Using (3.3) and (3.4),

$$\begin{aligned}\Delta_p(z_n, x^*) &\leq \Delta_p(v_n, x^*) - \rho_n p \frac{f^p(v_n)}{\|g(v_n)\|^p} + \frac{C_q}{q}\rho_n^q \frac{f^p(v_n)}{\|g(v_n)\|^p}\\ &= \Delta_p(v_n, x^*) - \left(\rho_n p - \frac{C_q}{q}\rho_n^q\right)\frac{f^p(v_n)}{\|g(v_n)\|^p}.\end{aligned} \tag{3.5}$$

Since $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$,

$$\Delta_p(z_n, x^*) \leq \Delta_p(v_n, x^*), \quad n \geq 1. \tag{3.6}$$

Step 2: We prove that $\{x_n\}$ is a Cauchy sequence.

Since, $\{\Delta_p(x_n, x_0)\}$ is nondecreasing and bounded. So, the limit $\lim_{n\to\infty} \Delta_p(x_n, x_0)$ exists and from (2.3) we have,

$$\begin{aligned}\Delta_p(x_{n+1}, x_n) = \Delta_p(x_{n+1}, \Pi_{C_n}x_0) &\leq \Delta_p(x_{n+1}, x_0) - \Delta_p(\Pi_{C_n}x_0, x_0)\\ &= \Delta_p(x_{n+1}, x_0) - \Delta_p(x_n, x_0),\end{aligned} \tag{3.7}$$

which implies that,

$$\lim_{n\to\infty} \Delta_p(x_{n+1}, x_n) = 0. \tag{3.8}$$

So, it follows from Lemma 2.1 that,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since $x_n = \Pi_{C_n}x_0 \subseteq C_m$ and from Lemma 2.1, for some positive integers m, n with $m \leq n$, we have

$$\begin{aligned}\Delta_p(x_m, x_n) = \Delta_p(x_m, \Pi_{C_n}x_0) &\leq \Delta_p(x_m, x_0) - \Delta_p(\Pi_{C_n}x_0, x_0)\\ &\leq \Delta_p(x_m, x_0) - \Delta_p(x_n, x_0).\end{aligned} \tag{3.10}$$

Since $\lim_{n\to\infty} \Delta_p(x_n, x_0)$ exists, it follows from (3.10) that $\lim_{n\to\infty} \|x_n - x_m\| = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence.

Step 3: We prove that $\lim_{n\to\infty} \|Tz_n - z_n\| = 0$, $\lim_{n\to\infty} \|(I - M_{\lambda_2}^{B_2})Ax_n\| = 0$ and $\lim_{n\to\infty} \|N_{\lambda_1}^{B_1}v_n - v_n\| = 0$. Since $v_n = J_{E_1^*}^q[J_{E_1}^p x_n + \theta_n(J_{E_1}^p x_n - J_{E_1}^p x_{n-1})]$. Then it follows that,

$$J_{E_1}^p v_n - J_{E_1}^p x_n = \theta_n(J_{E_1}^p x_n - J_{E_1}^p x_{n-1}).$$

By the uniform continuity of $J_{E_1}^p$ and from (3.9), we have

$$\|J_{E_1}^p v_n - J_{E_1}^p x_n\| = \|\theta_n(J_{E_1}^p x_n - J_{E_1}^p x_{n-1})\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, from the definition of C_{n+1}, we have

$$\Delta_p(x_{n+1}, z_n) \leq \Delta_p(x_{n+1}, v_n), \tag{3.12}$$

and

$$\Delta_p(x_{n+1}, y_n) \leq \Delta_p(x_{n+1}, v_n). \tag{3.13}$$

Hence, it follows from (3.12) and (3.13) that $\lim_{n\to\infty} \Delta_p(x_{n+1}, z_n) = 0$ and $\lim_{n\to\infty} \Delta_p(x_{n+1}, y_n) = 0$. By Lemma (2.1), we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \tag{3.14}$$

and so,

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.15}$$

From (3.5), we obtain

$$\begin{aligned} \left(\rho_n p - \frac{C_q}{q}\rho_n^q\right) \frac{f^p(v_n)}{\|g(v_n)\|^p} &\leq \Delta_p(v_n, x^*) - \Delta_p(z_n, x^*) \\ &= \langle J_{E_1}^p z_n - J_{E_1}^p v_n, x^* - v_n\rangle - \Delta_p(z_n, v_n) \\ &\leq \langle J_{E_1}^p z_n - J_{E_1}^p v_n, x^* - v_n\rangle \\ &\leq \|x^* - v_n\| \|J_{E_1}^p z_n - J_{E_1}^p v_n\|. \end{aligned} \tag{3.16}$$

Since E_1 is a p-uniformly convex and p-uniformly smooth real Banach space, thus $J_{E_1}^p$ is uniformly norm-to-norm continuous. By $\lim_{n\to\infty} \|v_n - z_n\| = 0$, we obtain $\|J_{E_1}^p z_n - J_{E_1}^p v_n\| \to 0$. From (3.16) and the fact that $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$, we have

$$\frac{f^p(v_n)}{\|g(v_n)\|^p} \to 0 \quad \text{as} \quad n \to \infty,$$

implies,

$$\lim_{n\to\infty} \|(I - M_{\lambda_2}^{B_2})Av_n\| = 0. \tag{3.17}$$

Also

$$\|A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Av_n\| \leq \|A\| \|(I - M_{\lambda_2}^{B_2})Av_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Thus

$$\|A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Av_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Again from (3.1), we get

$$\|J_{E_1}^p Tz_n - J_{E_1}^p z_n\| = \frac{1}{1-\alpha_n} \|J_{E_1}^p y_n - J_{E_1}^p z_n\|. \tag{3.18}$$

It follows from (3.15) that

$$\lim_{n\to\infty} \|J_{E_1}^p Tz_n - J_{E_1}^p z_n\| = 0, \tag{3.19}$$

which also implies that

$$\lim_{n\to\infty} \|Tz_n - z_n\| = 0. \tag{3.20}$$

By Lemma (2.7) and (3.16), we have

$$\begin{aligned}\Delta_p(z_n, w_n) &= \Delta_p(N_{\lambda_1}^{B_1} w_n, w_n) \leq \Delta_p(w_n, x^*) - \Delta_p(z_n, x^*)\\ &\leq \Delta_p(v_n, x^*) - \Delta_p(z_n, x^*) \ \to\ 0 \quad \text{as} \quad n \to\ \infty.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \|N_{\lambda_1}^{B_1} w_n - w_n\| = \lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.21}$$

Step 4: We show that $\{x_n\}$ converges strongly to an element $x^* = \Pi_{\Omega\cap F(T)} x_0$. Since $\{x_n\}$ is a Cauchy sequence, there exists $x^* \in E_1$ such that $\{x_n\}$ converges strongly to x^*. Since $z_n \to x^* \in E_1$, we also have $v_n \to x^* \in E_1$. From (3.21), we get $x^* \in F(N_{\lambda_1}^{B_1}) \in B_1^{-1}(0)$. From (3.20), $\lim_{n\to\infty} \|Tz_n - z_n\| = 0$ and the closeness of T that $x^* = Tx^*$ that is, $x^* \in F(T)$. Since A is a bounded linear operator, we have that $\lim_{n\to\infty} \|Ax_n - Ax^*\| = 0$. By (3.17) we get $\lim_{n\to\infty} \|(I - M_{\lambda_2}^{B_2})Ax_n\| = 0$, this implies that $Ax^* \in \hat{F}(M_{\lambda_2}^{B_2})$ and by Lemma 2.6 we have $Ax^* \in F(M_{\lambda_2}^{B_2})$. This means that $x^* \in \Omega \cap F(T)$.

Let $p \in \Omega \cap F(T) \subseteq C_n$ such that $p = \Pi_{\Omega\cap F(T)} x_0$ and by definition $x_n = \Pi_{C_n} x_0$, we have

$$\Delta_p(x_n, x_0) = \Delta_p(p, x_0). \tag{3.22}$$

This implies that

$$\Delta_p(x^*, x_0) \leq \lim_{n\to\infty} \Delta_p(x_n, x_0) \leq \Delta_p(p, x_0), \tag{3.23}$$

hence $x^* = p$. Therefore, $\{x_n\}$ converges strongly to $x^* \in \Omega \cap F(T)$, where $x^* = \Pi_{\Omega\cap F(T)} x_0$. This completes the proof. $\square$

We next present some consequences of our main results. Firstly, if $\theta_n = 0$, we obtain the following non-inertial shrinking projection result.

Corollary 3.2. *Let $\Omega \cap F(T) \neq \emptyset$. Select $x_0, x_1 \in E_1$ and the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = x_n \\ z_n = N_{\lambda_1}^{B_1}\left(J_{E_1^*}^q\left(J_{E_1}^p(v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n)\right)\right) \\ y_n = J_{E_1^*}^q[\alpha_n J_{E_1}^p(z_n) + (1-\alpha_n) J_{E_1}^p T(z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1. \end{cases} \tag{3.24}$$

where $f(v_n) = \frac{1}{p}\|(I - M_{\lambda_2}^{B_2})Av_n\|^p$, $g(v_n) = A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Av_n$ *and* $\{\rho_n\} \in (0,\infty)$ *satisfy* $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$. *Suppose that the set* $\Psi = \{n \in \mathbb{N} : (I - M_{\lambda_2}^{B_2})Av_n \neq 0\}$, *otherwise* $z_n = v_n$. *Then the sequence* $\{x_n\}$ *converges strongly to* $x^* = \Pi_{\Omega\cap F(T)}x_0$.

Also, by letting $M_{\lambda_2}^{B_2}$ be the metric projection mapping onto a closed convex subset Q of E_2 in Algorithm (3.1), *i.e.* $M_{\lambda_2}^{B_2} = P_Q$ and $N_{\lambda_1}^{B_1} = I$, we obtain the following result as a solution to split feasibility and fixed point problems.

Corollary 3.3. *With reference to the data in Algorithm* (3.1), *let* Q *be a nonempty closed convex subset of* E_2 *and* $M_{\lambda_2}^{B_2} = P_Q$. *Assuming* $\Gamma := \{x \in C : x \in F(T), Ax \in Q\} \neq \emptyset$. *Then the sequence* x_n *generated by Algorithm* (3.1) *converges strongly to* $u \in \Gamma$, *where* $u = \Pi_\Gamma x_0$.

4 A countable family of relatively nonexpansive mappings

In this section, we apply our result to the common fixed point problems of a family of relatively nonexpansive mappings and equilibrium problem.

Definition 4.1 ([7]). *Let* C *be a subset of a real* p*-uniformly convex and uniformly smooth Banach space* E. *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of mappings of* C *in to* E *such that* $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. *Then* $\{T_n\}_{n=1}^\infty$ *is said to satisfy the* $AKTT$*-condition if, for any bounded subset* B *of* C,

$$\sum_{n=1}^{\infty} \sup_{z\in B}\{\|J_p^E(T_{n+1}z) - J_p^E(T_n z)\|\} < \infty.$$

As in [36], we prove the following Proposition:

Proposition 4.2. *Let* C *be a nonempty, closed and convex subset of a real* p*-uniformly convex and uniformly smooth Banach space* E. *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of mappings of* C *such that* $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ *and* $\{T_n\}_{n=1}^\infty$ *satisfies the* $AKTT$*-condition. Then for any bounded subset* B *of* C *there exists a mapping* $T : B \to E$ *such that*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in B, \tag{4.1}$$

and

$$\lim_{n\to\infty} \sup_{z\in B} \|J_p^E(Tz) - J_p^E(T_n z)\| = 0.$$

Proof. To complete the proof we show that $\{T_n x\}$ is Cauchy sequence for each $x \in C$. Let $\epsilon > 0$ be given and by the $AKKT$-condition $\exists l_0 \in \mathbb{N}$ such that,

$$\sum_{l_0}^{\infty} \sup\{\|T_{n+1}y - T_n y\| : y \in C\} < \epsilon.$$

Let $k > l \geq l_0$, then

$$\begin{aligned}\|T_k x - T_l x\| &\leq \sup\{\|T_k y - T_l y\| : y \in C\} \\ &\leq \sup\{\|T_k y - T_{k-1} y\| : y \in C\} + \sup\{\|T_{k-1} y - T_l y\| : y \in C\} \\ &\quad \vdots \\ &\leq \sum_{l}^{k-1} \sup\{\|T_{n+1} y - T_n y\| : y \in C\} \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1} y - T_n y\| : y \in C\} < \epsilon.\end{aligned}$$

Therefore we have that $\{T_n x\}$ is Cauchy sequence. Moreover (3.4) implies that,

$$\|Tx - T_l x\| = \lim_{k\to\infty} \|T_k x - T_l x\| \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1} y - T_n y\| : y \in C\},$$

for all $x \in C$. So,

$$\sup \|Tx - T_l x\| \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1} y - T_n y\| : y \in C\},$$

therefore, we conclude that $\lim_{l_0\to\infty} \sup \|Tx - T_{l_0} x\| = 0$. $\square$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the $AKTT$-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition and T is defined by (4.1) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 4.3. *Let $\{T_n\}$ be a countable family of Bregman relatively nonexpansive mapping on E_1 such that $F(T_n) = \hat{F}(T_n)$ and assuming $\Omega_1 = \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \emptyset$. Select $x_0, x_1 \in E_1$ and the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = J_{E_1^*}^q [J_{E_1}^p x_n + \theta_n (J_{E_1}^p x_n - J_{E_1}^p x_{n-1})] \\ z_n = N_{\lambda_1}^{B_1} \left[J_{E_1^*}^q \left(J_{E_1}^p (v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n) \right) \right] \\ y_n = J_{E_1^*}^q [\alpha_n J_{E_1}^p (z_n) + (1-\alpha_n) J_{E_1}^p T_n (z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geq 1, \end{cases} \tag{4.2}$$

where $f(v_n) = \frac{1}{p}\|(I - M_{\lambda_2}^{B_2}) A v_n\|^p$, $g(v_n) = A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2}) A v_n$ and suppose that the set $\Psi = \{n \in \mathbb{N} : (I - M_{\lambda_2}^{B_2}) A v_n \neq 0\}$, otherwise $z_n = v_n$. Suppose that in addition $(\{T_n\}_{n=1}^{\infty}, T)$ satisfy AKTT-Condition and $F(T) = \hat{F}(T)$, then the sequence generated by $\{x_n\}$ converges strongly to $x^* \in \Omega_1$, where $x^* = \Pi_{\Omega_1} x_0$.

Proof. To this end, it suffices to show that $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. By following the method of proof in Theorem 3.1, we can show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0.$$

By Proposition 4.2, we see that

$$\begin{aligned}\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \|J_p^{E_1}(T_n x_n) - J_p^{E_1}(Tx_n)\| \\ &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_n x) - J_p^{E_1}(Tx)\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

Since $J_p^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

4.1 Equilibrium problem

Let E be a real Banach space and let E^* be the dual space of E. Let C be a closed convex subset of E. Let f be a bifunction from $C\times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find:

$$x^* \in C \text{ such that } f(x^*, y) \ge 0,\ \forall y \in C. \tag{4.3}$$

The set of solutions of (4.3) is denoted by $EP(f)$. For a given mapping $T : C \to E^*$, define $f(x,y) = \langle Tx, y-x\rangle$, for all $x, y \in C$. Then, $x^* \in EP(f)$ if and only if $\langle Tx^*, y - x^*\rangle \ge 0$, for all $y \in C$ *i.e.* is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (4.3).

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A_1) $f(x,x) = 0$ for all $x \in C$,

(A_2) f is monotone, *i.e.* $f(x,y) + f(y,x) \le 0$ for all $x, y \in C$,

(A_3) for all $x, y, z \in C$,

$$\limsup_{t\to 0} f(tz + (1-t)x, y) \le f(x,y),$$

(A_4) for all $x \in C, f(x,.)$ is convex and lower semicontinuous.

Lemma 4.4 ([10])**.** *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, J_E^P z - J_E^P x\rangle \geq 0 \quad \text{for all} \quad y \in C.$$

Lemma 4.5 ([39])**.** *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$, and let $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows*

$$T_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, J_E^P z - J_E^P x\rangle \geq 0 \quad \text{for all} \quad y \in C \right\},$$

for all $x \in E$. Then, the following hold:

1. *T_r^f is single-valued,*
2. *T_r^f is a firmly nonexpansive-type mapping [21], i.e., for all $x, y \in E$,*

$$\langle T_r^f x - T_r^f y, J_E^P T_r^f x - J_E^P T_r^f y\rangle \leq \langle T_r^f x - T_r^f y, J_E^P x - J_E^P y\rangle$$

3. *$F(T_r^f) = EP(f)$,*
4. *$EP(f)$ is closed and convex.*

We consider the following split equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{4.4}$$

and $y = Ax^* \in Q$ solves

$$f_2(y^*, y) \geq 0, \quad \forall y \in Q, \tag{4.5}$$

with the solution set $\Omega_2 = \{x^* \in EP(f_1) : Ax^* \in EP(f_2)\}$.

Theorem 4.6. *Let f_1, f_2 be bifunctions satisfying $(A_1) - (A_4)$ and assuming $\Omega_2 \cap F(T) \neq \emptyset$. Select $x_0, x_1 \in E_1$ and the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = J_{E_1^*}^q [J_{E_1}^p x_n + \theta_n (J_{E_1}^p x_n - J_{E_1}^p x_{n-1})] \\ z_n = T_r^{f_1} \left[J_{E_1^*}^q \left(J_{E_1}^p (v_n) - \rho_n \dfrac{f^{p-1}(v_n)}{\|g(v_n)\|^p} g(v_n) \right) \right] \\ y_n = J_{E_1^*}^q [\alpha_n J_{E_1}^p (z_n) + (1 - \alpha_n) J_{E_1}^p T(z_n)] \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(v_n, u)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{4.6}$$

where $f(v_n) = \frac{1}{p}\|(I - T_r^{f_2})Av_n\|^p$, $g(v_n) = A^* J_{E_2}^p (I - T_r^{f_2})Av_n$ *and* $\{\rho_n\} \in (0, \infty)$ *satisfy* $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$ *and suppose that the set* $\Psi = \{n \in \mathbb{N} : (I - T_r^{f_2})Av_n \neq 0\}$, *otherwise* $z_n = v_n$. *Then the sequence generated by* $\{x_n\}$ *converges strongly to* $x^* = \Pi_{\Omega_2 \cap F(T)} x_0$.

5 Numerical example

In this section, we present an example to show the behaviour of the Algorithm 3.1 presented in this paper and compare its performance with algorithm (1.6) of Cholamjiak *et al.* [14] and (1.5) of Bello Cruz *et al.* [9] by using MATLAB R2016(a). In numerical experiment, we will show that the sequence generated by Algorithm 3.1 via the self-adaptive technique converges faster than algorithms defined in (1.5) and (1.6) for different choices of the $\{\rho_n\}$ and initial values to see the convergence behaviour of Algorithm 3.1.

Example 1. *Let* $E_1 = E_2 = l_2(\mathbb{R})$, *where* $l_2(\mathbb{R}) := \left\{ r = (r_1, r_2, \ldots, r_i, \ldots), r_i \in \mathbb{R} : \sum_{i=1}^{\infty} |\, r_i \,|^2 < \infty \right\}$, $\|r_i\|_2 = \left(\sum_{i=1}^{\infty} |\, r_i \,|^2 \right)^{\frac{1}{2}}$, $\forall r \in E_1$. *Let* $C = C_1 := \{x \in E_1 : \|x\|_2 \leq 1\}$. *Let* $T : E_1 \to E_1$ *be defined by* $Tx = \dfrac{x}{2}$, $\forall x \in E_1$. *Let* $A : E_1 \to E_2$ *be a mapping defined by* $Ax = \dfrac{3x}{4}, \forall x \in E_1$. *Let* $\alpha_n = \dfrac{1}{2n}$ *and* $\theta_n = \dfrac{1+n}{5n}$ *and*

$$N_{\lambda_1}^{B_1} x = (1 + \lambda_1 B_1)^{-1} x = \frac{x}{1 + 3\lambda_1}, \quad \forall x \in E_1$$

and

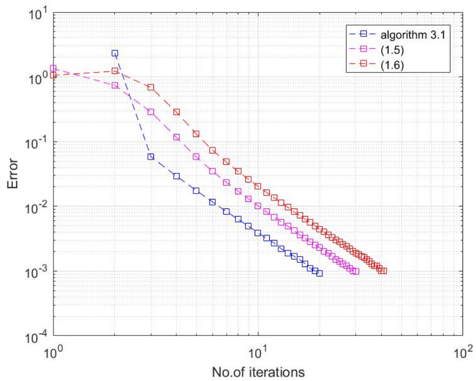
$$M_{\lambda_2}^{B_2} y = (1 + \lambda_2 B_2)^{-1} y = \frac{y}{1 + 5\lambda_2}, \quad \forall y \in E_2,$$

furthermore, it can be verified that for $\lambda_1, \lambda_2 \geq 0$.

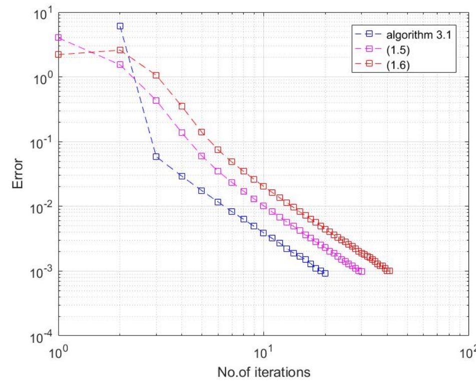
By choosing different ρ_n *and initial values with* $\lambda_1 = \lambda_2 = 1$ *for plotting the graphs of error* $= |x_{n+1} - x_n|$ *against number of iterations with stopping criteria* $|x_{n+1} - x_n| < 10^{-3}$ *for the following cases.*

1. $x_1 = x_0 = \left(2, 1, \frac{2}{3}, \ldots\right)$, $\rho_n = \dfrac{n}{n+1}$.
2. $x_1 = x_0 = \left(5, \frac{5}{2}, \frac{5}{3}, \ldots\right)$, $\rho_n = \dfrac{n}{n+1}$.
3. $x_1 = x_0 = \left(2, 1, \frac{2}{3}, \ldots\right)$, $\rho_n = \dfrac{3n}{n+1}$.
4. $x_1 = x_0 = \left(5, \frac{5}{2}, \frac{5}{3}, \ldots\right)$, $\rho_n = \dfrac{3n}{n+1}$.

Thus we see that sequences generated by our algorithm 3.1 converges to the solution set $\Omega \cap F(T)$. *The computational result can be found in Table 1 and Figure.1,2.*

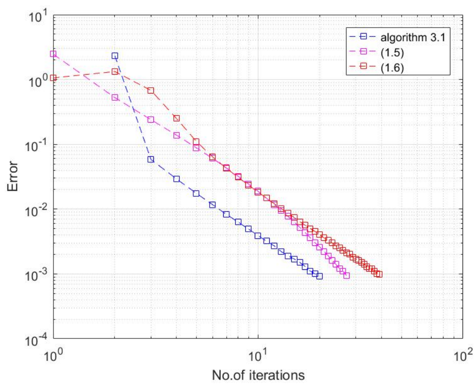


(a) Choice 1 in Example 1

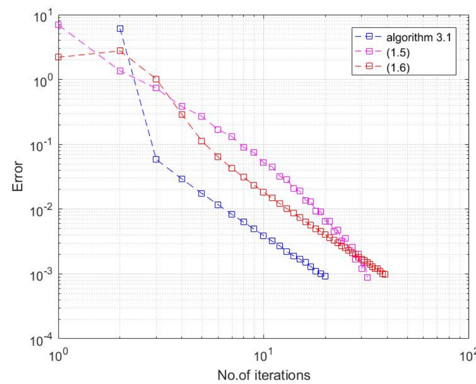


(b) Choice 2 in Example 1

Figure 1



(a) Choice 3 in Example 1



(b) Choice 4 in Example 1

Figure 2

Choice		Algorithm 3.1	(1.5)	(1.6)
1.	No. of Iteration	19	30	41
	CPU Time(s)	0.0313	0.0469	0.0564
2.	No. of Iteration	19	30	41
	CPU Time(s)	0.0524	0.0625	0.125
3.	No. of Iteration	18	27	39
	CPU Time(s)	0.1125	0.313	0.1250
4.	No. of Iteration	19	32	39
	CPU Time(s)	0.5938	0.0625	0.469

Table 1

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Cubic and quartic series with the tail of $\ln 2$

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ABSTRACT

In this paper we calculate some remarkable cubic and quartic series involving the tail of $\ln 2$. We also evaluate several linear and quadratic series with the tail of $\ln 2$.

RESUMEN

En este artículo calculamos algunas series cúbicas y cuárticas notables que involucran la cola de $\ln 2$. También evaluamos varias series lineales y cuadráticas con la cola de $\ln 2$.

Keywords and Phrases: Abel's summation formula, cubic series, quartic series, tail of $\ln 2$.

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1 Introduction and the main results

In this paper we calculate several remarkable cubic and quartic series involving the term $\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. The goal of this paper is to extend, to the case of cubic and quartic series, the results recorded in [3], in problems 3.15, 3.29 and 3.45, concerning the calculation of some quadratic series involving the tail of $\ln 2$. Our results are new in the literature and they are obtained based on a combination of techniques involving Abel's summation formula and shifting the index of summation, which allow us to reduce the calculation of a cubic or a quartic series to a linear or a quadratic series, respectively. We also solve an open problem posed in [5, Open problem, p. 107].

The main results of this paper are Theorems 1.1 and 1.2 below.

Theorem 1.1 (Remarkable cubic series with the tail of $\ln 2$)**.** *The following identities hold:*

(a) $$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\frac{5}{16}\zeta(3);$$

(b) $$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\frac{\zeta(2)}{4}-\frac{3}{2}\ln^2 2;$$

(c) $$\sum_{n=1}^{\infty}n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=-\frac{\zeta(2)}{4}-\frac{3}{4}\ln^2 2+\frac{3}{2}\ln 2+\frac{5}{32}\zeta(3).$$

We mention that the alternating version of the series in part (c) of Theorem 1.1

$$\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\frac{\ln 2}{4}-\frac{3\ln^2 2}{4}-\frac{\zeta(2)}{16}$$

was calculated in [4]. The results in parts (a) and (b) of Theorem 1.1 are due to C. I. Vălean, who communicated them to the first author, without proof, in an equivalent form in 2015.

Theorem 1.2 (Quartic series with the tail of $\ln 2$)**.** *The following identities hold:*

(a) $$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^4=2\ln^3 2+2\zeta(2)\ln 2-\frac{9}{4}\zeta(3);$$

(b) $$\sum_{n=1}^{\infty}n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^4=\ln^3 2-\frac{1}{2}\ln^2 2+\zeta(2)\ln 2-\frac{13}{16}\zeta(3).$$

We collect, in the next lemma, some results we need in proving Theorems 1.1 and 1.2.

Lemma 1.3 (A mosaic of linear and quadratic series with the tail of $\ln 2$)**.** *The following identities hold:*

Linear series

(a) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) = \frac{\ln^2 2}{2} - \frac{\zeta(2)}{2};$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) = \frac{\ln^2 2}{2} + \frac{\zeta(2)}{2};$$

(c) $$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{2n} - \frac{1}{2n+1} + \frac{1}{2n+2} - \cdots \right) = \ln^2 2;$$

(d) $$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) = \frac{13}{8}\zeta(3) - \zeta(2)\ln 2;$$

(e) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) = \frac{\ln 2}{2}\zeta(2) - \zeta(3);$$

Quadratic series

(f) $$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^2 = -\frac{\zeta(2)}{4};$$

(g) $$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^2 = \ln 2;$$

(h) $$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^2 = \frac{3}{2}\zeta(3) - \zeta(2)\ln 2 - \frac{\ln^3 2}{3};$$

(i) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^2 = \frac{\zeta(2)}{2}\ln 2 - \frac{3}{4}\zeta(3) - \frac{\ln^3 2}{3}.$$

Since

$$\ln 2 - \left[1 + \frac{(-1)^1}{2} + \frac{(-1)^2}{3} + \cdots + \frac{(-1)^{n-2}}{n-1} \right] = (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right),$$

we have that all the series in this paper involve the tail of $\ln 2$.

Before we prove the lemma, we observe that

$$\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots = \int_0^1 \left(x^{n-1} - x^n + x^{n+1} - \cdots \right) \mathrm{d}x = \int_0^1 \frac{x^{n-1}}{1+x}\, \mathrm{d}x. \tag{1.1}$$

This implies

$$\lim_{n\to\infty} n \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) = \frac{1}{2} \tag{1.2}$$

and it follows that

$$\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \sim \frac{1}{2n}.$$

This shows that the series in Theorems 1.1, 1.2 and Lemma 1.3 are all convergent.

We also need in our analysis Abel's summation formula, which states that: if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}),$$

or, the infinite version

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{1.3}$$

Now we are ready to prove Lemma 1.3.

2 Proof of Lemma 1.3

Proof. (a) We have, based on (1.1), that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 \frac{x^{n-1}}{1+x}\,\mathrm{d}x = \int_0^1 \frac{1}{x(1+x)} \left(\sum_{n=1}^{\infty} \frac{(-x)^n}{n} \right) \mathrm{d}x \\
&= -\int_0^1 \frac{\ln(1+x)}{x(1+x)}\,\mathrm{d}x = \int_0^1 \frac{\ln(1+x)}{1+x}\,\mathrm{d}x - \int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x \\
&= \frac{\ln^2 2}{2} - \int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x \\
&= \frac{\ln^2 2}{2} - \frac{\zeta(2)}{2}.
\end{aligned}$$

We used that $\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x = \frac{\zeta(2)}{2}$.

(b) We have, based on (1.1), that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \frac{x^{n-1}}{1+x}\,\mathrm{d}x = \int_0^1 \frac{1}{x(1+x)} \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \mathrm{d}x \\
&= -\int_0^1 \frac{\ln(1-x)}{x(1+x)}\,\mathrm{d}x = \int_0^1 \frac{\ln(1-x)}{1+x}\,\mathrm{d}x - \int_0^1 \frac{\ln(1-x)}{x}\,\mathrm{d}x \\
&= \frac{\ln^2 2}{2} + \frac{\zeta(2)}{2}.
\end{aligned}$$

We used that $\displaystyle\int_0^1 \frac{\ln(1-x)}{x}\,\mathrm{d}x = -\zeta(2)$ and $\displaystyle\int_0^1 \frac{\ln(1-x)}{1+x}\,\mathrm{d}x = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}$ (see [5, p. 203]).

(*c*) This nice result, which may be of independent interest, is obtained by adding the series in parts (*a*) and (*b*) of the lemma.

(*d*) We have, based on (1.1), that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right) &= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 \frac{x^{n-1}}{1+x}\,\mathrm{d}x \\ &= \int_0^1 \frac{1}{x(1+x)}\sum_{n=1}^{\infty}\frac{x^n}{n^2}\,\mathrm{d}x = \int_0^1 \frac{\mathrm{Li}_2(x)}{x(1+x)}\,\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2(x)}{1+x}\,\mathrm{d}x.\end{aligned} \tag{2.1}$$

We calculate the first integral in (2.1) and we have that

$$\int_0^1 \frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x = \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^3} = \zeta(3). \tag{2.2}$$

We calculate the second integral in (2.1). We integrate by parts, with $f(x) = \mathrm{Li}_2(x)$, $f'(x) = -\dfrac{\ln(1-x)}{x}$, $g'(x) = \dfrac{1}{1+x}$ and $g(x) = \ln(1+x)$, and we have that

$$\int_0^1 \frac{\mathrm{Li}_2(x)}{1+x}\,\mathrm{d}x = \ln(1+x)\mathrm{Li}_2(x)\Big|_0^1 + \int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\,\mathrm{d}x = \zeta(2)\ln 2 - \frac{5}{8}\zeta(3), \tag{2.3}$$

since $\displaystyle\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\,\mathrm{d}x = -\frac{5}{8}\zeta(3)$. For a proof of this result see [5, p. 328]. Combining (2.1), (2.2) and (2.3), the desired result holds and part (*d*) of the lemma is proved.

(*e*) We have, based on (1.1), that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 \frac{x^{n-1}}{1+x}\,\mathrm{d}x \\ &= \int_0^1 \frac{1}{x(1+x)}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\,\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{Li}_2(-x)}{x(1+x)}\,\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{Li}_2(-x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2(-x)}{1+x}\,\mathrm{d}x.\end{aligned} \tag{2.4}$$

We calculate the first integral in (2.4) and we have that

$$\int_0^1 \frac{\mathrm{Li}_2(-x)}{x}\,\mathrm{d}x = \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = -\frac{3}{4}\zeta(3). \tag{2.5}$$

We calculate the second integral in (2.4). We integrate by parts, with $f(x) = \mathrm{Li}_2(-x)$, $f'(x) = -\dfrac{\ln(1+x)}{x}$, $g'(x) = \dfrac{1}{1+x}$, $g(x) = \ln(1+x)$, and we have that

$$\int_0^1 \frac{\mathrm{Li}_2(-x)}{1+x}\,\mathrm{d}x = \ln(1+x)\mathrm{Li}_2(-x)\Big|_0^1 + \int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x = -\frac{\ln 2}{2}\zeta(2) + \frac{\zeta(3)}{4}, \tag{2.6}$$

since $\displaystyle\int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x = \frac{\zeta(3)}{4}$ (see [1, pp. 291–292]).

Combining (2.4), (2.5) and (2.6), the desired result holds and part (e) of the lemma is proved.

(f) We calculate the series by shifting the index of summation. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\
&= \ln^2 2 + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\
&\overset{n-1=m}{=} \ln^2 2 + \sum_{m=1}^{\infty}(-1)^{m}\left(\frac{1}{m+1}-\frac{1}{m+2}+\frac{1}{m+3}-\cdots\right)^2\\
&= \ln^2 2 - \sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{1}{m}-\left(\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+2}-\cdots\right)\right]^2\\
&= \ln^2 2 + \sum_{m=1}^{\infty}\frac{(-1)^m}{m^2} + 2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+2}-\cdots\right)\\
&- \sum_{m=1}^{\infty}(-1)^{m-1}\left(\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+2}-\cdots\right)^2
\end{aligned}$$

and it follows that

$$\begin{aligned}
&2\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\
&= \ln^2 2 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\\
&\overset{(a)}{=} \ln^2 2 - \frac{\zeta(2)}{2} + 2\left(\frac{\zeta(2)}{2}-\frac{\ln^2 2}{2}\right) = \frac{\zeta(2)}{2}.
\end{aligned}$$

We mention that this series was calculated by a different method in [3, problem 3.45].

(g) This result is proved, using an integration technique, in [3, problem 3.29]. Here we give another proof. We apply Abel's summation formula with $a_n = 1$ and $b_n = x_n^2$, where $x_n = \dfrac{1}{n} - \dfrac{1}{n+1} + \dfrac{1}{n+2} - \cdots$. Observe that $x_n + x_{n+1} = \dfrac{1}{n}$.

We have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2 = \lim_{n\to\infty} nx_{n+1}^2+\sum_{n=1}^{\infty} n\left(x_n^2-x_{n+1}^2\right)$$
$$= \sum_{n=1}^{\infty} n\left(x_n-x_{n+1}\right)\left(x_n+x_{n+1}\right)$$
$$= \sum_{n=1}^{\infty}\left(x_n-x_{n+1}\right)=x_1=\ln 2.$$

We used that $\lim_{n\to\infty} nx_{n+1}^2=0$, which follows based on (1.2).

(h) We need the following power series formula $\sum_{n=1}^{\infty}\frac{H_n}{n}x^n=\mathrm{Li}_2(x)+\frac{1}{2}\ln^2(1-x),\ x\in[-1,1)$. For a proof of this result see [5, p. 403].

We calculate the series by Abel's summation formula with $a_n=\frac{1}{n}$ and $b_n=x_n^2$, where $x_n=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$. Observe that $x_n+x_{n+1}=\frac{1}{n}$.

We have

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=\lim_{n\to\infty}H_nx_{n+1}^2+\sum_{n=1}^{\infty}H_n\left(x_n-x_{n+1}\right)\left(x_n+x_{n+1}\right)$$
$$=\sum_{n=1}^{\infty}\frac{H_n}{n}\left(2x_n-\frac{1}{n}\right)=2\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)-\sum_{n=1}^{\infty}\frac{H_n}{n^2}$$
$$\overset{(1.1)}{=}2\sum_{n=1}^{\infty}\frac{H_n}{n}\int_0^1\frac{x^{n-1}}{1+x}\,\mathrm{d}x-2\zeta(3)=2\int_0^1\frac{1}{x(1+x)}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}x^n\right)\mathrm{d}x-2\zeta(3)$$
$$=2\int_0^1\frac{1}{x(1+x)}\left(\mathrm{Li}_2(x)+\frac{1}{2}\ln^2(1-x)\right)\mathrm{d}x-2\zeta(3) \tag{2.7}$$
$$=2\int_0^1\left(\frac{1}{x}-\frac{1}{1+x}\right)\left(\mathrm{Li}_2(x)+\frac{1}{2}\ln^2(1-x)\right)\mathrm{d}x-2\zeta(3)$$
$$=2\int_0^1\frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x+\int_0^1\frac{\ln^2(1-x)}{x}\,\mathrm{d}x-2\int_0^1\frac{\mathrm{Li}_2(x)}{1+x}\,\mathrm{d}x-\int_0^1\frac{\ln^2(1-x)}{1+x}\,\mathrm{d}x-2\zeta(3).$$

We calculate

$$\int_0^1\frac{\ln^2(1-x)}{x}\,\mathrm{d}x=\int_0^1\frac{\ln^2 y}{1-y}\,\mathrm{d}y=\int_0^1\ln^2 y\left(\sum_{n=0}^{\infty}y^n\right)\mathrm{d}y$$
$$=\sum_{n=0}^{\infty}\int_0^1 y^n\ln^2 y\,\mathrm{d}y=\sum_{n=0}^{\infty}\frac{2}{(n+1)^3}=2\zeta(3). \tag{2.8}$$

We also have, see [5, p. 110], that

$$\int_0^1\frac{\ln^2(1-x)}{1+x}\,\mathrm{d}x=\frac{7}{4}\zeta(3)-\zeta(2)\ln 2+\frac{\ln^3 2}{3}. \tag{2.9}$$

It follows, based on (2.5), (2.6), (2.7), (2.8) and (2.9), that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=\frac{3}{2}\zeta(3)-\zeta(2)\ln 2-\frac{\ln^3 2}{3}.$$

We mention that the series in part (g) of Lemma 1.3 was calculated by a different method by Boyadzhiev in [2].

(i) This formula was proved by Boyadzhiev in [2, entry (19)]. □

Now we are ready to prove Theorem 1.1.

3 Proof of Theorem 1.1

Proof. (a) We calculate the series by shifting the index of summation. We have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3 \\
&= \left(1-\frac{1}{2}+\frac{1}{3}-\cdots\right)^3+\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3 \\
&\overset{n-1=i}{=} \ln^3 2+\sum_{i=1}^{\infty}\left(\frac{1}{i+1}-\frac{1}{i+2}+\frac{1}{i+3}-\cdots\right)^3 \\
&= \ln^3 2+\sum_{i=1}^{\infty}\left[\frac{1}{i}-\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\right]^3 \\
&= \ln^3 2+\sum_{i=1}^{\infty}\left[\frac{1}{i^3}-\frac{3}{i^2}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\right. \\
&\left. +\frac{3}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2-\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^3\right] \\
&= \ln^3 2+\zeta(3)-3\sum_{i=1}^{\infty}\frac{1}{i^2}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right) \\
&+3\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2-S.
\end{aligned}$$

It follows, based on parts (d) and (h) of Lemma 1.3, that

$$\begin{aligned}
2S &= \ln^3 2+\zeta(3)-3\sum_{i=1}^{\infty}\frac{1}{i^2}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)+3\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2 \\
&= \ln^3 2+\zeta(3)-3\left(\frac{13}{8}\zeta(3)-\zeta(2)\ln 2\right)+3\left(\frac{3}{2}\zeta(3)-\zeta(2)\ln 2-\frac{\ln^3 2}{3}\right)=\frac{5}{8}\zeta(3)
\end{aligned}$$

and part (a) of Theorem 1.1 is proved.

(b) We calculate the series using Abel's summation formula. We apply formula (1.3) with $a_n = 1$ and $b_n = (-1)^n x_n^3$, where $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^3 &= \lim_{n\to\infty}(-1)^{n+1} n x_{n+1}^3 + \sum_{n=1}^{\infty} n(-1)^n \left(x_n^3 + x_{n+1}^3\right)\\
&= \sum_{n=1}^{\infty}(-1)^n n \left(x_n + x_{n+1}\right)\left(x_n^2 - x_n x_{n+1} + x_{n+1}^2\right)\\
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=} \sum_{n=1}^{\infty}(-1)^n \left(x_n^2 - x_n x_{n+1} + x_{n+1}^2\right)\\
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=} \sum_{n=1}^{\infty}(-1)^n \left(3x_n^2 - \frac{3}{n}x_n + \frac{1}{n^2}\right)\\
&= 3\sum_{n=1}^{\infty}(-1)^n x_n^2 - 3\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x_n + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\\
&\overset{\text{Lemma 1.3 } (f),(a)}{=} 3\left(-\frac{\zeta(2)}{4}\right) - 3\left(\frac{\ln^2 2}{2} - \frac{\zeta(2)}{2}\right) - \frac{\zeta(2)}{2}\\
&= \frac{\zeta(2)}{4} - \frac{3}{2}\ln^2 2.
\end{aligned}$$

We used in the preceding calculations that $\lim_{n\to\infty} n x_{n+1}^3 = 0$, which follows from (1.2).

(c) Let $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. We calculate the series by shifting the index of summation. We have

$$\begin{aligned}
S = \sum_{n=1}^{\infty} n\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^3 &= \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots\right)^3 + \sum_{n=2}^{\infty} n x_n^3\\
&\overset{n-1=i}{=} \ln^3 2 + \sum_{i=1}^{\infty}(i+1)\left(\frac{1}{i+1} - \frac{1}{i+2} + \frac{1}{i+3} - \cdots\right)^3\\
&= \ln^3 2 + \sum_{i=1}^{\infty}(i+1)\left[\frac{1}{i} - \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots\right)\right]^3\\
&= \ln^3 2 + \sum_{i=1}^{\infty}(i+1)\left(\frac{1}{i^3} - \frac{3}{i^2}x_i + \frac{3}{i}x_i^2 - x_i^3\right)\\
&= \ln^3 2 + \sum_{i=1}^{\infty}\left(\frac{1}{i^2} + \frac{1}{i^3} - \frac{3}{i}x_i - \frac{3}{i^2}x_i + 3x_i^2 + \frac{3}{i}x_i^2 - i x_i^3 - x_i^3\right)\\
&= \ln^3 2 + \zeta(2) + \zeta(3) - 3\sum_{i=1}^{\infty}\frac{x_i}{i} - 3\sum_{i=1}^{\infty}\frac{x_i}{i^2} + 3\sum_{i=1}^{\infty}x_i^2 + 3\sum_{i=1}^{\infty}\frac{x_i^2}{i} - S - \sum_{i=1}^{\infty}x_i^3.
\end{aligned}$$

It follows, based on part (a) of Theorem 1.1 and parts (b), (d), (g) and (h) of Lemma 1.3, that

$$\begin{aligned}2S &= \ln^3 2 + \zeta(2) + \zeta(3) - 3\left(\frac{\ln^2 2}{2} + \frac{\zeta(2)}{2}\right) - 3\left(\frac{13}{8}\zeta(3) - \zeta(2)\ln 2\right)\\ &\quad + 3\ln 2 + 3\left(\frac{3}{2}\zeta(3) - \zeta(2)\ln 2 - \frac{\ln^3 2}{3}\right) - \frac{5}{16}\zeta(3)\\ &= -\frac{\zeta(2)}{2} - \frac{3}{2}\ln^2 2 + 3\ln 2 + \frac{5}{16}\zeta(3),\end{aligned}$$

and Theorem 1.1 is proved. □

Now we give the proof of Theorem 1.2.

4 Proof of Theorem 1.2

Proof. (a) We apply Abel's summation formula with $a_n = 1$ and $b_n = x_n^4$, where $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. We have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^4 = \lim_{n\to\infty} n x_{n+1}^4 + \sum_{n=1}^{\infty} n\left(x_n^4 - x_{n+1}^4\right)\\ &= \sum_{n=1}^{\infty} n\left(x_n - x_{n+1}\right)\left(x_n + x_{n+1}\right)\left(x_n^2 + x_{n+1}^2\right) \overset{x_n + x_{n+1} = \frac{1}{n}}{=} \sum_{n=1}^{\infty}\left(x_n - x_{n+1}\right)\left(x_n^2 + x_{n+1}^2\right)\\ &= \sum_{n=1}^{\infty}\left(x_n^3 - x_{n+1}^3 + \frac{x_n}{n^2} - \frac{3}{n}x_n^2 + 2x_n^3\right) = x_1^3 + \sum_{n=1}^{\infty}\frac{x_n}{n^2} - 3\sum_{n=1}^{\infty}\frac{x_n^2}{n} + 2\sum_{n=1}^{\infty}x_n^3\\ &\overset{(*)}{=} \ln^3 2 + \frac{13}{8}\zeta(3) - \zeta(2)\ln 2 - 3\left(\frac{3}{2}\zeta(3) - \zeta(2)\ln 2 - \frac{\ln^3 2}{3}\right) + \frac{5}{8}\zeta(3)\\ &= 2\ln^3 2 + 2\zeta(2)\ln 2 - \frac{9}{4}\zeta(3).\end{aligned}$$

We have applied at step $(*)$ parts (d) and (h) of Lemma 1.3 and part (a) of Theorem 1.1. We also used that $\lim_{n\to\infty} n x_{n+1}^4 = 0$, which follows from (1.2).

(b) We calculate the series by applying Abel's summation formula with $a_n = n$ and $b_n = x_n^4$, where $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$.

We have

$$\begin{aligned}&\sum_{n=1}^{\infty} n\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^4 = \frac{1}{2}\lim_{n\to\infty} n(n+1)x_{n+1}^4 + \frac{1}{2}\sum_{n=1}^{\infty} n(n+1)\left(x_n^4 - x_{n+1}^4\right)\\ &\overset{x_n + x_{n+1} = \frac{1}{n}}{=} \frac{1}{2}\sum_{n=1}^{\infty}(n+1)\left(x_n - x_{n+1}\right)\left(x_n^2 + x_{n+1}^2\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(n+1)\left(x_n^3 + x_n x_{n+1}^2 - x_{n+1}x_n^2 - x_{n+1}^3\right)\end{aligned}$$

$$\overset{x_n+x_{n+1}=\frac{1}{n}}{=} \frac{1}{2}\sum_{n=1}^{\infty}\left[nx_n^3-(n+1)x_{n+1}^3+2nx_n^3+3x_n^3-3x_n^2-\frac{3}{n}x_n^2+\frac{x_n}{n}+\frac{x_n}{n^2}\right]$$
$$=\frac{1}{2}\left[x_1^3+2\sum_{n=1}^{\infty}nx_n^3+3\sum_{n=1}^{\infty}x_n^3-3\sum_{n=1}^{\infty}x_n^2-3\sum_{n=1}^{\infty}\frac{x_n^2}{n}+\sum_{n=1}^{\infty}\frac{x_n}{n}+\sum_{n=1}^{\infty}\frac{x_n}{n^2}\right]$$
$$\overset{(*)}{=}\frac{\ln^3 2}{2}+\left(-\frac{\zeta(2)}{4}-\frac{3}{4}\ln^2 2+\frac{3}{2}\ln 2+\frac{5}{32}\zeta(3)\right)+\frac{15}{32}\zeta(3)-\frac{3}{2}\ln 2$$
$$-\frac{3}{2}\left(\frac{3}{2}\zeta(3)-\zeta(2)\ln 2-\frac{\ln^3 2}{3}\right)+\frac{\ln^2 2}{4}+\frac{\zeta(2)}{4}+\frac{13}{16}\zeta(3)-\frac{\zeta(2)\ln 2}{2}$$
$$=\ln^3 2-\frac{1}{2}\ln^2 2+\zeta(2)\ln 2-\frac{13}{16}\zeta(3).$$

We used at step $(*)$ parts (c) and (a) of Theorem 1.1 and parts (g), (h), (b) and (d) of Lemma 1.3. We also used that $\lim_{n\to\infty} n(n+1)x_{n+1}^4=0$, which follows from (1.2). □

The next corollary answers an open problem posed in [5, Open problem p. 107].

Corollary 4.1. *The following identities hold:*

(a) $$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3=4\ln^3 2+6\zeta(2)\ln 2-\frac{27}{4}\zeta(3);$$

(b) $$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3=4\ln^3 2+2\zeta(2)-12\ln^2 2-3\zeta(2)\ln 2+\frac{15}{4}\zeta(3).$$

Proof. (a) Let $x_n=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$ and observe that $\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots=x_n-x_{n+1}$ and $x_n+x_{n+1}=\frac{1}{n}$. It follows that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3=\sum_{n=1}^{\infty}(x_n-x_{n+1})^3$$
$$=\sum_{n=1}^{\infty}\left[x_n^3-x_{n+1}^3+3x_nx_{n+1}(x_{n+1}-x_n)\right]$$
$$=\sum_{n=1}^{\infty}\left[x_n^3-x_{n+1}^3+3x_n\left(\frac{1}{n}-x_n\right)\left(\frac{1}{n}-2x_n\right)\right]$$
$$=\sum_{n=1}^{\infty}\left[x_n^3-x_{n+1}^3+3\frac{x_n}{n^2}-\frac{9x_n^2}{n}+6x_n^3\right]$$
$$=x_1^3+3\sum_{n=1}^{\infty}\frac{x_n}{n^2}-9\sum_{n=1}^{\infty}\frac{x_n^2}{n}+6\sum_{n=1}^{\infty}x_n^3,$$

and the result follows based on part (a) of Theorem 1.1 and parts (d) and (g) of Lemma 1.3.

(b) We have, exactly as in the proof of part (a), that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3\\
&=\sum_{n=1}^{\infty}(-1)^n\left(x_n-x_{n+1}\right)^3=\sum_{n=1}^{\infty}(-1)^n\left[x_n^3-x_{n+1}^3+3x_nx_{n+1}(x_{n+1}-x_n)\right]\\
&=\sum_{n=1}^{\infty}(-1)^nx_n^3-\sum_{n=1}^{\infty}(-1)^nx_{n+1}^3+3\sum_{n=1}^{\infty}(-1)^n\frac{x_n}{n^2}-9\sum_{n=1}^{\infty}(-1)^n\frac{x_n^2}{n}+6\sum_{n=1}^{\infty}(-1)^nx_n^3\\
&=8\sum_{n=1}^{\infty}(-1)^nx_n^3+x_1^3+3\sum_{n=1}^{\infty}(-1)^n\frac{x_n}{n^2}-9\sum_{n=1}^{\infty}(-1)^n\frac{x_n^2}{n},
\end{aligned}$$

and the result follows based on part (b) of Theorem 1.1 and parts (e) and (i) of Lemma 1.3.

□

Remark 4.2. *The calculation of the quintic series* $\sum_{n=1}^{\infty} x_n^5$*, where* $x_n=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$*, which we believe it can be expressed in terms of well known constants, can be approached by reducing the series to the calculation of quadratic, cubic and quartic sums* $\sum_{n=1}^{\infty}\frac{x_n^2}{n^3}$, $\sum_{n=1}^{\infty}\frac{x_n^3}{n^2}$ *and* $\sum_{n=1}^{\infty}\frac{x_n^4}{n}$. *These series and other higher power sums involving the tail of* $\ln 2$ *are the topics of a research project that will be investigated by the authors.*

We mention that other challenging quadratic and cubic series involving the tail of various special functions, as well as open problems can be found in [5].

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Dirichlet series and series with Stirling numbers

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ABSTRACT

This paper presents a number of identities for Dirichlet series and series with Stirling numbers of the first kind. As coefficients for the Dirichlet series we use Cauchy numbers of the first and second kinds, hyperharmonic numbers, derangement numbers, binomial coefficients, central binomial coefficients, and Catalan numbers.

RESUMEN

Este artículo presenta identidades para series de Dirichlet y series con números de Stirling de primera especie. Como coeficientes de las series de Dirichlet usamos números de Cauchy de primera y segunda especie, números hiperarmónicos, subfactoriales, coeficientes binomiales, coeficientes binomiales centrales y números de Catalan.

Keywords and Phrases: Series identities, Stirling numbers of the first kind, harmonic numbers, hyperharmonic numbers, Cauchy numbers, derangement numbers, Catalan numbers, central binomial coefficients.

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1 Introduction

We consider the unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ defined by the equation

$$\frac{1}{x^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{x(x+1)\cdots(x+n)} \tag{1.1}$$

(see [2], [15, p. 29 and p. 171]) or by the well-known exponential generating function [10, p. 351]

$$\frac{(-1)^k}{k!} \ln^k(1-x) = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} \quad (|x| < 1). \tag{1.2}$$

These numbers have a very strong presence in combinatorics and also in classical analysis. For example, it is known that for $k \geq 1$

$$\zeta(k+1) = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!n} \tag{1.3}$$

where $\zeta(s)$ is Riemann's zeta function. This representation appears in Jordan's book [12] (see equation [6, p. 166] and equation (3.1) on [12, p. 194]; also the more general formula on p. 343 in this book). Interesting comments and a new proof can be found in Adamchik's paper [1].

The corresponding result for the Hurwitz zeta function $\zeta(s, a)$ was proved in [2]

$$\zeta(k+1, a) = \Gamma(a) \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n\Gamma(n+a)} \quad (a > 0\,,\ k \geq 1) \tag{1.4}$$

together with the representation

$$\sum_{p=1}^{\infty} \frac{H_p}{p^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\psi'(n)}{n!} \quad (k \geq 1) \tag{1.5}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, $H_0 = 0$, are the harmonic numbers and $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the digamma function. A representation for the polylogarithm $Li_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$ in the spirit of (1.3) was also proved in [2]. Other results in this line were obtained by Wang and Chen [16].

In this paper we want to develop a method of replacing $\zeta(k+1)$ by other Dirichlet series in the spirit of (1.4) and (1.5) and obtaining representations in terms of Stirling numbers. More precisely, we consider the following problem arising from the above representations: given a Dirichlet series of the form

$$F(s)=\sum_{n=1}^{\infty}\frac{a_{n-1}}{n^s} \tag{1.6}$$

we want to find the coefficients b_n so that

$$F(k+1)=\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix} b_n.$$

In the following section (Section 2) we state our main theorem which gives a solution to this problem for the case when the numbers a_k are the coefficients of an analytic function. Then in Section 3 we derive from this theorem a number of corollaries. For illustration, here are some identities proved in Section 3

$$k+1-\sum_{j=1}^{k}\zeta(j+1)=\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{1}{n!(n+1)^2} \quad (k\geq 1)$$

$$\sum_{p=0}^{\infty}\frac{1}{p!(p+1)^{k+1}}=\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\left(e-\sum_{j=0}^{n}\frac{1}{j!}\right) \quad (k\geq 0)$$

$$\sum_{p=0}^{\infty}\frac{C_p}{4^p(p+1)^{k+1}}=4\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{\beta(2n+2)}{n!} \quad (k\geq 0)$$

$$\sum_{p=1}^{\infty}\frac{C_{p-1}}{4^p(p+1)^{k+1}}+\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{1}{n!(2n+3)}=\frac{1}{2} \quad (k\geq 0)$$

where C_p are the Catalan numbers and $\beta(x)$ is Nielsen's beta function.

2 Main results

Theorem 2.1. *Suppose the function*

$$f(x)=\sum_{k=0}^{\infty}a_k x^k$$

is analytic on the unit disc $|x|<1$. Then we have the representations

$$\sum_{p=1}^{\infty}\frac{a_{p-1}}{p^{k+1}}=\sum_{p=0}^{\infty}\frac{a_p}{(p+1)^{k+1}}=\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{1}{n!}\int_0^1 f(x)(1-x)^n\,dx \tag{2.1}$$

$$\sum_{p=0}^{\infty}\frac{a_p}{(p+1)^{k+1}}=\frac{(-1)^k}{k!}\int_0^1 f(x)(\ln x)^k\,dx. \tag{2.2}$$

Proof. In the representation (1.1) let $x = p \geq 1$ be a positive integer and write

$$\frac{1}{p^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{p(p+1)\cdots(p+n)}. \tag{2.3}$$

We know that

$$\frac{1}{p(p+1)\cdots(p+n)} = \frac{1}{n!}\mathrm{B}(p, n+1) = \frac{1}{n!}\int_0^1 x^{p-1}(1-x)^n \, dx$$

where $\mathrm{B}(p, q)$ is Euler's beta function defined by

$$\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx.$$

Next, multiplying both sides of (2.3) by a_{p-1}, summing for $p = 1, 2, \ldots$, and exchanging the order of summation we have

$$\sum_{p=1}^{\infty} \frac{a_{p-1}}{p^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{p=1}^{\infty} \frac{a_{p-1}}{p(p+1)\cdots(p+n)} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!}\int_0^1 f(x)(1-x)^n \, dx.$$

This proves (2.1). The representation (2.2) follows from (2.1) by bringing the summation inside the integral and using equation (1.2). It can be proved also directly by expanding $f(x)$ and integrating term by term. The proof is completed. □

Some remarks. A Dirichlet series of the form (1.6) is called convergent, if it is absolutely convergent for some s_0. In this case it is absolutely and uniformly convergent for $Re(s) > s_0$ [9, 11]. Also, as shown by Charles Jordan in [12, p. 161] the Stirling numbers of the first kind have asymptotic growth

$$\begin{bmatrix} n \\ k \end{bmatrix} \sim \frac{(n-1)!}{(k-1)!}(\gamma + \ln n)^{k+1} \tag{2.4}$$

for $k \geq 1$ fixed. Here γ is Euler's constant. We will keep this asymptotic in mind for the corollaries that follow.

Taking the function

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1; \quad a_n = 1 \quad (n = 0, 1, \ldots)$$

we have for $n \geq 1$

$$\int_0^1 f(x)(1-x)^n \, dx = \int_0^1 (1-x)^{n-1} \, dx = \frac{1}{n}$$

and (1.3) follows immediately from (2.1). Our main applications are given in the next section.

3 Corollaries

We first give a new proof to equation (1.5). Applying (2.1) to the function

$$f(x) = -\frac{\ln(1-x)}{x(1-x)} = \sum_{n=0}^{\infty} H_{n+1}\, x^n \quad (|x| < 1)$$

we find

$$\sum_{p=1}^{\infty} \frac{H_p}{p^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!} \left\{ -\int_0^1 \frac{\ln(1-x)(1-x)^{n-1}}{x}\, dx \right\}.$$

With the substitution $1 - x = e^{-t}$ this integral becomes

$$\int_0^{\infty} \frac{te^{-nt}}{1-e^{-t}}\, dt = \sum_{m=0}^{\infty} \frac{1}{(m+n)^2} = \psi'(n)$$

and (1.5) follows. It is good to mention that

$$\psi'(n) = \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2}\right).$$

3.1 Dirichlet series with hyperharmonic numbers

Let $h_n^{(r)}$ be the hyperharmonic numbers defined by the equation

$$h_n^{(r+1)} = \binom{n+r}{r} (H_{n+r} - H_r)$$

for integers $n\,,\, r \geq 0$ [8]. When $r = 0$, $h_n^{(1)} = H_n$ are the harmonic numbers. The generating function for $h_n^{(r)}$ is given by

$$\sum_{n=0}^{\infty} h_n^{(r)} x^n = -\frac{\ln(1-x)}{(1-x)^r} \quad (|x| < 1).$$

Let $n > r$. We apply our theorem to the function $f(x) = -\dfrac{\ln(1-x)}{(1-x)^{r+1}}$ where the parameter $r+1$ is used for technical convenience. It is easy to compute

$$\begin{aligned} \int_0^1 f(x)(1-x)^n\, dx &= -\int_0^1 \ln(1-x)(1-x)^{n-r-1}\, dx = \frac{1}{n-r} \int_0^1 \ln(1-x)\, d(1-x)^{n-r} \\ &= \frac{1}{n-r} \ln(1-x)(1-x)^{n-r} \Big|_0^1 + \frac{1}{n-r} \int_0^1 (1-x)^{n-r-1}\, dx = \frac{1}{(n-r)^2}. \end{aligned}$$

This gives the following.

Corollary 3.1. . *For every $k > r \geq 0$ we have the identity*

$$\sum_{p=1}^{\infty} \frac{h_{p-1}^{(r+1)}}{p^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{n!(n-r)^2}. \tag{3.1}$$

In particular, for $r = 0$ we have (compare to (1.5))

$$\sum_{p=1}^{\infty} \frac{H_{p-1}}{p^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{n!n^2}. \tag{3.2}$$

3.2 On a result of Victor Adamchik

Adamchik in [1] discussed series of the form

$$G(k,q) = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{n!n^q}$$

and showed that

$$G(k,q) = G(q,k) \tag{3.3}$$

that is,

$$\sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{n!n^q} = \sum_{n=q}^{\infty} \left[\begin{matrix} n \\ q \end{matrix} \right] \frac{1}{n!n^k}.$$

With $q = 2$ this property implies equation (3.2) since $\left[\begin{matrix} n \\ 2 \end{matrix} \right] = (n-1)!H_{n-1}$. In the next result we will connect two different series of Stirling numbers of the first kind.

Corollary 3.2. *For any two integers $q \geq 0$, $k \geq 0$ we have*

$$\sum_{n=q}^{\infty} \left[\begin{matrix} n \\ q \end{matrix} \right] \frac{1}{n!(n+1)^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{1}{n!(n+1)^{q+1}} \tag{3.4}$$

Proof. We take the function

$$f(x) = \frac{(-1)^q}{q!} \ln^q(1-x) = \sum_{n=0}^{\infty} \left[\begin{matrix} n \\ q \end{matrix} \right] \frac{x^n}{n!}$$

where $q \geq 0$ is an integer. With

$$a_{p-1} = \frac{1}{(p-1)!} \left[\begin{matrix} p-1 \\ q \end{matrix} \right]$$

we find from the general formula (2.1)

$$\sum_{p=1}^{\infty} \begin{bmatrix} p-1 \\ q \end{bmatrix} \frac{1}{p^{k+1}(p-1)!} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^q}{n!q!} \int_0^1 \ln^q(1-x)(1-x)^n \, dx$$

and with the substitution $1-x = e^{-t}$ we compute

$$\int_0^1 \ln^q(1-x)(1-x)^n \, dx = (-1)^q \int_0^{\infty} t^q e^{-(n+1)t} \, dt = \frac{(-1)^q q!}{(n+1)^{q+1}}$$

which gives the representation

$$\sum_{p=1}^{\infty} \begin{bmatrix} p-1 \\ q \end{bmatrix} \frac{1}{p^{k+1}(p-1)!} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n+1)^{q+1}}.$$

Replacing p by $n+1$ gives the desired result for all $q \geq 0$, $k \geq 0$. □

This shows a symmetry like the one in (3.3) for $G(k,q)$.

For $q = 0$ we find for every $k \geq 0$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{(n+1)!} = 1. \tag{3.5}$$

Note that the same result follows from (1.1) for $x = 1$. The value of this remarkable series is independent of k. For $k = 0$ on the left hand side we have only one term, 1. For $k \geq 1$ fixed from (2.4) we find the asymptotic behavior for large n

$$\begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{(n+1)!} \sim \frac{(\gamma + \ln n)^{k-1}}{n(n+1)(k-1)!}$$

which shows a very slow convergence. For instance,

$$\sum_{n=5}^{1000} \begin{bmatrix} n \\ 5 \end{bmatrix} \frac{1}{(n+1)!} < 0.8$$

For $q = 1$, $k \geq 1$, we have $\begin{bmatrix} p \\ 1 \end{bmatrix} = (p-1)!$ and (2.1) implies the closed form evaluation

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n+1)^2} = \sum_{p=1}^{\infty} \frac{1}{p(p+1)^{k+1}} = k + 1 - \sum_{j=1}^{k} \zeta(j+1) \tag{3.6}$$

where the second equality follows from the recurrence relation

$$\sum_{p=1}^{\infty}\frac{1}{p(p+1)^{k+1}}=\sum_{p=1}^{\infty}\frac{1+p-p}{p(p+1)^{k+1}}=\sum_{p=1}^{\infty}\frac{1}{p(p+1)^{k}}-\zeta(k+1)+1$$
$$=\sum_{p=1}^{\infty}\frac{1}{p(p+1)^{k-1}}-\zeta(k+1)-\zeta(k)+2\quad\text{etc.}$$

For $q=2$ we have $\begin{bmatrix} p \\ 2 \end{bmatrix}=(p-1)!H_{p-1}$ and therefore,

$$\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{1}{n!(n+1)^3}=\sum_{p=1}^{\infty}\frac{H_{p-1}}{p(p+1)^{k+1}} \tag{3.7}$$

etc.

3.3 Dirichlet series with Cauchy numbers

The Cauchy numbers of the first kind c_n and second kind d_n are interesting combinatorial numbers. They are defined by the generating functions

$$\frac{x}{\ln(x+1)}=\sum_{n=0}^{\infty}\frac{c_n}{n!}x^n$$
$$\frac{-x}{(1-x)\ln(1-x)}=\sum_{n=0}^{\infty}\frac{d_n}{n!}x^n$$

where $|x|<1$ (see [7, p. 294] and [3, 14]). Now consider the function

$$f(x)=\frac{-x}{\ln(1-x)}=\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{n!}\,x^n\quad(|x|<1).$$

Using again the substitution $1-x=e^{-t}$ we compute

$$\int_0^1 f(x)(1-x)^n\,dx=\int_0^{\infty}\frac{1-e^{-t}}{t}\,e^{-(n+1)t}\,dt=\int_0^{\infty}\frac{e^{-(n+1)t}-e^{-(n+2)t}}{t}\,dt=\ln\frac{n+2}{n+1}$$

as this is a Frullani integral. After changing the index $p\to p+1$ we get the following result.

Corollary 3.3. *For every integer $k \geq 0$ we have the series identities*

$$\sum_{p=1}^{\infty} \frac{(-1)^{p-1} c_{p-1}}{p! p^k} = \sum_{p=0}^{\infty} \frac{(-1)^p c_p}{p!(p+1)^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!} \ln \frac{n+2}{n+1}. \tag{3.8}$$

$$\sum_{p=1}^{\infty} \frac{d_{p-1}}{p! p^k} = \sum_{p=0}^{\infty} \frac{d_p}{p!(p+1)^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!} \ln \left(1 + \frac{1}{n}\right). \tag{3.9}$$

Proof. The first identity has been proved above. For the second one we use the function

$$f(x) = \frac{-x}{(1-x)\ln(1-x)}$$

in the same way. □

Both series on the right hand sides of (3.8) and (3.9) are very slowly convergent series with positive terms. In (3.9), for instance, with $k \geq 1$ fixed

$$\begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!} \ln \left(1 + \frac{1}{n}\right) \sim \frac{1}{(k-1)!} \frac{(\gamma + \ln n)^{k-1}}{n} \ln \left(1 + \frac{1}{n}\right).$$

3.4 Dirichlet series with derangement numbers

The derangement numbers

$$D_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

are popular in combinatorics [7, p. 180] and [10, pp. 194–200]. We will relate them to Stirling numbers of the first kind. The generating function for the derangement numbers is given by

$$D(x) = \frac{e^{-x}}{1-x} = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!} \quad (|x| < 1).$$

In this case

$$\int_0^1 D(x)(1-x)^n dx = \int_0^1 e^{-x}(1-x)^{n-1} dx = \frac{1}{e} \int_0^1 e^t t^{n-1} dt = (-1)^n (n-1)! \left(e^{-1} - \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \right)$$

and therefore, we come to the series identity below.

Corollary 3.4. *For every integer $k \geq 0$*

$$\sum_{p=1}^{\infty} \frac{D_{p-1}}{(p-1)!p^{k+1}} = \sum_{p=0}^{\infty} \frac{D_p}{p!(p+1)^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{(-1)^n}{n} \left(e^{-1} - \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \right). \tag{3.10}$$

From Taylor's formula we get the estimate

$$\left| e^{-1} - \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \right| \leq \frac{1}{n!}$$

which assures convergence for the last series in view of (2.4).

The computation of the above integral shows that with $f(x) = e^{-x}$ in (2.1) we come to

$$\int_0^1 e^{-x}(1-x)^n \, dx = (-1)^{n+1} n! \left(e^{-1} - \sum_{j=0}^{n} \frac{(-1)^j}{j!} \right)$$

and this result implies the identity

$$\sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+1)^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] (-1)^{n+1} \left(e^{-1} - \sum_{j=0}^{n} \frac{(-1)^j}{j!} \right). \tag{3.11}$$

In the same way

$$\int_0^1 e^{x}(1-x)^n \, dx = n! \left(e - \sum_{j=0}^{n} \frac{1}{j!} \right)$$

and therefore,

$$\sum_{p=0}^{\infty} \frac{1}{p!(p+1)^{k+1}} = \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right] \left(e - \sum_{j=0}^{n} \frac{1}{j!} \right). \tag{3.12}$$

3.5 Identities for Dirichlet series with binomial and central binomial coefficients

Noticing that

$$\int_0^1 (1-x)^{\lambda} \, dx = \frac{1}{\lambda+1} \quad (\lambda > -1)$$

we take the function

$$f(x) = (1-x)^r = \sum_{n=0}^{\infty} \binom{r}{n} (-1)^n x^n, \quad |x| < 1, \;\; a_n = \binom{r}{n} (-1)^n$$

and from (2.1) we come to the next result:

Corollary 3.5. *For every integer $k \geq 0$ and $k+r+1>0$*

$$\sum_{p=0}^{\infty} \binom{r}{p} \frac{(-1)^p}{(p+1)^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n+r+1)}. \tag{3.13}$$

Here we need $n+r+1>0$. This will be true when $k+r+1>0$ or $r>-k-1$. This identity was obtained in [2, Example 10] by other means.

When r is a nonnegative integer, the sum on the left hand side is finite. For example, when $r=0$ equation (3.13) turns into (3.5). For $r=1$ and $r=2$ we have correspondingly as in [2, Example 10]

$$1 - \frac{1}{2^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n+2)} \tag{3.14}$$

$$1 - \frac{1}{2^k} + \frac{1}{3^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n+3)} \tag{3.15}$$

etc.

For $r=-1/2$ and $r=1/2$ the binomial coefficients take a special form

$$\binom{-1/2}{p} = (-1)^p \binom{2p}{p} \frac{1}{4^p}, \quad \binom{1/2}{p} = (-1)^{p+1} \binom{2p}{p} \frac{1}{4^p(2p-1)}$$

and (3.13) produces the two identities involving central binomial coefficients:

Corollary 3.6. *For any $k \geq 0$*

$$\sum_{p=0}^{\infty} \binom{2p}{p} \frac{1}{4^p(p+1)^{k+1}} = 2 \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(2n+1)} \tag{3.16}$$

$$\sum_{p=0}^{\infty} \binom{2p}{p} \frac{1}{4^p(2p-1)(p+1)^{k+1}} = -2 \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(2n+3)}. \tag{3.17}$$

In particular, with $k=0$ in (3.16) we have the evaluation

$$\sum_{p=0}^{\infty} \binom{2p}{p} \frac{1}{4^p(p+1)} = 2$$

coming from the generating function of the Catalan numbers for $x = \dfrac{1}{4}$ (see next subsection).

Wang and Xu [17] studied series similar to the one on the left hand side in (3.16) and evaluated them in terms of multiple zeta values.

3.6 Dirichlet series with Catalan numbers

We involve now the Catalan numbers

$$C_p = \binom{2p}{p} \frac{1}{1+p}$$

which are very popular in combinatorics and analysis [4], [7, p. 101], [8, p. 53], [10, p. 203]. It is easy to see that for $p \geq 1$

$$\binom{2p}{p} \frac{1}{2p-1} = 2C_{p-1}.$$

Our first identity with Catalan numbers comes from (3.17) written in the form

$$\sum_{p=1}^{\infty} \frac{C_{p-1}}{4^p (p+1)^{k+1}} = \frac{1}{2} - \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(2n+3)}. \tag{3.18}$$

Correspondingly, for $k = 0, 1, 2$ we have from (3.18)

$$\sum_{p=1}^{\infty} \frac{C_{p-1}}{4^p (p+1)} = \frac{1}{6} \tag{3.19}$$

$$\sum_{p=1}^{\infty} \frac{C_{p-1}}{4^p (p+1)^2} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n(2n+3)} = \frac{2\ln 2}{3} - \frac{7}{18} \tag{3.20}$$

$$\sum_{p=1}^{\infty} \frac{C_{p-1}}{4^p (p+1)^3} = \frac{1}{2} - \sum_{n=2}^{\infty} \frac{H_{n-1}}{n(2n+3)} = -\frac{77}{54} + \frac{\pi^2}{18} + \frac{16}{9}\ln 2 - \frac{2}{3}\ln^2 2 \tag{3.21}$$

where in equation (3.20) we used the evaluation (for example, from Wolfram Alpha)

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+3)} = \frac{8}{9} - \frac{2\ln 2}{3}.$$

and the second equality in (3.21) was found by Mathematica and was provided by one of the referees.

The generating function for the Catalan numbers is [4, 13]

$$\frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} C_n x^n \quad (|x| < 4)$$

Replacing x by $x/4$ we consider the function

$$f(x) = \frac{2}{1+\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{C_n}{4^n} x^n$$

and apply our theorem to it. Using the substitution $1-x=t^2$ we find

$$\int_0^1 f(x)(1-x)^n dx = 2\int_0^1 \frac{(1-x)^n}{1+\sqrt{1-x}}\, dx = 4\int_0^1 \frac{t^{2n+1}}{1+t}\, dt = 4\beta(2n+2)$$

where

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\, dt = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+x}$$

is Nielsen's beta function. We come to the curious companion to (3.18)

$$\sum_{p=0}^{\infty} \frac{C_p}{4^p(p+1)^{k+1}} = 4\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\beta(2n+2)}{n!}. \tag{3.22}$$

For $k=0$ this gives the known identity

$$\sum_{p=0}^{\infty} \frac{C_p}{4^p(p+1)} = 4\beta(2) = 4(1-\ln 2). \tag{3.23}$$

For $k=1$ in (3.22) we find

$$\sum_{p=0}^{\infty} \frac{C_p}{4^p(p+1)^2} = 4\sum_{n=1}^{\infty} \frac{\beta(2n+2)}{n} \tag{3.24}$$

and for $k=2$ we have a series identity involving Catalan, harmonic numbers, and beta values

$$\sum_{p=0}^{\infty} \frac{C_p}{4^p(p+1)^3} = 4\sum_{n=1}^{\infty} \frac{H_{n-1}\beta(2n+2)}{n}. \tag{3.25}$$

At the same time from (2.2)

$$\sum_{p=0}^{\infty} \frac{C_p}{4^p(p+1)^{k+1}} = \frac{2(-1)^k}{k!} \int_0^1 \frac{(\ln x)^k}{1+\sqrt{1-x}}\, dx \tag{3.26}$$

and for $k=0$ this confirms (3.23). For $k=1$ by computing the integral we find

$$\sum_{p=0}^{\infty} \frac{C_p}{4^p(p+1)^2} = 8 - 8\ln 2 + 4(\ln 2)^2 - \frac{\pi^2}{3} \tag{3.27}$$

which gives also the value of the series on the right hand side in (3.24)

$$\sum_{n=1}^{\infty} \frac{\beta(2n+2)}{n} = 2 - 2\ln 2 + (\ln 2)^2 - \frac{\pi^2}{12}.$$

The integral $\int_0^1 \frac{\ln x}{1+\sqrt{1-x}}\, dx$ can be computed by the substitution $1-x=t^2$ followed by the expansion of $\ln(1-t^2)$ in power series. More directly, one can use Maple or Mathematica.

3.7 Dirichlet series with even central binomial coefficients

The numbers $\binom{4n}{2n}$ appear in some interesting applications in mathematics [5, 6]. Their generating function for $0 \le x < 1$ is (see [6])

$$f(x) = \sum_{n=0}^{\infty} \binom{4n}{2n} \frac{x^n}{16^n} = \frac{1}{2}\left(\frac{1}{\sqrt{1-\sqrt{x}}} + \frac{1}{\sqrt{1+\sqrt{x}}}\right) = \frac{1}{2}\left(\frac{\sqrt{1-\sqrt{x}}+\sqrt{1+\sqrt{x}}}{\sqrt{1-x}}\right)$$

which is easily derived from the binomial series. The theorem implies the identity below.

Corollary 3.7. *For every integer $k \ge 0$ we have the identity*

$$\sum_{p=0}^{\infty} \binom{4p}{2p} \frac{1}{16^p (p+1)^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{A_n}{n!} \tag{3.28}$$

where

$$A_n = \frac{1}{2}\int_0^1 \left(\sqrt{1-\sqrt{x}} + \sqrt{1+\sqrt{x}}\right)(1-x)^{n-\frac{1}{2}}\, dx.$$

This interesting integral supposedly has the form $A_n = a_n - b_n\sqrt{2}$ with $a_n,\ b_n$ positive rational numbers. This hypothesis was suggested by several cases verified by Maple.

A rough estimate gives

$$A_n \le \frac{1}{2}\int_0^1 (1+2)\,(1-x)^{n-\frac{1}{2}}\, dx = \frac{3}{2n+1}$$

which in view of (2.4) provides good convergence for the right hand side in (3.28). Also, from (2.2) we find the integral representation

$$\begin{aligned}\sum_{p=0}^{\infty} \binom{4p}{2p} \frac{1}{16^p (p+1)^{k+1}} &= \frac{(-1)^k}{2k!}\int_0^1 \left(\frac{1}{\sqrt{1-\sqrt{x}}} + \frac{1}{\sqrt{1+\sqrt{x}}}\right)(\ln x)^k\, dx \qquad (3.29)\\ &= \frac{(-1)^k 2^k}{k!}\int_0^1 \left(\frac{t}{\sqrt{1-t}} + \frac{t}{\sqrt{1+t}}\right)(\ln t)^k\, dt\end{aligned}$$

for every $k \geq 0$. For example, when $k = 0, 1$

$$\sum_{p=0}^{\infty} \binom{4p}{2p} \frac{1}{16^p(p+1)} = \frac{8}{3} - \frac{2}{3}\sqrt{2} \tag{3.30}$$

$$\sum_{p=0}^{\infty} \binom{4p}{2p} \frac{1}{16^p(p+1)^2} = \frac{80}{9} - \frac{32}{9}(\ln 8 + \sqrt{2}) + \frac{16}{3} \ln (\sqrt{2} + 1) \tag{3.31}$$

etc. The integral $\int_0^1 \left(\frac{t}{\sqrt{1-t}} + \frac{t}{\sqrt{1+t}} \right) \ln t \, dt$ for (3.31) was computed here by using both Maple and Mathematica.

In conclusion, the author wants to express his deep gratitude to the referees for a number of valuable comments and suggestions that helped to improve the paper.

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New upper estimate for positive solutions to a second order boundary value problem with a parameter

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ABSTRACT

We consider a second order boundary value problem with a parameter. A new upper bound for positive solutions and Green's function of the problem is obtained.

RESUMEN

Consideramos un problema de valor en la frontera de segundo orden con un parámetro. Se obtiene una nueva cota superior para soluciones positivas y la función de Green del problema.

Keywords and Phrases: Boundary value problem with a parameter, positive solution, upper and lower estimates.

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1 Introduction

Fourth order boundary value problems arise from the study of elasticity. They are models for the deflection or bending of elastic beams (see [15, 16]). Recently, fourth order boundary value problems for differential equations with parameters have received quite some attention in the literature. For example, in 2003, Li [5] considered the fourth order boundary value problem

$$\begin{aligned} &u^{(4)} + \beta u'' - \alpha u = f(t,u), \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned}$$

where α, β are parameters. For a partial list of some recent papers on boundary value problems with parameters, we refer the reader to the papers [1, 2, 3, 4, 6, 7, 8, 9, 11, 12].

In 2011, Webb and Zima [10] studied the existence of multiple positive solutions for a class of fourth order boundary value problems. They also studied in [10] a class of second order boundary value problems with a parameter, which are closely related to the fourth order ones. One of the problems that were considered in [10] consists of the equation

$$u''(t) + k^2 u(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

and the boundary conditions

$$u(0) = u(1) = 0, \tag{1.2}$$

where $k \in (0, \pi)$ is a positive constant. It is well-known that second order boundary value problems are important in their own right. Second order problems arise in a wide variety of mathematical models and have been studied extensively.

When $k \in (0, \pi)$, the Green function $G : [0,1] \times [0,1] \to [0, \infty)$ for the problem (1.1)-(1.2) is given by (see [10])

$$G(t,s) = \begin{cases} \dfrac{\sin(kt)\sin(k(1-s))}{k \sin k}, & t \le s, \\ \dfrac{\sin(ks)\sin(k(1-t))}{k \sin k}, & s \le t. \end{cases}$$

The problem (1.1)-(1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds, \quad 0 \le t \le 1. \tag{1.3}$$

It is easy to see that $G(t,s) \ge 0$ for $0 \le t, s \le 1$. Webb and Zima proved a number of results in [10]. In particular, in the case of $k \in (\pi/2, \pi)$, they obtained the following upper and lower estimates for the Green function $G(t,s)$.

Lemma 1.1 ([10, Lemma 2.1]). *If $k \in (\pi/2, \pi)$, then it holds that*

$$c_T(t)\Phi_T(s) \leq G(t,s) \leq \Phi_T(s), \quad 0 \leq t, s \leq 1, \tag{1.4}$$

where

$$\Phi_T(s) = \frac{1}{k \sin k} \begin{cases} \sin(ks), & s < 1 - \pi/(2k), \\ \sin(ks)\sin(k(1-s)), & 1 - \pi/(2k) \leq s \leq \pi/(2k), \\ \sin(k(1-s)), & s > \pi/(2k), \end{cases}$$

and

$$c_T(t) = \min\left\{\sin(kt), \sin(k(1-t))\right\}, \quad 0 \leq t \leq 1.$$

There are different approaches to solutions for boundary value problems. One important way of finding positive solutions for boundary value problems is to apply fixed point index theorems on a positive cone. To define a positive cone in a function space (for example, the space $C[0,1]$), we need some *a priori* upper and lower estimates for positive solutions of the boundary value problem. Through the years, we have learned that sharper estimates can help define a smaller cone, and, it is easier to search for the positive solution(s) in a smaller cone than in a larger cone. In other words, finer upper and lower estimates can help us establish sharper existence and nonexistence conditions. We refer the reader to the recent papers [14, 15] in which the author used a fixed point theorem on cones to solve fourth order boundary value problems. In both papers, upper and lower estimates for positive solutions play a crucial role in finding solutions for the boundary value problems.

The main purpose of this paper is to further improve the upper estimate in (1.4). Throughout this paper, we assume that

(H) $k \in (\pi/2, \pi)$ is a real number, $f : [0,1] \times [0, \infty) \to [0, \infty)$ is a continuous function.

This paper is organized as follows. In Section 2, we establish a new upper estimate for the Green function $G(t,s)$. In Section 3, we prove an interval estimate for points where a positive solution to the problem (1.1)-(1.2) can achieve its maximum. In Section 4, we establish a new upper estimate for positive solutions to the problem (1.1)-(1.2). Here, by a positive solution, we mean a solution $u(t)$ to the the problem (1.1)-(1.2) such that $u(t) > 0$ for $0 < t < 1$. In Section 5, we present an example to illustrate that our new upper estimates can help us solve fourth order boundary value problems.

We remark that some authors like to base their study on estimates for the Green function (like the authors of [10]), and some other authors choose to base their study on estimates for positive solutions (like we will do in Section 5 of this paper). Since both types of estimates have applications, we in this paper will present both types (one type in Section 2, and a second type in Section 4).

Though the two types are similar in form, usually they do not imply each other. This is a second reason we choose to present both types of upper estimates in this paper.

2 New upper estimate for $G(t,s)$

In this section, we will prove a new upper estimate for the Green function $G(t,s)$. Since the analysis is on $G(t,s)$ only, we will not mention any positive solution $u(t)$ to the problem (1.1)-(1.2) in this section.

We define the function $b : [0,1] \to [0,1]$ by

$$b(t) = \begin{cases} \sin(k(1-t)), & 0 \le t \le 1 - \frac{\pi}{2k}, \\ 1, & 1 - \frac{\pi}{2k} \le t \le \frac{\pi}{2k}, \\ \sin(kt), & \frac{\pi}{2k} \le t \le 1. \end{cases}$$

The function $b(t)$ will be used to give a new upper estimate for the Green function of the problem (1.1)-(1.2). Also, we define the function $\tau : [0,1] \to [0,1]$ by

$$\tau(t) = \min\left\{\frac{\pi}{2k}, \max\left\{t, 1 - \frac{\pi}{2k}\right\}\right\}.$$

In other words,

$$\tau(t) = \begin{cases} 1 - \frac{\pi}{2k}, & 0 \le t \le 1 - \frac{\pi}{2k}, \\ t, & 1 - \frac{\pi}{2k} \le t \le \frac{\pi}{2k}, \\ \frac{\pi}{2k}, & \frac{\pi}{2k} \le t \le 1. \end{cases}$$

With this notation, we can rewrite the function $\Phi_T(s)$ in Lemma 1.1 into a new form.

Lemma 2.1. *We have*

$$\Phi_T(s) = G(\tau(s), s), \quad 0 \le s \le 1.$$

Proof. If $0 \le s \le 1 - \dfrac{\pi}{2k}$, we have

$$\tau(s) = 1 - \frac{\pi}{2k}.$$

In this case, we have $\tau(s) \ge s$, therefore,

$$\begin{aligned} G(\tau(s), s) &= \frac{\sin(ks)\sin(k(1-\tau(s)))}{k\sin k} = \frac{\sin(ks)\sin(k\cdot(\pi/(2k)))}{k\sin k} = \frac{\sin(ks)\sin(\pi/2)}{k\sin k} \\ &= \frac{\sin(ks)}{k\sin k} = \Phi_T(s). \end{aligned}$$

The other two cases — the case where $1 - \dfrac{\pi}{2k} \le s \le \dfrac{\pi}{2k}$ and the case where $\dfrac{\pi}{2k} \le s \le 1$ — can be handled in a similar way. The proof of the lemma is now complete. □

As a consequence of Lemma 2.1, we can now rewrite the upper estimate for $G(t,s)$ in Lemma 1.1 as

$$G(t,s) \leq G(\tau(s),s), \quad 0 \leq t,s \leq 1. \tag{2.1}$$

We will obtain a new upper estimate for $G(t,s)$, which is better than (2.1), in the next several lemmas.

Lemma 2.2. *If (H) holds and* $0 \leq t \leq s \leq 1$, *then* $G(t,s) \leq b(t)G(\tau(s),s)$.

Proof. We take six cases to prove the inequality.

Case 1: $0 \leq t \leq s \leq 1 - \pi/(2k)$. In this case, we have

$$0 \leq s - t \leq 1 - \frac{\pi}{2k},$$

and, consequently,

$$0 \leq k(s-t) \leq k - \frac{\pi}{2} < \frac{\pi}{2}.$$

Hence, in this case, we then have

$$b(t)G(\tau(s),s) - G(t,s) = \frac{\sin(k-kt)\sin(ks)}{k\sin k} - \frac{\sin(kt)\sin(k(1-s))}{k\sin k} = \frac{1}{k}\sin(k(s-t)) \geq 0.$$

Case 2: $0 \leq t \leq 1 - \pi/(2k) \leq s \leq \pi/(2k)$. In this case, we have

$$0 \leq kt \leq k - \frac{\pi}{2} < \frac{\pi}{2}.$$

It follows that

$$\frac{\pi}{2} \leq k - kt \leq k < \pi,$$

and therefore,

$$\sin(k-kt) \geq 0, \quad \cos(k-kt) \leq 0. \tag{2.2}$$

Also, since

$$k - \frac{\pi}{2} \leq ks \leq \frac{\pi}{2},$$

we have

$$\frac{\pi}{2} \leq \pi - ks \leq \frac{3\pi}{2} - k < \pi,$$

$$\sin(ks) = \sin(\pi - ks) \geq \sin\left(\frac{3\pi}{2} - k\right) = -\cos k. \tag{2.3}$$

By using (2.2) and (2.3), we have

$$\begin{aligned}
b(t)G(\tau(s),s) - G(t,s) &= \frac{\sin(k-ks)}{k\sin k}(\sin(k-kt)\sin(ks) - \sin(kt)) \\
&\geq \frac{\sin(k-ks)}{k\sin k}(-\sin(k-kt)\cos k - \sin(kt)) \\
&= -\frac{\sin(k-ks)}{k\sin k}\cdot\cos(k-kt)\sin k \\
&= -\frac{\sin(k-ks)}{k}\cdot\cos(k-kt) \geq 0.
\end{aligned}$$

Case 3: $0 \leq t \leq 1-\pi/(2k)$ and $\pi/(2k) \leq s \leq 1$. In this case, we have

$$0 \leq kt \leq k - \frac{\pi}{2} < \frac{\pi}{2},$$

from where it follows that

$$\frac{\pi}{2} \leq k - kt \leq k < \pi \quad \text{and} \quad \frac{\pi}{2} < \pi - kt \leq \pi.$$

In summary, we have

$$\frac{\pi}{2} \leq k - kt \leq \pi - kt \leq \pi,$$

which implies that

$$\sin(k-kt) \geq \sin(\pi - kt).$$

So, in this case, we have

$$\begin{aligned}
b(t)G(\tau(s),s) - G(t,s) &= \frac{\sin(k-ks)}{k\sin k}\left(\sin(k-kt) - \sin(kt)\right) \\
&= \frac{\sin(k-ks)}{k\sin k}\left(\sin(k-kt) - \sin(\pi-kt)\right) \geq 0.
\end{aligned}$$

Case 4: $1-\pi/(2k) \leq t \leq s \leq \pi/(2k)$. In this case, we have

$$0 \leq kt \leq ks \leq \frac{\pi}{2}$$

and

$$b(t)G(\tau(s),s) - G(t,s) = G(s,s) - G(t,s) = \frac{\sin(k-ks)}{k\sin k}(\sin(ks) - \sin(kt)) \geq 0.$$

Case 5: $1-\pi/(2k) \leq t \leq \pi/(2k)$ and $\pi/(2k) \leq s \leq 1$. In this case, we have

$$b(t)G(\tau(s),s) - G(t,s) = G(\pi/(2k),s) - G(t,s) = \frac{\sin(k-ks)}{k\sin k}(1 - \sin(kt)) \geq 0.$$

Case 6: $\pi/(2k) \le t \le s \le 1$. In this case,

$$b(t)G(\tau(s), s) - G(t, s) = 0.$$

The proof is now complete. □

Lemma 2.3. *If (H) holds and $0 \le s \le t \le 1$, then $G(t, s) \le b(t)G(\tau(s), s)$.*

Proof. First, we notice that, for all $t, s \in [0, 1]$,

$$G(t, s) = G(1-t, 1-s), \tag{2.4}$$

$$b(t) = b(1-t), \tag{2.5}$$

$$\tau(t) = \tau(1-t), \tag{2.6}$$

and

$$G(\tau(1-s), 1-s) = G(\tau(s), s). \tag{2.7}$$

Now, if $0 \le s \le t \le 1$, then $0 \le 1-t \le 1-s \le 1$, and, by Lemma 2.2,

$$G(1-t, 1-s) \le b(1-t)G(\tau(1-s), 1-s). \tag{2.8}$$

In this case, if we combine (2.8) together with the symmetry properties (2.4), (2.5), (2.6), and (2.7), we get

$$G(t, s) \le b(t)G(\tau(s), s), \quad \text{for } \ 0 \le s \le t \le 1.$$

The proof of the lemma is now complete. □

If we combine Lemmas 2.2 and 2.3, we get

Theorem 2.4. *If (H) holds, then, for all $t, s \in [0, 1]$, $G(t, s) \le b(t)G(\tau(s), s)$.*

Since $b(t) \le 1$ for $0 \le t \le 1$, it is clear that Theorem 2.4 improves the upper estimate (2.1) for $G(t, s)$ in Lemma 1.1.

3 Localization of the maximum

In this section, we shall prove some upper and lower estimates for the point where a solution to the problem (1.1)-(1.2) achieves its maximum on the interval $[0, 1]$. In other words, we shall find a subinterval of $[0, 1]$ which contains the point where the maximum is achieved by a solution.

Theorem 3.1. *Suppose that $k \in (\pi/2, \pi)$, and suppose that $u \in C^2[0,1]$. If*

$$u''(t) + k^2 u(t) \leq 0 \quad \text{for} \quad 0 \leq t \leq 1, \tag{3.1}$$

$u(0) = u(1) = 0$, and $u(t) \not\equiv 0$ on $[0,1]$, then $u(t) > 0$ on $(0,1)$, and there exists a unique $t_0 \in (0,1)$ such that $u(t_0) = \|u\|$. Here,

$$\|u\| := \max_{t \in [0,1]} |u(t)|.$$

Proof. For convenience, we define the auxiliary function

$$h(t) := -u''(t) - k^2 u(t), \quad 0 \leq t \leq 1.$$

Then, by (3.1), we have

$$u(t) = \int_0^1 G(t,s)(-u''(s) - k^2 u(s))\, ds \geq 0, \quad 0 \leq t \leq 1.$$

Since $u(t) \not\equiv 0$, we have $\|u\| > 0$. Combining (3.1) and the fact that $u(t) \geq 0$, we have

$$u''(t) \leq -k^2 u(t) \leq 0, \quad 0 \leq t \leq 1.$$

Since $u''(t) \leq 0$, by Theorem 1.2 of [13], we have

$$u(t) \geq \min\{t, 1-t\}\|u\|, \quad 0 \leq t \leq 1.$$

This implies that

$$u(t) > 0 \ \text{ for } \ 0 < t < 1. \tag{3.2}$$

Again, by virtue of (3.1), we have

$$u''(t) \leq -k^2 u(t) < 0, \quad 0 < t < 1.$$

This implies there exists a unique $t_0 \in (0,1)$ such that $u(t_0) = \|u\| > 0$. The proof of the theorem is now complete. □

Theorem 3.2. *Suppose that $k \in (\pi/2, \pi)$, and suppose that $u \in C^2[0,1]$ satisfies (3.1), $u(0) = u(1) = 0$, and $u(t) > 0$ on $(0,1)$. If $t_0 \in (0,1)$ is such that $u(t_0) = \|u\|$, then*

$$1 - \frac{\pi}{2k} \leq t_0 \leq \frac{\pi}{2k}.$$

Proof. We define the auxiliary function $h(t)$ the same way as in the proof of Theorem 3.1, that is,

$h(t) = -u''(t) - k^2u(t)$, $0 \le t \le 1$. It is clear that $h \in C[0,1]$ and $h(t) \ge 0$ on $[0,1]$, and

$$u(t) = \int_0^1 G(t,s)h(s)\,ds, \quad 0 \le t \le 1.$$

Since $u(t) \not\equiv 0$, we have $h(t) \not\equiv 0$ on $[0,1]$. Therefore, there exists a subinterval $[\alpha,\beta] \subset [0,1]$ such that

$$h(t) > 0, \quad \alpha \le t \le \beta. \tag{3.3}$$

It is clear that, for $0 \le t \le 1$,

$$u(t) = \int_0^1 G(t,s)h(s)\,ds = \int_0^t \frac{\sin(ks)\sin(k(1-t))}{k\sin k}h(s)\,ds + \int_t^1 \frac{\sin(kt)\sin(k(1-s))}{k\sin k}h(s)\,ds.$$

Taking the derivative, we get

$$u'(t) = -\int_0^t \frac{\sin(ks)\cos(k(1-t))}{\sin k}h(s)\,ds + \int_t^1 \frac{\cos(kt)\sin(k(1-s))}{\sin k}h(s)\,ds. \tag{3.4}$$

We note that

$$\sin(ks) > 0 \quad \text{for} \quad 0 < s < 1, \tag{3.5}$$

$$\sin(k(1-s)) > 0 \ \text{ for } \ 0 < s < 1, \tag{3.6}$$

$$-\cos(k(1-t)) > 0 \ \text{ for } \ 0 < t < 1 - \frac{\pi}{2k}, \tag{3.7}$$

$$\cos(kt) > 0 \quad \text{for} \quad 0 < t < 1 - \frac{\pi}{2k}. \tag{3.8}$$

If we apply (3.3), (3.5), (3.6), (3.7), and (3.8) in (3.4), we get

$$u'(t) > 0, \quad 0 \le t < 1 - \frac{\pi}{2k}. \tag{3.9}$$

So, if $t_0 \in (0,1)$ is such that $u(t_0) = \|u\|$, then $u'(t_0) = 0$ and therefore, in view of (3.9), it must hold that $t_0 \ge 1 - \frac{\pi}{2k}$. In a similar way, we can show that $t_0 \le \frac{\pi}{2k}$. The proof of the theorem is now complete. □

4 Upper estimate for positive solutions

In this section, we shall prove a new upper estimate for positive solutions to the problem (1.1)-(1.2). Note that this new upper estimate for positive solutions can not be derived directly from the upper estimate for the Green function $G(t,s)$ that was obtained in Section 2, though these upper estimates look similar.

Theorem 4.1. *Suppose that $k \in (\pi/2, \pi)$. If $u \in C^2[0,1]$ satisfies (3.1) and $u(0) = u(1) = 0$, then*

$$u(t) \le b(t)\|u\|, \quad 0 \le t \le 1. \tag{4.1}$$

Proof. Again, let $h(t) = -u''(t) - k^2 u(t)$. It is clear that $h(t) \ge 0$ for $0 \le t \le 1$.

If $u(t) \equiv 0$, then the theorem is trivially true. So, in the rest of the proof, we assume that $u(t) \not\equiv 0$ on $[0,1]$. In this case, by Theorem 3.1, we have $u(t) > 0$ on $(0,1)$, and there exists a unique $t_0 \in (0,1)$ such that $u(t_0) = \|u\| > 0$. By Theorem 3.2, the point t_0 satisfies

$$1 - \frac{\pi}{2k} \le t_0 \le \frac{\pi}{2k}.$$

Without loss of generality, we assume that $u(t_0) = \|u\| = 1$.

We will first show that

$$u(t) \le b(t)\|u\| = b(t), \quad 0 \le t \le 1 - \pi/(2k). \tag{4.2}$$

Assume, to the contrary, that there exists $\alpha \in (0, 1 - \pi/(2k))$ such that

$$u(\alpha) > b(\alpha) = \sin(k - k\alpha).$$

For easy reference, denote $\sigma = 1 - \pi/(2k)$. Then, we have $0 < \alpha < \sigma$. Define an auxiliary function

$$z(t) = \frac{u(t) - \sin(k - kt)}{\sin(kt + \frac{\pi - k}{2})}, \quad 0 \le t \le 1.$$

It is clear that

$$z(\alpha) > 0, \quad z(\sigma) \le 0, \quad z(1) = 0. \tag{4.3}$$

It follows that there exists $t_1 \in [\alpha, 1)$ such that $z'(t_1) = 0$, $z(t_1) \le 0$, and

$$z(t_1) \le z(t) \quad \text{for all} \quad \alpha \le t \le 1.$$

Direct calculations show that

$$z''(t) + p(t)z'(t) = q(t), \tag{4.4}$$

where

$$p(t) = \frac{2k\cos(kt + \frac{\pi - k}{2})}{\sin(kt + \frac{\pi - k}{2})}, \quad 0 \le t \le 1$$

and

$$q(t) = -\frac{h(t)}{\sin(kt + \frac{\pi - k}{2})}, \quad 0 \le t \le 1.$$

It is clear that $p(t)$ and $q(t)$ are continuous functions defined on $[0,1]$, and $q(t) \leq 0$ for $0 \leq t \leq 1$. Define

$$P(t) = \exp\left(\int_0^t p(s)ds\right), \quad 0 \leq t \leq 1.$$

Multiplying Equation (4.4) by $P(t)$, we get

$$(P(t)z'(t))' \leq 0, \quad 0 \leq t \leq 1.$$

Since $z'(t_1) = 0$, we have

$$P(t)z'(t) \geq 0, \quad 0 \leq t \leq t_1.$$

That is, $z(t)$ is non-decreasing on $[0, t_1]$. Since $z(t_1) \leq 0$ and $\alpha < t_1$, we have $z(\alpha) \leq 0$, which contradicts the first inequality in (4.3). Hence, (4.2) must be true.

In a similar way, we can show that

$$u(t) \leq b(t)\|u\|, \quad \pi/(2k) \leq t \leq 1.$$

And, it is obvious that

$$u(t) \leq \|u\| = b(t)\|u\|, \quad 1 - \pi/(2k) \leq t \leq \pi/(2k).$$

The proof of the theorem is now complete. □

Corollary 4.2. *Suppose that (H) holds. If* $u \in C^2[0,1]$ *is a positive solution for the problem* (1.1)-(1.2)*, then* $u(t)$ *satisfies* (4.1).

Proof. If $u \in C^2[0,1]$ is a positive solution for the problem (1.1)-(1.2), then $u(t)$ satisfies the boundary conditions (1.2), and, for $0 \leq t \leq 1$,

$$u''(t) + k^2u(t) = -f(t, u(t)) \leq 0.$$

That is, $u(t)$ satisfies the inequality (3.1). By Theorem 4.1, $u(t)$ satisfies (4.1). This completes the proof of the corollary. □

5 Example

We conclude this paper with a concrete example. Consider the fourth order boundary value problem

$$u''''(t) - \omega^4u(t) = f(t, u(t)), \quad 0 \leq t \leq 1, \tag{5.1}$$

$$u(0) = u''(0) = u''(1) = u(1) = 0. \tag{5.2}$$

Here, the function $f : [0, 1] \times [0, +\infty) \to [0, \infty)$ is defined as

$$f(t, u) = 15 \max\{(1 + 99u)/100, u^2\}, \quad u \geq 0. \tag{5.3}$$

It is clear that this function $f(t, u)$ is actually independent of t and continuous in u. Throughout the section, we fix $\omega = 3$.

We will adopt the same set of notations as in [10]. In particular, the symbols $m, \mu_1, f^0, f^\infty, f^{0,r}$ are all defined the same way as in [10] (see pages 233, 234 of [10]). Also, the Green's functions $G_0(t, s), G_T(t, s), G_H(t, s)$ are defined the same way as in [10] (see equations (2.18), (2.19), and (2.20) of [10]). Note that the function $G_T(t, s)$ of [10] is the same as the function $G(t, s)$ that was given in Section 1 of this paper. We know from [10] that all three functions $G_T(t, s), G_H(t, s)$, and $G_0(t, s)$ are non-negative functions.

For this special case (where $\omega = 3$), the following computational results are given in [10, page 235]:

$$m \approx 12.8961, \quad \mu_1 \approx 16.4091. \tag{5.4}$$

According to [10], these numerical values can be used together with the following existence result to solve the fourth order boundary value problem (5.1)-(5.2) for two positive solutions in the case where $\mu_1 < f^0, f^\infty \leq +\infty$.

Lemma 5.1 ([10, Theorem 2.4, Case (D_2)])**.** *If*

$$\mu_1 < f^0 \leq \infty, \quad f^{0,r} < m \quad \textit{for some} \quad r > 0 \quad \textit{and} \quad \mu_1 < f^\infty \leq \infty,$$

then the problem (5.1)-(5.2) *has at least two positive solutions.*

For the function $f(t, u)$ defined in (5.3), it is straightforward to verify that $f^0 = f^\infty = +\infty$ and, for each $r > 0$, $f^{0,r} \geq 15 > m$. Therefore, Lemma 5.1 does not apply to the problem (5.1)-(5.2).

On the other hand, by applying the new upper estimate that was obtained in this paper, we are able to show that the problem (5.1)-(5.2) has two positive solutions. For this purpose, we choose our function space $X = C[0, 1]$, which is equipped with the supremum norm $\|\cdot\|$. Define a positive cone P of X by

$$P = \{u \in X \mid b(t)u(1/2)/c_T(1/2) \geq u(t) \geq c_T(t)\|u\| \text{ for } 0 \leq t \leq 1\}.$$

Define the operator $T : P \to X$ by

$$(Tu)(t) = \int_0^1 G_0(t,s)f(s,u(s))\,ds, \quad \forall\, t \in [0,1],\, \forall\, u \in P.$$

It is clear that T is completely continuous. It is also clear that, in order to show that the problem (5.1)-(5.2) has two positive solutions, we need only to show that the operator T has two distinct nonzero fixed points in P. Next, we shall prove that, for this particular cone P, it holds that T maps P into P. We will need the upper estimate given in Theorem 4.1 in the proof of this fact.

Lemma 5.2. *For each $u \in X$ such that $u(t) \geq 0$ for $0 \leq t \leq 1$, it holds that $Tu \in P$. In particular, $T(P) \subset P$.*

Proof. Let $z(t) = (Tu)(t)$ and let $h(t) = z''(t) + \omega^2 z(t)$ for $0 \leq t \leq 1$. Then, we have

$$z''''(t) - \omega^4 z(t) = f(t,u(t)), \quad 0 \leq t \leq 1,$$

$$z(0) = z''(0) = z''(1) = z(1) = 0.$$

It follows that $h(0) = h(1) = 0$, and

$$h''(t) - \omega^2 h(t) - f(t,u(t)) = 0, \quad 0 \leq t \leq 1.$$

Hence,

$$h(t) = \int_0^1 G_H(t,s)(-f(s,u(s)))\,ds \leq 0, \quad 0 \leq t \leq 1.$$

Since $z''(t) + \omega^2 z(t) - h(t) = 0$ and $z(0) = z(1) = 0$, we have

$$z''(t) + \omega^2 z(t) \leq 0, \quad 0 \leq t \leq 1,$$

$$z(t) = \int_0^1 G_T(t,s)(-h(s))\,ds \geq 0, \quad 0 \leq t \leq 1.$$

Note that $\omega = 3 \in (\pi/2, \pi)$. If we apply Theorem 4.1, we get

$$z(t) \leq b(t)\|z\|, \quad 0 \leq t \leq 1.$$

For all $t_1, t_2 \in [0,1]$, by Lemma 1.1, we have

$$\begin{aligned} z(t_1) &= \int_0^1 G_T(t_1,s)(-h(s))\,ds \geq \int_0^1 c_T(t_1)\Phi_T(s)(-h(s))\,ds = c_T(t_1)\int_0^1 \Phi_T(s)(-h(s))\,ds \\ &\geq c_T(t_1)\int_0^1 G_T(t_2,s)(-h(s))\,ds = c_T(t_1)z(t_2). \end{aligned}$$

Since $t_2 \in [0,1]$ is arbitrary, we have

$$z(t_1) \geq c_T(t_1)\|z\|, \quad 0 \leq t_1 \leq 1.$$

In summary, we have, for all $0 \leq t \leq 1$,

$$z(t) \leq b(t)\|z\| \leq b(t)z(1/2)/c_T(1/2).$$

The proof of the lemma is now complete. □

Lemma 5.3. *For each $u \in P$ with $\|u\| = 1$, we have $\|Tu\| < \|u\|$.*

Proof. For each $u \in P$ with $\|u\| = 1$, we have $Tu \in P$, and

$$\begin{aligned}(G_T(1/2))\|Tu\| \leq (Tu)(1/2) &= \int_0^1 G_0(1/2,s)f(s,u(s))\,ds \\ &= \int_0^1 G_0(1/2,s)(15 \cdot (1 + 99u(s))/100)\,ds \\ &\leq \int_0^1 G_0(1/2,s)(15 \cdot (1 + 99b(s)/c_T(1/2))/100)\,ds.\end{aligned}$$

It follows that, for each $u \in P$ with $\|u\| = 1$,

$$\|Tu\| \leq (G_T(1/2))^{-1} \cdot (3/20) \cdot \int_0^1 G_0(1/2,s)(1 + 99b(s)/c_T(1/2))\,ds.$$

A direct calculation shows that the right hand side of the last inequality is approximately 0.978566. Thus, we have shown that, for each $u \in P$ with $\|u\| = 1$, it holds that

$$\|Tu\| < 0.979 < 1 = \|u\|.$$

The proof is complete. □

In a similar way, since $f^0 = f^\infty = +\infty$, we can show that

1. there exists a small positive number $\alpha \in (0, 1/2)$ such that, for each $u \in P$ with $\|u\| = \alpha$, it holds that $\|Tu\| \geq \|u\|$; and
2. there exists a positive number $\beta \in (2, +\infty)$ such that, for each $u \in P$ with $\|u\| = \beta$, it holds that $\|Tu\| \geq \|u\|$.

Now, by the norm type of the fixed point theorem of cone expansion and contraction (see Theorem 4 of [14]), the operator T has two fixed points u_1 and u_2 such that

$$0 < \alpha \leq \|u_1\| < 1 < \|u_2\| \leq \beta.$$

It follows that the problem (5.1)-(5.2) has two positive solutions. Note that we are able to achieve this because the new upper estimate (in terms of $b(t)$) from Section 4 can help us define a fine cone P, which makes the search for positive solution(s) easier.

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Surjective maps preserving the reduced minimum modulus of products

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ABSTRACT

Suppose $\mathfrak{B}(H)$ is the Banach algebra of all bounded linear operators on a Hilbert space H with $\dim(H) \geq 3$. Let $\gamma(.)$ denote the reduced minimum modulus of an operator. We charaterize surjective maps φ on $\mathfrak{B}(H)$ satisfying

$$\gamma(\varphi(T)\varphi(S)) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)).$$

Also, we give the general form of surjective maps on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of Jordan triple products of operators.

RESUMEN

Suponga que $\mathfrak{B}(H)$ es el álgebra de Banach de todos los operadores lineales acotados en un espacio de Hilbert H con $\dim(H) \geq 3$. Denote por $\gamma(.)$ el módulo mínimo reducido de un operador. Caracterizamos las aplicaciones sobreyectivas φ en $\mathfrak{B}(H)$ que satisfacen

$$\gamma(\varphi(T)\varphi(S)) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)).$$

También entregamos la forma general de las aplicaciones sobreyectivas en $\mathfrak{B}(H)$ que preservan el módulo mínimo reducido de productos triples de Jordan de operadores.

Keywords and Phrases: Reduced minimum modulus, operator product, Jordan triple product, nonlinear preservers.

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1 Introduction and Preliminaries

Throughout the paper all Banach spaces are assumed over the field of complex numbers $\mathbb{C}$. For a given Banach space X, $\mathfrak{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. For $T \in \mathfrak{B}(X)$, $R(T)$ and $\ker(T)$ denote the range and the null space of T, respectively. The unit circle in $\mathbb{C}$ will be denoted by $\mathbb{T}$.

Mappings between Banach algebras or operator algebras who preserve various spectral properties have been widely studied. Suppose H is a Hilbert space. Mbekhta [10] characterized surjective linear maps on $\mathfrak{B}(H)$ preserving the generalized spectrum, and then deduced the form of all surjective unital linear maps on $\mathfrak{B}(H)$ preserving the reduced minimum modulus. See also the paper by Bourhim [2], the Banach space case is settled. This result was generalized by Skhiri [13] who, for an arbitrary Banach space X, determined the structure of surjective linear maps φ on $\mathfrak{B}(X)$ preserving the reduced minimum modulus, provided that $\varphi(I)$ is invertible. Bourhim *et. al.* [3] showed that a surjective linear map between C^*-algebras which preserves the reduced minimum modulus is a Jordan $*$-isomorphism multiplied by a unitary element. Consequently, the invertiblity assumption of $\varphi(I)$ in [13] is superfluous.

Let X and Y be Banach spaces. Mashreghi and Stepanyan [9], described a bicontinuous bijective (with no linearity assumption) map $\varphi : \mathfrak{B}(X) \to \mathfrak{B}(Y)$ which leaves invariant the reduced minimum modulus of sum/difference of operators. Later, Costara [5] showed that a bijective map on $M_n(\mathbb{C})$ which preserves the reduced minimum modulus of difference of operators is automatically bicontinuous. Cui and Hou [6] characterized maps on standard operator algebras on a Hilbert space H preserving functional values of operator products, where by a functional value on a standard operator algebra $\mathcal{A}$ we mean a function $F : \mathcal{A} \to [0, +\infty]$ satisfying the following conditions:

(i) $F(T) < \infty$ for each rank one $T \in \mathcal{B}(H)$,

(ii) F is unitary (and conjugate unitary) similarity invariant,

(iii) $F(\lambda T) = |\lambda| F(T)$ for all $T \in \mathfrak{B}(H)$ and $\lambda \in \mathbb{C}$,

(iv) $F(T) = 0$ if and only if $T = 0$.

The *reduced minimum modulus* of an operator $T \in \mathfrak{B}(X)$ is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, \ker(T)) \geq 1\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0. \end{cases} \tag{1.1}$$

(see *e.g.* [3, 8, 12]). This quantity measures the closeness of the range of an operator, that is for $T \in \mathfrak{B}(X)$, $\gamma(T) > 0$ if and only if $R(T)$, the range of T, is closed (see [12, Part 10, Chapter II]). It is proved that if T is invertible then $\gamma(T) = \|T^{-1}\|^{-1}$, see [3, 12]. Suppose H is a Hilbert space.

For $T \in \mathfrak{B}(H)$, let $\sigma(T)$ denote the spectrum of T, then

$$\gamma(T)^2 = \inf\{\lambda : \lambda \in \sigma(T^*T) \setminus \{0\}\}, \tag{1.2}$$

see [8, Theorem 4]. Consequently, $\gamma(T) = \gamma(T^*T)^{\frac{1}{2}} = \gamma(TT^*)^{\frac{1}{2}} = \gamma(T^*)$. So, $\gamma(T)^2 = \gamma(T^2)$ whenever $T = T^*$. Moreover, if $U, V \in \mathfrak{B}(H)$ are unitary operators, then $\gamma(UTV) = \gamma(T)$ for all $T \in \mathfrak{B}(H)$.

We denote by $\mathfrak{R}_1(H)$ the set of all bounded rank one operators on H. We recall that every rank one operator T in $\mathfrak{B}(H)$ is of the form $T = x \otimes y$ for some nonzero vectors $x, y \in H$, and $(x \otimes y)^* = y \otimes x$. So, $(x \otimes y)^*(x \otimes y) = (y \otimes x)(x \otimes y) = \|x\|^2 y \otimes y$. Thus, $\sigma((x \otimes y)^*(x \otimes y)) = \{0, \|x\|^2\|y\|^2\}$, and $\gamma(x \otimes y) = \|x\|\|y\|$.

In this paper, we study surjective maps preserving the reduced minimum modulus of products and Jordan triple products. Obviously, such maps preserve zero product/Jordan triple product in both directions. So, preserving zero product/Jordan triple product plays an important role in our arguments.

Recall that, another definition of $\gamma(\cdot)$ was given by C. Apostol in [1] which is different at $T = 0$. The advantage of Definition (1.1) is that it separates the zero operator from the others. So we would be able to use the results for zero product (resp. zero Jordan triple product) preservers. Therefore, in this article, we shall work with the definition of $\gamma(\cdot)$ given by (1.1).

In Section 2, we assume that H is a complex Hilbert space of dimension greater than or equal 3 and study surjective maps (no linearity and continuity are assumed) on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products. Note that the reduced minimum modulus is not a functional value in the sense of [6], as it does not satisfy Condition (iv) in the definition of a functional value. However, Condition (iv) in [6] is used to show zero product preserving property for the maps under consideration. So, the characterization given in [6] works here. We use this characterization to find a finer characterization for surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products. We show that a surjective map ϕ on $\mathfrak{B}(H)$ preserves the reduced minimum modulus of products if and only if ϕ is a linear or conjugate linear $*$-automorphism multiplied by partial isometries. More precisely, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$ for all $T \in \mathfrak{B}(H)$, where ψ is a linear or conjugate linear $*$-automorphism and U_T, V_T are partial isometries on $\overline{R(\psi(T))}$ and $\overline{R(\psi(T)^*)}$, respectively. We recall that by the general characterization of $*$-automorphisms (resp. $*$-anti-automorphisms) on $\mathfrak{B}(H)$ (see [11, Theorem A.8]), $\psi(T) = UTU^*$ (resp. $\psi(T) = UT^*U^*$), where U is a unitary (resp. anti-unitary) operator on H. Finally in Section 3, we consider surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of Jordan triple products of operators. If H is infinite dimensional, we prove that a surjective map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ preserves the reduced minimum modulus of Jordan triple products if and only if there is a unitary operator U on H and a function $\mu : \mathfrak{B}(H) \to \mathbb{T}$ such that either $\phi(T) = \mu(T)UTU^*$ or

$\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$. In finite dimensional case, we will show that such a map on $M_n(\mathbb{C})$ $(n \geq 3)$, has one of the forms $\phi(A) = \mu(A)Uf(A)U^*$ or $\phi(A) = \mu(A)Uf(A)^{tr}U^*$ for all $A \in M_n(\mathbb{C})$, where μ is a function from $M_n(\mathbb{C})$ to $\mathbb{T}$ and for a matrix $A = [a_{ij}]$, $f(A) = [f_0(a_{ij})]$, where $f_0 : \mathbb{C} \to \mathbb{C}$ is the identity or the conjugation map.

2 Preserving reduced minimum modulus of operator products

Let H be a complex Hilbert space of dimension ≥ 3 and let $\mathcal{U}(H)$ denote the set of unitaries on H. In this section we describe a surjective (with no linearity and continuity assumption) map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfying

$$\gamma\big(\phi(T)\phi(S)\big) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)). \tag{2.1}$$

Then obviously, for $T, S \in \mathfrak{B}(H)$, $TS = 0 \Rightarrow \phi(T)\phi(S) = 0$. So, ϕ preserves zero product. We recall that $\gamma(.)$ does not satisfy Condition (iv) in the definition of a functional value. However, in arguments leading to [6, Theorem 2.3 and Theorem 3.2], the only use of this condition is zero product preserving property. In addition, $\gamma(p) = \inf\{\lambda : \lambda \in \sigma(p^*p) \setminus \{0\}\}^{\frac{1}{2}} = 1$ for all projections $p \in \mathfrak{B}(H)$. Particularly, $\gamma(.)$ is constant on the set of all rank one projections. So, we have the same characterization as in [6, Theorem 2.3] on $\mathfrak{R}_1(H)$. Hence by a similar discussion leading to [6, Theorem 3.2], we see that a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (2.1) if and only if there exist a unitary or an anti-unitary U_0 in $\mathfrak{B}(H)$ and functions $h_1, h_2 : \mathfrak{B}(H) \to \mathcal{U}(H)$ satisfying $h_1(T)T = Th_2(T)$ for all $T \in \mathfrak{B}(H)$, such that

$$\phi(T) = U_0 h_1(T) T {U_0}^* = U_0 T h_2(T) {U_0}^*, \tag{2.2}$$

for all $T \in \mathfrak{B}(H)$.

Here by using properties of γ, we are going to find further necessary and sufficient conditions for ϕ to satisfy (2.1).

To prove our main results, we need the following lemma.

Lemma 2.1. *Let $A, B \in \mathfrak{B}(H)$. Then the following statements are equivalent.*

(i) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{B}(H)$.

(ii) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii) $|A| = |B|$.

Similarly, the following statements are also equivalent.

(i)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{B}(H)$.

(ii)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii)′ $|A^*| = |B^*|$.

Proof. The implication (i) ⇒ (ii) is obvious. Assume that $\gamma(AT) = \gamma(BT)$ for all $T \in \mathfrak{R}_1(H)$. Let $x, y \in H$ and $y \neq 0$, then

$$\|Ax\|\|y\| = \gamma(A(x \otimes y)) = \gamma(B(x \otimes y)) = \|Bx\|\|y\|.$$

Thus, $\|Ax\| = \|Bx\|$ for all $x \in H$. So, $\langle A^*Ax, x\rangle = \langle B^*Bx, x\rangle$ for all $x \in H$. Consequently, $|A| = |B|$ that is (ii) implies (iii). If $|A| = |B|$, then $A^*A = B^*B$ and

$$\gamma(AT)^2 = \gamma(T^*A^*AT) = \gamma(T^*B^*BT) = \gamma(BT)^2, \tag{2.3}$$

for all $T \in \mathfrak{B}(H)$. Thus, $\gamma(AT) = \gamma(BT)$ for all $T \in \mathfrak{B}(H)$.

Since $\gamma(T) = \gamma(T^*)$ for all $T \in \mathfrak{B}(H)$, the equivalence of the last three statements is an immediate consequence of the one of the previous statements. □

Proposition 2.2. *Let H be a complex Hilbert space with* $\dim H \geq 3$, *and* $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ *a surjective map. Then ϕ satisfies* (2.1) *if and only if there exists a linear or conjugate linear $*$-automorphism $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ such that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$.*

Proof. Assume that ϕ satisfies (2.1). Using (2.2), it is easy to see that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$, where $\psi(T) = U_0TU_0{}^*$ and U_0 is a unitary or anti-unitary operator on H.

Conversely, suppose that there exists a linear or conjugate linear $*$-automorphism ψ on $\mathfrak{B}(H)$ such that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$. Let $T \in \mathfrak{B}(H)$ be an arbitrary but fixed element, then by the implication (iii) ⇒ (i) in Lemma 2.1, we have

$$\gamma(\phi(T)\phi(S)) = \gamma(\psi(T)\phi(S)) \quad (S \in \mathfrak{B}(H)). \tag{2.4}$$

On the other hand, since $|\phi(S)^*| = |\psi(S)^*|$ for all $S \in \mathfrak{B}(H)$, by the implication (iii)′ ⇒ (i)′ in Lemma 2.1, we get

$$\gamma(\psi(T)\phi(S)) = \gamma(\psi(T)\psi(S)) = \gamma(\psi(TS)) = \gamma(TS) \quad (S \in \mathfrak{B}(H)). \tag{2.5}$$

Comparing (2.4) and (2.5) implies that $\gamma(\phi(T)\phi(S)) = \gamma(TS)$ for all $T, S \in \mathfrak{B}(H)$, and we are done. $\square$

Lemma 2.3. *Let $T \in \mathfrak{B}(H)$ and U be a partial isometry on $\overline{R(T)}$. Then, $\gamma(UTS) = \gamma(TS)$ for all $S \in \mathfrak{B}(H)$.*

Proof. Since U is a partial isometry on $\overline{R(T)}$,

$$\|UTx\| = \|Tx\| \tag{2.6}$$

for all $x \in H$ and $\ker UT = \ker T$. Let $S \in \mathfrak{B}(H)$, then

$$x \in \ker(UTS) \Longleftrightarrow UTSx = 0 \Longleftrightarrow Sx \in \ker UT \Longleftrightarrow Sx \in \ker T \Longleftrightarrow x \in \ker TS.$$

By (2.6), we get $\|UTSx\| = \|TSx\|$ for all $x \in H$. Now, using the above argument we have

$$\gamma(UTS) = \inf\{\|UTSx\| : \operatorname{dist}(x, \ker UTS) \geq 1\} = \inf\{\|TSx\| : \operatorname{dist}(x, \ker TS) \geq 1\} = \gamma(TS).$$

$\square$

Now we are ready to give a slightly finer characterization for surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products.

Theorem 2.4. *Let H be a complex Hilbert space with $\dim H \geq 3$, and $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ a surjective map. Then ϕ satisfies (2.1) if and only if*

$$\phi(T) = U_T\psi(T) = \psi(T)V_T^* \quad (T \in \mathfrak{B}(H)),$$

where ψ is a linear or conjugate linear $*$-automorphism on $\mathfrak{B}(H)$ and for each $T \in \mathfrak{B}(H)$, U_T , V_T are partial isometries on $\overline{R(\psi(T))}$, $\overline{R(\psi(T)^*)}$, respectively. As a consequence, there is a unitary or anti-unitary operator U on H such that

$$\phi(T) = U_TUTU^* = UTU^*V_T^* \quad (T \in \mathfrak{B}(H)).$$

Proof. First, we assume that ϕ satisfies (2.1). By Proposition 2.2, there exists a linear or conjugate linear $*$-automorphism $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ such that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$. Choose an arbitrary but fixed $T \in \mathfrak{B}(H)$. Note that $\|\phi(T)x\| = \|\psi(T)x\|$ and $\|\phi(T)^*x\| = \|\psi(T)^*x\|$ for all $x \in H$. Define

$$\begin{aligned} U_1 : \psi(T)(H) &\to \phi(T)(H) \\ \psi(T)x &\mapsto \phi(T)x, \end{aligned}$$

for all $x \in H$. Then, U_1 is well-defined. Indeed, if $y_1, y_2 \in \psi(T)(H)$, then there exist $x_1, x_2 \in H$ such that $\psi(T)x_1 = y_1$ and $\psi(T)x_2 = y_2$. Also,

$$\begin{aligned}\|U_1y_1 - U_1y_2\| &= \|U_1(\psi(T)x_1) - U_1(\psi(T)x_2)\| = \|\phi(T)x_1 - \phi(T)x_2\| \\ &= \|\phi(T)(x_1 - x_2)\| = \|\psi(T)(x_1 - x_2)\| = \|\psi(T)x_1 - \psi(T)x_2\| \\ &= \|y_1 - y_2\|.\end{aligned}$$

So $U_1y_1 = U_1y_2$, whenever $y_1 = y_2$ which means that U_1 is well-defined. It is easy to see that U_1 is a linear isometry. Hence, it has a linear isometric extension $\overline{U_1}$ to $\overline{R(\psi(T))}$. Define $U_T : H \to H$ by $U_T(x) = \overline{U_1}(x)$ whenever $x \in \overline{R(\psi(T))}$, and $U_T(x) = 0$ for $x \in \overline{R(\psi(T))}^{\perp}$. Therefore, U_T is a partial isometry with $\ker U_T = \overline{R(\psi(T))}^{\perp}$ and we have $\phi(T) = U_T\psi(T)$. By a similar argument, we find a partial isometry V_T such that $\ker V_T = \overline{R(\psi(T)^*)}^{\perp}$ and $\phi(T)^* = V_T\psi(T)^*$. So, $\phi(T) = \psi(T)V_T^*$. Consequently, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$ for all $T \in \mathfrak{B}(H)$.

Conversely, suppose that for $T \in \mathfrak{B}(H)$, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$, where $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ is a linear or conjugate linear $*$-automorphism and U_T , V_T are partial isometries on $\overline{R(\psi(T))}$, $\overline{R(\psi(T)^*)}$, respectively. Then by Lemma 2.3, for $T, S \in \mathfrak{B}(H)$,

$$\begin{aligned}\gamma\big(\phi(T)\phi(S)\big) &= \gamma(U_T\psi(T)\psi(S)V_S^*\big) = \gamma\big(\psi(T)\psi(S)V_S^*\big) = \gamma\big(V_S\psi(S)^*\psi(T)^*\big) = \gamma\big(\psi(S)^*\psi(T)^*\big) \\ &= \gamma\big(\psi(T)\psi(S)\big) = \gamma(TS).\end{aligned}$$

The last assertion follows by [11, Theorem A.8]. □

3 Preserving reduced minimum modulus of Jordan triple product

In [7], authors studied preservers of zero Jordan triple products and found a characterization through some certain subsets of $\mathfrak{B}(X)$. We recall that the Jordan triple product of operators T, S is TST. In the sequel, we consider surjective maps ϕ on $\mathfrak{B}(H)$ satisfying

$$\gamma(\phi(T)\phi(S)\phi(T)) = \gamma(TST) \quad (T, S \in \mathfrak{B}(H)). \tag{3.1}$$

It is easily seen that such a map preserves zero Jordan triple product in both directions, that is

$$TST = 0 \Longleftrightarrow \phi(T)\phi(S)\phi(T) = 0. \tag{3.2}$$

We apply the characterization of maps satisfying (3.2), in [7], to find a finer characterization for maps satisfying (3.1).

Remark 3.1. *(1) Applying [7, Theorem 2.2], we conclude that if H is infinite dimensional and a surjection ϕ on $\mathfrak{B}(H)$ satisfies* (3.2)*, then there is a function $\mu : \mathfrak{B}(H) \to \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator $A : H \to H$ such that either*

$$(a)\ \phi(T) = \mu(T)ATA^{-1} \quad (T \in \mathfrak{B}(H)) \quad or \quad (b)\ \phi(T) = \mu(T)AT^{\star}A^{-1} \quad (T \in \mathfrak{B}(H)).$$

Here $T^{\star}$ denotes the Banach space adjoint of $T \in \mathfrak{B}(H)$. If J is the conjugate linear isomorphism from H onto its dual H^*, then it is easily seen that $T^{\star} = JT^*J^{-1}$, for all $T \in \mathfrak{B}(H)$. Therefore,

$$\phi(T) = \mu(T)AJT^*J^{-1}A^{-1} \quad (T \in \mathfrak{B}(H)).$$

Clearly, AJ is linear or conjugate linear depending on A is conjugate linear or linear, respectively. Renaming AJ by A, we arrive at

(b)′ $\phi(T) = \mu(T)AT^*A^{-1}$*, for all $T \in \mathfrak{B}(H)$,*

where A is a linear or conjugate linear invertible operator.

(2) Suppose that $H = \mathbb{C}^n$, $n \geq 3$, and that $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a surjective map satisfying (3.2)*. Applying [7, Theorem 2.1] shows that there exist an invertible matrix $S \in M_n(\mathbb{C})$, a field automorphism $f_0 : \mathbb{C} \to \mathbb{C}$, and a scalar function $\mu : M_n(\mathbb{C}) \to \mathbb{C} \setminus \{0\}$ such that one of the following holds:*

(c) $\phi(A) = \mu(A)Sf(A)S^{-1} \quad (A \in M_n(\mathbb{C}))$,
or

(d) $\phi(A) = \mu(A)Sf(A)^{tr}S^{-1} \quad (A \in M_n(\mathbb{C}))$*, where* $f([a_{ij}]) = [f_0(a_{ij})]$.

In the two following theorems, we show that if a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (3.1), then the invertible operators A and S in Remark 3.1 (1)-(2) can be replaced by unitaries and moreover, $|\mu| = 1$. As a consequence, ϕ is norm preserving.

Let H be a complex Hilbert space and let $\{e_i\}_i$ be a fixed orthonormal basis for H. If $x = \sum_i \langle x, e_i\rangle e_i$ is an arbitrary element in H, we define $Cx = \sum_i \overline{\langle x, e_i\rangle} e_i$ which is called the *conjugation* operator on H. It is evident that C is an anti-unitary operator with $C^* = C$. Hence, $C^{-1} = C$ and $C^2 = I$. Since $\sigma(CTC) = \sigma(T)$, we have $\sigma((CTC)^*(CTC)) = \sigma(CT^*TC) = \sigma(T^*T)$. Thus, $\gamma(CTC) = \gamma(T)$ for all $T \in \mathfrak{B}(H)$.

Theorem 3.2. *Let H be an infinite dimensional complex Hilbert space. A surjective map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfies* (3.1) *if and only if there exist a function $\mu : \mathfrak{B}(H) \to \mathbb{T}$ and a unitary or anti-unitary operator U on H such that either $\phi(T) = \mu(T)UTU^*$ or $\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$.*

Proof. The "if" part holds in an obvious way. Suppose that ϕ satisfies (3.1), then ϕ satisfies (3.2). Thus by Remark 3.1 (1)-(a), (b)$'$, there exists an invertible linear or conjugate linear operator $A \in \mathfrak{B}(H)$ such that either

$$\text{(i) } \phi(T) = \mu(T)ATA^{-1} \quad (T \in \mathfrak{B}(H)) \quad \textit{or} \quad \text{(ii) } \phi(T) = \mu(T)AT^*A^{-1} \quad (T \in \mathfrak{B}(H)).$$

It follows that for each $T \in \mathfrak{B}(H)$, $\gamma(\phi(T)) = \gamma(T)$. Indeed, $1 = \gamma(I) = \gamma(\phi(I)^3) = |\mu(I)^3|$ and so $|\mu(I)| = 1$. Therefore,

$$\gamma(T) = \gamma(\phi(I)\phi(T)\phi(I)) = |\mu(I)|^2\gamma(\phi(T)) = \gamma(\phi(T)) \quad (T \in \mathfrak{B}(H)).$$

Case 1. In either case, assume that A is linear and that $A = U|A|$ is the polar decomposition of A. Then U is unitary. Set $\phi_{_U}(T) = U^*\phi(T)U$ $(T \in \mathfrak{B}(H))$ and $R = |A|$, then

$$\phi_{_U}(T) = \mu(T)RTR^{-1} \quad (T \in \mathfrak{B}(H)) \quad \textit{or} \quad \phi_{_U}(T) = \mu(T)RT^*R^{-1} \quad (T \in \mathfrak{B}(H)).$$

For a unit vector $x \in H$, we have

$$1 = \gamma(x \otimes x) = \gamma\big(\phi(x \otimes x)\big) = \gamma\big(U^*\phi(x \otimes x)U\big) = \gamma\big(\phi_{_U}(x \otimes x)\big) = |\mu(x \otimes x)|\|Rx\|\|R^{-1}x\|.$$

On the other hand,

$$1 = \gamma\big((x \otimes x)I(x \otimes x)\big) = \gamma\big(\phi_{_U}(x \otimes x)^2\big) = |\mu(x \otimes x)|^2\|Rx\|\|R^{-1}x\|.$$

Therefore, $|\mu(x\otimes x)|^2 = |\mu(x\otimes x)|$. Since $\gamma(\phi_{_U}(x\otimes x)) = \gamma(x\otimes x) = 1$ is nonzero, $|\mu(x\otimes x)| = 1$. It follows that $|\mu| = 1$ on the set of rank one projections on H. Consequently, $\|Rx\|\|R^{-1}x\| = 1$ for all unit vectors $x \in H$. By [6, Lemma 2.4], there is $\alpha > 0$ such that $R = \alpha I$. So,

$$\phi_{_U}(T) = \mu(T)\alpha IT\alpha^{-1}I = \mu(T)T \quad (T \in \mathfrak{B}(H))$$

or

$$\phi_{_U}(T) = \mu(T)\alpha IT^*\alpha^{-1}I = \mu(T)T^* \quad (T \in \mathfrak{B}(H)).$$

In addition, for $T \in \mathfrak{B}(H)$

$$\gamma(T) = \gamma(\phi(T)) = \gamma(\phi_{_U}(T)) = |\mu(T)|\gamma(T).$$

Thus, $|\mu(T)| = 1$ for every $T \in \mathfrak{B}(H)$, and we infer that

$$\phi(T) = \mu(T)UTU^* \quad (T \in \mathfrak{B}(H)) \quad \textit{or} \quad \phi(T) = \mu(T)UT^*U^* \quad (T \in \mathfrak{B}(H)).$$

Case 2. Let A be conjugate linear (in (i) or (ii)), and let C be the conjugation operator on H. Define $\phi_C(T) = C\phi(T)C$ for all $T \in \mathfrak{B}(H)$. Then, ϕ_C satisfies (3.1). Since ϕ satisfies one of the conditions (i) or (ii) above, $\phi_C(T) = \mu(T)CATA^{-1}C$, or $\phi_C(T) = \mu(T)CAT^*A^{-1}C$, where CA is linear with inverse $A^{-1}C$. Now, by the first part of the proof, there is a unitary operator V on H such that either $\phi_C(T) = \mu(T)VTV^*$ or $\phi_C(T) = \mu(T)VT^*V^*$, for all $T \in \mathfrak{B}(H)$ and $|\mu(T)| = 1$ for all T. Putting $U = CV$, then U is an anti-unitary operator and either $\phi(T) = \mu(T)UTU^*$ or $\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$. □

The proof of the following theorem follows the same line as the proof of [7, Theorem 4.1]. We recall that A^{tr} denotes the transpose of a matrix A.

Theorem 3.3. *Suppose $n \geq 3$. Then $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies (3.1) if and only if there exists a unitary matrix U and a function $\mu : M_n(\mathbb{C}) \to \mathbb{T}$ such that either*

$$\text{(i) } \phi(A) = \mu(A)Uf(A)U^* \quad \textit{or} \quad \text{(ii) } \phi(A) = \mu(A)U(f(A))^{tr}U^*,$$

for all $A = [a_{ij}] \in M_n(\mathbb{C})$. We have $f([a_{ij}]) = [f_0(a_{ij})]$ where, $f_0 : \mathbb{C} \to \mathbb{C}$ is the identity or the complex conjugate on $\mathbb{C}$.

Remark 3.4. *(i) As we mentioned in Section 1, another definition of the reduced minimum modulus was given by C. Apostol in [1] which differs from (1.1) at $T = 0$. Let T be a bounded linear operator on a Banach space X. According to [1], the reduced minimum modulus of T which we denote by $\gamma_a(T)$, is defined by*

$$\gamma_a(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, \ker(T)) \geq 1\} & \textit{if } T \neq 0, \\ 0 & \textit{if } T = 0. \end{cases} \tag{3.3}$$

It is natural to ask whether our results remain valid when we replace (1.1) by (3.3). The advantage of Definition (1.1) is that it separates the zero operator from the others. So we would be able to use the properties of zero product (resp. zero Jordan triple product) preservers. Since positivity of $\gamma(T)$ (resp. $\gamma_a(T)$) is equivalent to the closeness of the range of T, and since in finite dimensional case every operator has closed range, so in this case $\gamma_a(T) = 0$ if and only if $T = 0$. Hence, our results hold true with convention (3.3). However, in the inifinite dimensional case, we still do not know whether the same characterizations remain valid with convention (3.3), and the problem remains open.

(ii) One of our main assumptions in this article is that $\dim H \geq 3$. In fact a principal key in our arguments is the characterization of zero product (resp. zero Jordan triple product) preservers on certain subalgebras of $\mathcal{B}(X)$ when X is a Banach space with $\dim X \geq 3$, given in [6, 7]. In general, this assumption on dimension is crucial for characterizing zero product preservers, see [4, Example 3.1]. It seems that characterizing the maps preserving the reduced

minimum modulus of products (resp. Jordan triple product) of complex 2×2 *matrices needs different arguments.*

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Fixed points of set-valued mappings satisfying a Banach orbital condition

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ABSTRACT

In this note, we prove a fixed point existence theorem for set-valued functions by extending the usual Banach orbital condition concept for single valued mappings. As we show, this result applies to various types of set-valued contractions existing in the literature.

RESUMEN

En esta nota, demostramos un teorema de existencia de un punto fijo para funciones a valores en conjuntos extendiendo el concepto de la condición orbital de Banach usual para funciones univaluadas. Como mostramos, este resultado aplica a diversos tipos de contracciones a valores en conjuntos existentes en la literatura.

Keywords and Phrases: Banach orbital condition; continuity of set-valued mappings; fixed point; Hausdorff upper semicontinuity; set-valued contraction.

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1 Introduction

Several authors, among others, Berinde [1], Berinde and Păcurar [3], Cho [4], Hicks and Rhoades [8], Kasahara [9] and Kirk and Shahzad [10] studied the existence of fixed points of single and set-valued operators, by stating conditions on the orbits of these operators. In the current work, we are interested in investigating the existence of fixed points, for set-valued mappings or correspondences, by a type of the so called Banach orbital condition. This condition is an adaptation of the usual one, which we introduce motivated by the work of Hicks and Rhoades in [8].

The main result of this note establishes the existence of fixed points for set-valued mappings satisfying the mentioned condition. Moreover, we show that this result and variants of it apply to various multi-valued mappings existing in the literature.

The presentation of this work is subdivided into three sections. Apart of this introduction, in Section 2, some notations and preliminary definitions are presented. The main result and its consequences are introduced in Section 3. Finally, Section 4 is devoted to some examples existing in the literature and satisfying the Banach orbital condition for set-valued mappings.

2 Preliminaries

In the sequel, (X,d) stands for a complete metric space and, for $a \in X$ and $r > 0$, we denote $B(a,r) = \{x \in X : d(x,a) < r\}$. A subset A is said to be bounded, whenever there exist $a \in X$ and $r > 0$ such that $A \subset B(a,r)$. We denote by $\mathcal{B}(X)$ the family of all bounded sets of X and by $\mathcal{C}(X)$ the family of all nonempty and closed subsets of X. In what follows, $\mathcal{CB}(X) = \mathcal{C}(X) \cap \mathcal{B}(X)$ and $B(A,r) = \bigcup_{a \in A} B(a,r)$, for each $A \in \mathcal{B}(X)$ and $r > 0$.

Let $T : X \to \mathcal{CB}(X)$ be a set-valued mapping, $x \in X$ and B be a subset of X. We denote $T(B) = \bigcup_{y \in B} Ty$ and for each $n \in \mathbb{N}$, $T^{n+1}x = T(T^n x)$, with $T^0 x = \{x\}$. The *orbit* of x under T is defined as

$$\mathcal{O}(x,T) = \bigcup_{n=0}^{\infty} T^n x.$$

Let $x_0 \in X$. A function $G : X \to \mathbb{R}$ is said to be (x_0,T)-*orbitally lower semicontinuous* at $x^* \in X$, if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathcal{O}(x_0,T)$ converging to x^*, we have $G(x^*) \le \liminf G(x_n)$. In the sequel, $G_T : X \to \mathbb{R}$ stands for the function defined as $G_T(x) = d(x,Tx)$ and for $\xi : X \to X$, we denote $G_\xi = G_{\{\xi\}}$.

Given a set-valued mapping $T : X \to \mathcal{CB}(X)$, $x_0 \in X$, and $k \in [0,1)$, we say T satisfies the *multivalued Banach orbital* (MBO) condition at x_0 with constant k, whenever for all $x \in \mathcal{O}(x_0,T)$, $\inf\limits_{y \in Tx} d(y,Ty) \le k d(x,Tx)$, and that, T satisfies the *strong multivalued Banach orbital* (SMBO) condition at x_0 with constant k, whenever for all $x \in \mathcal{O}(x_0,T)$, $\sup\limits_{y \in Tx} d(y,Ty) \le k d(x,Tx)$.

3 Main results

Theorem 3.1. *Let $T : X \to \mathcal{CB}(X)$ be a set-valued mapping satisfying the MBO condition at $x_0 \in X$ with constant k. Then, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that, for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$, and the following two conditions hold:*

(i) $d(x_n, Tx_n) \le d(x_n, x_{n+1}) \le k^n d(x_0, Tx_0)$ and

(ii) $d(x^*, Tx_n) \le \{k^{n+1}/(1-k)\}d(x_0, Tx_0)$, for all $n \in \mathbb{N}$.

Moreover, the following conditions are equivalent:

(iii) $x^* \in Tx^*$

(iv) G_T is (x_0, T)-orbitally lower semicontinuous at x^*, and

(v) the function $h : X \to \mathbb{R}$, defined by $h(x) = d(x, Tx)$, is lower semicontinuous at x^*.

Proof. Let $\rho \in (k, 1)$. If $d(x_0, Tx_0) = 0$, we define $x_n = x_0$, for all $n \ge 1$. Otherwise, from assumption, there exists $x_1 \in Tx_0$ such that $d(x_1, Tx_1) < \rho d(x_0, Tx_0)$. If $d(x_1, Tx_1) = 0$, we define $x_n = x_1$, for all $n \ge 2$. Otherwise, there exists $x_2 \in Tx_1$ such that $d(x_2, Tx_2) < \rho d(x_1, Tx_1) < \rho^2 d(x_0, Tx_0)$. It follows by induction that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that, for all $n \in \mathbb{N}$, $d(x_n, Tx_n) \le d(x_n, x_{n+1}) \le \rho^n d(x_0, Tx_0)$ and $x_{n+1} \in Tx_n$. Hence, condition (i) holds.

For all $n \in \mathbb{N}$ and $m \ge 1$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\le \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) \le \sum_{k=0}^{m-1} \rho^{n+k} d(x_0, Tx_0) = \rho^n \sum_{k=0}^{m-1} \rho^k d(x_0, Tx_0) \\ &\le \rho^n \sum_{k=0}^{m-1} \rho^k d(x_0, Tx_0). \end{aligned}$$

Hence, $d(x_n, x_{n+m}) \le \{\rho^n/(1-\rho)\}d(x_0, Tx_0)$. In particular, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and consequently there exists $x^* \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. By taking limit, as $m \to \infty$, in the last inequality, we have

$$d(x^*, Tx_{n-1}) \le d(x^*, x_n) \le \{\rho^n/(1-\rho)\}d(x_0, Tx_0), \text{ for all } n \ge 1,$$

and consequently condition (ii) holds.

Suppose $x^* \in Tx^*$. Since $G_T(x^*) = 0$, it is clear that G_T is (x, T)-orbitally lower semicontinuous at x^*, for all $x \in X$. This proves that condition (iii) implies condition (iv). Next, conditions (iv) and (v) are equivalent, by the first axiom of countability. Finally, by assuming the lower semicontinuity of h, we have $d(x^*, Tx^*) = h(x^*) \leq \liminf h(x_n) = 0$, by condition (i). Since Tx^* is closed, this proves that condition (v) implies condition (iii) and the proof is complete. $\square$

Remark 3.2. Any lower semicontinuity set-valued mapping, $T : X \to \mathcal{CB}(X)$, satisfying assumptions of Theorem 3.1, also satisfies the equivalent conditions $(iii)-(v)$. Indeed, let h be the function defined in condition (v) and $a > 0$. Hence, $\{x \in X : h(x) < a\} = \{x \in X : Tx \cap B(x, a) \neq \emptyset\}$. That is, h is upper semicontinuous.

Given $x_0 \in X$ and a single valued function, $f : X \to X$, we denote $\mathcal{O}(x_0, f) = \mathcal{O}(x_0, \{f\})$. As usual, $\{f^n\}_{n\in\mathbb{N}}$ denotes the sequence of functions defined recursively as f^0 the identity function and $f^{n+1} = f \circ f^n$, for all $n \in \mathbb{N}$. The following corollary is an equivalent version of the main result of Hicks and Rhoades in [8].

Corollary 3.3. *Let $\xi : X \to X$ be a function and $k \in [0, 1)$. Suppose there exists $x_0 \in X$ such that, for all $x \in \mathcal{O}(x_0, \xi)$, $d(\xi(x), \xi^2(x)) \leq kd(x, \xi(x))$. Then, there exists $x^* \in X$ such that the following two conditions hold:*

(i) $\lim\limits_{n\to\infty} d(x^*, \xi^n(x_0)) = 0$ *and*

(ii) $d(x^*, \xi^n(x_0)) \leq \{k^n/(1-k)\}d(x_0, \xi(x_0))$, *for all* $n \in \mathbb{N}$.

Moreover, $x^* = \xi(x^*)$, if and only if, the function $x \in X \mapsto d(x, \xi(x)) \in \mathbb{R}$ is (x_0, ξ)-orbitally lower semicontinuous at x^*.

Proof. By Theorem 3.1, there exist $x_k^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x_k^* such that $x_{n+1} = \xi(x_n) = \xi^n(x_0)$. Since the sequence $\{x_n\}_{n\in\mathbb{N}}$ only depends on x_0 and not on k, neither does x_k^* depend on k. Therefore, conditions (i) and (ii) follow from Theorem 3.1 and the proof is complete. $\square$

A set-valued mapping $T : X \to \mathcal{CB}(X)$ is said to be *Hausdorff upper semicontinuous*, if for each $x \in X$ and $\epsilon > 0$, there exists a neighborhood U of x such that $Ty \subset B(Tx, \epsilon)$, for all $y \in U$. This concept is weaker that the upper semicontinuity ot T. However, as we see below, it contributes to obtaining orbital lower semicontinuity for T.

Theorem 3.4. *Let $T : X \to \mathcal{CB}(X)$ be a Hausdorff upper semicontinuous set-valued mapping and suppose T satisfies the MBO condition at $x_0 \in X$ with constant k. Then, there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Proof. By Theorem 3.1, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{O}(x_0, T)$, converging to x^* such that, for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$. Let $\epsilon > 0$. From assumption, there exists a neighborhood U

of x^* such that $Tx \subset B(Tx^*, \epsilon)$, for all $x \in U$. Let $N \in \mathbb{N}$ such that $x_n \in U$, for all $n \geq N$. Hence $Tx_n \subset B(Tx^*, \epsilon)$, which implies that $\sup_{y \in Tx_n} d(y, Tx^*) \leq \epsilon$, for all $n \geq N$. We have

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \leq d(x^*, x_{n+1}) + \epsilon, \text{ for all } n \geq N.$$

By taking inf-limit in n and considering that $\epsilon > 0$ is arbitrary, we obtain $d(x^*, Tx^*) = 0$. Since Tx^* is closed, we have $x^* \in Tx^*$, which completes the proof. □

We denote by $\mathcal{H}$ the Pompeiu-Hausdorff metric (see [3]) associate to d, *i.e.*, $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ is defined as

$$\mathcal{H}(U, V) = \inf \{\epsilon > 0 : U \subset B(V, \epsilon) \text{ and } V \subset B(U, \epsilon)\}.$$

Corollary 3.5. *Let $T : X \to \mathcal{CB}(X)$ be a continuous set-valued mapping with respect to the Pompeiu-Hausdorff metric, i.e. $\lim_{n\to\infty} \mathcal{H}(Tx_n, Tx) = 0$, for all sequence, $\{x_n\}_{n\in\mathbb{N}}$, in X converging to $x \in X$. Suppose T satisfies the MBO condition at $x_0 \in X$ with constant k. Then, there exist $x^* \in X$ such that $x^* \in Tx^*$.*

Proof. It is a consequence of Theorem 3.4, and the Pompeiu-Hausdorff continuity of T implies its Hausdorff upper semicontinuity. □

Remark 3.6. Let $T : X \to \mathcal{CB}(X)$ be a set-valued mapping, $x_0 \in X$ and $k \in [0, 1)$. Notice that, a sufficient condition to T satisfies the MBO condition is $d(y, Ty) \leq kd(x, y)$, for all $x \in \mathcal{O}(x_0, T)$ and $y \in Tx$, and a sufficient condition to T satisfies the SMBO condition is $d(y, Ty) \leq kd(x, Tx)$, for all $y \in Tx$.

4 Some examples

In this section, we introduce some special types of set-valued mappings, which satisfy the MBO condition.

1. (Nadler contraction [6, 11]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ is a Nadler contraction, if for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq kd(x, y)$, for some $k \in [0, 1)$. Let $x \in X$ and $y \in Tx$. Hence,

$$d(y, Ty) \leq \sup_{z\in Tx} d(z, Ty) \leq \mathcal{H}(Tx, Ty) \leq kd(x, y),$$

   and consequently T satisfies the MBO condition. In this case, there exists $x^* \in X$ such that $x^* \in Tx^*$, by Corollary 3.5.

2. (Kannan contraction [12]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Kannan contraction, if and only if, there exists $k \in [0, 1/2)$ such that $\mathcal{H}(Tx, Ty) \leq k(d(x, Tx) +$

$d(y, Ty))$, for all $x, y \in X$. Let $k \in [0, 1/2)$ such that $\mathcal{H}(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty))$, for all $x, y \in X$. We have

$$d(y, Ty) \leq \mathcal{H}(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)),$$

and hence, $(1 - k)d(y, Ty) \leq kd(x, Tx)$. Accordingly,

$$d(y, Ty) \leq \{k/(1 - k)\}d(x, Tx), \text{ for all } x \in X \text{ and } y \in Tx.$$

Since $k/(1 - k) \in [0, 1)$, we have T satisfies the SMBO condition with constant $k/(1 - k)$.

3. (Kannan generalized contraction [7, 12]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the generalized Kannan contraction, if and only if, there exists $k \in [0, 1)$ such that $\mathcal{H}(Tx, Ty) \leq k \max\{d(x, Tx), d(y, Ty)\}$, for all $x, y \in X$. In this case, if for some $y \in Tx$, $d(x, Tx) \leq d(y, Ty)$, then $d(y, Ty) = 0$, otherwise $d(y, Ty) \leq kd(x, Tx)$, for all $y \in Tx$. Consequently, T satisfies the SMBO condition with constant k.

4. (Chatterjea contraction [13]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Chatterjea contraction, if there exists $k \in [0, 1/2)$ such that for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx))$. Let $x \in X$ and $y \in Tx$. Hence,

$$d(y, Ty) \leq \mathcal{H}(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx)) = kd(x, Ty).$$

This fact along with the inequality $d(x, Ty) \leq d(x, y) + d(y, Ty)$ implies that

$$d(y, Ty) \leq \{k/(1 - k)\}d(x, y), \text{ for all } x \in X \text{ and } y \in Tx.$$

Consequently, T satisfies the multivalued Banach orbital condition with constant $k/(1-k) \in [0, 1)$.

5. (Chatterjea generalized contraction) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the generalized Chatterjea contraction, if there exists $k \in [0, 1/2)$ such that, for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq k \max\{d(x, Ty), d(y, Tx)\}$. Let $x \in X$ and $y \in Tx$. Hence,

$$d(y, Ty) \leq \mathcal{H}(Tx, Ty) \leq kd(x, Ty),$$

and accordingly, T satisfies the SMBO condition with constant $k/(1 - k) \in [0, 1)$.

6. (Berinde contraction [2]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Berinde contraction if there exist $k \in [0, 1)$ and $L \geq 0$ such that, for all $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq$

$kd(x,y) + Ld(y,Tx)$. Let $x \in X$ and $y \in Tx$. We have

$$\mathcal{H}(Tx,Ty) \leq kd(x,y) + L(y,Tx) = kd(x,y), \text{ for all } x \in X \text{ and } y \in Tx,$$

and since $y \in Tx$, we obtain $d(y,Ty) \leq kd(x,y)$ and hence T satisfy the MBO condition with constant k.

7. (Ciric-Reich-Rus contraction [2]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ is said to verify the Ciric-Reich-Rus contraction if and only if, there exists $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha+\beta+\gamma \in [0,1)$ and, for all $x, y \in X$, $\mathcal{H}(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty)$. Let $x \in X$, $y \in Tx$. We will prove that any Ciric-Reich-Rus contraction is a Berinde contraction. Let $x, y \in X$. As we observed previously, we have the inequality

$$d(y,Ty) \leq d(y,z) + (z,Ty),$$

for all $z \in Tx$. Replacing this, and by the fact $d(x,Tx) \leq d(x,z)$, we have

$$\begin{aligned}
\mathcal{H}(Tx,Ty) &\leq \alpha d(x,y) + \beta d(x,z) + \gamma d(y,Ty)) \\
&\leq \alpha d(x,y) + \beta(d(x,y) + d(y,z)) + \gamma(d(y,z) + d(z,Ty)) \\
&= (\alpha+\beta)d(x,y) + (\beta+\gamma)d(y,z) + \gamma d(z,Ty) \\
&\leq (\alpha+\beta)d(x,y) + (\beta+\gamma)d(y,z) + \gamma\mathcal{H}(Tx,Ty).
\end{aligned}$$

Hence,

$$\mathcal{H}(Tx,Ty) \leq (\{\alpha+\beta)/(1-\gamma)\}d(x,y) + \{(\beta+\gamma)/(1-\gamma)\}d(y,Tx), \text{ for all } x \in X \text{ and } y \in Tx,$$

and since $\alpha+\beta+\gamma < 1$, it follows that $(\alpha+\beta)/(1-\gamma) < 1$ and $(\beta+\gamma)/(1-\gamma) \geq 0$. Therefore, T is a Berinde contraction, and accordingly T satisfies the MBO condition.

8. (Ciric contraction [5]) A set-valued mapping $T : X \to \mathcal{CB}(X)$ satisfies the Ciric contraction, if there exist $\alpha \in [0,1/2)$ such that for all $x, y \in X$,

$$\mathcal{H}(Tx,Ty) \leq \alpha \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

We have T satisfies the multivalued Banach orbital condition. Indeed, let $x \in X$ and $y \in Tx$. Hence, for some $\alpha \in [0,1/2)$, we have

$$\mathcal{H}(Tx,Ty) \leq \alpha \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

but, since $y \in Tx$ and $d(x,Tx) \leq d(x,y)$, we obtain

$$\mathcal{H}(Tx,Ty) \leq \alpha \max\{d(x,y), d(y,Ty), d(x,Ty)\} \leq k(d(x,y)+d(y,Ty)).$$

Consequently,

$$d(y,Ty) \leq \{k/(1-k)\}d(x,y), \text{ for all } x \in X \text{ and } y \in Tx,$$

and therefore, T satisfies the MBO condition.

9. We introduce a new type of contraction, which satisfies the SMBO condition. Indeed, let $T : X \to \mathcal{CB}(X)$ be given as follows:

$$\mathcal{H}(Tx,Ty) \leq \alpha(d(x,Ty)+d(y,Ty)), \text{ for all } x,y \in X,$$

where $\alpha \in [0,1)$. Observe that, for all $y \in Tx$ and $x \in X$, we have $d(y,Ty) \leq \alpha d(x,Tx)$. Consequently, T satisfies the SMBO condition with constant α.

It is worth noting that the existence of a fixed point for contractions (1)-(6) was proved in [2].

Remark 4.1. Although the nine contraction set-valued mappings in this section satisfy the MBO condition, only the Nadler contraction has a fixed point without additional assumptions. The MBO condition for the other contractions is insufficient to have a fixed point.

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