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On a class of fractional $p(\cdot,\cdot)-$Laplacian problems with sub-supercritical nonlinearities

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ABSTRACT

This paper is devoted to study a class of nonlocal variable exponent problems involving fractional $p(\cdot,\cdot)$-Laplacian operator. Under appropriate conditions, some new results on the existence and nonexistence of solutions are established via variational approach and Pohozaev's fibering method.

RESUMEN

Este artículo está dedicado al estudio de una clase de problemas no locales con exponente variable que involucran al operador $p(\cdot,\cdot)$-Laplaciano fraccionario. Bajo condiciones apropiadas se establecen algunos resultados nuevos sobre la existencia y no existencia de soluciones a través de un enfoque variacional y el método de fibración de Pohozaev.

Keywords and Phrases: Fractional $p(\cdot,\cdot)$-Laplacian operator; sub-supercritical nonlinearities; variational methods; Pohozaev's fibering method.

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1 Introduction

In the present paper, we are interested in the existence of solutions for the following problem

$$\begin{cases} M\left(T_u\right)\left(-\Delta_{p(\cdot,\cdot)}\right)^s u + w(x)|u|^{p(x,x)-2}u = \lambda a(x)|u|^{q(x)-2}u - \varepsilon b(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \qquad (P_{\lambda,\varepsilon}^M)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$, $q, r : \overline{\Omega} \to (1, +\infty)$ are continuous functions, $s \in (0, 1)$ with $N > sp(x, y)$ for all $(x, y) \in \overline{\Omega}$, $\lambda, \varepsilon > 0$ are parameters, $a, b, w \in L^\infty(\Omega)$, M models a Kirchhoff coefficient,

$$T_u = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x, y)|x - y|^{N+sp(x,y)}} dx\, dy$$

and $\left(-\Delta_{p(\cdot,\cdot)}\right)^s$ is the fractional $p(\cdot, \cdot)$-Laplacian defined as

$$\left(-\Delta_{p(x,\cdot)}\right)^s u(x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy, \quad x \in \mathbb{R}^N,$$

where $p.v.$ is used as abbreviation in the principal value sense.

In the past few decades, nonlinear problems involving nonlocal and pseudo-differential operators have gained considerable popularity and importance. The interest in investigating such problems is stimulated by their applications in numerous fields of applied sciences, such as the description of some phenomena in physics and engineering, population dynamics, finance, chemical reaction design, optimization, minimal surfaces and game theory (see [12, 29, 32, 38]). Moreover, differential equations and variational problems with variable exponent have a strong physical motivation. As can be seen in [5, 22, 35], they emerge from the mathematical description of the dynamics fluids like the electrorhelogical and the thermorheological. They also appear in elastic mechanics, image restoration and biology (see [14, 16, 37, 43]). Some recent results on $p(\cdot, \cdot)$-Laplacian problems can be found in [1, 4, 6, 13, 15, 19, 25, 27, 30, 36, 42].

Recently, great attention has been focused in extending some results on $p(\cdot, \cdot)$-Laplacian problems to the fractional case. For example, we cite [11, 26]. In [26] Kaufmann *et al.* introduced the fractional Sobolev space with variable exponent, and established the existence and uniqueness of solutions for the fractional $p(\cdot, \cdot)$-Laplacian problem

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^s u + |u|^{q(x)-2}u = f(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Bahrouni *et al.* [11] established some results on the following fractional $p(\cdot, \cdot)$-Laplacian equation with the nonlocal Robin boundary condition

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^s u + |u|^{p(x,x)-2}u = f(x,u) & \text{in} \quad \Omega \\ \mathcal{N}_{s,p(\cdot,\cdot)}u + \beta(x)|u|^{p(x,x)-2}u = g(x) & \text{on} \quad \mathbb{R}^N\backslash\overline{\Omega}, \end{cases}$$

where $\mathcal{N}_{s,p(\cdot,\cdot)}$ is the nonlinear modification of the following Neumann boundary condition

$$\mathcal{N}_s u(x) := c_{N,s}\int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \quad x \in \mathbb{R}^N\backslash\overline{\Omega},$$

which was first introduced by Dipierro *et al.* in [17]. The latter nonlocal normal derivative is used in [18] to describe the diffusion of a biological population living in an ecological niche and subject to both local and nonlocal dispersals.

We also refer the reader to [9, 10, 23, 24] for more information.

Problem $(P^M_{\lambda,\varepsilon})$ is a fractional version related to the following hyperbolic equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which was initially introduced by Kirchhoff [28] as a generalization of the classical D'Alembert wave equation taking into consideration the change in length of the strings produced by transverse vibrations. For additional discussions and physical phenomena described by nonlinear vibration theory, we mention [31]. It was mainly after the work [21], where Fiscella and Valdinoci proposed a stationary fractional Kirchhoff model, that the existence and multiplicity of solutions for Kirchhoff-type fractional p-Laplacian and $p(\cdot,\cdot)$-Laplacian problems were well investigated by many authors, one can see [8, 34, 39, 41, 44]. In particular, Zhang *et al.* [41] studied the following problem

$$\begin{cases} M\left(T_u\right)\left(-\Delta_{p(\cdot,\cdot)}\right)^s u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N\backslash\Omega. \end{cases} \tag{1.1}$$

By means of variational methods and mountain pass theorem, they proved the existence of at least one nontrivial solution for (1.1). In [2], Akkoyunlu and Ayazoglu considered the following fractional p-Kirchhoff problem with potential

$$M(||u||^p)\left((-\Delta)^s_p u + V(x)|u|^{p-2}u\right) = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where

$$||u||^p = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u|^p dx.$$

By using the variational approach, (S_+) mapping theory and Krasnoselskii's genus theory, the authors have established the existence of infinitely many nontrivial weak solutions. After that, the equation (1.2) was generalized by Ayazoglu *et al.* in [7] considering the following fractional Schrödinger-Kirchhoff equation

$$M(A_{s,q(\cdot),p(\cdot,\cdot)}(u))\left((-\Delta)^s_{p(\cdot,\cdot)}u + V(x)|u|^{q(x)-2}u\right) = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where

$$A_{s,q(\cdot),p(\cdot,\cdot)}(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy + \int_{\mathbb{R}^N} \frac{V(x)}{q(x)}|u|^{q(x)}dx,$$

$N \geq 2$, $M : (0,+\infty) \to (1,\infty)$ is a continuous and monotone Kirchhoff function, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and V is a potential function. They obtained the existence and multiplicity of solutions by applying the variational approach combined with Mountain Pass Theorem and Krasnoselskii's genus theory.

Inspired by the above cited papers, we will consider problem $(P^M_{\lambda,\varepsilon})$ with sub-supercritical nonlinearities, and prove the existence of solutions via the variational methods combined with the fibering method that was introduced by Pohozaev [33]. We also give the behavior of the solution for problem $(P_{\lambda,\varepsilon})$, and so of the energy functional associated, as $\varepsilon \to 0$. The Pohozaev's fibering method is centered on representing solutions in the form $u = tv$, where t is a real number ($t \neq 0$), and $v \in X \setminus \{0\}$, satisfying the condition:

$$\frac{\partial \Phi}{\partial t}(t,v) = 0. \tag{1.3}$$

Here, Φ denotes a functional defined on $\mathbb{R} \times X$. Consequently, the fundamental concept of the Pohozaev's fibering method involves embedding the space X of the original problem within the larger space $\mathbb{R} \times X$ and subsequently exploring the new problem of conditional solvability within the $\mathbb{R} \times X$ space, subject to the condition (1.3).

2 Preliminaries

At first, we give some useful notations and basic results on variable exponent Lebesgue spaces that will be used in proving the main theorems (see [20]). We denote by $C_+(\overline{\Omega})$ the set of all continuous functions $q : \overline{\Omega} \to (1,\infty)$. For $q \in C_+(\overline{\Omega})$, we write

$$q^+ := \max_{x\in\overline{\Omega}} q(x) \quad \text{and} \quad q^- := \min_{x\in\overline{\Omega}} q(x).$$

Define the variable exponent Lebesgue space as follows:

$$L^{q(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u|^{q(x)}dx < \infty \right\}.$$

$L^{q(\cdot)}(\Omega)$ endowed with the norm

$$||u||_{q(\cdot)} = \inf\left\{\tau > 0 : \int_\Omega \left|\frac{u}{\tau}\right|^{q(x)} dx \leq 1\right\}.$$

is a separable and reflexive Banach space. Let $L^{q'(\cdot)}(\Omega)$ be the conjugate space of $L^{q(\cdot)}(\Omega)$ with $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$. Then the following Hölder-type inequality holds.

Lemma 2.1 ([20]). *Let $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$. Then*

$$\int_\Omega |uv|\, dx \leq \left(\frac{1}{q^-} + \frac{1}{(q')^-}\right) ||u||_{q(\cdot)} ||v||_{q'(\cdot)}.$$

On the space $L^{q(\cdot)}(\Omega)$, we consider the modular function given by

$$\rho_{q(\cdot)}(u) = \int_\Omega |u|^{q(x)} dx.$$

Lemma 2.2 ([20]). *For any $u \in L^{q(\cdot)}(\Omega)$, we have*

$$\min\left(||u||_{q(\cdot)}^{q^-}, ||u||_{q(\cdot)}^{q^+}\right) \leq \rho_{q(\cdot)}(u) \leq \max\left(||u||_{q(\cdot)}^{q^-}, ||u||_{q(\cdot)}^{q^+}\right).$$

Lemma 2.3 ([20]). *Let $u \in L^{q(\cdot)}(\Omega)$ and $\{u_n\} \subset L^{q(\cdot)}(\Omega)$. Then the following properties are equivalent:*

(1) $\lim_{n\to\infty} ||u_n - u||_{q(\cdot)} = 0$;

(2) $\lim_{n\to\infty} \rho_{q(\cdot)}(u_n - u) = 0$.

Lemma 2.4 ([3]). *Let $q, r \in C_+(\overline{\Omega})$ with $q(x) \leq r(x)$ in Ω and $u \in L^{r(\cdot)}(\Omega)$. Then $|u|^{q(\cdot)} \in L^{\frac{r(\cdot)}{q(\cdot)}}(\Omega)$ and*

$$\||u|^{q(\cdot)}\|_{\frac{r(\cdot)}{q(\cdot)}} \leq \max\left(\|u\|_{r(\cdot)}^{q^+}, \|u\|_{r(\cdot)}^{q^-}\right).$$

Next, we define the convenient variable exponent fractional Sobolev space to supply a variational structure for handling our problems. Let $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ be as mentioned above and put

$$\overline{p}(x) = p(x, x) \quad \text{for all} \quad x \in \overline{\Omega}.$$

Let $W^{s,p(\cdot,\cdot)}(\Omega)$ be the variable exponent fractional Sobolev space defined as follows:

$$\mathbb{W} := W^{s,p(\cdot,\cdot)}(\Omega) = \left\{u \in L^{\overline{p}(\cdot)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dx\, dy < \infty, \text{ for some } \xi > 0\right\}.$$

Equip $\mathbb{W}$ with the norm

$$||u||_{\mathbb{W}} = [u]_{\mathbb{W}} + ||u||_{\overline{p}(\cdot)},$$

where

$$[u]_{\mathbb{W}} = \inf\left\{\xi > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \le 1\right\}.$$

Then $(\mathbb{W}, ||u||_{\mathbb{W}})$ is a Banach space. For any $u \in \mathbb{W}$, we set

$$\rho_{p,\overline{p}}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega |u|^{\overline{p}(x)}dx$$

and

$$||u||_{p,\overline{p}} = \inf\left\{\xi > 0 : \rho_{p,\overline{p}}\left(\frac{u}{\xi}\right) \le 1\right\}.$$

The norm $||\cdot||_{p,\overline{p}}$ is equivalent to $||\cdot||_{\mathbb{W}}$. Furthermore, from [41, Lemma 2.2], $(\mathbb{W}, ||\cdot||_{\mathbb{W}})$ is uniformly convex and hence $\mathbb{W}$ is a reflexive Banach space. The following lemma states the compactness of the embedding $\mathbb{W}$ into the variable exponent Lebesgue spaces.

Lemma 2.5 ([40, 41])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $s \in (0,1)$. Assume that $p : \overline{\Omega}\times\overline{\Omega} \to (1,\infty)$ is continuous and symmetric (i.e. $p(x,y) = p(y,x)$) with $sp(x,y) < N$ for all $x, y \in \overline{\Omega}$. Let $q \in C_+(\overline{\Omega})$ such that*

$$q(x) < p_s^*(x) := \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \quad \textit{for all} \quad x \in \overline{\Omega}.$$

Then, there exists $C = C(N, s, p, q, \Omega)$ such that

$$||u||_{q(\cdot)} \le C||u||_{\mathbb{W}} \quad \textit{for all} \quad u \in \mathbb{W}.$$

Therefore, the space $\mathbb{W}$ is continuously embedded into $L^{q(\cdot)}(\Omega)$. Moreover, this embedding is compact.

Due to the presence of the Dirichlet boundary condition $u = 0$ in $\mathbb{R}^N\backslash\Omega$, we need to encode this condition in the weak formulation of $(P^M_{\lambda,\varepsilon})$ and $(P_{\lambda,\varepsilon})$. For this, let us define the new space

$$\mathbb{X} := X^{s,p(\cdot,\cdot)}(\Omega) = \left\{u : \mathbb{R}^N \to \mathbb{R},\ u|_\Omega \in L^{\overline{p}(\cdot)}(\Omega), \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < \infty,\ \text{for some } \xi > 0\right\},$$

where $\mathcal{Q} = \mathbb{R}^N \times \mathbb{R}^N\backslash(\Omega^c \times \Omega^c)$. Endow $\mathbb{X}$ with the norm

$$||u||_{\mathbb{X}} = [u]_{\mathbb{X}} + ||u||_{\overline{p}(\cdot)},$$

where

$$[u]_{\mathbb{X}} = \inf\left\{\xi > 0 : \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \le 1\right\}.$$

In the same way $(\mathbb{X}, ||\cdot||_{\mathbb{X}})$ is a separable reflexive Banach space.

Since the variable exponent p, $\overline{p}$ and q are continuous, we can extend p to $\mathbb{R}^N\times\mathbb{R}^N$ and $\overline{p}, q$ to

$\mathbb{R}^N$ continuously with conditions given in Lemma 2.5. Let $\mathbb{X}_0$ be the linear space:

$$\mathbb{X}_0 = \{u \in \mathbb{X} : u = 0 \quad \text{a.e. in } \mathbb{R}^N \backslash \Omega\}$$

equipped with the norm

$$||u||_{\mathbb{X}_0} = [u]_{\mathbb{X}} = \inf \left\{ \xi > 0 : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy \leq 1 \right\}.$$

Obviously, $(\mathbb{X}_0, ||\cdot||_{\mathbb{X}_0})$ is a reflexive Banach space. Set

$$\rho_0(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy \quad \text{for all } u \in \mathbb{X}_0.$$

Lemma 2.6 ([41])**.** *For all $u, u_n \in \mathbb{X}_0$, the following properties hold true:*

(1) $||u||_{\mathbb{X}_0} > 1 \implies ||u||_{\mathbb{X}_0}^{p^-} \leq \rho_0(u) \leq ||u||_{\mathbb{X}_0}^{p^+}$;

(2) $||u||_{\mathbb{X}_0} \leq 1 \implies ||u||_{\mathbb{X}_0}^{p^+} \leq \rho_0(u) \leq ||u||_{\mathbb{X}_0}^{p^-}$;

(3) $||u_n - u||_{\mathbb{X}_0} \to 0 \iff \rho_0(u_n - u) \to 0.$

Lemma 2.7 ([41])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $s \in (0,1)$. Assume that $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ is continuous and symmetric with $sp(x,y) < N$ for all $x, y \in \overline{\Omega}$. Let $q \in C_+(\overline{\Omega})$ such that*

$$q(x) < p_s^*(x) := \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \quad \textit{for all} \quad x \in \overline{\Omega}.$$

Then, there exists $C = C(N, s, p, q, \Omega) > 0$ such that

$$||u||_{q(\cdot)} \leq C||u||_{\mathbb{X}_0} \quad \textit{for all} \quad u \in \mathbb{X}_0.$$

Therefore, the space $\mathbb{X}_0$ is continuously embedded into $L^{q(\cdot)}(\Omega)$. Moreover, this embedding is compact.

Remark 2.8. *Since $1 < \overline{p}(x) = p(x,x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, by Lemma 2.7, the norms $||\cdot||_{\mathbb{X}_0}$ and $||\cdot||_{\mathbb{X}}$ are equivalent in $\mathbb{X}_0$.*

We look for solutions of problems $(P_{\lambda,\varepsilon}^M)$ and $(P_{\lambda,\varepsilon})$ in the separable reflexive Banach space $X = \mathbb{X}_0 \cap L^{r(\cdot)}(\Omega)$ which is equipped with the norm

$$||u||_X = ||u||_{\mathbb{X}} + ||u||_{r(\cdot)}.$$

3 Hypotheses and main results

Before stating what we believe that are the main contributions, we first list some assumptions on the data of $(P^M_{\lambda,\varepsilon})$. Concerning the Kirchhoff function $M:\mathbb{R}^+\to\mathbb{R}^+$, we use the following two assumptions:

(M_0) M is a C^1 nondecreasing function;

(M_1) M is a continuous function such that $M(t)\geq m_0>0$ for all $t>0$.

For the functions a,b,w,p,q and r, we make the following hypotheses:

(H_1) $q,r:\overline{\Omega}\to(1,\infty)$ and $p:\overline{\Omega}\times\overline{\Omega}\to(1,\infty)$ are continuous such that $sp(x,y)<N$, $p(x,y)=p(y,x)$ and $q(x)<p_s^*(x)<r^-:=\min\limits_{x\in\overline{\Omega}} r(x)$ for all $(x,y)\in\overline{\Omega}\times\overline{\Omega}$, where

$$p_s^*(x):=\frac{Np(x,x)}{N-sp(x,x)};$$

(H_2) $a,b,w\in L^\infty(\Omega)$ with b and w are nonnegative and $|\Omega_a^+|>0$, where $\Omega_a^+=\{x\in\Omega : a(x)>0\}$;

(H_3) $ab^{-\frac{q(\cdot)}{r(\cdot)}}\in L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\Omega_a^+)$;

(H_4) $q^-(r^--q^+)<p^+(r^--p^-)$ and $r^+\leq\min\left\{\frac{q^-p^+(q^+-p^-)}{p^+(r^--p^-)-q^-(r^--q^+)},\frac{q^-(r^--p^-)}{q^+-p^-}\right\}$;

The main results can be stated as follows.

Theorem 3.1. *Assume that $(M_0)-(M_1)$ and $(H_1)-(H_2)$ hold. If $q^+<p^-$, then problem $(P^M_{\lambda,\varepsilon})$ admits at least one nontrivial solution.*

Theorem 3.2. *Assume that (M_1) and $(H_1)-(H_2)$ hold. If $p^+<q^-$, $a(x)\geq 0$ for a.e. $x\in\Omega$ and $b(x)>b_0>0$ for a.e. $x\in\Omega$, then for all $\varepsilon>0$ there exists $\lambda_\varepsilon>0$ such that problem $(P^M_{\lambda,\varepsilon})$ has no nontrivial solution for all $\lambda\in(0,\lambda_\varepsilon)$.*

The following two theorems concern problem $(P^M_{\lambda,\varepsilon})$ with $M\equiv 1$, that is,

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^s u+w(x)|u|^{p(x,x)-2}u=\lambda a(x)|u|^{q(x)-2}u-\varepsilon b(x)|u|^{r(x)-2}u & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega.\end{cases}\qquad (P_{\lambda,\varepsilon})$$

Theorem 3.3. *Assume that $(H_1)-(H_4)$ hold. If $q(\cdot)=q$ is constant with $p^+<q$ or $p(\cdot)$ and $r(\cdot)$ are constants, then for all $\varepsilon>0$ there exists $\lambda^*_\varepsilon>0$ such that problem $(P_{\lambda,\varepsilon})$ admits at least one nontrivial solution provided $\lambda>\lambda^*_\varepsilon$.*

Theorem 3.4. *Assume that $(H_1)-(H_4)$ hold and $q(\cdot)=q$ is constant with $p^+<q$ or $p(\cdot)$ and $r(\cdot)$ are constants. Let $\varepsilon_0>0$ and $\lambda>\lambda^*_{\varepsilon_0}$. Then, there exists $\varepsilon_1\in(0,\varepsilon_0)$ such that for all*

$\varepsilon \in (0, \varepsilon_1)$, problem $(P_{\lambda,\varepsilon})$ admits at least one nontrivial solution u_ε verifying $||u_\varepsilon||_X \to +\infty$ and $\mathcal{I}_\varepsilon(u_\varepsilon) \to -\infty$ as $\varepsilon \to 0$, where $\mathcal{I}_\varepsilon$ is the associated energy functional to $(P_{\lambda,\varepsilon})$.

Remark 3.5. *The conclusions of Theorems 3.1 and 3.2 also hold for problem $(P_{\lambda,\varepsilon})$.*

4 Proof of theorems

Proof of Theorem 3.1. It is well known that the weak solution of $(P^M_{\lambda,\varepsilon})$ corresponds to the critical point of the energy functional defined on X by

$$\begin{aligned}\mathcal{I}_\varepsilon(u) &= \widehat{M}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right) + \int_\Omega \frac{w(x)}{\overline{p}(x)}|u|^{\overline{p}(x)}dx \\ &\quad -\lambda\int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}dx + \varepsilon\int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)}dx, \end{aligned} \tag{4.1}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$. By standard arguments, one can verify that $\mathcal{I}_\varepsilon \in C^1(X,\mathbb{R})$. For any $(t,v) \in (0,\infty)\times X$, we define

$$\begin{aligned}\Phi_\varepsilon(t,v) &:= \mathcal{I}_\varepsilon(tv) \\ &= \widehat{M}\left(\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right) + \int_\Omega \frac{w(x)}{\overline{p}(x)}t^{\overline{p}(x)}|v|^{\overline{p}(x)}dx \\ &\quad -\lambda\int_\Omega \frac{a(x)}{q(x)}t^{q(x)}|v|^{q(x)}dx + \varepsilon\int_\Omega \frac{b(x)}{r(x)}t^{r(x)}|v|^{r(x)}dx.\end{aligned}$$

Observe that if $u = tv$ is a nontrivial critical of $\mathcal{I}_\varepsilon$, then $\dfrac{\partial\Phi_\varepsilon}{\partial t}(t,v) = 0$. Moreover, if for each $v \in X\setminus\{0\}$, there is a unique $t = t(v)$ satisfying

$$\frac{\partial\Phi_\varepsilon}{\partial t}(t,v) = 0 \tag{4.2}$$

and $t : v \mapsto t(v)$ is continuously differentiable on $X\setminus\{0\}$, we can infer that

$$\widetilde{\mathcal{I}}_\varepsilon(v) := \mathcal{I}_\varepsilon(t(v)v)$$

is a well-defined C^1 functional. The following result plays a key role in the proof of our main theorem.

Lemma 4.1 ([33])**.** *Let $\Psi : X \to \mathbb{R}$ be a functional of class C^1 on $X\setminus\{0\}$ verifying*

$$\langle \Psi'(v), v\rangle \neq 0 \quad \textit{if} \quad \Psi(v) = 1.$$

If v is a conditional critical point of $\widetilde{I}_\varepsilon$ under the constraint $\Psi(v) = 1$, then $u := t(v)v$ is a critical point of $\mathcal{I}_\varepsilon$.

Consider the functional $\Psi_\varepsilon : X \to \mathbb{R}$ given by

$$\begin{aligned}\Psi_\varepsilon(v) &= M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + \int_\Omega w(x)|v|^{\overline{p}(x)}dx + \varepsilon\int_\Omega b(x)|v|^{r(x)}dx. \qquad (4.3)\end{aligned}$$

It is obvious that Ψ_ε satisfies hypotheses of Lemma 4.1, therefore the problem of finding solutions of $(P^M_{\lambda,\varepsilon})$ will be reduced to that of locating the critical points of $\widetilde{I}_\varepsilon$ on the set

$$\mathcal{U}_\varepsilon = \{v \in X : \Psi_\varepsilon(v) = 1\}.$$

Note that (4.2) is equivalent to

$$\begin{aligned}\varphi_v(t) &:= M\left(\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + \int_\Omega t^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx + \varepsilon\int_\Omega t^{r(x)}b(x)|v|^{r(x)}dx - \lambda\int_\Omega t^{q(x)}a(x)|v|^{q(x)}dx \\ &= 0. \qquad (4.4)\end{aligned}$$

Let

$$\Theta_a := \left\{v \in X : \int_\Omega a(x)|v|^{q(x)}dx > 0\right\}.$$

Claim 4.2. *For any $v \in \Theta_a$, equation (4.4) admits a unique positive solution $t(v)$. Moreover, $\varphi_v(t) < 0$ for all $t < t(v)$ and $\varphi_v(t) > 0$ for all $t > t(v)$.*

Indeed, by (M_0), for all $t \geq 1$,

$$\begin{aligned}\varphi_v(t) &\geq t^{p^-}M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + t^{p^-}\int_\Omega w(x)|v|^{\overline{p}(x)}dx + \varepsilon t^{r^-}\int_\Omega b(x)|v|^{r(x)}dx - \lambda t^{q^+}\int_\Omega a(x)|v|^{q(x)}dx\end{aligned}$$

and for all $0 < t \leq 1$,

$$\begin{aligned}\varphi_v(t) &\leq t^{p^-}M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + t^{p^-}\int_\Omega w(x)|v|^{\overline{p}(x)}dx + \varepsilon t^{r^-}\int_\Omega b(x)|v|^{r(x)}dx - \lambda t^{q^+}\int_\Omega a(x)|v|^{q(x)}dx\end{aligned}$$

Since $q^+ < p^-$, we can choose $t_\infty > 1$ such that $\varphi_v(t_\infty) > 0$ and by (H_2), we can find $0 < t_0 \leq 1$ satisfying $\varphi_v(t_0) \leq 0$. Therefore, by virtue of the continuity of φ_v, equation (4.4) has at least one solution $t(v) > 0$. The uniqueness of $t(v)$ follows from (H_2) and using the fact that $q^+ < p^-$ and

M is nondecreasing. Furthermore, for all $t < t(v)$,

$$\begin{aligned}
&M\left(\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&+\int_\Omega t^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t^{r(x)}b(x)|v|^{r(x)}dx \\
&<\lambda\int_\Omega t^{q(x)}a(x)|v|^{q(x)}dx
\end{aligned} \tag{4.5}$$

and for all $t > t(v)$,

$$\begin{aligned}
&M\left(\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&+\int_\Omega t^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t^{r(x)}b(x)|v|^{r(x)}dx \\
&>\lambda\int_\Omega t^{q(x)}a(x)|v|^{q(x)}dx.
\end{aligned} \tag{4.6}$$

Then, the function $t : v \mapsto t(v)$ is well defined, and by applying the implicit function theorem, we deduce that $t(\cdot) \in C^1(X\setminus\{0\},(0,+\infty))$. If $v \in \mathcal{U}_\varepsilon \cap \Theta_a$ and $t(v) \geq 1$, it holds from (H_1), the nondecreasing of M and (4.4) that

$$\begin{aligned}
t(v)^{p^-} &= t(v)^{p^-}\Psi_\varepsilon(v) \\
&= M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t(v)^{p^-}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&\quad +t(v)^{p^-}\int_\Omega w(x)|v|^{\overline{p}(x)}dx+\varepsilon t(v)^{p^-}\int_\Omega b(x)|v|^{r(x)}dx \\
&\leq M\left(\int_{\mathbb{R}^{2N}} t(v)^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t(v)^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&\quad +\int_\Omega t(v)^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t(v)^{r(x)}b(x)|v|^{r(x)}dx \\
&= \lambda\int_\Omega t(v)^{q(x)}a(x)|v|^{q(x)}dx \\
&\leq \lambda t(v)^{q^+}\int_\Omega a(x)|v|^{q(x)}dx,
\end{aligned}$$

thus

$$t(v)^{p^- - q^+} \leq \lambda\int_\Omega a(x)|v|^{q(x)}dx.$$

This shows that $t(\cdot)$ is bounded in $\mathcal{U}_\varepsilon \cap \Theta_a$. Since M is nondecreasing, $\widehat{M}(\tau) \leq \tau M(\tau)$ for all $\tau \geq 0$.

Then, by (H_1) and (4.4) for any $v \in \mathcal{U}_\varepsilon \cap \Theta_a$, we have

$$\begin{aligned}
\widetilde{I}_\varepsilon(v) &\leq \frac{1}{p^-} M\left(\int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right) \int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&\quad + \frac{1}{p^-}\int_\Omega t^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)}dx + \frac{\varepsilon}{r^-}\int_\Omega t^{r(x)} b(x)|v|^{r(x)}dx - \frac{\lambda}{q^+}\int_\Omega t^{q(x)} a(x)|v|^{q(x)}dx \\
&= \left(\frac{1}{p^-} - \frac{1}{q^+}\right) M\left(\int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right) \int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&\quad + \left(\frac{1}{p^-} - \frac{1}{q^+}\right)\int_\Omega t^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)}dx + \varepsilon\left(\frac{1}{r^-} - \frac{1}{q^+}\right)\int_\Omega t^{r(x)} b(x)|v|^{r(x)}dx \\
&< 0.
\end{aligned}$$

Then

$$\alpha_0 := \inf_{v \in \mathcal{U}_\varepsilon \cap \Theta_a} \widetilde{I}_\varepsilon(v) < 0.$$

Let $\{v_n\} \subset \mathcal{U}_\varepsilon \cap \Theta_a$ be a sequence such $\widetilde{I}_\varepsilon(v_n) \to \alpha_0$. From (M_1), we have

$$1 = \Psi_\varepsilon(v_n) \geq m_0 \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy,$$

thus from Lemma 2.6, we deduce that $\{v_n\}$ is bounded in $\mathbb{X}_0$. Therefore, up to a subsequence, we may assume that

$$\begin{cases} v_n \rightharpoonup v_0 \text{ in } \mathbb{X}_0, \\ v_n \to v_0 \text{ in } L^{\overline{p}(\cdot)}(\Omega) \text{ and } L^{q(\cdot)}(\Omega), \\ v_n \to v_0 \text{ a.e. in } \Omega. \end{cases} \tag{4.7}$$

We may also assume that $t(v_n) \to t_0$, since $\{t(v_n)\}$ is bounded. Then

$$\widehat{M}\left(\int_{\mathbb{R}^{2N}} t_0^{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right) \leq \liminf_{n\to+\infty} \widehat{M}\left(\int_{\mathbb{R}^{2N}} t(v_n)^{p(x,y)} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right),$$

$$\lim_{n\to+\infty}\int_\Omega \frac{t(v_n)^{\overline{p}(x)} w(x)}{p(x)}|v_n|^{\overline{p}(x)}dx = \int_\Omega \frac{t_0^{\overline{p}(x)} w(x)}{\overline{p}(x)}|v_0|^{\overline{p}(x)}dx,$$

$$\lim_{n\to+\infty}\int_\Omega \frac{t(v_n)^{q(x)} a(x)}{q(x)}|v_n|^{q(x)}dx = \int_\Omega \frac{t_0^{q(x)} a(x)}{q(x)}|v_0|^{q(x)}dx$$

and

$$\int_\Omega \frac{t_0^{r(x)} b(x)}{r(x)}|v_0|^{r(x)}dx \leq \liminf_{n\to+\infty}\int_\Omega \frac{t(v_n)^{r(x)} b(x)}{r(x)}|v_n|^{r(x)}dx.$$

Therefore

$$\mathcal{I}_\varepsilon(t_0 v_0) \leq \liminf_{n\to+\infty} \mathcal{I}_\varepsilon(t(v_n)v_n) = \liminf_{n\to+\infty} \widetilde{I}_\varepsilon(v_n) = \alpha_0 < 0, \tag{4.8}$$

from which, we deduce that $v_0 \neq 0$ and $t_0 > 0$. Recall that the pair $(t(v_n), v_n)$ verifies (4.4), so by sending n to $+\infty$ and using (4.7), we arrive at

$$M\left(\int_{\mathbb{R}^{2N}} t_0^{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t_0^{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \tag{4.9}$$

$$+\int_\Omega t_0^{\overline{p}(x)} w(x)|v_0|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t_0^{r(x)} b(x)|v_0|^{r(x)}dx \tag{4.10}$$

$$\le \lambda\int_\Omega t_0^{q(x)} a(x)|v_0|^{q(x)}dx. \tag{4.11}$$

Thus $\int_\Omega a(x)|t_0v_0|^{q(x)}dx>0$. Furthermore, $t_0v_0\in L^{r(\cdot)}(\Omega)$, and hence $t_0v_0\in X$. In view of Claim 4.2 and (4.9), we have $t_0\le t(v_0)$. Suppose by contradiction that $t_0<t(v_0)$. Let $\psi_{v_0}: t\mapsto \mathcal{I}_\varepsilon(tv_0)$. Then $t\psi'_{v_0}(t)=\varphi_{v_0}(t)$, therefore by Claim 4.2, $t\psi'_{v_0}(t)<0$ for all $0<t<t(v_0)$, which yields that the function ψ_{v_0} is decreasing on $[0,t(v_0)]$. It follows from (4.8) that

$$\widetilde{\mathcal{I}}_\varepsilon(v_0)=\mathcal{I}_\varepsilon(t(v_0)v_0)<\mathcal{I}_\varepsilon(t_0v_0)\le\alpha_0. \tag{4.12}$$

By definition of $t(\cdot)$, for any $\tau>0$, we have

$$\begin{aligned}&M\left(\int_{\mathbb{R}^{2N}} \frac{t(\tau v_0)^{p(x,y)}|\tau(v_0(x)-v_0(y))|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t(\tau v_0)^{p(x,y)}\frac{|\tau(v_0(x)-v_0(y))|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\&+\int_\Omega t(\tau v_0)^{\overline{p}(x)}w(x)|\tau v_0|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t(\tau v_0)^{r(x)}b(x)|\tau v_0|^{r(x)}dx\\&=\lambda\int_\Omega t(\tau v_0)^{q(x)}a(x)|\tau v_0|^{q(x)}dx,\end{aligned}$$

so that

$$\begin{aligned}&M\left(\int_{\mathbb{R}^{2N}} \frac{(\tau t(\tau v_0))^{p(x,y)}|v_0(x)-v_0(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} (\tau t(\tau v_0))^{p(x,y)}\frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\&+\int_\Omega (\tau t(\tau v_0))^{\overline{p}(x)}w(x)|v_0|^{\overline{p}(x)}dx+\varepsilon\int_\Omega (\tau t(\tau v_0))^{r(x)}b(x)|v_0|^{r(x)}dx\\&=\lambda\int_\Omega (\tau t(\tau v_0))^{q(x)}a(x)|v_0|^{q(x)}dx.\end{aligned}$$

Hence, by the uniqueness of the solution $t(v_0)$ of equation (4.4), we have

$$\tau t(\tau v_0)=t(v_0). \tag{4.13}$$

We next choose $\tau>0$ such that $\tau v_0\in\mathcal{U}_\varepsilon$. From (4.12) and (4.13), we obtain

$$\widetilde{\mathcal{I}}_\varepsilon(\tau v_0)=\mathcal{I}_\varepsilon(t(\tau v_0)\tau v_0)=\mathcal{I}_\varepsilon(t(v_0)v_0)=\widetilde{\mathcal{I}}_\varepsilon(v_0)<\alpha_0,$$

which contradicts the definition of α_0, and consequently $t_0=t(v_0)$. By (4.8) and (4.13), we have

$$\alpha_0\le\widetilde{\mathcal{I}}_\varepsilon(\tau v_0)=\mathcal{I}_\varepsilon(t(\tau v_0)\tau v_0)=\mathcal{I}_\varepsilon(t(v_0)v_0)=\widetilde{\mathcal{I}}_\varepsilon(v_0)\le\alpha_0,$$

thus $\widetilde{\mathcal{I}}_\varepsilon(v_0)=\alpha_0$. Hence v_0 is a conditional critical point of $\widetilde{\mathcal{I}}_\varepsilon$. Applying Lemma 4.1, we conclude that $u:=t(v_0)v_0$ is a solution of $(P^M_{\lambda,\varepsilon})$. The proof of Theorem 3.1 is finished.

Proof of Theorem 3.2. Suppose that problem $(P^M_{\lambda,\varepsilon})$ has a nontrivial solution u. Then, taking u as a test function,

$$\begin{aligned}
&M\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&+\int_\Omega w(x)|u|^{\overline{p}(x)}dx+\varepsilon\int_\Omega b(x)|u|^{r(x)}dx=\lambda\int_\Omega a(x)|u|^{q(x)}dx \qquad (4.14)
\end{aligned}$$

Since $b(x)>b_0>0$, for a.e. $x\in\Omega$, by Young's inequality, we can write

$$\begin{aligned}
\lambda\int_\Omega a(x)|u|^{q(x)}dx&\le\varepsilon\int_\Omega\frac{q(x)}{r(x)}b(x)|u|^{r(x)}dx+\int_\Omega\frac{r(x)-q(x)}{r(x)}\varepsilon^{\frac{-q(x)}{r(x)-q(x)}}(\lambda a(x))^{\frac{r(x)}{r(x)-q(x)}}b(x)^{\frac{q(x)}{q(x)-r(x)}}\,dx\\
&\le\frac{\varepsilon q^+}{r^-}\int_\Omega b(x)|u|^{r(x)}dx+\frac{r^+-q^-}{r^-}\int_\Omega\varepsilon^{\frac{-q(x)}{r(x)-q(x)}}(\lambda a(x))^{\frac{r(x)}{r(x)-q(x)}}b(x)^{\frac{q(x)}{q(x)-r(x)}}\,dx\\
&\le\frac{\varepsilon q^+}{r^-}\int_\Omega b(x)|u|^{r(x)}dx+\frac{r^+-q^-}{r^-}\varepsilon^{-\kappa}\lambda^\varrho||a||_\infty^\gamma\int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}}\,dx,
\end{aligned}$$

where

$$\kappa:=\begin{cases}\frac{q^+}{r^--q^+} & \text{if}\quad \varepsilon\le1\\ \frac{q^-}{r^+-q^-} & \text{if}\quad \varepsilon>1,\end{cases}\qquad \varrho:=\begin{cases}\frac{r^-}{r^+-q^-} & \text{if}\quad \lambda<1\\ \frac{r^+}{r^--q^+} & \text{if}\quad \lambda\ge1\end{cases}$$

and

$$\gamma:=\begin{cases}\frac{r^-}{r^+-q^-} & \text{if}\quad ||a||_\infty<1\\ \frac{r^+}{r^--q^+} & \text{if}\quad ||a||_\infty\ge1.\end{cases}$$

It holds then from (4.14) that

$$\begin{aligned}
&M\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\le\frac{\varepsilon(q^+-r^-)}{r^-}\int_\Omega b(x)|u|^{r(x)}dx+\frac{r^+-q^-}{r^-}\varepsilon^{-\kappa}\lambda^\varrho||a||_\infty^\gamma\int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}}\,dx\\
&\le\frac{r^+-q^-}{r^-}\varepsilon^{-\kappa}\lambda^\varrho||a||_\infty^\gamma\int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}}\,dx, \qquad (4.15)
\end{aligned}$$

since $q^+<r^-$. On the other hand, by Lemmas 2.2, 2.6 and 2.7, for some $C_0>0$, we have

$$\int_\Omega|u|^{q(x)}dx\le C_0\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right)^\vartheta, \qquad (4.16)$$

where

$$\vartheta := \begin{cases} \frac{q^-}{p^+} & \text{if} \quad ||u||_{q(x)} \leq 1 \text{ and } ||u||_{\mathbb{X}_0} \leq 1 \\ \frac{q^+}{p^+} & \text{if} \quad ||u||_{q(x)} > 1 \text{ and } ||u||_{\mathbb{X}_0} \leq 1 \\ \frac{q^-}{p^-} & \text{if} \quad ||u||_{q(x)} \leq 1 \text{ and } ||u||_{\mathbb{X}_0} > 1 \\ \frac{q^+}{p^-} & \text{if} \quad ||u||_{q(x)} > 1 \text{ and } ||u||_{\mathbb{X}_0} > 1. \end{cases}$$

Note that $\vartheta > 1$, since $p^+ < q^-$. From (M_1), (4.14) and (4.16), we get

$$\begin{aligned} m_0 \left(\frac{1}{C_0||a||_\infty} \int_\Omega a(x)|u|^{q(x)}dx \right)^{\frac{1}{\vartheta}} &\leq M \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy \right) \\ &\quad \times \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy \\ &\leq \lambda \int_\Omega a(x)|u|^{q(x)}dx, \end{aligned} \tag{4.17}$$

which implies

$$\left(\frac{m_0^\vartheta}{\lambda C_0||a||_\infty} \right)^{\frac{1}{\vartheta-1}} \leq m_0 \left(\frac{1}{C_0||a||_\infty} \int_\Omega a(x)|u|^{q(x)}dx \right)^{\frac{1}{\vartheta}}. \tag{4.18}$$

Combining (4.15), (4.17) and (4.18), we obtain

$$\left(\frac{m_0^\vartheta}{\lambda C_0||a||_\infty} \right)^{\frac{1}{\vartheta-1}} \leq \frac{r^+-q^-}{r^-} \varepsilon^{-\kappa} \lambda^{\varrho} ||a||_\infty^{\gamma} \int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}} dx,$$

hence

$$\lambda \geq \lambda_\varepsilon := \left(\frac{r^- \varepsilon^{\kappa} m_0^{\frac{\vartheta}{\vartheta-1}}}{C_0^{\frac{1}{\vartheta-1}} ||a||_\infty^{\frac{\gamma(\vartheta-1)+1}{\vartheta-1}} (r^+ - q^-) \int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}} dx} \right)^{\frac{\vartheta-1}{\varrho(\vartheta-1)+1}},$$

and the proof of Theorem 3.2 is completed.

Proof of Theorem 3.3. Assume $q(\cdot) = q$ is constant. For $v \in \Theta_a$ and $t > 0$, we set

$$\Upsilon_{\varepsilon,v}(t) := \frac{\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)-q}|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy + \int_\Omega t^{\overline{p}(x)-q} w(x)|v|^{\overline{p}(x)}dx + \varepsilon \int_\Omega t^{r(x)-q} b(x)|v|^{r(x)}dx}{\int_\Omega a(x)|v|^q dx},$$

$$F(v) := \frac{\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy + \int_\Omega w(x)|v|^{\overline{p}(x)}dx}{\int_\Omega a(x)|v|^q dx}$$

and

$$H(v) := \frac{\varepsilon \int_\Omega b(x)|v|^{r(x)}dx}{\int_\Omega a(x)|v|^q dx}.$$

Then

$$\begin{cases} t^{p^- - q}F(v) + t^{r^- - q}H(v) \leq \Upsilon_{\varepsilon,v}(t) \leq t^{p^+ - q}F(v) + t^{r^+ - q}H(v) & \text{if} \quad t \geq 1 \\ t^{p^+ - q}F(v) + t^{r^+ - q}H(v) \leq \Upsilon_{\varepsilon,v}(t) \leq t^{p^- - q}F(v) + t^{r^- - q}H(v) & \text{if} \quad t < 1. \end{cases} \tag{4.19}$$

Having in mind that $p^+ < q < r^-$, it follows that

$$\lim_{t\to 0^+} \Upsilon_{\varepsilon,v}(t) = \lim_{t\to +\infty} \Upsilon_{\varepsilon,v}(t) = +\infty. \tag{4.20}$$

On the other hand, it is not difficult to see that the function $\Upsilon_{\varepsilon,v}$ admits a global minimum $t^*(v)$, which is a unique solution of the equation

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{(q-p(x,y))\, t^{p(x,y)} |v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy + \int_\Omega (q-\overline{p}(x))\, t^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)} dx \\ &= \varepsilon \int_\Omega (r(x)-q)\, t^{r(x)} b(x)|v|^{r(x)} dx.\end{aligned} \tag{4.21}$$

By (4.20), for $\lambda > 0$ large enough, there are exactly two positive reals $t_1(v) < t^*(v) < t_2(v)$ such that $\Upsilon_{\varepsilon,v}(t_1(v)) = \Upsilon_{\varepsilon,v}(t_2(v)) = \lambda$. Clearly $t_1(v)$ and $t_2(v)$ satisfy (4.4) with $M \equiv 1$, and $t(v) := t_2(v)$ increases as λ increases or ε decreases. Let

$$\Theta_a^\varepsilon(\lambda) := \{v \in \Theta_a : \lambda > \Upsilon_{\varepsilon,v}(t^*(v))\}.$$

Then, for λ sufficiently large, $\Theta_a^\varepsilon(\lambda) \neq \emptyset$. By (4.21), for $v \in \Theta_a^\varepsilon(\lambda)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{p(x,y) t^*(v)^{p(x,y)} |v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy + \int_\Omega \overline{p}(x) t^*(v)^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)} dx \\ &+\varepsilon \int_\Omega r(x) t^*(v)^{r(x)} b(x)|v|^{r(x)} dx < \lambda q t^*(v)^q \int_\Omega a(x)|v|^q dx,\end{aligned}$$

it holds then

$$t^*(v) < \begin{cases} \left(\frac{\lambda q \int_\Omega a(x)|v|^q dx}{\varepsilon r^- \int_\Omega b(x)|v|^{r(x)} dx}\right)^{\frac{1}{r^- - q}} & \text{if} \quad t^*(v) \geq 1 \\ \left(\frac{\lambda q \int_\Omega a(x)|v|^q dx}{\varepsilon r^- \int_\Omega b(x)|v|^{r(x)} dx}\right)^{\frac{1}{r^+ - q}} & \text{if} \quad t^*(v) < 1. \end{cases} \tag{4.22}$$

Claim 4.3. *If $v \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$, then*

$$1 < \varepsilon \int_\Omega b(x)|v|^{r(x)} dx + \beta \left(\int_\Omega b(x)|v|^{r(x)} dx\right)^\theta.$$

for some $\beta > 0$ and

$$\theta := \begin{cases} \frac{q(r^- - p^-) - r^+(q-p^-)}{r^+(r^- - q)} & \textit{if} \quad ||v||_q < 1 \textit{ and } t^*(v) \geq 1, \\ \frac{p^+}{r^+} & \textit{if} \quad ||v||_q < 1 \textit{ and } t^*(v) < 1, \\ \frac{q(r^+ - p^+) - r^+(q-p^+)}{r^+(r^+ - q)} & \textit{if} \quad ||v||_q \geq 1 \textit{ and } t^*(v) < 1, \\ \frac{q(r^- - p^-) - r^+(q-p^-)}{r^+(r^- - q)} & \textit{if} \quad ||v||_q \geq 1 \textit{ and } t^*(v) \geq 1. \end{cases}$$

We just prove the case $||v||_q < 1$ and $t^*(v) \geq 1$, since others cases can be treated similarly. In fact, we have $\Upsilon_{\varepsilon,v}(t^*(v)) < \lambda$, thus

$$\int_{\mathbb{R}^{2N}} \frac{t^*(v)^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t^*(v)^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)}dx < \lambda t^*(v)^q \int_\Omega a(x)|v|^q dx, \tag{4.23}$$

which yields

$$t^*(v)^{p^-}\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega w(x)|v|^{\overline{p}(x)}dx\right) < \lambda t^*(v)^q \int_\Omega a(x)|v|^q dx. \tag{4.24}$$

Taking into account that $\Psi_\varepsilon(v) = 1$, from (4.3) with $M \equiv 1$ and (4.24), we get

$$1 - \varepsilon \int_\Omega b(x)|v|^{r(x)}dx < \lambda t^*(v)^{q-p^-} \int_\Omega a(x)|v|^q dx$$

and hence in view of (4.22),

$$\left(\varepsilon \int_\Omega b(x)|v|^{r(x)}dx\right)^{\frac{q-p^-}{r^--q}} \left(1 - \varepsilon \int_\Omega b(x)|v|^{r(x)}dx\right) < \left(\frac{q}{r^-}\right)^{\frac{q-p^-}{r^--q}} \left(\lambda \int_\Omega a(x)|v|^q dx\right)^{\frac{r^--p^-}{r^--q}}. \tag{4.25}$$

By Lemmas 2.1, 2.4 and (H_3), we can find $C_1 > 0$ such that

$$\int_\Omega a(x)|v|^q dx \le C_1 \left(\int_\Omega b(x)|v|^{r(x)}dx\right)^{\frac{q}{r^+}}. \tag{4.26}$$

Combining this inequality with (4.25), we deduce

$$1 < \varepsilon \int_\Omega b(x)|v|^{r(x)}dx + \beta \left(\int_\Omega b(x)|v|^{r(x)}dx\right)^{\frac{q(r^--p^-)-r^+(q-p^-)}{r^+(r^--q)}}$$

and the claim follows. Therefore, for some $C_2 > 0$,

$$\int_\Omega b(x)|v|^{r(x)}dx > C_2 \quad \text{for all} \quad v \in \Theta_a^\varepsilon(\lambda).$$

So, according to (4.22) and (4.26), the set $\{t(v) : v \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)\}$ is bounded above. Let v_1 be fixed in $\mathcal{U}_\varepsilon$. Then, $v_1 \in \Theta_a^\varepsilon(\lambda)$ for all $\lambda > \lambda_\varepsilon^1 := \Upsilon_{\varepsilon,v_1}(t^*(v_1))$. From (4.4) with $M \equiv 1$, we have

$$\begin{aligned}\widetilde{\mathcal{I}}_\varepsilon(v_1) \le \left(\frac{1}{p^-} - \frac{1}{r^-}\right) \int_{\mathbb{R}^{2N}} t(v_1)^{p(x,y)} \frac{|v_1(x)-v_1(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ + \left(\frac{1}{p^-} - \frac{1}{r^-}\right) \int_\Omega t(v_1)^{\overline{p}(x)} w(x)|v_1|^{\overline{p}(x)}dx - \lambda \left(\frac{1}{q} - \frac{1}{r^-}\right) \int_\Omega t(v_1)^q a(x)|v_1|^q dx.\end{aligned} \tag{4.27}$$

Recalling that $\lambda \mapsto t_\lambda(v_1) := t(v_1)$ increases as λ increases and $p^- \le p^+ < q < r^-$, we choose $\lambda_\varepsilon^2 > 0$ large enough such that for all $\lambda > \lambda_\varepsilon^2$, $\widetilde{\mathcal{I}}_\varepsilon(v_1) < 0$. Hence, for all $\lambda > \lambda_\varepsilon^* := \max(\lambda_\varepsilon^1, \lambda_\varepsilon^2)$, $\alpha_1 := \inf_{v \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)} \widetilde{\mathcal{I}}_\varepsilon(v) < 0$. Now, we show that the minimum of $\widetilde{\mathcal{I}}_\varepsilon$ is achieved in $\mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$ with

$\lambda > \lambda_\varepsilon^*$. Indeed, let $\{v_n\} \subset \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$ such that $\widetilde{\mathcal{I}}_\varepsilon(v_n) \to \alpha_1$. Since $\{v_n\}$ is bounded in $\mathbb{X}_0$, going to a subsequence if necessary, there exists $v_0 \in \mathbb{X}_0$ satisfying (4.7). As previously argued in the proof of Theorem 3.1, we deduce that $v_0 \neq 0$, $v_0 \in L^{r(\cdot)}(\Omega)$ and $\{t(v_n)\}$ converges to $t_0 = t(v_0) > 0$ with

$$\widetilde{\mathcal{I}}_\varepsilon(v_0) = \mathcal{I}_\varepsilon(t(v_0)v_0) = \mathcal{I}_\varepsilon(t_0 v_0) \leq \alpha_1. \tag{4.28}$$

Since $\{t^*(v_n)\}$ is also bounded, up to a subsequence, $t^*(v_n) \to t_0^*$. By (4.19) and direct computation, we obtain

$$\begin{aligned}
\Upsilon_{\varepsilon,v_0}(t^*(v_n)) &\geq \min_{t>0}\left(t^{p^- - q}F(v_0) + t^{r^- - q}H(v_0)\right)\\
&= \left[\left(\frac{q-p^-}{r^- - q}\right)^{\frac{p^- - q}{r^- - p^-}} + \left(\frac{q-p^-}{r^- - q}\right)^{\frac{r^- - q}{r^- - p^-}}\right] F(v_0)^{\frac{r^- - q}{r^- - p^-}} H(v_0)^{\frac{q-p^-}{r^- - p^-}} \text{ if } t^*(v_n) \geq 1\\
\Upsilon_{\varepsilon,v_0}(t^*(v_n)) &\geq \min_{0<t<1}\left(t^{p^+ - q}F(v_0) + t^{r^+ - q}H(v_0)\right)\\
&= \left[\left(\frac{q-p^+}{r^+ - q}\right)^{\frac{p^+ - q}{r^+ - p^+}} + \left(\frac{q-p^+}{r^+ - q}\right)^{\frac{r^+ - q}{r^+ - p^+}}\right] F(v_0)^{\frac{r^+ - q}{r^+ - p^+}} H(v_0)^{\frac{q-p^+}{r^+ - p^+}} \text{ if } t^*(v_n) < 1.
\end{aligned}$$

Therefore, passing to the limit as $n \to +\infty$, we get $\Upsilon_{\varepsilon,v_0}(t_0^*) > 0$, thus $t_0^* > 0$. On the other hand, by (4.7) and Fatou's lemma, we entail $\lambda \geq \Upsilon_{\varepsilon,v_0}(t_0^*) \geq \Upsilon_{\varepsilon,v_0}(t^*(v_0))$. Suppose by contradiction that $\lambda = \Upsilon_{\varepsilon,v_0}(t^*(v_0))$. We have $\lambda = \Upsilon_{\varepsilon,v_n}(t(v_n))$, thus

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{t(v_n)^{p(x,y)}|v_n(x) - v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t(v_n)^{\overline{p}(x)}w(x)|v_n|^{\overline{p}(x)}dx\\
&+\varepsilon\int_\Omega t(v_n)^{r(x)}b(x)|v_n|^{r(x)}dx = \lambda t(v_n)^q\int_\Omega a(x)|v_n|^q dx,
\end{aligned}$$

and so, by (4.7),

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{t(v_0)^{p(x,y)}|v_0(x) - v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t(v_0)^{\overline{p}(x)}w(x)|v_0|^{\overline{p}(x)}dx\\
&+\varepsilon\int_\Omega t(v_0)^{r(x)}b(x)|v_0|^{r(x)}dx \leq \lambda t(v_0)^q\int_\Omega a(x)|v_0|^q dx,
\end{aligned}$$

which means that $\Upsilon_{\varepsilon,v_0}(t^*(v_0)) = \lambda \geq \Upsilon_{\varepsilon,v_0}(t(v_0))$. Therefore,

$$t^*(v_0) = t(v_0) = t_0. \tag{4.29}$$

From (4.21), we have

$$\begin{aligned}
&\left(q-p^-\right)\left(\int_{\mathbb{R}^{2N}} \frac{t^*(v_0)^{p(x,y)}|v_0(x) - v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t^*(v_0)^{\overline{p}(x)}w(x)|v_0|^{\overline{p}(x)}dx\right)\\
&\geq \left(r^- - q\right)\varepsilon\int_\Omega t^*(v_0)^{r(x)}b(x)|v_0|^{r(x)}dx.
\end{aligned} \tag{4.30}$$

By virtue of (4.4) with $M \equiv 1$, (4.29)-(4.30) and (H_4), we get

$$
\begin{aligned}
\alpha_1 &= \lim_{n\to+\infty} \mathcal{I}_\varepsilon(t(v_n)v_n) \geq \mathcal{I}_\varepsilon(t(v_0)v_0) = \mathcal{I}_\varepsilon(t^*(v_0)v_0) \\
&\geq \frac{\varepsilon}{q}\left(\frac{(q-p^+)(r^- - q)}{p^+(q-p^-)} - \frac{r^+ - q}{r^+}\right)\int_\Omega t^*(v_0)^{r(x)} b(x)|v_0|^{r(x)}dx \\
&= \frac{\varepsilon\left(qp^+(q-p^-) - r^+\left[p^+(r^- - q) - q(r^- - q)\right]\right)}{r^+p^+q(q-p^-)}\int_\Omega t^*(v_0)^{r(x)} b(x)|v_0|^{r(x)}dx \\
&\geq 0,
\end{aligned}
$$

which contradicts $\alpha_1 < 0$. Then $\lambda > \Upsilon_{\varepsilon,v_0}(t^*(v_0))$, and consequently $v_0 \in \Theta_a^\varepsilon(\lambda)$. We choose $\tau > 0$ such that $\tau v_0 \in \mathcal{U}_\varepsilon$. Using the uniqueness of the solution $t^*(v_0)$ of equation (4.21), we infer $\tau t^*(\tau v_0) = t^*(v_0)$. Therefore $\Upsilon_{\varepsilon,\tau v_0}(t^*(\tau v_0)) = \Upsilon_{\varepsilon,v_0}(t^*(v_0)) < \lambda$, thus $\tau v_0 \in \Theta_a^\varepsilon(\lambda)$. Hence $\tau v_0 \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$. It holds from (4.13) and (4.28) that

$$
\alpha_1 \leq \widetilde{\mathcal{I}}_\varepsilon(\tau v_0) = \mathcal{I}_\varepsilon(t(\tau v_0)\tau v_0) = \mathcal{I}_\varepsilon(t(v_0)v_0) = \widetilde{\mathcal{I}}_\varepsilon(v_0) \leq \alpha_1,
$$

thus $\widetilde{\mathcal{I}}_\varepsilon(v_0) = \alpha_1$. Thanks again to Lemma 4.1, we see that $u := t(v_0)v_0$ is a solution of $(P_{\lambda,\varepsilon})$.

Suppose now that $p(x,y) = p$, $r(x) = r$ are constant and $q(x)$ varies. Let

$$
\Gamma_v(t) := A(v) + \varepsilon t^{r-p} B(v) - \lambda \int_\Omega t^{q(x)} a(x)|v|^{q(x)-p}dx,
$$

where

$$
A(v) := \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega w(x)|v|^p dx
$$

and

$$
B(v) := \int_\Omega b(x)|v|^r dx.
$$

Then Γ_v is continuous, $\Gamma_v(0) = A(v) > 0$ and $\Gamma_v(t) \to +\infty$ as $t \to +\infty$, since $p < q(x) < r$ for all $x \in \Omega$. On the other hand, for λ large enough, we have $\inf\limits_{t>0} \Gamma_v(t) < 0$. Therefore, by (H_2), there are exactly two positive reals $t_1(v) < t_2(v)$ such that $\Gamma_v(t_1(v)) = \Gamma_v(t_2(v)) = 0$. So, by using the same arguments as above, we obtain a solution of $(P_{\lambda,\varepsilon})$. The proof of Theorem 3.3 is completed.

Proof of Theorem 3.4. Let $\varepsilon_0 > 0$. In view of Theorem 3.3, for $\lambda > \lambda^*_{\varepsilon_0}$, problem $(P_{\lambda,\varepsilon})$ with $\varepsilon = \varepsilon_0$ admits a solution $u_{\varepsilon_0} = t(v_{\varepsilon_0})v_{\varepsilon_0}$ with $v_{\varepsilon_0} \in \Theta_a^{\varepsilon_0}(\lambda)$. In the case $q(x) = q$, for all $\varepsilon \in (0,\varepsilon_0)$, problem $(P_{\lambda,\varepsilon})$ has a solution $u_\varepsilon = t(v_\varepsilon)v_\varepsilon$. In fact, from (4.19), we have

$$
\begin{aligned}
\Upsilon_{\varepsilon,v}(t^*(v)) \quad &\leq \quad \left[\left(\frac{q-p^+}{r^+-q}\right)^{\frac{p^+-q}{r^+-p^+}} + \left(\frac{q-p^+}{r^+-q}\right)^{\frac{r^+-q}{r^+-p^+}}\right] F(v)^{\frac{r^+-q}{r^+-p^+}} \\
&\quad \times \left(\frac{\int_\Omega b(x)|v|^{r(x)}dx}{\int_\Omega a(x)|v|^q dx}\right)^{\frac{q-p^+}{r^+-p^+}} \varepsilon^{\frac{q-p^+}{r^+-p^+}} \quad \text{if} \quad t^*(v) \geq 1
\end{aligned}
$$

and

$$\Upsilon_{\varepsilon,v}(t^*(v)) \leq \left[\left(\frac{q-p^-}{r^--q}\right)^{\frac{p^--q}{r^--p^-}} + \left(\frac{q-p^-}{r^--q}\right)^{\frac{r^--q}{r^--p^-}}\right] F(v)^{\frac{r^--q}{r^--p^-}}$$
$$\times \left(\frac{\int_\Omega b(x)|v|^{r(x)}dx}{\int_\Omega a(x)|v|^q dx}\right)^{\frac{q-p^-}{r^--p^-}} \varepsilon^{\frac{q-p^-}{r^--p^-}} \quad \text{if} \quad t^*(v) < 1.$$

Since $p^- \leq p^+ < q < r^-$, $\Upsilon_{\varepsilon,v}(t^*(v)) \downarrow 0$ as $\varepsilon \downarrow 0$. Thus $\lambda > \Upsilon_{\varepsilon,v_{\varepsilon_0}}(t^*(v_{\varepsilon_0}))$ for any $\varepsilon \in (0,\varepsilon_0)$. Hence $v_{\varepsilon_0} \in \Theta_a^\varepsilon(\lambda)$. By (4.21), we have

$$\min\left((t_\varepsilon^*(v_{\varepsilon_0}))^{p^-}, (t_\varepsilon^*(v_{\varepsilon_0}))^{p^+}\right) \int_{\mathbb{R}^{2N}} \frac{(q-p(x,y))\,|v_{\varepsilon_0}(x)-v_{\varepsilon_0}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$
$$\leq \varepsilon \max\left((t_\varepsilon^*(v_{\varepsilon_0}))^{r^-}, (t_\varepsilon^*(v_{\varepsilon_0}))^{r^+}\right) \int_\Omega (r(x)-q)\,b(x)|v_{\varepsilon_0}|^{r(x)}dx,$$

which yields

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^{2N}} \frac{(q-p(x,y))\,|v_{\varepsilon_0}(x)-v_{\varepsilon_0}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$
$$\leq \max\left((t_\varepsilon^*(v_{\varepsilon_0}))^{r^+-p^-}, t_\varepsilon^*(v_{\varepsilon_0})^{r^--p^+}\right) \int_\Omega (r(x)-q)\,b(x)|v_{\varepsilon_0}|^{r(x)}dx.$$

It holds that $t_\varepsilon^*(v_{\varepsilon_0}) \to +\infty$ as $\varepsilon \to 0$, since $p^+ < r^-$. Noting that $t_\varepsilon^*(v_{\varepsilon_0}) < t_\varepsilon(v_{\varepsilon_0})$, we deduce that $t_\varepsilon(v_{\varepsilon_0}) \to +\infty$ as $\varepsilon \to 0$. Therefore, in view of (4.27), for some $\varepsilon_1 \in (0,\varepsilon_0)$ small enough, $\widetilde{\mathcal{I}}_\varepsilon(v_{\varepsilon_0}) < 0$ for all $\varepsilon \in (0,\varepsilon_1)$. Let $\tau > 0$ such that $\tau v_{\varepsilon_0} \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$. Since $\widetilde{\mathcal{I}}_\varepsilon(\tau v_{\varepsilon_0}) = \widetilde{\mathcal{I}}_\varepsilon(v_{\varepsilon_0}) < 0$,

$$\inf_{v\in\mathcal{U}_\varepsilon\cap\Theta_a^\varepsilon(\lambda)} \widetilde{\mathcal{I}}_\varepsilon(v) < 0 \quad \text{for all} \quad \varepsilon \in (0,\varepsilon_1).$$

Through a similar reasoning to that of Theorem 3.1, we can show that for any $\varepsilon \in (0,\varepsilon_1)$, problem $(P_{\lambda,\varepsilon})$ has a solution $u_\varepsilon = t_\varepsilon(v_\varepsilon)v_\varepsilon$, with $v_\varepsilon \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$. Moreover, $\mathcal{I}_\varepsilon(u_\varepsilon) = \widetilde{\mathcal{I}}_\varepsilon(v_\varepsilon) \to -\infty$ as $\varepsilon \to 0$. By (4.1) with $M \equiv 1$ and (4.16), we conclude that $||u_\varepsilon||_X \to +\infty$ as $\varepsilon \to 0$. The proof of Theorem 3.4 is completed.

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