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On the solution of $\mathcal{T}-$controllable abstract fractional differential equations with impulsive effects

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ABSTRACT

In this research article, we delimitate the definition of mild solution for abstract fractional differential equations with state-dependent delay (AFDEw/SDD) of order $\alpha \in (1, 2)$ with impulsive effects and compare the solution to the second-order impulsive differential equations. Further, we obtain sufficient conditions of the existence of mild solution for instantaneous and non-instantaneous impulsive fractional functional differential inclusions with state-dependent delay (IFDIw/SDD) using the multi-valued fixed point theory and operator techniques. Furthermore, we study the trajectory controllability ($\mathcal{T}-$controllability) of the AFDEw/SDD. At last, we present some examples to illustrate the sufficient conditions involving partial and ordinary derivatives.

RESUMEN

En este artículo de investigación, delimitamos la definición de solución mild para ecuaciones diferenciales fraccionarias con retardo dependiente del estado (AFDEw/SDD) de orden $\alpha \in (1, 2)$ con efectos impulsivos y comparamos la solución con aquellas de ecuaciones diferenciales impulsivas de segundo orden. Además obtenemos condiciones suficientes para la existencia de soluciones mild de inclusiones funcionales diferenciales fraccionales instantánea y no-instantáneamente impulsivas con retardo dependiente del estado (IFDIw/SDD) usando la teoría de punto fijo multivaluados y técnicas de operadores. Más aún, estudiamos la controlabilidad por trayectoria ($\mathcal{T}-$controlabilidad) de los AFDEw/SDD. Finalmente, presentamos algunos ejemplos para ilustrar las condiciones suficientes que involucran derivadas parciales y ordinarias.

Keywords and Phrases: Fractional differential equation, functional-differential equations with fractional derivatives, initial value problems, fixed point theorems, controllability.

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1 Introduction

In the last few decades, many researchers paid attention on impulsive differential equations, because the models subject to abrupt changes are not described by classical models, so such type equations simulated in term of impulsive models. In the nature, there are lots of systems in which the time evolution of the state variable depends on the past history in some arbitrary way subject to abrupt changes are modeled in impulsive functional differential equations, see [12–14, 16, 19, 41, 43] for update. These equations arise in several fields of science and engineering which describe the evolution processes. The impulsive effects may be instantaneous or non-instantaneous (more details [2, 25, 37]) which is shown in many biological phenomena involving thresholds, optimal control models in economics, etc.

The reason of receiving great attention of fractional calculus is that it describes the memory and hereditary property. Due to this property fractional mathematical models give the more realistic and practical results than the ordinary models. For the fractional calculus and its applications see the monographs and papers [7, 30, 31, 34, 38–40] and references therein. Further, more specific type of functional differential equations are state dependent delay equations which arise in applied model when traditional simplifications are abandoned. For recent development theory of functional differential equations with state dependent delay reader can see the papers [1, 6, 8, 17, 18, 21] and references therein.

In additional, fractional differential inclusion is the generalization of fractional differential equation; therefore, all problems which contain the property of solution such as existence, uniqueness, stability, periodicity and controllability are presented in the theory of inclusion. A differential inclusion usually has many solutions which start from a given point and pass through others. It is recently seen that new issue appear in the differential inclusion for the investigation of topological properties of the set of solution, and selection of solutions. One can see the articles [9, 10, 15] for more info about this hot topic.

In this appraise, we describe the existence of solution for fractional order case. Feckan *et al.* [19] gave the suitable definition of solution for impulsive nonlinear fractional differential equation of order $\alpha \in (0,1)$, and Wang [43] extended the problem considered in [19] for the order $\alpha \in (1,2)$. Wang *et al.* [41] defined the mild solution using the probability density function for impulsive fractional evolution equations of order $\alpha \in (0,1)$, and motivated by [41] authors [16] extended the definition of mild solution for neutral impulsive fractional functional differential equation with order $\alpha \in (0,1)$ using analytic operator theory. Shu *et al.* [40] determined the definition of mild solution for fractional differential equations with nonlocal conditions to order $\alpha \in (1,2)$ without impulse. The existence results of mild solution for impulsive fractional differential inclusions with nonlocal conditions investigated by Wang *et al.* [42] when the linear part is a fractional sectorial operator for convex and nonconvex of nonlinear term. Liu and Ahmad [32] analyzed an impulsive multiterm fractional differential equations with single and multiple base points for Caputo's fractional derivative. Recently, Feckan *et al.* [20] proposed two type Caputo's fractional derivative named as generalized Caputo's derivative for single base point with the lower bound at zero and classical Caputo's derivative for multiple base points with lower bounded at non-zero.

Controllability is one of the contemplated properties of fractional dynamical systems (FDSs) that confirm the steering of a FDS from an arbitrary initial state to a desired arbitrary final state via a set of certain admissible control. In 1963, Kalman [28], first time gave the notion of controllability. Based on the available literature, we found that there are various concepts of controllability, some like

- approximate controllability (any state vector may be steered arbitrarily close to another state vector)
- exact controllability (any pair of state vectors may be connected by a trajectory)
- the null controllability (any state vector may be steered to 0)
- $\mathcal{T}-$controllability (we look for a control which steers the system along a prescribed trajectory rather than a control steering a given initial state to desired final state.)

It is obvious that $\mathcal{T}-$controllability is a stronger notion than other controllability notions. For example: To launch a rocket in space sometimes it may be desirable a precise path along with desired destination for cost effectiveness and so on, which is based on $\mathcal{T}-$controllability notation. For more details on $\mathcal{T}-$controllability one can see the papers [11, 23, 27, 35] and reference therein.

We found that there is no literature available on existence of mild solution for instantaneous and non-instantaneous impulsive fractional differential inclusion of order $\alpha \in (1,2)$. By inspiration of works [11, 16, 19, 23, 27, 29, 33, 35, 36, 40, 41, 43–45], we consider the following fractional functional differential inclusion with instantaneous and non-instantaneous impulsive effects.

First, we obtain the sufficient conditions of existence of mild solution for the following problem with instantaneous impulse

$$\begin{aligned}
{}_0^C D_t^\alpha u(t) &\in Au(t) + f(t, u_{\rho(t,u_t)}), \quad 0 < t \le T, \quad t \neq t_k, \quad k = 1, 2, \ldots, m, && (1.1)\\
u(t) &= \phi(t), \quad t \in (-\infty, 0]; \quad u'(0) = u_0 \in X, && (1.2)\\
\Delta u(t_k) &= I_k(u(t_k^-)); \quad \Delta u'(t_k) = J_k(u(t_k^-)), && (1.3)
\end{aligned}$$

where ${}_0^C D_t^\alpha$ denotes the generalized Caputo's fractional derivative of order $\alpha \in (1,2)$ for the state $u(t)$ belong to complex Banach space X and $A : D(A) \subset X \to X$ is the closed linear densely defined operator of sectorial type defined on X. The functions $f : [0,T] \times \mathfrak{B}_e \to \mathcal{F}(X)$; $\rho : [0,T] \times \mathfrak{B}_e \to (-\infty, T]$; $\phi(t) : (-\infty, 0] \to X$ satisfy some assumptions, and $\phi(t)$ in to a abstract phase space $\mathfrak{B}_e$.

The notation $(0,T]$ denotes operational interval such that $0 \leq t_0 < t_1 < \cdots < t_m < t_{m+1} \leq T < \infty$. The history function $u_t : (-\infty, 0] \to X$ defined by $u_t(\theta) = u(t+\theta),\ \theta \in (-\infty, 0]$ belongs to $\mathfrak{B}_e$ and $u'(t)$ denotes the ordinary derivative of $u(t)$. The jump functions $I_k, J_k \in C(X,X),\ k = 1,2,\ldots,m$, are bounded and $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$ where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand and left-hand limits of $u(t)$ at $t = t_k$ with $u(t_k^-) = u(t_k)$. Also, we have $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$ where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right-hand and left-hand limits of $u'(t)$ at $t = t_k$, also we take $u'(t_k^-) = u'(t_k)$ respectively.

Second, we give the sufficient conditions for problem with non-instantaneous impulsive fractional functional differential equation

$$ {}_0^C D_t^{\alpha} u(t) = Au(t) + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}), \quad t \in (s_i, t_{i+1}] \subseteq (0,T], \quad i = 0,1,\ldots,N, \tag{1.4} $$

$$ u(t) = g_i(t, u(t)), \quad u'(t) = q_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,N, \tag{1.5} $$

$$ u(t) + G(u) = \phi(t), \quad t \in (-\infty, 0] \quad u'(0) = u_1 \in X, \tag{1.6} $$

where ${}_0^C D_t^{\alpha}$ is classical Caputo's fractional derivative. $f : [0,T] \times \mathfrak{B}_e \times \mathfrak{B}_e \to X, G : X \to X$ are given functions and satisfy some assumptions and the term $Bu_{\rho(t,u_t)}$ is given by $Bu_{\rho(t,u_t)} = \int_0^t K(t,s)(u_{\rho(s,u_s)})\, ds$ where $K \in C(D, \mathbb{R}^+)$ is the set of all positive functions which are continuous on $D = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t < T\}$ and $B^* = \sup_{t \in [0,t]} \int_0^t K(t,s)\, ds < \infty$. Here $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = T$ are pre-fixed numbers, and $g_i, q_i \in C((t_i, s_i] \times X; X)$ for all $i = 1,2,\ldots,N$. The nonlocal condition $G(u)$ defined as $G(u) = \sum_{k=1}^{r} c_k u(t_k),$ where $c_k, k = 1,\ldots,r$, are given constants and $0 < t_1 < t_2 < \cdots < t_r < T$ respectively.

Finally, we consider nonlinear fractional delay differential equation with non-local condition and provide some sufficient conditions for $\mathcal{T}-$controllability for the equation of the form:

$$ {}_0^C D_t^{\alpha} u(t) = Au(t) + \text{Ḅ}\varpi(t) + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}),\ t \in (s_i, t_{i+1}] \subseteq (0,T],\ i = 0,1,\ldots,N, \tag{1.7} $$

$$ u(t) = g_i(t, u(t)), \quad u'(t) = q_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,N, \tag{1.8} $$

$$ u(t) + G(u) = \phi(t), \quad t \in (-\infty, 0] \quad u'(0) = u_1 \in X, \tag{1.9} $$

The linear operator Ḅ $: \mathcal{U}(\text{Banach space}) \to X$ is a bounded operator and $\varpi(t) \in \mathcal{L}^2(J,\mathcal{U})$ is a control function of the system.

Moreover, a strong motivation to study the model problem (1.1), (1.4) and, (1.7) with aftereffect and subject to impulsive conditions (1.3), (1.5) and (1.9) comes from physics because this model represents the inverse heat condition problem. In this paper, we have used the standard fixed point technique taking generalized and classical Caputo's fractional derivative in abstract phase space to established the results.

Further, motivation is that in dynamical models, generally we assume that the linear or non-linear terms are smooth or continuous functions. However, in many modern models, the underlying dynamical models are not necessarily even continuous. For examples, models of friction and Low dimensional climate models do not belong to above models so to remove the restriction or for non-smooth systems with the discontinuous terms are frequently remodeled as a differential inclusion. This is the advantage to study the qualitative analysis of this paper.

A strong motivation to prove the existence results that the knowledge of existence does not prove the uniqueness of solutions also. For example, we have some fractional differential equation model like ${}_0^C D_t^{1/2} x(t) = x^{1/2}(t)$ with initial condition $x(0) = 0$ for $t \in [0, T]$ has a trivial solution $x \equiv 0$ and non trivial solution $x(t) = \frac{\pi}{4} t$. This shows that the solution obviously exists and is not unique because it fails to satisfy the Lipschitz continuity condition. Hence, in a differential equation, solution can exist and can be not unique. In other words, the knowledge of existence does not ensure the uniqueness of the solution.

Further information about this work, it has five sections. Section 2 provides some basic definitions, theorems, notations and lemmas. Section 3 is equipped with existence results of the mild solution for the considered problems (1.1)-(1.6). Section 4 contributes to the Trajectory controllability results for the considered fractional delay differential equation. In Section 5 examples are provided to illustrate our results.

2 Preliminaries

Let X be a arbitrary complex Banach space with norm $\|\cdot\|_X$ and $L(X)$ denotes the Banach space of bounded linear operators from X into X with norm $\|\cdot\|_{L(X)}$ and both are equipped with its natural topology. Let $C([0,T], X)$ be the space of all real valued (or complex valued) continuous functions from $[0, T]$ into X with the sup norm

$$\|u\|_{C([0,T],X)} = \sup_{t \in [0,T]} \{\|u(t)\|_X : u \in C([0,T], X)\}.$$

is a Banach space.

For the general setting of abstract phase space $\mathfrak{B}_e, \mathfrak{B}'_e$ with impulse effects we refer the work [16,24] and for further notations like ${}_a^C D_t^\alpha$ (Caputo's derivative), ${}_a\mathcal{J}_t^\alpha$ (Riemann-Liouville integral) and $E_{\alpha,\beta}(\cdot)$ (Mittag-Leffler function) we refer [34, 38]. For $A : D(A) \subseteq X \to X$ (Sectorial operator) see [40], and for $S_\alpha(t), T_\alpha(t)$ (Operators) [40] particular case of $W_{\alpha,\beta}(t)$ (Operator functions) we refer [22] respectively.

Let $\mathcal{T}$ be the set of all functions $\vartheta(\cdot) \in \mathfrak{B}'_e$ defined on $J = [0, T]$ such that $\vartheta(0) = \phi(0)$, $\vartheta'(0) = u_1$ and $\vartheta(T) = \phi_T$, $\vartheta'(T) = u_T$ for all $t \in J$ and the fractional derivative ${}^C D_t^\alpha \vartheta(t)$ exist almost everywhere. The set $\mathcal{T}$ is called the set of all feasible trajectories for the fractional dynamical system.

Lemma 2.1 ([24]). *Let $u : (-\infty, T] \to X$ be a function such that $u_0 = \phi, u\mid_{(t_k,t_{k+1}]} \in C^2((t_k, t_{k+1}], X)$, then for all $t \in (t_k, t_{k+1}]$, the following conditions hold:*

(C_1) $u_t \in \mathfrak{B}_e$.

(C_2) $\|u(t)\|_X \leq H\|u_t\|_{\mathfrak{B}_e}$.

(C_3) $\|u_t\|_{\mathfrak{B}_e} \leq K(t)\sup\{\|u(s)\| : 0 \leq s \leq t\} + M(t)\|\phi\|_{\mathfrak{B}_e}$, *where $H > 0$ is constant; $K, M : [0, \infty) \to [0, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and K, M are independent of $u(t)$.*

(C_4) *The function $t \mapsto \phi_t$ is well defined and continuous from the set*

$$\Re(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in [0, T] \times \mathfrak{B}_e\}$$

into $\mathfrak{B}_e$ and there exists a continuous and bounded function $J^\phi : \Re(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}_e} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}_e}$ for every $t \in \Re(\rho^-)$.

Lemma 2.2 ([8]). *Let $u : (-\infty, T] \to X$ be function such that $u_0 = \phi$, $u\mid_{(t_k,t_{k+1}]} \in C^2((t_k, t_{k+1}], X)$ and if (C_4) hold, then*

$$\|u_s\|_{\mathfrak{B}_e} \leq (M_e + J^\phi)\|\phi\|_{\mathfrak{B}_e} + K_e \sup\{\|u(\theta)\|;\ \theta \in [0, \max\{0, s\}]\},\ s \in \Re(\rho^-) \cup (t_k, t_{k+1}],$$

where $J^\phi = \sup_{t\in\Re(\rho^-)} J^\phi(t)$, $M_e = \sup_{s\in[0,T]} M(s)$ and $K_e = \sup_{s\in[0,T]} K(s)$.

To use the multi-valued analysis that is discussed in reference [9], we have some properties which are required to prove our main result. Denote by $\mathcal{F}(X) = \{Y \subset X : Y \neq \emptyset\}, \mathcal{F}_{cl}(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is closed}\}, \mathcal{F}_b(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is bounded}\}, \mathcal{F}_{cv}(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is convex}\}, \mathcal{F}_{cp}(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is compact}\}$.

A multi-valued map $\mathcal{G} : X \to \mathcal{F}(X)$ is convex (closed) valued if $\mathcal{G}(x)$ is convex (closed) for all $x \in X$. $\mathcal{G}$ is bounded on bounded sets if $\mathcal{G}(B) = \cup_{x\in B}\mathcal{G}(x)$ is bounded in X for any bounded set B of $\mathcal{F}(X)$ (*i.e.* $\sup_{x\in B}\{\sup\{\|y\| : y \in \mathcal{G}(x)\}\} < \infty$).

A multi-valued map $\mathcal{G} : [0, T] \to P_{cl}(X)$ is said to be measurable if for each $y \in X$ the function $Y : [0, T] \to \mathbb{R}$ defined by

$$Y(t) = d(y, G(t)) = \inf\{|y - z| : z \in \mathcal{G}(t)\},$$

belongs to $L^1([0, T], \mathbb{R})$.

Definition 2.3 ([9]). *A multi-valued map $F : [0,T] \times X \to \mathcal{F}(X)$ is Carátheódory if*

(i) $t \to F(t,u)$ is measurable for each $u \in X$, and

(ii) $u \to F(t,u)$ is upper semi continuous (u.s.c.) for almost all $t \in [0,T]$.

For each $y \in C([0,T],X)$, define the set of selections for F by

$$S_{Fy} = \{v \in L^1([0,T],X) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Let (X,d) be a metric space induced by the norm space $(X, \|\cdot\|_X)$. Consider $H_d : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}_+ \cup \infty$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{F}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{F}_{cl}(X), H_d)$ is a generalized metric space.

Definition 2.4 ([9]). *A multi-valued operator $\mathcal{N} : X \to \mathcal{F}_{cl}(X)$ is called:*

(i) $\gamma-$Lipschitz if there exists $\gamma > 0$ such that

$$H_d(\mathcal{N}(x), \mathcal{N}(y)) \le \gamma d(x,y) \quad \text{for all } x,y \in X;$$

(ii) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2.5 ([9]). *Let (X,d) be a complete metric space. If $\mathcal{N} : X \to \mathcal{F}_{cl}(X)$ is a contraction, then Fix $\mathcal{N} \neq \emptyset$.*

Lemma 2.6 ([9]). *Let f satisfy the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator of the type (M,θ,α,μ). Consider differential equation of order $\alpha \in (1,2)$ with instantaneous impulse*

$$\begin{aligned}
{}_0^C D_t^\alpha u(t) &= Ay(t) + f(t), \quad t \in [0,T], \quad t \neq t_k, && (2.1)\\
u(0) &= u_0 \in X; \quad u'(0) = u_1 \in X, && (2.2)\\
\Delta u(t_k) &= I_k(u(t_k^-)); \quad \Delta u'(t_k) = J_k(u(t_k^-)), \quad t \neq t_k, \quad k = 1,2,\ldots,m. && (2.3)
\end{aligned}$$

and with non-instantaneous impulse

$$\begin{aligned}
{}_a^C D_t^\alpha u(t) &= Au(t) + f(t), \quad t \in (s_i, t_{i+1}] \subset J = (a,T], \quad a \ge 0, \quad i = 0,1,\ldots,N, && (2.4)\\
u(a) &= u_0 \in X; \quad u'(a) = u_1 \in X, && (2.5)\\
u(t) &= g_i(t,u(t)); \quad u'(t) = q_i(t,u(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,N. && (2.6)
\end{aligned}$$

Then a function $u(t) \in PC([0,T],X)$ is a solution of the system (2.1)-(2.3) if it satisfies following integral equation

$$u(t) = \begin{cases} S_\alpha(t)u_0 + u_1 \int_0^t S_\alpha(s)ds + \int_0^t T_\alpha(t-s)f(s)ds, & t \in (0,t_1] \\ S_\alpha(t)u_0 + u_1 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) & \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)f(s)ds, & t \in (t_k, t_{k+1}], \end{cases} \tag{2.7}$$

and a function $u(t) \in PC([a,T],X)$ is a solution of system (2.4)-(2.6) if it satisfies the following integral equation

$$u(t) = \begin{cases} S_\alpha(t-a)u_0 + u_1 \int_a^t S_\alpha(s-a)ds + \int_a^t T_\alpha(t-s)f(s)ds & t \in (a,t_1], \\ S_\alpha(t-s_i)g_i(s_i,u(s_i)) + q_i(s_i,u(s_i)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_{s_i}^t T_\alpha(t-s)f(s)ds & t \in (s_i, t_{i+1}] \end{cases} \tag{2.8}$$

Remark 2.7. *The α-resolvent family $T_\alpha(t)$ associated with solution operator $S_\alpha(t)$ can be defined as*

$$\int_0^t S_\alpha(\theta)x\,d\theta = {}_0\mathcal{J}_t^1 S_\alpha(\theta)x\,d\theta; \quad T_\alpha(t)x = {}_0\mathcal{J}_t^{\alpha-1} S_\alpha(\theta)x\,d\theta, \quad x \in X, \quad t \in [0,T].$$

For the special case when $\alpha \to 2$, we get following results

(1) *$T_\alpha(t)$ is the cosine function $C(t)$ and $S_\alpha(t)$ is the sine function $S(t)$ defined as*

$$S(t)x = \int_0^t C(\theta)x\,d\theta, \quad x \in X, \quad t \in [0,T]$$

(2) *Solution of system (2.1)-(2.3) for $t \in (0,T]$ can be reduced as*

$$u(t) = \begin{cases} C(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)ds & t \in (0,t_1] \\ C(t)u_0 + S(t)u_1 + \sum_{i=1}^k C(t-t_i)I_i(u(t_i^-)) & \\ + \sum_{i=1}^k S(t-t_i)J_i(u(t_i^-)) + \int_0^t S(t-s)f(s)ds & t \in (t_k, t_{k+1}], \end{cases}$$

which is the same as Definition 2.1 in [26].

(3) *Solution of system (2.4)-(2.6) for $t \in (a,T]$ can be reduced as*

$$u(t) = \begin{cases} C(t-a)u_0 + u_1 \int_a^t S(s-a)ds + \int_a^t S(t-s)f(s)ds & t \in (a,t_1], \\ C(t-s_i)g_i(s_i,u(s_i)) + q_i(s_i,u(s_i)) \int_{t_i}^t S(s-t_i)ds + \int_{s_i}^t S(t-s)f(s)ds & t \in (s_i, t_{i+1}] \end{cases}$$

which is the same as Definition 2.1 in [25].

Definition 2.8. *A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}'_e$, is called a mild solution of problem* (1.1)-(1.3) *if $u(0) = \phi(0)$ and it satisfies the following integral equation*

$$u(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \int_0^t T_\alpha(t-s)f(s, u_{\rho(s,u_s)})ds, & t \in (0, t_1] \\ S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) & \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)f(s, u_{\rho(s,u_s)})ds, & t \in (t_k, t_{k+1}]. \end{cases}$$

Definition 2.9. *A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}'_e$ is called a mild solution of the problem (1.4)-(1.6) if $u(0) = \phi(0) - G(u)$ and satisfies the following integral equation*

$$u(t) = \begin{cases} (\phi(0) - G(u))S_\alpha(t) + u_1 \int_0^t S_\alpha(s)ds & \\ + \int_0^t T_\alpha(t)f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})ds, & t \in (0, t_1], \\ g_i(s_i, u(s_i))S_\alpha(t-s_i) + q_i(s_i, u(s_i)) \int_{t_i}^t S_\alpha(s-t_i)ds & \\ + \int_{s_i}^t T_\alpha(t-s)f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})ds, & t \in (s_i, t_{i+1}], \end{cases}$$

for $i = 1, 2, \ldots, N$.

Definition 2.10. *The system* (1.1) *is said to be $\mathcal{T}-$*controllable *if for any $u(\cdot) \in \mathcal{T}$ there exists a control function $\varpi(t) \in \mathcal{L}^2(J, \mathcal{U})$ such that the corresponding solution $u(\cdot)$ of Eq.* (1.1) *satisfies $u(t) = \vartheta(t)$ almost everywhere.*

Definition 2.11. *A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}'_e$ is called a mild solution of the problem* (1.7)-(1.9) *if $u(0) = \phi(0) - G(u)$ and satisfies the following integral equation*

$$u(t) = \begin{cases} (\phi(0) - G(u))S_\alpha(t) + u_1 \int_0^t S_\alpha(s)ds & \\ + \int_0^t T_\alpha(t-s)[\mathcal{B}\varpi(s) + f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})]ds, & t \in (0, t_1], \\ g_i(s_i, u(s_i))S_\alpha(t-s_i) + q_i(s_i, u(s_i)) \int_{t_i}^t S_\alpha(s-t_i)ds & \\ + \int_{s_i}^t T_\alpha(t-s)[\mathcal{B}\varpi(s) + f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})]ds, & t \in (s_i, t_{i+1}], \end{cases}$$

for $i = 1, 2, \ldots, N$.

3 Existence result of mild solution

In this section, we shall establish the existence result of solution for the problems (1.1)-(1.6) for the both case of impulsive effects and also prove the continuous dependent of solution on initial conditions. Further, if A is a sectorial operator then strongly continuous functions are bounded - *i.e.*,

$$\|S_\alpha(t)\|_{L(X)} \leq M; \quad \|T_\alpha(t)\|_{L(X)} \leq M.$$

3.1 Instantaneous case

In this case, we prove the existence of mild solution for problem (1.1)-(1.3) with a non-convex valued right-hand side. Due to this analysis we can make the following assumptions:

(H_1) $f:[0,T]\times\mathfrak{B}_e\to\mathcal{F}_{cp}(X)$ is Caratheódory and has the property that $f(\cdot,\psi):[0,T]\to\mathcal{F}_{cp}(X)$ is measurable, for each $\psi\in\mathfrak{B}_e$.

(H_2) There exists $l\in L^1([0,T],\mathbb{R}^+)$ such that

$$H_d(f(t,\psi),f(t,\xi))\le l(t)\|\psi-\xi\|_{\mathfrak{B}_e}\quad \text{for every}\quad \psi,\xi\in\mathfrak{B}_e$$

and

$$d(0,f(t,0))\le l(t)\quad \text{a.e.}\quad t\in[0,T].$$

Our result is based on contraction multi-valued fixed point theorem given by Covitz and Nadler [15].

Theorem 3.1. *Let the assumptions* (H_1) *and* (H_2) *hold. Then problem* (1.1)-(1.3) *has at least one mild solution* $u(t)$ *on* $[0,T]$.

Proof. Consider the space $\mathfrak{B}''_e=\{u\in\mathfrak{B}'_e: u(0)=\phi(0)\}$ and $y(t)=\phi(t)$ for $t\in(-\infty,0]$ endowed with the uniform convergence topology. We shall show that $\mathcal{P}$ has fixed points, where the multi-valued operator $\mathcal{P}:\mathfrak{B}''_e\to\mathcal{F}(\mathfrak{B}''_e)$ defined as $\mathcal{P}(u)=\{\bar{e}\in\mathfrak{B}''_e\}$ with

$$\bar{e}(t)=\begin{cases} S_\alpha(t)\phi(0)+u_0\int_0^t S_\alpha(s)ds+\int_0^t T_\alpha(t-s)v(s)ds, & t\in(0,t_1],\\ S_\alpha(t)\phi(0)+u_0\int_0^t S_\alpha(s)ds+\sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) & \\ +\sum_{i=1}^k J_i(u(t_i^-))\int_{t_i}^t S_\alpha(s-t_i)ds+\int_0^t T_\alpha(t-s)v(s)ds, & t\in(t_k,t_{k+1}],\end{cases}$$

where $v(s)\in S_{f,\bar{u}_{\rho(s,\bar{u}_s)}}$ for $t\in[0,T]$ and $\bar{u}:(-\infty,T]\to X$ is such that $\bar{u}(0)=\phi(0)$ and $\bar{u}=u$ on $[0,T]$. We shall show that $\mathcal{P}$ has fixed points. Let $\mathcal{P}(u)\in\mathcal{F}_{cl}(\mathfrak{B}''_e)$ for all $u\in\mathfrak{B}''_e$. Let $\{u_n\}_{n\ge0}\in\mathcal{P}(u)$ be such that $u_n\to u\in\mathfrak{B}'''_e$. Then there exists $v_n\in S_{f,\bar{u}_{\rho(s,\bar{u}_s)}}$ such that, for each $t\in(t_k,t_{k+1}]$,

$$u_n(t)=\begin{cases} S_\alpha(t)\phi(0)+u_0\int_0^t S_\alpha(s)ds+\sum_{i=1}^k S_\alpha(t-t_i)I_i(u_n(t_i^-))\\ +\sum_{i=1}^k J_i(u_n(t_i^-))\int_{t_i}^t S_\alpha(s-t_i)ds+\int_0^t T_\alpha(t-s)v_n(s)ds.\end{cases}$$

Using the fact that f has compact values, we may pass to a subsequence if necessary to obtain that v_n converges to v in $L^1([0,T],X)$ and hence $v\in S_{f,\bar{u}_{\rho(s,\bar{u}_s)}}$. Thus, for each $t\in(t_k,t_{k+1}]$

$$u_n(t)\to u(t)=\begin{cases} S_\alpha(t)\phi(0)+u_0\int_0^t S_\alpha(s)ds+\sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-))\\ +\sum_{i=1}^k J_i(u(t_i^-))\int_{t_i}^t S_\alpha(s-t_i)ds+\int_0^t T_\alpha(t-s)v(s)ds,\end{cases}$$

which implies that $u \in \mathcal{P}(u)$.

There exist $\gamma < 1$ such that

$$H_d(f(u_1), f(u_2)) \leq \gamma \|u_1 - u_2\|_{\mathfrak{B}_e'''} \quad \text{for all} \quad u_1, u_2 \in \mathfrak{B}_e''.$$

Let $u_1, u_2 \in \mathfrak{B}_e''$ and $\bar{e} \in \mathcal{P}(u)$. Then there exists $v(t) \in f(t, \bar{u}_{\rho(t,\bar{u}_t)})$ such that, for each $t \in (t_k, t_{k+1}]$,

$$\bar{e}(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i) I_i(u(t_i^-)) \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)v(s)ds. \end{cases}$$

From (H_2) it follows that

$$H_d(f(t, \bar{u_1}_{\rho(t,\bar{u_1}_t)}), f(t, \bar{u_2}_{\rho(t,\bar{u_2}_t)})) \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

Hence, there exists $w \in f(t, \bar{u}_{\rho(t,\bar{u}_t)})$ such that

$$\|v - w\|_{\mathfrak{B}_e''} \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

Consider $U : [0, T] \to \mathcal{F}(X)$ given by

$$U(t) = \{w \in X : \|v - w\| \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}\}.$$

Since the multi-valued operator $V(t) = U(t) \cap f(t, \bar{u_2}_{\rho(t,\bar{u_2}_t)})$ is measurable [10], there exists a function $v_2(t)$ which is a measurable selection for V. Thus, $\bar{v}(t) \in f(t, \bar{u_2}_{\rho(t,\bar{u_2}_t)})$ and for each $t \in (t_k, t_{k+1}]$,

$$v(t) - \bar{v}(t) \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

For each $t \in (t_k, t_{k+1}]$ we define

$$\bar{e}(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i) I_i(u(t_i^-)) \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)\bar{v}(s)ds. \end{cases}$$

Then, we have

$$\begin{aligned} \|e(t) - \bar{e}(t)\|_{\mathfrak{B}_e''} &\leq \int_0^t \|T_\alpha(t-s)\|_{L(X)} \|v(s) - \bar{v}(s)\| ds \leq M \int_0^t l(s)\|u_1 - u_2\| ds \\ &\leq \int_0^t \bar{l}(s)\|u_1 - u_2\| ds \leq \frac{1}{\tau} e^{\tau L(t)} \|u_1 - u_2\|_{\mathfrak{B}_e''}, \end{aligned}$$

where $\tau > 1$, $L(t) = \int_0^t Ml(s)ds$ and $\|\cdot\|_{\mathfrak{B}_e''}$ is the Bielecki-type norm on $\mathfrak{B}_e''$ defined by

$$\|u\|_{\mathfrak{B}_e''} = \sup\{e^{-\tau L(t)}\|u(t)\| : t \in [0,T]\}.$$

Therefore

$$\|e(t) - \bar{e}(t)\|_{\mathfrak{B}_e''} \leq \frac{1}{\tau}\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

Obtained by interchanging of u_1 and u_2, and by an analogous relation, it follows that

$$H_d(\mathcal{P}(u_1), \mathcal{P}(u_2)) \leq \frac{1}{\tau}\|u_1 - u_2\|_{\mathfrak{B}_e''},$$

which implies that $\mathcal{P}$ is a contraction, and thus, by Lemma 2.5 there exists a fixed point $u(t) \in \mathfrak{B}_e''$, which is a mild solution to the problem (1.1)-(1.3). This completes the proof. □

3.2 Non-instantaneous Case

In this case, we shall establish the existence result of solution for the problem (1.4)-(1.6). Now, we introduce the following assumption.

(H_3) The function f is jointly continuous and there exist positive constants L_{f1}, L_{f2} such that

$$\|f(t,\psi,\mu) - f(t,\xi,\nu)\|_X \leq L_{f1}\|\psi - \xi\|_{\mathfrak{B}_e} + L_{f2}\|\mu - \nu\|_{\mathfrak{B}_e}, \quad \forall\, \psi, \xi, \mu, \nu \in \mathfrak{B}_e.$$

(H_4) The functions g_i, q_i and G are continuous and there exist positive constants L_{g_i}, L_{q_i} and L_G such that

$$\|g_i(t,x) - g_i(t,y)\|_X \leq L_{g_i}\|x - y\|_X; \quad \|q_i(t,x) - q_i(t,y)\|_X \leq L_{q_i}\|x - y\|_X;$$

$$\|G(x) - G(y)\|_X \leq L_G\|x - y\|_X,$$

for all $x, y \in X$, $t \in (t_i, s_i]$ and each $i = 1, 2, \ldots, N$.

Theorem 3.2. *If the assumptions (H_3) and (H_4) hold and constant*

$$\Delta = (\delta + TMK_e(L_{f1} + B^* L_{f2})) < 1,$$

where $\delta = \max\{L_G M, L_{g_i}M + L_{q_i}MT\}$ for $i = 1, \ldots, N$. Then there exists a unique mild solution $u(t)$ of the problem (1.4)-(1.6) on $[0,T]$.

Proof. Consider the space $\mathfrak{B}_e''$ as given in Theorem 3.1 and we define an operator $\mathcal{P}:\mathfrak{B}_e''\to\mathfrak{B}_e''$ as

$$\mathcal{P}u(t)=\begin{cases}(\phi(0)-G(\bar{u}))S_\alpha(t)+u_1\int_0^t S_\alpha(s)ds\\ +\int_0^t T_\alpha(t-s)f(s,\bar{u}_{\rho(s,\bar{u}_s)},B\bar{u}_{\rho(s,\bar{u}_s)})ds, & t\in(0,t_1],\\ g_i(s_i,\bar{u}(s_i))S_\alpha(t-s_i)+q_i(s_i,\bar{u}(s_i))\int_{t_i}^t S_\alpha(s-t_i)ds\\ +\int_{s_i}^t T_\alpha(t-s)f(s,\bar{u}_{\rho(s,\bar{u}_s)},B\bar{u}_{\rho(s,\bar{u}_s)})ds, & t\in(s_i,t_{i+1}],\end{cases}\tag{3.1}$$

where $\bar{u}:(-\infty,T]\to X$ is such that $\bar{u(0)}=\phi(0)-G(\bar{u}),\bar{u'(0)}=u_1$ and $\bar{u}=u$ on $[0,T]$. We shall show that the operator $\mathcal{P}$ has a fixed point. So let $u(t),u^*(t)\in\mathfrak{B}_e'$ for $t\in(0,t_1]$, we get

$$\begin{aligned}\|\mathcal{P}u-\mathcal{P}u^*\|_{\mathfrak{B}_e'} &\le \|G(\bar{u})-G(\bar{u}^*)\|\|S_\alpha(t)\|_{L(X)}+\int_0^t\|T_\alpha(t-s)\|_{L(X)}\\ &\quad\times\|f(s,\bar{u}_{\rho(s,\bar{u}_s)},B\bar{u}_{\rho(s,\bar{u}_s)})-f(s,\bar{u}^*_{\rho(s,\bar{u}^*_s)},B\bar{u}^*_{\rho(s,\bar{u}^*_s)})\|_X ds,\\ \|\mathcal{P}u-\mathcal{P}u^*\|_X &\le \{L_GM+TMK_e(L_{f1}+B^*L_{f2})\}\|u-u^*\|_X.\end{aligned}$$

For $t\in(s_i,t_{i+1}]$, we have

$$\begin{aligned}\|\mathcal{P}u-\mathcal{P}u^*\|_{\mathfrak{B}_e'} &\le \|g_i(s_i,\bar{u}(s_i))-g_i(s_i,\bar{u}^*(s_i))\|_X\|S_\alpha(t-s)\|_{L(X)}\\ &\quad+\|q_i(s_i,\bar{u}(s_i))-q_i(s_i,\bar{u}^*(s_i))\|_X\int_0^t\|S_\alpha(t-s)\|_{L(X)}ds\\ &\quad+\int_{s_i}^t\|T_\alpha(t-s)\|_{L(X)}\|f(s,\bar{u}_{\rho(s,\bar{u}_s)},B\bar{u}_{\rho(s,\bar{u}_s)})-f(s,\bar{u}^*_{\rho(s,\bar{u}^*_s)},B\bar{u}^*_{\rho(s,\bar{u}^*_s)})\|_X ds,\\ \|\mathcal{P}u-\mathcal{P}u^*\|_X &\le (L_{g_i}M+L_{q_i}MT+TMK_e(L_{f1}+B^*L_{f2}))\|u-u^*\|_X.\end{aligned}$$

Let $\delta=\max\{L_GM,L_{g_i}M+L_{q_i}MT\}$, then for all $t\in[0,T]$, we obtain

$$\|\mathcal{P}u-\mathcal{P}u^*\|_X \le (\delta+TMK_e(L_{f1}+B^*L_{f2}))\|u-u^*\|_X.$$

We have

$$\|\mathcal{P}u-\mathcal{P}u^*\|_X \le \Delta\|u-u^*\|_X.$$

Since $\Delta<1$, which implies that $\mathcal{P}$ is a contraction map and there exists a unique fixed point $u(t)$ which is the mild solution of system (1.4)-(1.6) on $[0,T]$. $\square$

3.3 Continuous Dependence of Mild Solutions

This section is concerned with continuous dependence of mild solutions consider the system (1.4)-(1.6).

Theorem 3.3. *Suppose that the assumptions* (H_3) *and* (H_4) *are satisfied and the following condition hold:*

$$\left[\max\{ML_G, ML_{g_i} + MTL_{q_i}\} + MT(L_{f_1} + L_{f_2}B^*)(M_e + J^\phi)\right] < 1.$$

Then for each ϕ, ϕ^*, *let* u, u^* *be the corresponding mild solutions of the system* (1.4)-(1.6), *then the following inequalities hold:*

$$\begin{aligned}
\|u - u^*\|_X &\leq \frac{MT(M + L_{f_1} + L_{f_2}B^*)}{1 - [ML_G + MT(L_{f_1} + L_{f_2}B^*)(M_e + J^\phi)]}\|\phi - \phi^*\|, \quad t \in (0, t_1], \\
\|u - u^*\|_X &\leq \frac{MT(M + L_{f_1} + L_{f_2}B^*)}{1 - [ML_{g_i} + MTL_{q_i} + MT(L_{f_1} + L_{f_2}B^*)(M_e + J^\phi)]}\|\phi - \phi^*\|, \quad t \in (s_i, t_{i+1}],
\end{aligned}$$

for $i = 1, 2, \ldots, N$.

Proof. The proof is similar as Theorem 3.2. □

4 Trajectory Controllability

This section deals with the $\mathcal{T}-$controllability results of the considered nonlinear fractional delay differential equation with non-local condition and non-instantaneous impulses.

Theorem 4.1. *Let the assumption* (H_3) *and* (H_4) *hold, then problem* (1.7)-(1.9) *is* $\mathcal{T}-$*controllable on* $[0, T]$.

Proof. Let $\vartheta(t)$ be any given trajectory in $\mathcal{T}$ and we choose the feedback control $\varpi(t)$ given as

$$\varpi(t) = Ḅ^{-1}[{}_0^C D_t^\alpha \vartheta(t) - A\vartheta(t) - f(t, \vartheta_{\rho(t,\vartheta_t)}, B\vartheta_{\rho(t,\vartheta_t)})], \quad t \in (s_i, t_{i+1}] \subseteq (0, T]. \tag{4.1}$$

Plugging the control $\varpi(t)$ from Eq. (4.1) in Eq. (1.7) and we get

$$\begin{aligned}
{}_0^C D_t^\alpha u(t) = Au(t) + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}) + {}_0^C D_t^\alpha \vartheta(t) - A\vartheta(t) - f(t, \vartheta_{\rho(t,\vartheta_t)}, B\vartheta_{\rho(t,\vartheta_t)}), \\
t \in (s_i, t_{i+1}] \subseteq (0, T].
\end{aligned}$$

From the equation above, we have

$$ {}_0^C D_t^\alpha [u(t)-\vartheta(t)] = A[u(t)-\vartheta(t)] + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}) - f(t, \vartheta_{\rho(t,\vartheta_t)}, B\vartheta_{\rho(t,\vartheta_t)}), \quad t \in (s_i, t_{i+1}] \subseteq (0, T]. $$

Again, if we choose $\chi(t) = u(t) - \vartheta(t)$, without loss of generality, then our original problem (1.7)-(1.9) is modified as follows:

$$ \begin{aligned} {}_0^C D_t^\alpha \chi(t) &= A\chi(t) + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}) - f(t, \vartheta_{\rho(t,\vartheta_t)}, B\vartheta_{\rho(t,\vartheta_t)}), \quad (4.2) \\ & \qquad t \in (s_i, t_{i+1}] \subseteq (0,T], \quad i = 0, 1, \ldots, N, \\ \chi(t) &= g_i(t, u(t)) - g_i(t, \vartheta(t)), \quad \chi'(t) = q_i(t, u(t)) - q_i(t, \vartheta(t)), \quad (4.3) \\ & \qquad t \in (t_i, s_i], \quad i = 1, 2, \ldots, N, \quad (4.4) \\ \chi(t) &= -G(u) + G(\vartheta), \quad t \in (-\infty, 0], \quad \chi'(0) = 0. \quad (4.5) \end{aligned} $$

The mild solution of the problem (4.2)-(4.5) is given by

$$ \chi(t) = \begin{cases} (-G(u)+G(\vartheta))S_\alpha(t) + \int_0^t T_\alpha(t-s)[f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)}) - f(s, \vartheta_{\rho(s,\vartheta_s)}, B\vartheta_{\rho(s,\vartheta_s)})]ds, & t \in (0, t_1], \\ S_\alpha(t-s_i)[g_i(t,u(t)) - g_i(t,\vartheta(t))] + \int_{t_i}^t S_\alpha(s-t_i)ds[q_i(t,u(t)) - q_i(t,\vartheta(t))] \\ \quad + \int_{s_i}^t T_\alpha(t-s)[f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)}) - f(s, \vartheta_{\rho(s,\vartheta_s)}, B\vartheta_{\rho(s,\vartheta_s)})]ds, & t \in (s_i, t_{i+1}], \end{cases} $$

For the trajectory control, we will show that $\|\chi(t)\| = 0$. Now, without loss of generality, we consider the subinterval $(s_i, t_{i+1}]$, to estimate

$$ (L_{g_i}M + L_{q_i}MT + TMK_e(L_{f1} + B^* L_{f2}))\|u - u^*\|_X. $$

$$ \begin{aligned} \|\chi(t)\| &\le \|S_\alpha(t-s_i)\|\|g_i(t,u(t)) - g_i(t,\vartheta(t))\| + \int_{t_i}^t \|S_\alpha(s-t_i)\|ds\|q_i(t,u(t)) - q_i(t,\vartheta(t))\| \\ &\quad + \int_{s_i}^t \|T_\alpha(t-s)\|\|f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)}) - f(s, \vartheta_{\rho(s,\vartheta_s)}, B\vartheta_{\rho(s,\vartheta_s)})\|ds, \\ &\le L_{g_i}M\|\chi(t)\| + L_{q_i}M\int_{t_i}^t |\chi(t)\|ds + MK_e(L_{f1} + B^*L_{f2})\int_{t_i}^t \|\chi(t)\|ds \\ &= L_{g_i}M\|\chi(t)\| + [L_{q_i}M + MK_e(L_{f1} + B^*L_{f2})]\int_{t_i}^t \|\chi(t)\|ds \\ &= \Phi\|\chi(t)\| + \Psi\int_{t_i}^t \|\chi(s)\|ds, \end{aligned} $$

where $\Phi = L_{g_i}M$, $\Psi = [L_{q_i}M + MK_e(L_{f1} + B^*L_{f2})]$ are constants. Now, applying Gronwall's inequality, we get

$$\chi(t) = 0.$$

Hence $u(t) = \vartheta(t)$ almost everywhere. Thus, the control problem (1.7)-(1.9) is $\mathcal{T}-$controllable. $\square$

5 Examples

This section contains examples to validate the derived results (existence and $\mathcal{T}-$controllability) of the considered systems.

5.1 Example

To prove the theoretical existence result, we shall consider the following impulsive fractional order partial differential inclusion of the form

$$\begin{aligned}
\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} &\in \frac{\partial^2 u(t,x)}{\partial y^2} + \int_{-\infty}^{t} e^{2(s-t)} \cos\left(\frac{u(s-\rho_1(s)\rho_2(\|u\|),x)}{16}\right) ds, \quad t \neq \frac{1}{2}, && (5.1)\\
u(t,0) &= u(t,\pi) = 0; \quad u'(t,0) = u'(t,\pi) = 0, \quad t \geq 0, && (5.2)\\
u(t,x) &= \phi(t,x); \quad u'(0,x) = u_0, \quad t \in (-\infty, 0], \quad x \in [0,\pi], && (5.3)\\
\Delta u|_{t=\frac{1}{2}} &= \int_{-\infty}^{\frac{1}{2}} g\left(\frac{1}{2}-s\right) u(s,x)\, ds; \quad \Delta u'|_{t=\frac{1}{2}} = \int_{-\infty}^{\frac{1}{2}} q\left(\frac{1}{2}-s\right) u(s,x)\, ds, && (5.4)
\end{aligned}$$

are fixed numbers and $\phi(t) \in \mathfrak{B}_e$. Let $X = L^2[0,\pi]$ and define the operator $A : D(A) \subset X \to X$ by $Aw = w''$ with the domain $D(A) := \{w \in X : \ w, w'$ are absolutely continuous, $w'' \in X$, $w(0) = 0 = w(\pi)\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

where $w_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in X and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} (\omega, \omega_n)\omega_n, \quad \text{for all } \omega \in X, \quad \text{and every } t > 0.$$

Let $h(s) = e^{2s},\ s < 0$ then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$ and define

$$\|\phi\|_{\mathfrak{B}_e} = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} \|\phi(\theta)\|_{L^2} ds.$$

Hence for $(t,\phi) \in [0,1] \times \mathfrak{B}_e$, where $\phi(\theta)(x) = \phi(\theta,x),\ (\theta,x) \in (-\infty,0] \times [0,\pi]$. We assume that $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous functions.

Set $u(t)(x) = u(t,x)$, and $\rho(t,\phi) = \rho_1(t)\rho_2(\|\phi(0)\|)$ we have

$$f(t,\phi)(x) = \int_{-\infty}^{0} e^{2(s)} \cos\left(\frac{\phi}{16}\right) ds.$$

Then with above setting the problem (5.1)-(5.4) can be written in the abstract form of equation (1.1)-(1.3). Further, we can estimate

$$\begin{aligned} \|f(t,\phi)(x) - f(t,\varphi)(x)\|_{L^2} &= \left[\int_0^{\pi} \left\{\int_{-\infty}^{0} e^{2(s)} \left\|\cos\left(\frac{\phi}{16}\right) - \cos\left(\frac{\varphi}{16}\right)\right\| ds\right\}^2 dx\right]^{\frac{1}{2}} \\ &\leq \frac{1}{16}\left[\int_0^{\pi} \left\{\int_{-\infty}^{0} e^{2(s)}(\|\phi-\varphi\|_{L^2}) ds\right\}^2 dx\right]^{\frac{1}{2}} \leq \frac{\sqrt{\pi}}{16}\|\phi-\varphi\|_{\mathfrak{B}_e}. \end{aligned}$$

This shows that the multivalued map f follows the assumption H_2. This implies that there exists at least one mild solution of problem (5.1)-(5.4).

5.2 Example

Consider the following fractional order functional differential equation

$$\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha} u(t,x) &= \frac{\partial^2}{\partial y^2} u(t,x) + \int_{-\infty}^{t} e^{2(\nu - t)} \frac{u(\nu - \sigma(\|u\|), x)}{24}\, d\nu \\ &+ \int_0^t \cos(t-s) \int_{-\infty}^{\xi} e^{2(\nu-\xi)} \frac{u(\nu - \sigma(\|u\|), x)}{25}\, d\nu\, ds, \\ &\quad (t,x) \in \cup_{i=1}^{N} [s_i, t_{i+1}] \times [0,\pi], \qquad (5.5) \\ u(t,0) &= u(t,\pi) = 0, \quad t \geq 0, \qquad (5.6) \\ u(t,x) + \sum_{k=1}^{r} c_k u(s_k,x) &= \phi(t,x), \quad t \in (-\infty, 0]; \quad u'(t,x) = 0, \quad x \in [0,\pi], \qquad (5.7) \\ u(t,x) &= G_i(t,y); \quad u'(t,x) = H_i(t,y), \quad t \in (t_i, s_i], \qquad (5.8) \end{aligned}$$

are fixed numbers and $\phi \in \mathfrak{B}_e$. Setting $u(t)(x) = u(t,x)$, and

$$\rho(t,\phi) = t - \sigma(\|\phi(0)\|), \quad (t,\phi) \in [0,T] \times \mathfrak{B}_e,$$

we have

$$f(t,\phi,B\phi) = \int_{-\infty}^{0} e^{2(\nu)} \frac{\phi}{24} d\nu + \int_0^t \cos(t-s) \int_{-\infty}^{0} e^{2(\nu)} \frac{\phi}{25}\, d\nu\, ds,$$

$$g_i(t,y) = G_i(t,y); \quad q_i(t,y) = H_i(t,y), \quad G(y) = \sum_{k=1}^{r} c_k u(s_k,x).$$

Then the above equations (5.5)-(5.8) can be written in the abstract form as (1.4)-(1.6). Furthermore, we can see that for $(t, \phi, B\phi), (t, \psi, B\psi) \in [0, T] \times \mathfrak{B}_e \times \mathfrak{B}_e$, may verify that

$$\begin{aligned}
\|f(t,\phi,B\phi) - f(t,\psi,B\psi)\|_{L^2} \leq & \Bigg[\int_0^\pi \left\{\int_{-\infty}^0 e^{2(s)} \left\|\frac{\phi}{24} - \frac{\psi}{24}\right\| ds\right\}^2 dy \\
& + \int_0^\pi \left\{\left\|\int_0^t \cos(t-s) \int_{-\infty}^0 e^{2(\nu)} \frac{\phi}{25} - \frac{\psi}{25} d\nu\, ds\right\|\right\} dy\Bigg]^{1/2} \\
& \leq \Bigg[\int_0^\pi \left\{\frac{1}{24}\int_{-\infty}^0 e^{2(s)} \sup\|\phi - \psi\| ds\right\}^2 dy \\
& + \int_0^\pi \left\{\frac{1}{25}\int_{-\infty}^0 e^{2(s)} \sup\|\phi - \psi\| ds\right\}^2 dy\Bigg]^{1/2} \\
& \leq \frac{\sqrt{\pi}}{24}\|\phi - \psi\| + \frac{\sqrt{\pi}}{25}\|\phi - \psi\|.
\end{aligned}$$

Hence, function f satisfies (H_3). Similarly, we can show that the functions g_i, q_i, $h(y)$ satisfy (H_4). All the condition of Theorem 3.2 have fulfilled, so we deduced that the system (5.5)-(5.8) has a unique mild solution on $[0, T]$.

5.3 Example

Consider the following example for fractional functional ordinary differential equation

$$\begin{aligned}
{}_0^C D_t^\alpha u(t) &= u(t) + \frac{e^t u(t - \sigma(u(t))) + 2}{1 + u^2(t - \sigma(u(t)))} + \int_0^t \sin(t-s) u(s - \sigma(u(s))) ds, \quad t \in (0,1], \quad (5.9) \\
u(t) &+ \sum_{k=1}^r c_k u(s_k) = \frac{1}{2}, \quad t \in (-\infty, 0], \quad u'(t) = 0, \quad (5.10) \\
u(t) &= \frac{u(t)}{16(1 + u(t))}; \quad u'(t) = \frac{u(t)}{25(1 + u(t))}, \quad t \in (1, 2], \quad (5.11)
\end{aligned}$$

where ${}_0^C D_t^\alpha$ is classical Caputo's fractional derivative of order $\alpha \in (1,2), 0 = t_0 = s_0 < t_1 = 1 < s_1 = 2$ are prefixed numbers and $\frac{1}{2} \in \mathfrak{B}_e$. Setting

$$\begin{aligned}
\rho(t, \varphi) &= t - \sigma(\varphi(0)), \\
f(t, \varphi, B\varphi) &= \frac{e^t \varphi + 2}{1 + \varphi^2} + \int_0^t \sin(t-s)\varphi\, ds, \\
g_i(t, y) &= \frac{u(t)}{16(1 + u(t))}; \quad q_i(t, y) = \frac{u(t)}{25(1 + u(t))}, \quad G(y) = \sum_{k=1}^r c_k u(s_k),
\end{aligned}$$

then the problem (5.9)-(5.11) can be written in the abstract form as (1.4)-(1.6), which implies that the system (5.9)-(5.11) has a unique mild solution on $[0, 2]$.

5.4 Example

Consider the following control system

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\int_{-\infty}^{t} e^{4(\nu-t)}\frac{u(\nu-\sigma(\|u\|),x)}{12}\,d\nu+14\varpi(t,x) \\ &+ \int_0^t \sin(t-s)\int_{-\infty}^{\xi} e^{4(\nu-\xi)}\frac{u(\nu-\sigma(\|u\|),x)}{28}\,d\nu\,ds, \quad (t,x)\in\cup_{i=1}^{N}[s_i,t_{i+1}]\times[0,\pi],\end{aligned} \tag{5.12}$$

with initial, history and impulsive conditions given as (5.6)-(5.8). With these settings as given in example 5.2, the problem (5.12) with conditions (5.6)-(5.8) can be written in the abstract form of equation (1.7)-(1.9). Therefore the problem (5.12) is $\mathcal{T}-$controllable on J.

Thus, examples provided in this paper demonstrate the authenticity of our results. In first example, we considered fractional order partial differential inclusion with instantaneous impulsive and showed that considered problem has least one mild solution. Non-instantaneous impulse with partial derivative and nonlocal condition is taken in second examples and proved that there exists a unique mild solution for it. In third example, we considered the functional ordinary differential equation with infinite delay and demonstrate the uniqueness of mild solution for the system.

6 Conclusion

In this investigation, we observed that the Definition 2.8 is more reasonable and suitable by using the generalized Caputo's derivative in compare to classical and it is generalized form. Furthermore, we have proved the existence, uniqueness and continuous dependence results of mild solutions for fractional differential inclusion and equations with state dependent delay subject to instantaneous and non-instantaneous impulse. We showed $\mathcal{T}-$controllability. Also, we have illustrated the existence and $\mathcal{T}-$controllability theory from some examples.

7 Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this article.

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